My journey in mathematics: multiple integrals

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1 Introduction

In the present text I want to tell the story of my own professional career in Mathematics and also a summary of my results. The reader of my articles may be bothered at first glance with the fact that they range from Differential Equations to Hodge Theory and from Topology to Analytic Number Theory. I would like to explain in this text that the unifying fact is the notion of multiple integral, also called period or Abelian integral, that is present in almost all of my articles. I have explained my mathematical publication in a chronological order and not as a mathematical classification. Explaining an article at a time does mean that the basic idea of that article took place in my mind at that time, however, the final publication may have occured many years later because of filling details, writing labor, rejection and publication delay. I hope that for a person who wants to have a flavor of the mathematics which I have done and I am doing, this text will be a nice and fast substitute.

After studying algebraic numbers, one naturally starts to study transcendental numbers and among them the numbers obtained by integration. Of particular interest is the case in which the integrand is a differential form obtained by algebraic operations and the integration takes place over a topological cycle of an algebraic affine variety. The first examples are elliptic integrals. Since the 19th century, many people have worked on the theory of elliptic integrals, including Gauss, Abel, Bernoulli, Ramanujan and many others, and still it is an active area due to its application in differential equations and arithmetic of elliptic curves. Going to higher genus, one has the theory of Jacobian and Abelian varieties and in higher dimension one has Hodge theory. The later recently plays a fundamental role in Mathematical Physics, and in particular mirror symmetry. My main research in mathematics turns around such integrals and related topics. From now on I will use the term multiple integral, period or Abelian integral.

2 October 1993-October 1997 (Undergraduate)

This period formed the basis for my career in mathematics. In the beginning I thought that the study of logic is a must in order to have a good overview on the whole of mathematics. As a consequence, logic and model theory were my favorites in mathematics, besides the university courses that I also enjoyed. I started to understand Gödel's proof of the consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory, Cohen's proof of independency of the mentioned axioms and Gödel's incompleteness theorems, all of which I enjoyed a lot. But I learned an important lesson in the initial steps of my mathematical career: If I would not know what is happening in other parts of mathematics, for instance in algebraic geometry, or say differential equations, then I would never obtain a holistic view. The last interesting thing which I learned in this period, in a talk given be Ali Enayat at IPM Tehran, was the construction of the field of complex numbers by an ultra-product of the algebraic closure of finite fields and an amazing proof of a theorem in complex analysis through this construction. At the same time I was doing a course in complex dynamics by Prof. Siavash Shahshahani at Sharif University. All these gave me the inspiration to pursue professional mathematics in complex holomorphic foliations.

3 January 1998-December 2000 (Ph.D.)

I started my professional mathematical career on complex dynamical systems at IMPA Rio de Janeiro. One can divide this area into two main parts: (1) The study of holomorphic foliations in a complex manifold, in particular $\mathbb{P}^2$ (the complex projective space of dimension two) and (2) the iteration of a holomorphic function on a complex manifold. The concept of holonomy along a real one-dimensional cycle in a leaf of a foliation gives a first connection between these two areas. The former was the main interest of study in the Ph.D. program. Before starting to work on my thesis, I learned about singular holomorphic foliations on complex manifolds, dynamics of iteration of functions and hyperbolic dynamics.

The Hilbert's 16-th problem expects that the number of limit cycles of a real polynomial equation

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases} \tag{1}$$

where P and Q are two polynomials of degree $\leq d$, is uniformly bounded by a number depending on d. The least of such numbers is called the Hilbert number and it is an open question whether the Hilbert number is finite or not. One of the useful methods in studying such a differential equation is to complexify and compactify it in $\mathbb{P}^2$ (holomorphic foliations). There are many works that attempt to get lower bounds for the Hilbert number. What is fundamental to many of these approaches is the following: To consider the deformation of a Hamiltonian equation and to use a certain type of Abelian integral; the zeros of the Abelian integral represents limit cycles (not all of them) being born from family of cycles (underlying cycles of integration) after a deformation. Let us explain this by a simple example. Take the polynomial $f = y^2 - x^3 + 3x$ in two variables x and y and the family of elliptic curves $E_t : \{f - t = 0\},\ t \in \mathbb{R}$. Only for $t = -2, 2$ the curve E_t is singular. For t a real number between 2 and -2 the level surface of f in $\mathbb{R}^2$ contains an oval δ_t. The polynomial f is a first integral of the differential equation

$$\mathcal{F}_0 : \begin{cases} \dot{x} = 2y \\ \dot{y} = 3x^2 - 3 \end{cases} . \tag{2}$$

We make a perturbation of $\mathcal{F}_0$

$$\mathcal{F}_\epsilon : \begin{cases} \dot{x} = 2y + \epsilon\frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon s y \end{cases} , \epsilon \in (\mathbb{R}, 0), s \in \mathbb{R}. \tag{3}$$

If $\int_{\delta_0} (\frac{x^2}{2} dy - sydx) = 0$ or equivalently

$$s := \frac{-\int_{\Delta_0} xdx \wedge dy}{\int_{\Delta_0} dx \wedge dy} = \frac{5}{7}\frac{\Gamma(\frac{5}{12})\Gamma(\frac{13}{12})}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})} \sim 0.9025,$$

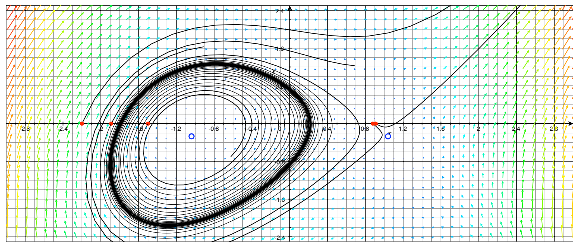

Figure 1: A limit cycle crossing $(x, y) \sim (-1.79, 0)$

where Δ_0 is the bounded open set in $\mathbb{R}^2$ with the boundary δ_0, then for ϵ near to 0, $\mathcal{F}_\epsilon$ has a limit cycle near δ_0. In fact for $\epsilon = 1$ and $s = 0.9$ such a limit cycle still exists and it is depicted in Figure (1).

We may ask for deformation of pencils in $\mathbb{P}^2$. The first natural candidates are generic Lefschetz pencils and pencils of the type $\frac{F^p}{G^q} = t, t \in \mathbb{P}^1$, where $\frac{deg(F)}{deg(G)} = \frac{q}{p}$, $g.c.d.(p, q) = 1$. These cases are treated in my doctoral thesis ([10, 9]). The Picard-Lefschetz theory of rational functions and the classification of relatively exact forms with respect to a rational function are two basic subjects which appear in my thesis. The result of my thesis can be interpreted in a complete Algebraic Geometry context. The space of holomorphic foliations in $\mathbb{P}^2$, of projective degree d and with a Morse type (isolated with Milnor number one) singularity is an algebraic set and I proved that the set of foliations with a first integral of type $\frac{F^p}{G^q}$ forms an irreducible component of it. In order to prove this, I had to prove that for a generic pencil of the above type, any two vanishing cycles are mapped to each other (up to sign) by a monodromy map, see [10]. This result in the case $p = q = 1$ and an arbitrary projective variety is equivalent to hard Lefschetz theorem (see K. Lamotke 1981).

4 January 2001-September 2001

In this period I tried to find the common features of Abelian integrals in holomorphic foliations (differential equations) and those in algebraic geometry. In this direction I started to learn complex algebraic geometry based on some works of H. Grauert and P. Griffiths.

Abelian integrals in holomorphic foliations and the integrals of meromorphic 1-forms in $\mathbb{P}^2$ over cycles lying in the fibers of a pencil in $\mathbb{P}^2$ are important tools in the local study of Hilbert's 16-th problem. Restricting the meromorphic 1-forms to fibers, we see that these integrals are of the third type, i.e. the 1-forms have poles with nonzero residues. In [11] I work with an arbitrary two dimensional complex manifold (instead of $\mathbb{P}^2$) and with an arbitrary pencil. In this article I have studied Abelian integrals, Melninkov functions, relatively exact forms, Picard-Lefschetz theory and so on, in an algebro-geometric context.

During this period I understood the importance of Global Brieskorn modules in the study of the Picard-Lefschetz theory of rational functions and I started to generalize Brieskorn's results in 1970 for a Lefschetz pencil on a projective manifold. The article [15] was the result of this generalization. Later I completed this work at the Max-Planck Institute in Bonn. This was a period in which I was interested in complex algebraic geometry due to its own problems than to its immediate applications to differential equations and holomorphic foliations.

During this period I got acquainted with Grauert's article in 1962 and this article became one of the foundation stones for my knowledge of complex algebraic geometry. In the work [2] based on Grauert's article jointly C. Camacho and P. Sad, I studied the germ of holomorphic foliations around a negatively embedded variety. Roughly speaking, Grauert proves that the classification of negative embeddings leads to finite dimensional spaces. In [2] we investigate the same problem for the germs of holomorphic foliations around negatively embedded varieties. In particular, we prove that if the self-intersection number of a curve is negative and smaller than $2-2g$, where g is the genus of the curve, then the germ of any transverse foliation is equivalent to the linear one that exists in the normal bundle. Later in 2003 with my coauthor Prof. Camacho, I investigated further about negatively embedded varieties (see [1]). This text is mainly expository and it contains many developments on negatively embedded varieties after Grauert's work. This piece of mathematics seems to be different from my main interest. However, many years later in 2010 (see [24]), I found it in a good connection with multiple integrals. Manipulations with integrals on Calabi-Yau three folds give us a generating function which conjecturally counts the number of rational curves in a generic quintic three fold. It is conjectured that these curves have normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$.

5 October 2001-September 2002 (MPIM Bonn)

During my work on Abelian integrals I understood the importance of global Brieskorn modules. The first version of [15] came out of a temptation to generalize Brieskorn's arguments to a generic Lefschetz pencil on a projective variety. When I was writing it I was not acquainted with the language of meromorphic connections on holomorphic sheaves. At the Max-Planck Institute I learned this language by studying Hertling's book in 2002 and its references (especially some articles of B. Malgrange) and in this way I succeeded to rewrite that article with up-to-date notations. The main result in that article is as follows:

Let M be a complex projective manifold of dimension $n+1$ and f a meromorphic function on M obtained by a generic pencil of hyperplane sections $\{M_t\}_{t\in\mathbb{P}^1}$ of M. Here by the word generic I mean that the axis of the pencil intersects M transversally and f has only isolated singularities in its domain of definition. The n-th cohomology vector bundle of $f_0=f|_{M-\mathcal{R}}$, where $\mathcal{R}$ is the set of indeterminacy points of f, is defined on the set of regular values of f_0 and I have the usual Gauss-Manin connection on it. In [15] following Brieskorn's methods, I extend the n-th cohomology vector bundle of f_0 and the associated Gauss-Manin connection to $\mathbb{P}^1$ by means of differential forms. I call it the Brieskorn type extension. The new connection turns out to be meromorphic on the critical values of f_0. Denote by $c_1,c_2,\ldots,c_r$ the critical values of f_0 and suppose that $\infty\in\mathbb{P}^1$ is not one of them or equivalently that $D:=M_\infty$ is a smooth hyperplane section. In modern terminology I obtain a meromorphic connection $\nabla: V\to\Omega^1_{\mathbb{P}^1}(\sum_{i=1}^r k_ic_i)\otimes V$, where the k_i's are positive numbers, $\Omega^1_{\mathbb{P}^1}(\sum_{i=1}^r k_ic_i)$ is the sheaf of meromorphic 1-forms in $\mathbb{P}^1$ with poles of order less than or equal to k_i at c_i and V is the extended vector bundle. The number k_i depends on the type of singularities of f within M_{c_i}, for instance it is one if all the singularities within M_{c_i} are nondegenerate. I prove in a natural way that the meromorphic global sections of the vector bundle V with poles of arbitrary order at $\infty\in\mathbb{P}^1$ is isomorphic to the Brieskorn module of f:

$$H'=\frac{\Omega^n(*D)}{df\wedge\Omega^{n-1}(*D)+d\Omega^{n-1}(*D)}$$

in a natural way, where $\Omega^i(*D)$ is the set of meromorphic i-forms in M with poles of arbitrary order along D. Therefore using Grothendieck's decomposition theorem for V I conclude that the Brieskorn module in this case is a free $\mathbb{C}[t]$-module of rank β_n, where $\mathbb{C}[t]$ is the ring of polynomials in t and β_n is the dimension of n-th cohomology group of a regular fiber of f_0.

In [14] I look at the multiplicity of Abelian integrals in differential equations. The main result in [15] enables us to look at this problem in the context of meromorphic connections on vector bundles on $\mathbb{P}^1$. Let V be a vector bundle on $\mathbb{P}^1$ and $\nabla : V \to \Omega^1_{\mathbb{P}^1}(\sum_{i=1}^r k_i c_i) \otimes V$ be a meromorphic connection on V. The dual vector space V^* is equipped with the dual connection. Now let δ be a constant section of V^* in $U \subset \mathbb{P}^1 - \{c_1, \dots, c_r\}$ and ω be a global meromorphic section of V with a pole of order less than or equal n at $\infty \in \mathbb{P}^1$. We ask for the maximum multiplicity of $\delta(\omega)$ at a fixed point of $b \in U$ with a fixed δ and varying ω. $\delta(\omega)$ like any other Abelian integral satisfies a Picard-Fuchs differential equation. An upper bound for this number is given in [14] but it is far from being realistic.

I consider [12] as my main result at MPIM. I wrote this text in a language which is more appropriate for the context of limit cycles and so it was well accepted among specialists, at least better than my Ph.D. results. There I study a problem in differential equations called center conditions. Let $\mathcal{F}(d)$ be the projectivization of the space of polynomial 1-forms $Pdx + Qdy$ in $\mathbb{C}^2$ with $deg(P), deg(Q) \leq d$. An $\omega \in \mathcal{F}(d)$ has a center singularity at $p \in \mathbb{C}^2$ if (1) $P(p) = Q(p) = 0$ 2. $D(P,Q)(p) \neq 0$ and (3) in a coordinate system (x, y) around p we have $d(x^2 + y^2) \wedge \omega = 0$. It is not difficult to see that the space of 1-forms in $\mathcal{F}(d)$ with at least one center singularity, namely $\mathcal{M}(d)$, is an algebraic subset of $\mathcal{F}(d)$. Now the problem of identifying the irreducible components of $\mathcal{M}(d)$ arises. Yu. Ilyashenko in 1969 identifies an irreducible component of $\mathcal{M}(d)$ whose points are of the form $df = f_x dx + f_y dy$, where f is a polynomial of degree less than or equal to $d+1$ in $\mathbb{C}^2$. Let $d+1 = \sum_{i=1}^s d_i$. In [12] I identify an irreducible component of $\mathcal{M}(d)$ whose points are of the form $f_1 f_2 \cdots f_s \sum_{i=1}^s \lambda_i \frac{df_i}{f_i}$, where f_i is a polynomial of degree less than or equal d_i. An interesting fact about this result is that we can state it for an arbitrary algebraically closed field instead of $\mathbb{C}$. Now one may be curious to know whether this result would be true in this generality or not.

One of the basic tools in the proof of the above result is the notion of global Gauss-Manin connection. Let f be a polynomial in $\mathbb{C}^2$ and $H = \Omega^1/(df \wedge \Omega^0 + d\Omega^0)$ be the associated Brieskorn module (=Petrov module in differential equations). H is a $\mathbb{C}[t]$-module and we can consider the localization of H, namely $\tilde{H}$, by polynomials vanishing only on the critical values of f. Now the global Gauss-Manin connection ∇ is an operator from H to $\tilde{H}$. If we restrict H to the regular fibers then we obtain the usual Gauss-Manin connection. By Leibniz rule one can define $\nabla : \tilde{H} \to \tilde{H}$ and may be interested in studying the action of ∇. For instance in [12] we need to identify the kernel of $\nabla^n (= \nabla \circ \cdots \circ \nabla$ n-times) for an arbitrary n. These kind of problems with respect to ∇ are closely related to the the type of critical fibers of f. Another basic tool in the proof of the above result is Gusein-Zade/A'Campo theorem on calculating the Dynkin diagram of polynomials.

6 October 2002-June 2003 (Technische Universität Clausthal)

In this period I started learning literature on dynamical zeta functions and modular forms. The main production in this period was [7]. This article is concerned about a multidimensional version of the Lewis equation

$$\phi(z) = \phi(z+1) + \lambda z^{-2s} \phi\left(1 + \frac{1}{z}\right) \tag{4}$$

with $\lambda = \pm 1$. The holomorphic solutions of the above equation in the cut plane $\mathbb{C} - (-\infty, 0]$ are called periods, in similarity with the periods of modular forms in number theory. We prove that Hecke type operators act on the solution space of such an equation and we give an explicit formula for such actions. A kind of Atkin-Lehner theory is also developed.

I have to confess that because of a job problem I started to work on the theory of modular forms which was apparently distant from my main interest. But many years later, modular forms became one of my main interest in mathematics, and surprisingly I found them in the heart of my interest on multiple integrals. In this period I made a two-month visit to IMPA-Rio de Janeiro and IMCA-Lima and the monograph [1] was also written.

7 July 2003-September 2004 (Universität Göttingen)

In this period I dedicated my time to Hodge theory. In 2001 when I started to learn Hodge theory I had always in my mind how to describe a Hodge cycle by means of equations involving integrals. This is because I understood that the origins of Hodge theory comes from the study of double and multiple integrals due to Poincaré and Picard. My first attempt in this direction led to [17]. All Hodge cycles of a smooth hypersurface live essentially in its affine part and each cohomology class in an affine variety is given by algebraic differential forms. After some work for determining algebraic differential forms without residue at infinity and of certain (p,q)-type in the original variety, which is the main result of [17] and which is formulated in terms of mixed Hodge structures, we can write the property of being a Hodge cycle in terms of multiple integrals. For instance, for

$$f := x_1^3 + x_2^3 + \cdots + x_5^3 - x_1 - x_2 \tag{5}$$

and for all $c \in \mathbb{C} - \{\pm\frac{4}{3\sqrt{3}}, 0\}$ a cycle $\delta \in H_4(f^{-1}(c), \mathbb{Z})$ is Hodge if and only if

$$(972c^2 - 192)\int_\delta x_1x_2\eta + (-405c^3 - 48c)\int_\delta x_2\eta + (-405c^3 - 48c)\int_\delta x_1\eta + (243c^4 - 36c^2 + 64)\int_\delta \eta = 0 \tag{6}$$

In [17] I started also to develop algorithms for simplifying multiple integrals and calculating their derivations with respects to parameters. In a modern language this is equivalent to find an explicit basis for algebraic de Rham cohomology of varieties and calculating the Gauss-Manin connection. I wrote the paper [16] and during the next years I wrote the monograph [22].

A combination of Atiyah-Hodge theorem and Kodaira vanishing theorem implies that the de Rham cohomologies of an affine variety are finite dimensional and they are given by polynomial differential forms. This implies that every Abelian integral can be written in terms of simpler ones. Let us explain this by a simple example. Take $f = y^2 - x^3 + 3x$ and $E_t : \{f = t\}$. The arithmetic algebraic geometers usually use the differential forms $\frac{dx}{y}$, $\frac{xdx}{y}$, which restricted to the regular fibers of f are holomorphic and form a basis of the corresponding de Rham cohomology. The relation of these differential forms and those in the previous paragraphs is given by:

$$\int_{\delta_t} (\frac{x^2}{2}dy - sydx) = (-\frac{3}{5}st + \frac{6}{7})\int_{\delta_t}\frac{dx}{y} + (\frac{6}{5}s - \frac{3}{7}t)\int_{\delta_t}\frac{xdx}{y} \tag{7}$$

The Abelian integral $\int_{\delta_t}\frac{dx}{y}$ (resp. $\int_{\delta_t}\frac{xdx}{y}$) satisfies the differential equation

$$(8)\qquad \frac{5}{36}I + 2tI' + (t^2-4)I'' = 0 \qquad (\text{ resp. } \frac{-7}{36}I + 2tI' + (t^2-4)I'' = 0)$$

which is called a Picard-Fuchs equation. One also says that such a linear differential equation comes from geometry. If we choose another cycle $\delta'_t \in H_1(E_t,\mathbb{Z})$ which together with δ_t form a basis of $H_1(E_t,\mathbb{Z})$ then the matrix $Y = \begin{pmatrix} \int_{\delta_t}\frac{dx}{y} & \int_{\delta'_t}\frac{dx}{y} \\ \int_{\delta_t}\frac{xdx}{y} & \int_{\delta'_t}\frac{xdx}{y} \end{pmatrix}$ form a fundamental system of the linear differential equation:

$$(9)\qquad Y' = \frac{1}{t^2-4}\begin{pmatrix} \frac{-1}{6}t & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6}t \end{pmatrix} Y$$

which is essentially the Gauss-Manin connection of the family $E_t,\ t \in \mathbb{C}$. The main point behind the calculation of Picard-Fuchs equations and Gauss-Manin connections is the techniques of derivation of an integral with respect to a parameter and simplifying the result in a similar way as in (7). All these calculations for tame polynomials are explained in [17, 16, 22]. In this period also I started to write the library [13] in SINGULAR which does these kind of claculations. It reached to a complete form during the next five years.

I have always tried to keep my contact with the people who works in limit cycles of differential equations. During this period I visited L. Gavrilov in Toulouse. The results of this visit was [5]. In this article we study the analogue of the infinitesimal 16th Hilbert problem in dimension zero. Lower and upper bounds for the number of the zeros of the corresponding Abelian integrals (which are algebraic functions) are found. We study the relation between the vanishing of an Abelian integral defined over $\mathbb{Q}$ and its arithmetic properties. We also give necessary and sufficient conditions for an Abelian integral to be identically zero. Similar problems in the case of one dimensional Abelian integrals are mainly open.

One of the ambitious projects that I started during my stay at Göttingen was to look for a problem of Griffiths on automorphic functions for the moduli of polarized Hodge structures. Two years later I gave a mini course which contained essentially a summary of what I studied for this project and also some of my ideas. Later, I published the lecture notes of this mini course in [19]. In the forthcoming years I had to reformulate this problem many times.

8 October 2004-September 2005 (Technische Universität Darmstadt)

The most important event of this period for me was to meet Stefan Reiter with whom I enjoyed Abelian integrals from another point of view. Below I explain the origins of my paper with him [28] which was worked out during this period.

The reader may transfer the singularities $-2, 2$ of (8) to 0 and 1 and obtain a recursive formula for the coefficients of the Taylor series around 0 of its solutions. Since the integrals $\int_{\delta_t}\frac{dx}{y}$ and $\int_{\delta_t}\frac{xdx}{y}$ are holomorphic around $t=-2$ doing in this way we get:

$$\int_{\delta_t}\frac{dx}{y} = \frac{-2\pi}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{t+2}{4}),\ \int_{\delta_t}\frac{xdx}{y} = \frac{2\pi}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{t+2}{4})$$

where

$$F(a,b,c|z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,\ c\not\in\{0,-1,-2,-3,\ldots\},$$

is the (Gauss) hypergeometric function and $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$. Therefore, Abelian integrals give us a rich class of special functions. On of the main problems in the theory of transcendental numbers is to take a special function, for instance Γ-function, Riemann zeta function or hypergeometric function, and evaluate it in a rational or algebraic number. The resulting number is in general a new transcendental number (usually hard to prove), however, for some cases it can be calculated in terms of previously known numbers. For instance, the evaluation of the Riemann zeta function on an even positive number $2k$ is π^{2k} up to a rational number. One of the basic ideas to study this kind of phenomenon is to associate a geometry and then to use the machinery of Algebraic Geometry to investigate it. The hypergeometric function F with rational a,b,c parameters and up to some Γ factors, is an abelian integral associated to families of curves. This simple observation has been used by F. Beukers, J. Shiga, J. Wolfart and many others, to investigate the special values of the Gauss hypergeometric function. What we do essentially in [28] is to show what Hodge theory of cubic four fold hypersurfaces has to do with the hypergeometric function. For instance, consider the assertion:

$$F\left(\frac{5}{6},\frac{1}{6},1\Big|\pm\frac{\sqrt{\frac{54001}{15}}}{120}+\frac{1}{2}\right)\frac{\pi^2}{\Gamma(\frac{1}{3})^3}\in\bar{\mathbb{Q}}, \tag{10}$$

Let M_c be the projectivization of $f^{-1}(c)$, where f is given in (5), and α is the differential equation such that $\int_\delta\alpha$ is the sum in the left hand side of (6). It turns out that the $\mathbb{Q}$-vector space of the periods of α is spanned over algebraic numbers by $\Gamma(\frac{1}{3})^3\mathbb{Q}(\zeta_3)$ times the periods of $\frac{dx}{y}$ over the elliptic curve $L_t : y^2-x^3+3x-t,\ t=2-\frac{27}{4}c^2$. For $j=\frac{1}{t^2-4}=2^4\cdot3^4\cdot5^3$, L_t has a complex multiplication by $\mathbb{Q}(\zeta_3)$ and this gives us a Hodge cycle δ in $H_4(M_c,\mathbb{Q})$. One of the consequences of the Hodge conjecture is that for $c\in\bar{\mathbb{Q}}$ the integration over δ of any 4-differential form in $\mathbb{C}^5$, which is defined over $\bar{\mathbb{Q}}$ and is without residue at infinity, belongs to $\pi^2\bar{\mathbb{Q}}$. Since the Hodge conjecture is proved for cubic hypersurfaces of dimension 4 (S. Zucker 1977), we get a proof of (10).

My first reformulation of the Griffiths problem on automorphic functions for the moduli of polarized Hodge structures was just a reformulation without any contribution. Many years later, I wrote it in the text [8] because I had a solution at least in the case of Hodge structures arising from elliptic curves and Calabi-Yau varieties. The case of elliptic curves came first and produced a mathematics different from the original problem. It resulted in two of my articles [20, 21].

For the family of elliptic curves

$$E_t : y^2-4(x-t_1)^3+t_2(x-t_1)+t_3,\ t\in T:=\mathbb{C}^3\backslash\{27t_3^2-t_2^3=0\} \tag{11}$$

the abelian integral $\int\frac{xdx}{y}$ is constant along the solutions of the Ramanujan ordinary differential equation

$$\begin{cases}\dot{t}_1=t_1^2-\frac{1}{12}t_2\\ \dot{t}_2=4t_1t_2-6t_3\\ \dot{t}_3=6t_1t_3-\frac{1}{3}t_2^2\end{cases}. \tag{12}$$

I calculated this differential equation through the Gauss-Manin connection of the family (11) and at that time I was not aware of the work of S. Ramanujan in 1961. What he did was to show that the Eisenstein series

$$E_k(z) = a_k(1 + (-1)^k \frac{4k}{B_k} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n), \quad k = 1, 2, 3, \ z \in \mathbb{H},$$

$$B_1 = \frac{1}{6}, \ B_2 = \frac{1}{30}, \ B_3 = \frac{1}{42}, \ \ldots, \ \sigma_i(n) := \sum_{d|n} d^i,$$

$$(a_1, a_2, a_3) = (\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3),$$

satisfy the ODE (12), as it is expected because he was an expert of power series. I realized that one can define an algebra which contains classical modular forms and it is closed under derivation. I called its element a differential modular form. Through the referee of my article [20] I understood that such an object is known under the name quasi modular form. My article went under publication, because of the geometric methods, such as Gauss-Manin connection, that I introduced in the area. For instance, I was able to reprove a classical but hard theorem in analytic number theory, namely, the Eisenstein series are algebraically independent over $\mathbb{C}$. When I went to IMPA, I learned through E. Ghys that such differential equations are even older and they go back to Darboux and Halphen. In modern language the spirit of the argument of Halphen can be summarized in the following:

Let $\{\delta_1, \ \delta_2\}$ be a basis of $H_1(E_t, \mathbb{Z}), \ \Delta(t) \neq 0$ with $\langle \delta_1, \delta_2 \rangle = 1$. The functions

$$I_1 := t_1(\int_{\delta_2} \frac{dx}{y})^2 - (\int_{\delta_2} \frac{xdx}{y})(\int_{\delta_2} \frac{dx}{y}), \ I_2 := t_2(\int_{\delta_2} \frac{dx}{y})^4, \ I_3 := t_3(\int_{\delta_2} \frac{dx}{y})^6$$

can be written in terms of the variable

$$z := \frac{\int_{\delta_1} \frac{dx}{y}}{\int_{\delta_2} \frac{dx}{y}}$$

The vector $I(z) = (I_1, I_2, I_3)$ viewed as a function of z is a solution of the vector field R. More precisely, we have

$$I_1 = E_1, \ I_2 = E_2, \ I_3 = E_3.$$

Instead of integrals Halphen used the Gauss-hypergeometric function and so get an ordinary differential equation depending in three parameters. Reformulating Halphen's argument by elliptic integrals, enables us to do the same for other integrals, as I did it in [24] for integrals on Calabi-Yau varieties.

In [21] I studied (12) from dynamical point of view. A good description of accumulation of the leaves of (12) is done in this article. For me the most fascinating theorem of this article is the following: Any leaf of (12) outside $\Delta = 0$ crosses a point with algebraic coordinates at most once. I do not know any other example of an ODE defined over $\mathbb{Q}$ and with this property. I have to confess that I used a very strong theorem in the theory of transcendental numbers, namely the abelian subvariety theorem. I just state a consequence of this theorem for curves:

Let S be a curve of genus two and ω be a differential form of the first kind on S, both defined over a number field. A direct consequence of the Abelian subvariety theorem, see the Bourbaki article of Bost 1994 and the references there, says that if $\int_\delta \omega = 0$ for some homologically non trivial topological cycle δ, then there exists a morphism $f : S \to E$ from S into an elliptic curve E, where f and E are defined over a finite extension of the original number field, such that δ is mapped to zero under f and ω is the pull-back of some differential form of the first kind on E. This statement is trivially false when we do not assume that our objects are defined over a number field. For an arbitrary genus, one can say that the Jacobian of S is not simple but to obtain the contraction of S we need more hypothesis.

9 October 2005-September 2006 (Ochanomizu Univerisity at Tokyo)

The first draft of the articles [26] and [6] were written in this period. In the context of planar differential equations, if the abelian integral is identically zero then the birth of limit cycles is controlled by iterated integrals. This kind of integrals were worked out intensively by K.-T. Chen and R. Hain.

In [26] we study deformations of a holomorphic foliation with a generic non-rational first integral in the complex plane. We consider two vanishing cycles in a regular fiber of the first integral with a non-zero self intersection and with vanishing paths which intersect each other only at their start points. We prove that if the deformed holonomies of such vanishing cycles commute then the deformed foliation has also a first integral. Our result generalizes a similar result of Ilyashenko on the rigidity of holomorphic foliations with a persistent center singularity. The main tools of the proof are Picard-Lefschetz theory and the theory of iterated integrals for such deformations.

In [6] we express the leading term of the holonomy map of a perturbed plane polynomial Hamiltonian foliation in terms of explicit iterated integrals. Similar to the case of Abelian integrals, the non-vanishing of this term implies the non-persistence of the corresponding Hamiltonian identity cycle. We prove that this does happen for generic perturbations and cycles, as well for cycles which are commutators in Hamiltonian foliations of degree two.

10 October 2006-Present

In 2008 I invited Stefan Reiter to IMPA and the articles [29] and [27] were the fruit of our collaboration in this period. For a linear differential equation which depends on some parameters, it is a natural question to ask for which values of the parameters the specialized differential equation is Picard-Fuchs. We say that the linear differential equation is Picard-Fuchs or it comes from geometry if there is a proper family of algebraic varieties $X \to \mathbb{P}^1$ over $\mathbb{C}$ and a relatively closed differential i-form ω in X such that the periods $\int_{\delta_z} \omega$, where $\delta_z \in H_i(X_z, \mathbb{Z})$ is a continuous family of cycles, spans the solutions space of the linear differential equation. In [29] we analyzed this problem for the linear differential equation:

$$y'' + p_1(z)y' + p_2(z)y = 0 \tag{13}$$

where

$$p_1(z) := \frac{1-\theta_1}{z-t} + \frac{1-\theta_2}{z} + \frac{1-\theta_3}{z-1} - \frac{1}{z-\lambda},$$

$$p_2(z) := \frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}$$

$$t(t-1)K=\lambda(\lambda-1)(\lambda-t)\mu^2-(\theta_2(\lambda-1)(\lambda-t)+\theta_3\lambda(\lambda-t)+(\theta_1-1)\lambda(\lambda-1))\mu+\kappa(\lambda-t),$$
$$\kappa=\frac{1}{4}((\sum_{i=1}^{3}\theta_i-1)^2-\theta_4^2),$$
and $\theta=(\theta_1,\theta_2,\theta_3,\theta_4)$ is a fixed multi-parameter. The isomonodromic deformations of this linear differential equation in (t,λ,μ)-space leads to the so called Painlevé VI equation. We were able to classify essentially five such families coming from geometry. Surprisingly these give us common algebraic solutions of Painlevé VI equation discovered by Ch. Doran 2001 and B. Ben Hamed-L. Gavrilov 2005. In [27] we analyzed the same problem for the Heun differential equation:

$$y''+(\frac{1-\theta_1}{z-t}+\frac{1-\theta_2}{z}+\frac{1-\theta_3}{z-1})y'+(\frac{\theta_{41}\theta_{42}z-q}{z(z-1)(z-t)})y=0 \tag{14}$$

with
$$\theta_{41}=-\frac{1}{2}(\theta_1+\theta_2+\theta_3-2+\theta_4),\ \theta_{42}=-\frac{1}{2}(\theta_1+\theta_2+\theta_3-2-\theta_4).$$
If (14) (and also (13)) come from geometry then the exponents $\theta_i,\ i=1,2,\ldots,4$, are rational numbers but not vice versa. Based on the classification of families of elliptic curves with exactly four singular fibers done by Herfurtner in 1991, we were able to distinguish 39 families of Heun equations coming from geometry.

Since 2008 I am learning some Mathematical Physics, and in particular mirror symmetry, in order to understand certain calculations involving triple integrals associated to families of Calabi-Yau varieties. After two years of intensive investigation, I wrote the paper [24] bringing my own ideas to Mathematical Physics. In 1991 there appeared the article of Candelas, de la Ossa, Green and Parker, in which they calculated in the framework of mirror symmetry a generating function, called the Yukawa coupling, which predicts the number of rational curves of a fixed degree in a generic quintic three fold. From mathematical point of view, the finiteness is still a conjecture carrying the name of Clemens. Since then there was some effort to express the Yukawa coupling in terms of classical modular or quasi modular forms, however, there was no success. The Yukawa coupling is calculated from the periods of a one parameter family of Calabi-Yau varieties and this suggests that there must be a theory of quasi modular forms attached to this family. The main goal of the paper [24] to realize the construction of such a theory. My main contribution to the area is that many generating functions, instead of functional equations satisfy nice differential equations which characterize any other generating function of the same class. For the mentioned Calabi-Yau varieties I prove that the corresponding differential equation is:

$$\begin{cases}\dot{t}_0=\frac{1}{t_5}(\frac{6}{5}t_0^5+\frac{1}{3125}t_0t_3-\frac{1}{5}t_4)\\ \dot{t}_1=\frac{1}{t_5}(-125t_0^6+t_0^4t_1+125t_0t_4+\frac{1}{3125}t_1t_3)\\ \dot{t}_2=\frac{1}{t_5}(-1875t_0^7-\frac{1}{5}t_0^5t_1+2t_0^4t_2+1875t_0^2t_4+\frac{1}{5}t_1t_4+\frac{2}{3125}t_2t_3)\\ \dot{t}_3=\frac{1}{t_5}(-3125t_0^8-\frac{1}{5}t_0^5t_2+3t_0^4t_3+3125t_0^3t_4+\frac{1}{5}t_2t_4+\frac{3}{3125}t_3^2)\\ \dot{t}_4=\frac{1}{t_5}(5t_0^4t_4+\frac{1}{625}t_3t_4)\\ \dot{t}_5=\frac{t_6}{t_5}\\ \dot{t}_6=(-\frac{72}{5}t_0^8-\frac{24}{3125}t_0^4t_3-\frac{3}{5}t_0^3t_4-\frac{2}{1953125}t_3^2)+\frac{t_6}{t_5}(12t_0^4+\frac{2}{625}t_3)\end{cases} \tag{15}$$

where
$$\dot{t}=5q\frac{\partial t}{\partial q}.$$

We write each t_i as a formal power series in q, $t_i = \sum_{n=0}^{\infty} t_{i,n} q^n$ and substitute in the above differential equation and we see that it determines all the coefficients $t_{i,n}$ uniquely with the initial values:

$$t_{0,0} = \frac{1}{5},\ t_{0,1} = 24,\ t_{4,0} = 0 \tag{16}$$

and assuming that $t_{5,0} \neq 0$. After substitution we get the two possibilities $0, \frac{-1}{3125}$ for $t_{5,0}$, and $t_{i,n},\ n \geq 2$ is given in terms of $t_{j,m},\quad j = 0, 1, \ldots, 6,\quad m < n$. We calculate the expression $\frac{-(t_4 - t_0^5)^2}{625 t_5^3}$ and write it in Lambert series form. It turns out that

$$\frac{-(t_4 - t_0^5)^2}{625 t_5^3} = 5 + 2875 \frac{q}{1-q} + 609250 \cdot 2^2 \frac{q^2}{1-q^2} + \cdots + n_d d^3 \frac{q^d}{1-q^d} + \cdots$$

is the Yukawa coupling studied in mirror symmetry. The numbers n_d are called instanton numbers or PBS states degeneracies and using them we can calculate the Gromov-Witten invariants. The differential equation (15) is a generalization of the Ramanujan differential equation and as usual I have derived it from the Gauss-Manin connection of a family of Calabi-Yau varieties.

During my stay at IMPA I continued my collaborations with my colleagues and in particular, Prof. César Camacho and Prof. Paulo Sad. The papers [3, 30] are the continuation of my works [1, 2] originated by Grauert's work in 1962 and they came out through fruitful discussions with them. In [3] we study $\mathbb{C}^*$ actions on Stein surfaces and we construct their moduli by means of the resolution data of the dicritical singularity of the action. In particular, we prove that $\mathbb{C}^*$ transversal actions around a curve embedded in a two dimensional manifold is biholomorphic to its linear part, without any condition on the self intersection of the curve. In [30] we prove the existence of regular foliations with a prescribed tangency divisor in neighborhoods of negatively embedded holomorphic curves. This result together with our previous result in [2] gives a simple proof of the linearization theorem due to Grauert.

My collaboration with my Ph.D. student E. Vieira resulted in the article [31]. In this article we study projective cycles in $\mathbb{P}^2_{\mathbb{R}}$. Our inspiring example was the Jouanolou foliation of odd degree which has a hyperbolic projective limit cycle. We proved that only odd degree foliations may have projective cycles and foliations with exactly one real simple singularity have a projective cycle. The main result of this article is the following: After a perturbation of a generic Hamiltonian foliation with a projective cycle, we have a projective limit cycle if and only if the perturbation is not Hamiltonian.

During a conference in Hodge Theory at ICTP, I meet Ch. Doran and he informed me about his article with Clingher and our first observation was that we can combine his and my ideas in [24]. The result is now being being written in [4]. Another text which I wrote throughout the last seven years, but never in an appropriate format to divulge it, is [8].

During 2008-2009 I invested a good amount of my time to write two lecture notes. One is about arithmetic of elliptic curves and modular forms [23] and the other is Hodge theory [18]. As it is the case with many mathematicians I wanted to deepen my knowledge in these areas and I also wanted to write my own ideas in both topics.

I wrote the article [20] without knowing that its main object is known under the name quasi modular form and it is already studied in analytic number theory and even it is rediscovered in Mathematical Physics by Atiyah and Hitchin. However, this article was well accepted because of its new ideas like using elliptic integrals and Gauss-Manin connection. During June 2010 I was invited to give a mini course in Clermont-Ferrand about this and as a result I wrote my lecture notes [25] developing more the notion of a quasi modular form in an Algebraic Geometry context.

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