Commuting holonomies and rigidity of holomorphic foliations [1]

Hossein Movasati[2], Isao Nakai

Abstract

In this article we study deformations of a holomorphic foliation with a generic non-rational first integral in the complex plane. We consider two vanishing cycles in a regular fiber of the first integral with a non-zero self intersection and with vanishing paths which intersect each other only at their start points. It is proved that if the deformed holonomies of such vanishing cycles commute then the deformed foliation has also a first integral. Our result generalizes a similar result of Ilyashenko on the rigidity of holomorphic foliations with a persistent center singularity. The main tools of the proof are Picard-Lefschetz theory and the theory of iterated integrals for such deformations.

1 Introduction

In a deformation of an integrable foliation one obtains the first Melnikov function as an Abelian integral whose zeros give rise to limit cycles in the deformed foliation, see for instance [9]. In the case where the Abelian integral is identically zero such limit cycles are controlled by higher order Melnikov functions and L. Gavrilov in [3] has shown that such functions can be expressed in terms of iterated integrals and so they satisfy certain Picard-Fuchs equations. In a different context, the second named author and K. Yanai in [11] have used iterated integrals to investigate the existence of relations between formal diffeomorphisms. Basic properties of iterated integrals were established by A. N. Parsin in 1969 and a systematic approach for de Rham cohomology type theorems for iterated integrals was made by K.-T. Chen around 1977. In the present text we use iterated integrals and investigate the non-existence of non-trivial commutator relations between deformed holonomies.

Let us consider a polynomial in two variables $f(x,y) \in \mathbb{C}[x,y], \deg(f) \geq 3$ and perform a perturbation

$$(1) \qquad \mathcal{F}_\epsilon : df + \epsilon\omega, \ \epsilon \in (\mathbb{C},0), \ \deg(\omega) \leq \deg(f) - 1,$$

where $\omega = Pdx + Qdy$ is a polynomial differential form, $(\mathbb{C},0)$ is a small neighborhood of the origin in $\mathbb{C}$ and $\deg(\omega)$ is the maximum of $\deg(P)$ and $\deg(Q)$. We take a path $\delta \in \pi_1(f^{-1}(b), p)$, where b is a regular value of f and p is a point in $f^{-1}(b)$, and ask for the conditions on ω such that the deformed holonomy $h_\epsilon : \Sigma \to \Sigma$, where Σ is a transversal section to $\mathcal{F}_0$ at p, is identity. The first result in this direction is due to Yu. Ilyashenko:

Assume that the last homogeneous piece of f is a product of $\deg(f)$ distinct lines and the critical points of f are non-degenerate with distinct images. These are generic conditions on f. For simplicity assume that the coefficients of f are real numbers and take δ one of the oriented ovals which lies in the level curves of the map $f : \mathbb{R}^2 \to \mathbb{R}$ and call δ a vanishing cycle.

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Theorem.(Ilyashenko, [5]) *If h_ϵ is the identity map for all $\epsilon \in (\mathbb{C}, 0)$ and the homology class of δ is a vanishing cycle then there is a polynomial $g \in \mathbb{C}[x, y]$, $\deg(g) \leq \deg(f)$ such that $\omega = dg$ and so $\mathcal{F}_\epsilon : d(f + \epsilon g) = 0$ is again Hamiltonian.*

A generalization of this result for pencils of type $\frac{F^p}{G^q}$ in $\mathbb{P}^2$ and pencils in arbitrary projective manifolds and logarithmic foliations is done in the articles [9, 10]. The theory of iterated integrals gives us further generalizations of the above theorem for cycles with zero homology classes. For a, b in a group G let $(a, b) := aba^{-1}b^{-1}$ be the commutator of a and b.

Theorem 1. *For a generic polynomial f as before, let us assume that h_ϵ is the identity map for all $\epsilon \in (\mathbb{C}, 0)$, $\delta = (\delta_1, \delta_2)$ and the homology classes of δ_1 and δ_2 vanish along two paths which do not intersect again except at b. Further assume that the homology classes of δ_1 and δ_2 have non-zero intersection. Then ω is an exact form and so $\mathcal{F}_\epsilon$ is again Hamiltonian.*

In §3 we state Theorem 2 for strongly tame functions. Two special cases of Theorem 2 are Theorem 1 and the following: Let M be a projective compact manifold of dimension two and $\mathcal{F}(\omega_0)$ be a holomorphic foliation in M obtained by a generic non-rational Lefschetz pencil (see [6] and §2). Here ω_0 is a global holomorphic section of $\Omega^1_M \otimes L$ such that the zero locus of ω_0 is a finite set in M, where L is a line bundle on M and Ω^1_M is the sheaf of holomorphic 1-forms in M. Let

$$\mathcal{F}_\epsilon = \mathcal{F}(\omega_0 + \epsilon\omega_1), \epsilon \in (\mathbb{C}, 0),\ \omega_1 \in H^0(M, \Omega^1_M \otimes L)$$

be a linear deformation of $\mathcal{F}(\omega_0)$. If δ_1, δ_2 are two vanishing cycles with the same properties as in Theorem 1, $H_1(M, \mathbb{Q}) = 0$ and the holonomies associated to δ_1 and δ_2 commute then $\mathcal{F}_\epsilon$ has a first integral.

A typical example of the situation of Theorem 1 is the following: Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ has two non-degenerate critical points: p_1 a center singularity and p_2 a saddle singularity. Assume that there is no more critical value of f between $f(p_1)$ and $f(p_2)$. The real vanishing cycle around p_1 and the complex vanishing cycle around p_2 satisfy the hypothesis of Theorem 1. For a more explicit example take f to be the product of d degree 1 real polynomials which are in general position and deform it in order to obtain a generic polynomial required by Theorem 1. For a precise description of the generic properties we have posed on f see §2.

The paper is organized as follows: In §2 we define a strongly tame function in an affine variety. In §3 we state and prove Theorem 2 which is a general form of Theorem 1.

2 Deformation of Holomorphic foliations

In this section we consider a smooth affine variety U and a polynomial function f in U and look at it as a morphism $f : U \to \mathbb{C}$ of algebraic varieties. There is a compact projective manifold M and a divisor D in M such that $U = M \backslash D$. There is also a rational morphism $\bar{f} : M \to \mathbb{P}^1$ which coincides with $f : U \to \mathbb{C}$ when restricted to U. For $t \in \mathbb{C}$ we define $U_t := f^{-1}(t)$.

Definition 1. The morphism $f : U \to \mathbb{C}$ is tame if

1. The divisor at infinity D is smooth and connected and further we have $H_1(U, \mathbb{Q}) = 0$;

2. The foliation $\mathcal{F}(df)$ is not rational, i.e. the closure of a generic fiber of f is not isomorphic to $\mathbb{P}^1$.

3. f has non-degenerate singularities $p_i,\ i = 1, 2, 3 \dots, \mu$ with distinct images $c_i := f(p_i)$ (critical values of f);

4. A generic fiber of f is connected and its closure in M intersects D transversally.

One usually calls μ the Milnor number of f and $C := \{c_1, c_2, c_3, \dots, c_\mu\}$ the set of critical values of f. Ehresmann's theorem implies that a tame morphism is topologically trivial over $\mathbb{C} \backslash C$. We have two main examples in mind. The first is a generic Lefschetz pencil (see [6]) in a projective manifold $M \subset \mathbb{P}^m$. The first and second conditions become intrinsic properties of the pair $(M, \mathbb{P}^m)$. For this example, one can take D in such a way that $\bar{f}$ is also topologically trivial over ∞. The second example is mainly used in planar differential equations. Let f be a polynomial in two variables with $\deg(f) = d \geq 3$. We may compactify $\mathbb{C}^2$ inside $\mathbb{P}^2$, and look at $\mathcal{F}(df)$ as a foliation in $\mathbb{P}^2$. For a generic choice of the coefficients, the polynomial f is tame. For instance, to obtain the fourth condition one assumes that $\{[x; y] \in \mathbb{P}^1_\infty \mid f_d(x, y) = 0\}$ has d distinct points, where f_d is the last homogeneous piece of f. In this case $D \cong \mathbb{P}^1$ is not a regular fiber of f. Geometrically seen, d sheets of a regular fiber of f accumulate at D.

We take a distinguished system of paths $\gamma_i,\ i = 1, 2, \dots, \mu$ in $\mathbb{C}$ (see [1]). The path γ_i connects a regular value b of f to c_i and does not intersect other paths except at b. Let $\delta_i \in H_1(U_b, \mathbb{Z})$ be the vanishing cycle along γ_i. One calls $\delta_i,\ i = 1, 2, \dots, \mu$ a distinguished basis of vanishing cycles. The Dynkin diagram of f is a graph whose vertexes are vanishing cycles δ_i. The vertex δ_i is connected to δ_j if and only if $\langle \delta_i, \delta_j \rangle \neq 0$, where

$$\langle \cdot, \cdot \rangle : H_1(U_b, \mathbb{Z}) \times H_1(U_b, \mathbb{Z}) \to \mathbb{Z}$$

is the intersection form in $H_1(U_b, \mathbb{Z})$. The morphism f is called strongly tame if f is tame and its Dynkin diagram is connected. A generic Lefschetz pencil and a generic polynomial in two variables discussed above are strongly tame. For a proof see [6] 7.3.5 and [8] Theorem 2.3.2, 2. The polynomial case has been proved in [5]. It follows also from the following: If a tame polynomial f is obtained by a topologically trivial deformation of a morphism $g : U \to \mathbb{C}$ with only one singularity then the Dynkin diagram of f is connected and so it is strongly tame (see [7, 2, 4]). By a topologically trivial deformation we mean the one in which the topological structure of the smooth fiber does not change.

Proposition 1. *If f is a strongly tame morphism then*

1. *A distinguished basis of vanishing cycles generate $H_1(U_b, \mathbb{Q})$;*

2. *For a cycle $\delta \in H_1(U_b, \mathbb{Q})$ such that $H_1(U_b, \mathbb{Q}) \to \mathbb{Q},\ \delta' \mapsto \langle \delta, \delta' \rangle$ is not the zero map, the action of the monodromy on δ generates $H_1(U_b, \mathbb{Q})$. In particular, this is true for vanishing cycles.*

Proof. The first part can be proved by a slight modifications of the arguments of [6], §5. For a precise proof see [8] Theorem 2.2.1. The second part follows from the first part, the connectivity of the Dynkin diagram and Picard-Lefschetz formula. □

Let Ω^i_U be the set of meromorphic differential i-forms in M with poles along D. A peculiar property of a tame polynomial is that if $\int_{\delta_t} \omega = 0$ for a continuous family of vanishing cycles δ_t and $\omega \in \Omega^1_U$ then ω is relatively exact, i.e. $\int_\delta \omega = 0, \ \forall \delta \in H_1(U_t, \mathbb{Z}), t \in \mathbb{C} \backslash C$ or equivalently ω is of the form $dP + Qdf$ for some $P, Q \in \Omega^0_U$ (see [9] Theorem 5.1).

The Brieskorn module

$$H = \frac{\Omega^1_U}{df \wedge \Omega^0_U + d\Omega^0_U},$$

is a $\mathbb{C}[t]$-module in a canonical way: $t[\omega] := [f\omega]$. The Gauss-Manin connection $\nabla_{\frac{\partial}{\partial t}} = \frac{d}{df}$ on H takes the form

$$\frac{d}{df} : H \to H_C, \ \omega \mapsto \omega' := \frac{d\omega}{df},$$

where H_C is the localization of H on the multiplicative group generated by $t - c_i, \ i = 1, 2, \ldots, \mu$ (see [10]).

Let $\mathcal{F} = \mathcal{F}(df)$ be the foliation in U with the first integral f. We consider the holomorphic foliation

$$\mathcal{F}_\epsilon : df + \epsilon\omega, \ \epsilon \in (\mathbb{C}, 0), \ \omega \in \Omega^1_U. \tag{2}$$

Let b be a regular point of f, $p \in U_b$ and Σ be a transversal section at p to $\mathcal{F}(df)$ parameterized by the image t of f. Let also $\delta \in G := \pi_1(U_b, p)$ and $h_\epsilon(t) : \Sigma \to \Sigma$ be the holonomy of $\mathcal{F}_\epsilon$ along the path δ. We write the Taylor expansion of $h_\epsilon(t)$ in ϵ:

$$h_\epsilon(t) - t = M_1(t)\epsilon + M_2(t)\epsilon^2 + \cdots + M_i(t)\epsilon^i + \cdots, \ M_i(t) := \frac{1}{i!}\frac{\partial^i h_\epsilon}{\partial \epsilon^i} |_{\epsilon=0} .$$

M_i is called the i-th Melnikov function of the deformation along the path δ. For $\delta_1, \delta_2 \in G$ we denote by $(\delta_1, \delta_2) = \delta_1\delta_2\delta_1^{-1}\delta_2^{-1}$ the commutator of δ_1 and δ_2 and for two sets $A, B \subset G$ by (A, B) we mean the group generated by $(a, b), \ a \in A, b \in B$. Let

$$G_r := (G_{r-1}, G), \ r = 1, 2, 3, \ldots, \ G_1 := G.$$

Using the methods introduced in [3] it can be proved that if $\delta \in G_k$ then $M_1 = M_2 = \cdots = M_{k-1} = 0$ and

$$M_k(t) = \int_{\delta_t} \underbrace{\omega_1(\omega_1(\cdots(\omega_1(}_{k-1 \text{ times } (}\omega_1)'\cdots)')'. \tag{3}$$

In particular, for $k = 2$ and $\delta = (\delta_1, \delta_2)$ we have

$$M_2(t) = \int_{(\delta_1, \delta_2)} \omega\omega' = \det \begin{pmatrix} \int_{\delta_1} \omega & \int_{\delta_2} \omega \\ \int_{\delta_1} \omega' & \int_{\delta_2} \omega' \end{pmatrix}. \tag{4}$$

3 Main theorem

For $\omega \in \Omega^1_U$ define $\deg(\omega)$ to be the pole order of ω along D, where D is the compactification divisor of U as it is explained in §2.

Theorem 2. *In the deformation (2) with* $\deg(\omega) \leq \deg(df)$ *assume that f is a strongly tame polynomial. Consider $\delta_1, \delta_2 \in \pi_1(U_b, p)$ such that the corresponding cycles in $H_1(U_b, \mathbb{Z})$ vanishes along the paths which do not intersect each other except at b. Also assume that*

$$\forall \delta \in H_1(U_b, \mathbb{Z}) \tag{5}$$

$$\langle \delta, \delta_1 \rangle = 0 \ \textit{or} \ \langle \delta, \delta_2 \rangle = 0 \ \textit{or} \ \exists \delta' \in H_1(U_b, \mathbb{Z}) \ \textit{s.t.} \ \langle \delta_1, \delta \rangle\langle \delta_2, \delta' \rangle - \langle \delta_2, \delta \rangle\langle \delta_1, \delta' \rangle \neq 0.$$

If the deformed monodromies along δ_1 and δ_2 commute then $\mathcal{F}_\epsilon$ has a first integral.

Note that if $\langle \delta_1, \delta_2\rangle \neq 0$ then the condition in (5) is fulfilled. Theorem 1 is therefore a special case of Theorem 2.

Lemma 1. *Consider a strongly tame morphism f, a differential 1-form $\omega \in \Omega^1_U$ and a family of vanishing cycles $\delta = \delta_t$ such that $P(t) := \int_\delta \omega$ has the following property: At each $c \in C$, P can be written locally in the form $P(t) = (t-c)^\alpha \cdot p(t)$ for some $\alpha \in \mathbb{C}$ and a single-valued holomorphic function p in $(\mathbb{C}, c)\backslash\{c\}$. Then ω is a relatively exact 1-form and so $\int_\delta \omega$ is identically zero.*

Proof. Take a vanishing cycle δ' with the corresponding critical value c' of f and the vanishing path γ'. If $\langle \delta, \delta'\rangle \neq 0$ then by the Picard-Lefschetz formula along the path γ' and for the cycle δ we have:

$$\int_{\delta'} \omega = c_{\delta'} P(t), \tag{6}$$

where $c_{\delta'}$ is some constant depending on δ'. Since the Dynkin diagram of f is connected, the equality in (6) holds for all vanishing cycles δ'. Let δ be a vanishing cycle along the path γ in the critical value c. Since $\langle \delta, \delta\rangle = 0$, the value of the integral $\int_\delta \omega$ after the monodromy along the path γ and around c does not change and so the corresponding α must be integer.

We conclude that $\int_\delta \omega$ for any vanishing cycle δ, is a single-valued function in $\mathbb{C}\backslash C$. Using the Picard-Lefschetz formula for two vanishing cycles δ_i, δ_j with non zero intersection number, we conclude that ω is a relatively exact 1-form. □

Proof of Theorem 2: The first Melnikov function associated to the path (δ_1, δ_2) is trivially zero. By (4) we have:

$$\frac{\int_{\delta_1} \omega'}{\int_{\delta_1} \omega} = \frac{\int_{\delta_2} \omega'}{\int_{\delta_2} \omega}. \tag{7}$$

If for a continuous family of vanishing cycles δ, we have $\int_\delta \omega = 0$ then by Proposition 1 the 1-form ω is relatively exact and so $\omega = Pdf + dQ$ for two meromorphic function in M with poles along D. The hypothesis $\deg(\omega) \leq \deg(df)$ implies that $P = 0$ and so $\omega = dQ$.

Therefore, we can assume that that $\int_{\delta_i} \omega,\ i = 1, 2$ are not identically zero. Then the multi-valued function (7) is well-defined. We denote it by P and claim that P is a rational function. Since integrals have finite growth at critical points and at infinity, it is enough to prove that P is single-valued. By the hypothesis on the vanishing paths $\gamma_i,\ i = 1, 2$ of δ_i, we can put γ_i inside a distinguished system of paths Γ. Let $c \in C$ and δ be the corresponding vanishing cycle along the path $\gamma \in \Gamma$. By the Picard-Lefschetz formula along the path γ we have:

$$\frac{\int_{\delta_1} \omega' + r_1 \int_\delta \omega'}{\int_{\delta_1} \omega + r_1 \int_\delta \omega} = \frac{\int_{\delta_2} \omega' + r_2 \int_\delta \omega'}{\int_{\delta_2} \omega + r_2 \int_\delta \omega},$$

where $r_i := \langle \delta_i, \delta\rangle,\ i = 1, 2$. This and (7) imply together that either $P(t) = \frac{\int_\delta \omega'}{\int_\delta \omega}$ or $\int_{r_1\delta_2 - r_2\delta_1} \omega = 0$. If one of r_i's is zero then P is single-valued along γ. If both are non-zero then the second case cannot happen because of Proposition 1 and the hypothesis in (5). In the first case we conclude that P is again single-valued in a neighborhood of γ.

We have proved that P is a rational function. Now $\ln(\int_{\delta_1} \omega)' = P(t)$ and so

$$\int_{\delta} \omega = e^{\int P(t)dt} = Q \prod_{c \in K} (t-c)^{\alpha_c}, \ Q \in \mathbb{C}(t), \ \alpha_c \in \mathbb{C}$$

where K is a finite subset of $\mathbb{C}$. Lemma 1 finishes the proof. $\square$

Concerning the comments after Theorem 1 note that for a hyperplane section D of M we have the long exact sequence

$$\cdots \to H_2(M, \mathbb{Q}) \xrightarrow{s_1} H_2(M, U, \mathbb{Q}) \to H_1(U, \mathbb{Q}) \to H_1(M, \mathbb{Q}) \to \cdots$$

and the Leray-Thom-Gysin isomorphism $s_2 : H_2(M, U, \mathbb{Q}) \to H_0(D, \mathbb{Q}) \cong \mathbb{Q}$. The composition $s_2 \circ s_1$ is the intersection with D and so s_1 is not the zero map. This implies that if $H_1(M, \mathbb{Q})$ is zero then $H_1(U, \mathbb{Q}) = 0$.

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Hossein Movasati
IMPA-Estrada D. Castorina, 110
Jardim Botânico
Rio de Janeiro - RJ
CEP. 22460-320
Brazil
hossein@impa.br

Isao Nakai
Ochanomizu University
Department of Mathematics
2-1-1 Otsuka, Bunkyo-ku
Tokyo 112-8610
Japan
nakai@math.ocha.ac.jp