Eisenstein type series for Calabi-Yau varieties

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Abstract

In this article we introduce an ordinary differential equation associated to the one parameter family of Calabi-Yau varieties which is mirror dual to the universal family of smooth quintic three folds. It is satisfied by seven functions written in the $q$-expansion form and the Yukawa coupling turns out to be rational in these functions. We prove that these functions are algebraically independent over the field of complex numbers, and hence, the algebra generated by such functions can be interpreted as the theory of (quasi) modular forms attached to the one parameter family of Calabi-Yau varieties. Our result is a reformulation and realization of a problem of Griffiths around seventies on the existence of automorphic functions for the moduli of polarized Hodge structures. It is a generalization of the Ramanujan differential equation satisfied by three Eisenstein series.

1 Introduction

Modular and quasi-modular forms as generating functions count very unexpected objects beyond the scope of analytic number theory. There are many examples for supporting this fact. The Shimura-Taniyama conjecture, now the modularity theorem, states that the generating function for counting $\mathbb{F}_p$-rational points of an elliptic curve over $\mathbb{Z}$ for different primes $p$, is essentially a modular form (see [32] for the case of semi-stable elliptic curves and [5] for the case of all elliptic curves). Monstrous moonshine conjecture, now Borcherds theorem, relates the coefficients of the $j$-function with the representation dimensions of the monster group (see [4]). Counting ramified coverings of an elliptic curve with a fixed ramification data leads us to quasi-modular forms (see [9] and [18]).

In the context of Algebraic Geometry, the theory of modular forms is attached to elliptic curves and in a similar way the theory of Siegel and Hilbert modular forms is attached to polarized abelian varieties. One may dream of other modular form theories attached to other varieties of a fixed topological type. An attempt to formulate such theories was first done around seventies by P. Griffiths in the framework of Hodge structures, see [15]. However, such a formulation leads us to the notion of automorphic cohomology which has lost the generating function role of modular forms. Extending the algebra of any type of modular forms into an algebra, which is closed under canonical derivations, seems to be indispensable for further generalizations.

In 1991 there appeared the article [6] of Candelas, de la Ossa, Green and Parkes, in which they calculated in the framework of mirror symmetry a generating function, called the Yukawa coupling, which predicts the number of rational curves of a fixed degree in a generic quintic three fold. From mathematical point of view, the finiteness is still a conjecture carrying the name of Clemens (see [7]). Since then there was some effort

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\end{flushright}
to express the Yukawa coupling in terms of classical modular or quasi-modular forms, however, there was no success. The Yukawa coupling is calculated from the periods of a one parameter family of Calabi-Yau varieties and this suggests that there must be a theory of (quasi) modular forms attached to this family (this is also predicted in [1], page 2). The main aim of the present text is to realize the construction of such a theory.

Consider the following ordinary differential equation in seven variables $t_0, t_1, \ldots, t_4, t_5, t_6$: \[ \begin{cases} i_0 = \frac{1}{t_5} (\frac{6}{5} t_0^5 + \frac{1}{625} t_0 t_3 - \frac{1}{5} t_4) \\ i_1 = \frac{1}{t_5} (-125 t_0^6 + t_4^2 t_1 + 125 t_0 t_4 + \frac{1}{3125} t_1 t_3) \\ i_2 = \frac{1}{t_5} (-1875 t_0^3 - \frac{1}{5} t_0^2 t_2 + 2 t_0^2 t_2 + 1875 t_0 t_4 + \frac{2}{3125} t_2 t_3) \\ i_3 = \frac{1}{t_5} (-3125 t_0^8 - \frac{1}{5} t_0^3 t_2 + 3 t_0^3 t_3 + 3125 t_0^3 t_4 + \frac{3}{3125} t_3 t_4) \\ i_4 = \frac{1}{t_5} (5 t_0^4 t_4 + 1 \frac{1}{625} t_3 t_4) \\ i_5 = \frac{1}{t_5} (5 t_0^4 t_4 + 1 \frac{1}{625} t_3 t_4) \\ i_6 = (-\frac{72}{5} t_0^8 - 24 \frac{1}{3125} t_0^3 t_3 - \frac{3}{2} t_0^3 t_4 + \frac{1}{1953125} t_3^2 t_4) + \frac{t_6}{t_5} (12 t_0^4 + \frac{1}{625} t_3 t_4) \end{cases} \]

where \[ i = 5q \frac{\partial t}{\partial q}. \]

We write each $t_i$ as a formal power series in $q$, $t_i = \sum_{n=0}^{\infty} t_{i,n} q^n$ and substitute in the above differential equation and we see that it determines all the coefficients $t_{i,n}$ uniquely with the initial values:

\[ \begin{align*} t_{0,0} &= \frac{1}{5}, \quad t_{0,1} = 24, \quad t_{4,0} = 0 \end{align*} \]

and assuming that $t_{5,0} \neq 0$. After substitution we get the two possibilities $0, \frac{-1}{3125}$ for $t_{5,0}$, and $t_{i,n}$, $n \geq 2$ is given in terms of $t_{j,m}$, $j = 0, 1, \ldots, 6$, $m < n$. See §17 for the first eleven coefficients of $t_i$’s. We calculate the expression $\frac{-(t_{i,m})^2}{625 t_5^3}$ and write it in Lambert series form. It turns out that

\[ \frac{-(t_{4} - \frac{5}{6})^2}{625 t_5^3} = 5 + \frac{2875}{1 - q} + \frac{609250 \cdot 2^2 \cdot q^2}{1 - q^2} + \cdots + n_d q^3 \frac{q^d}{1 - q^d} + \cdots. \]

Let $W_{\psi}$ be the variety obtained by the resolution of singularities of the following quotient:

\[ W_{\psi} := \left\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \mid x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5 \psi x_0 x_1 x_2 x_3 x_4 = 0 \right\}/G, \]

where $G$ is the group

\[ G := \{(\zeta_1, \zeta_2, \cdots, \zeta_5) \mid \zeta_5^5 = 1, \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5 = 1\} \]

acting in a canonical way. The family $W_{\psi}$ is Calabi-Yau and it is mirror dual to the universal family of quintic varieties in $\mathbb{P}^4$.

**Theorem 1.** The quantity $\frac{-(t_{4} - \frac{5}{6})^2}{625 t_5^3}$ is the Yukawa coupling associated to the family of Calabi-Yau varieties $W_{\psi}$.

The $q$-expansion of the Yukawa coupling is calculated by Candelas, de la Ossa, Green, Parkes in [6], see also [24]. Using physical arguments they showed that $n_d$ must be the number of degree $d$ rational curves inside a generic quintic three fold. However, from mathematical point of view we have the Clemens conjecture which claims that there are
finite number of such curves for all \( d \in \mathbb{N} \). This conjecture is established for \( d \leq 9 \) and remains open for \( d \) equal to 10 or bigger than it (see [17] and the references within there). The Gromov-Witten invariants \( N_d \) can be calculated using the well-known formula 

\[ N_d = \sum_{k|d} \frac{n_d}{k^4} \]

The numbers \( n_d \) are called instanton numbers or BPS degeneracies. The \( \mathbb{C} \)-algebra generated by \( t_i \)'s can be considered as the theory of (quasi) modular forms attached to the family \( W_\psi \). We prove:

**Theorem 2.** The functions \( t_i, i = 0, 1, \ldots, 6 \) are algebraically independent over \( \mathbb{C} \), this means that there is no polynomial \( P \) in seven variables and with coefficients in \( \mathbb{C} \) such that \( P(t_0, t_1, \cdots, t_6) = 0 \).

Calculation of instanton numbers by our differential equation (1) or by using periods, see [6, 24], or by constructing moduli spaces of maps from curves to projective spaces, see [21], leads to the fact that they are rational numbers. It is conjectured that all \( n_d \)'s are integers (Gopakumar-Vafa conjecture, see [13]). Some partial results regarding this conjecture is established recently by Kontsevich-Schwarz-Vologodsky (see [22]) and Krattenthaler-Rivoal (see [23]).

All the quantities \( t_i, i = 0, 1, \ldots, 6 \) and \( q \) can be written in terms of the periods of the family \( W_\psi \). The differential form

\[
\Omega = \frac{x_4 dx_0 \wedge dx_1 \wedge dx_2}{\frac{\partial Q}{\partial x_3}},
\]

where \( Q \) is the defining polynomial of \( W_\psi \), induces a holomorphic 3-form in \( W_\psi \) which we denote by the same letter \( \Omega \). Note that \( 5\psi \Omega \) is the standard choice of a holomorphic differential 3-form on \( W_\psi \) (see [6], p. 29). Let also \( \delta_1, \delta_2, \delta_3, \delta_4 \) be a particular basis of \( H_3(W_\psi, \mathbb{Q}) \) which will be explained in §11, and

\[
x_{ij} = \frac{\partial^{j-1}}{\partial \psi^{j-1}} \int_{\delta_i} \Omega, \quad i, j = 1, 2, 3, 4.
\]

**Theorem 3.** The \( q \)-expansion of \( t_i \)'s are convergent and if we set \( q = e^{2\pi i \frac{t}{t_{11}}} \) then

\[
t_0 = a^{-3} \psi x_{11}, \\
t_1 = a^{-9} 625 x_{11} (5\psi^3 x_{12} + 5\psi^4 x_{13} + (\psi^5 - 1) x_{14}), \\
t_2 = a^{-9} (-625) x_{12}^2 (5\psi^3 x_{11} + (\psi^5 - 1) x_{13}), \\
t_3 = a^{-12} 625 x_{11}^3 (-5\psi^4 x_{11} + (\psi^5 - 1) x_{12}), \\
t_4 = a^{-15} x_{11}^5, \\
t_5 = a^{-11} \frac{-1}{5} (\psi^5 - 1) x_{12}^2 (x_{12} x_{21} - x_{11} x_{22}), \\
t_6 = a^{-23} \frac{1}{25} (\psi^5 - 1) x_{11}^5 (-5\psi^4 x_{11} x_{12} x_{21} - 2(\psi^5 - 1) x_{12} x_{21} - (\psi^5 - 1) x_{11} x_{13} x_{21} + 5\psi^4 x_{12}^2 x_{22} + 2(\psi^5 - 1) x_{11} x_{12} x_{22} + (\psi^5 - 1) x_{11}^2 x_{23}),
\]

where \( a = \frac{2\pi i}{5} \).

Once all the above quantities are given, using the Picard-Fuchs of \( x_{i1} \)'s, see (11), one can check easily that they satisfy the ordinary differential equation (1). However, how we have calculated them, and in particular moduli interpretation of \( t_i, i = 0, 1, \ldots, 6 \), will be explained throughout the present text.
This work can be considered as a realization of a problem of Griffiths around 1970's on the automorphic form theory for the moduli of polarized Hodge structures, see [15]. In our case $H^3(W,\mathbb{C})$ is of dimension 4 and it carries a Hodge decomposition with Hodge numbers $h^{30} = h^{21} = h^{12} = h^{03} = 1$. As far as I know, this is the first case of automorphic function theory for families of varieties for which the corresponding Griffiths period domain is not Hermitian symmetric. It would be of interest to see how the results of this paper fit into the automorphic cohomology theory of Griffiths or vice versa.

Here, I would like to say some words about the methods used in the present text and whether one can apply them to other families of varieties. We construct affine coordinates for the moduli of the variety $W$ enhanced with elements in its third de Rham cohomology, see §3, §6 and §16. Such a moduli turns out to be of dimension seven and such coordinates, say $t_i$, $i = 0, 1, \ldots, 6$, have certain automorphic properties with respect to the action of an algebraic group (the action of discrete groups in the classical theory of automorphic functions is replaced with the action of algebraic groups). We use the Picard-Fuchs equation of the periods of $\Omega$ and calculate the Gauss-Manin connection (see for instance [20]) of the universal family of Calabi-Yau varieties over the mentioned moduli space. The ordinary differential equation (1), seen as a vector field on the moduli space, has some nice properties with respect to the Gauss-Manin connection which determines it uniquely. A differential equation of type (1) can be introduced for other type of varieties, see [27], however, whether it has a particular solution with a rich enumerative geometry behind, depends strongly on some integral monodromy conditions, see §9, §8 and §11. For the moment I suspect that the methods introduced in this article can be generalized to arbitrary families of Calabi-Yau varieties and even to some other cases where the geometry is absent, see for instance the list of Calabi-Yau operators in [3, 31] and a table of mirror consistent monodromy representations in [10]. Since the theory of Siegel modular forms is well developed and in light of the recent work [8], see also the references within there, the case of K3 surfaces is quite promising. For local Calabi-Yau manifolds we need Siegel modular forms and this is explained in [1].

Here, I would like to discuss about the physics underlying the mathematical results and calculations of the present text. Higher genus topological string partition functions $F_g(t)$, $g \geq 1$ and their calculations play an important role in quantum field theory. Mirror symmetry conjecture leads to directly calculable predictions for the functions $F_g$ for Calabi-Yau varieties using the periods of the mirror manifold. To understand better the importance of the quantities $t_0, t_1, \ldots, t_6$ in quantum field theory, one has to understand the role of three classical Eisenstein series in the calculation of partition functions for elliptic curves (for a detailed discussion see [9]). The original definition of partition functions $F_g$ in quantum Yang-Mills theory does not calculate them explicitly. In the case of elliptic curves M. R. Douglas in [11] finds explicit calculable formulas for $F_g$ in terms of generalized theta functions. This leads in a natural way to the polynomial structure of partition functions in terms of three Eisenstein series. Since we have the Ramanujan differential equation between Eisenstein series, we conclude that derivatives of partition functions in the Kähler modulus have still such polynomial structures, and hence, they can be calculated in an effective way. In the case of mirror symmetry for quintic hypersurfaces the polynomial structure of partition functions are described by Yamaguchi and Yau in [33] (see also [2, 16] for further generalization of their results for other type of Calabi-Yau varieties). They show that partition functions are homogeneous polynomials in five quantities. A simple comparison of the period expressions in Theorem 3 and those of [33] end of §3.3, shows that these quantities can be computed explicitly in terms
of \( t_i, \ i = 0, 1, \ldots, 7 \). In fact, Theorem 2 implies that five is the minimum number of generators for the partition functions. The particular appearance of \( t_5 \) in the differential equation (1) together with Theorem 3 imply that apart from the \( \tau \)-frame, where in our case \( \tau := \frac{2\pi i}{\pi} \), and \( \psi \)-frame in the mirror manifold, we have a \( y \)-frame which is given by \( \frac{\partial \tau}{\partial y} = \frac{1}{t_6} \). Looking \( t_i, \ i = 0, \ldots, 4 \)'s as functions in \( y \) we have an ordinary differential equation in dimension five which is essentially the first five lines of (1) without the factor \( \frac{1}{t_6} \). Using the result of Yamaguchi and Yau, this completely describes the polynomial and differential structure of the partition functions in the \( y \)-frame. I am not aware of any significance of the physical interpretation of the \( y \)-frame. Theorem 2 says that in order to understand the differential structure of partition functions in the \( \tau \)-frame we need exactly seven functions. The ordinary differential equation in the \( y \)-frame will not result in the calculation of instanton type expansions of \( t_i \)'s as we have explained in the beginning of Introduction. The quantities \( t_5 \) and \( t_6 \) are necessary for such calculations.

The monodromy group \( \Gamma \) of the mirror manifold acts naturally and non trivially on the Kähler modulus \( \tau \) and this leads to the modular properties of the partition functions with respect to \( \Gamma \). From physical point of view this action is important and it describes the interchange of large area with small area which is a well-known example of the so called duality transformation in string theory (see [12]). In the case of elliptic curves this action is described in terms of the classical functional equations of Eisenstein series, and hence any polynomial of them (see [9]). In order to understand the modular properties of partition functions attached to quintic Calabi-Yau manifolds, we have to understand the modular properties of \( t_i \)'s. The modular properties of \( t_i \)'s are first described in terms of the action of an algebraic group, and then, using the period map this is translated into the modular properties of \( t_i \)'s with respect to \( \Gamma \), see §17.

We have calculated the differential equation (1) and the first coefficients of \( t_i \) by Singular, see [14]. The reader who does not want to calculate everything by his own effort can obtain the corresponding Singular code from my web page.\(^2\)

In the final steps of the present article Charles Doran informed me of his joint article [8] and the results obtained by Yamaguchi and Yau in [33]. Here, I would like to thank him for his interest. I would also like to thank the referee whose comments and suggestions improved the paper and who introduced me with the article [1]. Finally, sincere thanks goes to Satoshi Yamaguchi who informed me about further developments [2, 16] of his results with Prof. Yau.

### 2 Quasi-modular forms

The differential equation (1) is a generalization of the Ramanujan differential equation

\[
\begin{align*}
\dot{g}_1 &= g_2^2 - \frac{1}{12} g_2 \\
\dot{g}_2 &= 4g_1 g_2 - 6g_3 \\
\dot{g}_3 &= 6g_1 g_3 - \frac{1}{3} g_2^2 \\
\dot{g} &= 12q \frac{\partial g}{\partial q}
\end{align*}
\]

which is satisfied by the Eisenstein series:

\[
g_i = a_k \left( 1 + b_k \sum_{d=1}^{\infty} d^{2k-1} \frac{q^d}{1-q^d} \right), \quad k = 1, 2, 3,
\]

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\(^2\)w3.impa.br/~hossein/manyfiles/singular-haftarticle.txt
where
\[(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = (1, 12, 8).\]

We have calculated (1) using the Gauss-Manin connection of the family \(W_\psi\) which is essentially the Picard-Fuchs differential equation of the holomorphic differential form of the family \(W_\psi\). This is done in a similar way as we calculate (4) from the Gauss-Manin connection of a family of elliptic curves, see [25, 26]. The general theory of differential equations of type (1) and (4) is developed in [27]. Relations between the Gauss-Manin connection and Eisenstein series appear in the appendix of [19]. Let \(g_1, g_2, g_3\) be the Eisenstein series (5). The \(\mathbb{C}\)-algebra \(\mathbb{C}[g_1, g_2, g_3]\) is freely generated by \(g_1, g_2, g_3\). With \(\deg(g_i) = i, i = 1, 2, 3\), its homogeneous pieces are quasi-modular forms over \(\text{SL}(2, \mathbb{Z})\). It can be shown that any other quasi-modular form for subgroups of \(\text{SL}(2, \mathbb{Z})\) with finite index, is in the algebraic closure of \(\mathbb{C}(g_1, g_2, g_3)\).

3 Moduli space, I

In the affine coordinates \(x_0 = 1\), the variety \(W_\psi\) is given by:
\[
\{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid f = 0\}/G,
\]
where
\[
f = -z - x_1^5 - x_2^5 - x_3^5 - x_4^5 + 5x_1x_2x_3x_4
\]
and we have introduced a new parameter \(z := \psi^{-5}\). We also use \(W_{1,z}\) to denote the variety \(W_\psi\). For \(z = 0, 1\), \(\infty\) the variety \(W_{1,z}\) is singular and for all others it is a smooth variety of complex dimension 3. From now on, by \(W_{1,z}\) we mean a smooth one. Up to constant there is a unique holomorphic three form on \(W_{1,z}\) which is given by
\[
\eta = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{df}.
\]

Note that the pair \((W_{1,z}, 5\eta)\) is isomorphic to \((W_\psi, 5\psi\Omega)\), with \(\Omega\) as in the Introduction. The later is used in [6] p. 29. The third de Rham cohomology of \(W_{1,z}\), namely \(H^3_{dR}(W_{1,z})\), carries a Hodge decomposition with Hodge numbers \(h^{30} = h^{21} = h^{12} = h^{03} = 1\). By Serre duality \(H^2(W_{1,z}, \Omega^1) \cong H^1(W_{1,z}, \Theta)\), where \(\Omega^1\) (resp. \(\Theta\)) is the sheaf of holomorphic differential 1-forms (resp. vector fields) on \(W_{1,z}\). Since \(h^{21} = \dim_\mathbb{C} H^2(W_{1,z}, \Omega^1) = 1\), the deformation space of \(W_{1,z}\) is one dimensional. This means that \(W_{1,z}\) can be deformed only through the parameter \(z\). In fact \(z\) is the classifying function of such varieties. Note that the finite values of \(z\) does not cover the smooth variety \(W_\psi, \psi = 0\).

Let us take the polynomial ring \(\mathbb{C}[t_0, t_4]\) in two variables \(t_0, t_4\) (the variables \(t_1, t_2\) and \(t_3\) will appear later). It can be seen easily that the moduli \(S\) of the pairs \((W, \omega)\), where \(W\) is as above and \(\omega\) is a holomorphic differential form on \(W\), is isomorphic to
\[
S \cong \mathbb{C}^2 \setminus \{(t_0^5 - t_4)t_4 = 0\},
\]
where we send the pair \((W_{1,z}, a\eta)\) to \((t_0, t_4):= (a^{-1}, za^{-5})\). The multiplicative group \(G_m := \mathbb{C}^*\) acts on \(S\) by:
\[
(W, \omega) \cdot k = (W, k^{-1}\omega), \quad k \in G_m, \quad (W, \omega) \in S.
\]
In coordinates \((t_0, t_4)\) this corresponds to
\[
(t_0, t_4) \cdot k = (kt_0, k^5t_4), \quad (t_0, t_4) \in S, \quad k \in G_m.
\]
We denote by \((W_{t_0,t_4}, \omega_1)\) the pair \((W_{t_0,t_4}, \omega_1)\). The one parameter family \(W_{1,z}\) (resp. \(W_{\psi}\)) can be recovered by putting \(t_0 = 1\) and \(t_4 = z\) (resp. \(t_0 = \psi\) and \(t_4 = 1\)). In fact, the pair \((W_{t_0,t_4}, \omega_1)\) in the affine chart \(x_0 = 1\) is given by:

\[
\left\{f_{t_0,t_4}(x) = 0\right\}/G, \quad \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{df_{t_0,t_4}},
\]

where

\[f_{t_0,t_4} := -t_4 - x_1^5 - x_2^5 - x_3^5 - x_4^5 + 5t_0x_1x_2x_3x_4.\]

4 Gauss-Manin connection, I

We would like to calculate the Gauss-Manin connection

\[
\nabla : H^3_{\text{dR}}(W/S) \to \Omega^3_{S} \otimes_{\mathcal{O}_S} H^3_{\text{dR}}(W/S).
\]

of the two parameter proper family of varieties \(W_{t_0,t_4}, (t_0,t_4) \in S\). By abuse of notation we use \(\frac{\partial}{\partial t_i}\), \(i = 0, 4\) instead of \(\nabla \frac{\partial}{\partial t_i}\). We calculate \(\nabla\) with respect to the basis

\[\omega_i = \frac{\partial^{i-1}}{\partial t_0^{i-1}}(\omega_1), \quad i = 1, 2, 3, 4\]

of global sections of \(H^3_{\text{dR}}(W/S)\). For this purpose we return back to the one parameter case. We set \(t_0 = 1\) and \(t_4 = z\) and calculate the Picard-Fuchs equation of \(\eta\) with respect to the parameter \(z\):

\[
\frac{\partial^4 \eta}{\partial z^4} = \sum_{i=1}^{4} a_i(z) \frac{\partial^{i-1} \eta}{\partial z^{i-1}} \text{ modulo relatively exact forms.}
\]

This is in fact the linear differential equation

\[
I^{''''} = \frac{-24}{625z^4 - 625z^3} I + \frac{-24z + 5}{5z^4 - 5z^3} I' + \frac{-72z + 35}{5z^3 - 5z^2} I'' + \frac{-8z + 6}{z^2 - z} I^{'''}
\]

which is calculated in [6], see also [27] for some algorithms which calculate such differential equations. It is satisfied by the periods \(I(z) = \int_{\delta} \eta, \delta \in H_3(W_1,z,\mathbb{Q})\) of the differential form \(\eta\) on the family \(W_{1,z}\). In the basis \(\frac{\partial \eta}{\partial z^i}, i = 0, 1, 2, 3\) the Gauss-Manin connection matrix has the form

\[
A(z)dz := \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
a_1(z) & a_2(z) & a_3(z) & a_4(z)
\end{pmatrix} dz.
\]

Now, consider the identity map

\[g : W_{(t_0,t_4)} \to W_{1,z},\]

which satisfies \(g^* \eta = t_0 \omega_1\). Under this map

\[
\frac{\partial}{\partial z} = \frac{-1}{5} \frac{\partial}{\partial t_0} \left( = \frac{t_0}{5} \frac{\partial}{\partial t_4} \right).
\]
From these two equalities we obtain a matrix $S = S(t_0, t_4)$ such that

$$\left[ t_0, \frac{\partial \eta}{\partial z}, \frac{\partial^2 \eta}{\partial z^2}, \frac{\partial^3 \eta}{\partial z^3} \right] = S^{-1} [\omega_1, \omega_2, \omega_3, \omega_4]^t,$$

where $t$ denotes the transpose of matrices, and the Gauss-Manin connection in the basis $\omega_i, \ i = 1, 2, 3, 4$ is:

$$(dS + S \cdot A(t_4/t_0^4) \cdot d(t_4/t_0^4)) \cdot S^{-1}$$

which is the following matrix after doing explicit calculations:

$$\begin{pmatrix}
-\frac{1}{5t_0^4} dt_4 & dt_0 + \frac{2t_0}{5t_0^4} dt_4 & dt_0 + \frac{t_0}{5t_0^4} dt_4 & 0 \\
0 & \frac{2t_0}{5t_0^4} dt_4 & \frac{t_0}{5t_0^4} dt_4 & 0 \\
-\frac{1}{5t_0^4} dt_0 + \frac{r_2}{5t_0^4+r_4} dt_4 & -\frac{15t_0^2}{5t_0^4} dt_0 + \frac{3r_3}{5t_0^4+r_4} dt_4 & -\frac{25t_0^3}{5t_0^4} dt_0 + \frac{5r_4}{5t_0^4+r_4} dt_4 & \frac{10t_0^5}{5t_0^4} dt_0 + \frac{2r_4}{5t_0^4+4r_4} dt_4
\end{pmatrix}$$

From the above matrix or directly from (8) one can check that the periods $x_{i1}, i = 1, 2, 3, 4$ in the Introduction satisfy the Picard-Fuchs equation:

$$(11) \quad I''' = -\frac{\psi}{\psi^5} - I + \frac{15\psi^2}{\psi^5} - I' + \frac{25\psi^3}{\psi^5} - I'' + \frac{10\psi^4}{\psi^5} - I'''', \quad \psi = \partial \eta.$$

### 5 Intersection form and Hodge filtration

For $\omega, \alpha \in H^3_{\text{dR}}(\Omega_{t_0,t_4})$ let

$$\langle \omega, \alpha \rangle := \frac{1}{(2\pi i)^3} \int_{\Omega_{t_0,t_4}} \omega \cup \alpha.$$

This is Poincaré dual to the intersection form in $H_3(\Omega_{t_0,t_4}, \mathbb{Q})$. In $H^3_{\text{dR}}(\Omega_{t_0,t_4})$ we have the Hodge filtration

$$\{0\} = F^4 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3_{\text{dR}}(\Omega_{t_0,t_4}), \quad \text{dim}_{\mathbb{C}}(F^i) = 4 - i.$$

There is a relation between the Hodge filtration and the intersection form which is given by the following collection of equalities:

$$\langle F^i, F^j \rangle = 0, \ i + j \geq 4.$$

The Griffiths transversality is a property combining the Gauss-Manin connection and the Hodge filtration. It says that the Gauss-Manin connection sends $F^i$ to $\Omega^5_5 \otimes F^{i-1}$ for $i = 1, 2, 3$. Using this we conclude that:

$$\omega_i \in F^{4-i}, \ i = 1, 2, 3, 4.$$

**Proposition 1.** *The intersection form in the basis $\omega_i$ is:*

$$\begin{pmatrix}
0 & 0 & 0 & -\frac{1}{625} (t_4 - t_0^5)^{-1} & -\frac{1}{125} t_0^3 (t_4 - t_0^5)^{-2} \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{125} t_0^3 (t_4 - t_0^5)^{-1} & 0 & -\frac{1}{125} t_0^3 (t_4 - t_0^5)^{-2} \\
-\frac{1}{625} (t_4 - t_0^5)^{-1} & 0 & -\frac{1}{125} t_0^3 (t_4 - t_0^5)^{-2} & 0 & 0
\end{pmatrix}$$
where \( \omega \) coordinates (see [6], (4.6)). From this we get:

\[
\langle \omega_1, \omega_4 \rangle = 5^{-4} \frac{1}{t_4 - t_0^4}.
\]

The corresponding calculations are as follows: In \((t_0, t_4)\) coordinates we have \( \psi = t_0 t_4^{-\frac{1}{5}} \) and \( \frac{\partial}{\partial \psi} = t_4^\frac{1}{4} \frac{\partial}{\partial t_0} \) and

\[
\langle \psi \Omega, \frac{\partial^3 \psi \bar{\eta}}{\partial \psi^3} \rangle = \langle t_0 \omega_1, (t_4^{\frac{1}{5}} \frac{\partial}{\partial t_0})^3 (t_0 \omega_1) \rangle = \langle t_0 \omega_1, t_0 t_4^\frac{3}{4} \omega_4 \rangle = t_0^2 t_4^\frac{3}{4} \langle \omega_1, \omega_4 \rangle.
\]

From another side \( \frac{1}{5^2} \frac{\psi^2}{1 - \psi^3} = \frac{1}{5^2} \frac{t_4^\frac{2}{5}}{t_4 - t_0^4} \).

We make the derivation of the equalities \( \langle \omega_1, \omega_3 \rangle = 0 \) and (12) with respect to \( t_0 \) and use the Picard-Fuchs equation of \( \omega_1 \) with respect to the parameter \( t_0 \) and with \( t_4 \) fixed:

\[
\frac{\partial \omega_1}{\partial t_0} = M_{11} \omega_1 + M_{42} \omega_2 + M_{43} \omega_3 + M_{44} \omega_4
\]

Here, \( M_{ij} \) is the \((i, j)\)-entry of (10) after setting \( dt_4 = 0, \ dt_0 = 1 \). We get

\[
\langle \omega_2, \omega_3 \rangle = -\langle \omega_1, \omega_4 \rangle, \quad \langle \omega_2, \omega_4 \rangle = \frac{\partial \langle \omega_1, \omega_4 \rangle}{\partial t_0} - M_{44} \langle \omega_1, \omega_4 \rangle
\]

Derivating further the second equality we get:

\[
\langle \omega_3, \omega_4 \rangle = \frac{\partial \langle \omega_2, \omega_4 \rangle}{\partial t_0} - M_{45} \langle \omega_2, \omega_3 \rangle - M_{44} \langle \omega_2, \omega_4 \rangle.
\]

\[\square\]

6 Moduli space, II

Let \( T \) be the moduli of pairs \((W, \omega)\), where \( W \) is a Calabi-Yau variety as before and \( \omega \in H^3_{\text{dR}}(W) \setminus F^1 \) and \( F^1 \) is the biggest non trivial piece of the Hodge filtration of \( H^3_{\text{dR}}(W) \). In this section, we construct good affine coordinates for the moduli space \( T \).

Let \( G_m \) be the multiplicative group \((\mathbb{C} - \{0\}, \cdot)\) and let \( G_a \) be the additive group \((\mathbb{C}, +)\). Both these algebraic groups act on the moduli spaces \( T \):

\[
(W, \omega) \cdot k = (W, k \omega), \quad k \in G_m, \quad (W, \omega) \in T,
\]

\[
(W, \omega) \cdot k = (W, \omega + k \omega'), \quad k \in G_a, \quad (W, \omega) \in T,
\]

where \( \omega' \) is uniquely determined by \( \langle \omega', \omega \rangle = 1, \ \omega' \in F^3 \). We would like to have affine coordinates \((t_0, t_1, t_2, t_3, t_4)\) for \( T \) such that:

1. We have a canonical map

\[
\pi : T \to S, \quad (W, \omega) \mapsto (W, \omega'),
\]

where \( \omega' \) is determined uniquely by \( \langle \omega', \omega \rangle = 1, \ \omega' \in F^3 \). In terms of the coordinates \( t_i \)'s it is just the projection on \( t_0, t_4 \) coordinates.
2. With respect to the action of $G_m$, $t_i$'s behave as below:

$$t_i \cdot k = k^{i+1} t_i, \ i = 0, 1, \ldots, 4.$$ 

3. With respect to the action of $G_a$, $t_i$'s behave as below:

$$t_i \cdot k = t_i, \ i = 0, 2, 3, 4, \ k \in G_a,$$

$$t_1 \cdot k = t_1 + k, \ k \in G_a.$$ 

In order to construct $t_i$'s we take the family $W_{t_0,t_4}$ as before and three new variable $t_1, t_2, t_3$. One can verify easily that

$$\{(t_0, t_1, t_2, t_3, t_4) \in \mathbb{C}^5 \mid t_4(t_4 - t_0^5) \neq 0\} \cong T,$$

$$(t_0, t_1, t_2, t_3, t_4) \mapsto (W_{t_0,t_{n+1}}, \omega),$$

where

$$\omega = t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 + \frac{\omega_4}{\langle \omega_1, \omega_4 \rangle}.$$ 

7 Gauss-Manin connection, II

For the five parameter family $W_t, t := (t_0, t_1, t_2, t_3, t_4) \in T$, we calculate the differential forms $\alpha_i, \ i = 1, 2, 3, 4$ in $T$ which are defined by the equality:

$$\nabla \omega = \sum_{i=1}^{4} \alpha_i \otimes \omega_i,$$

where $\omega$ is defined in (13), and we check that the $\mathbb{Q}(t)$ vector space spanned by $\alpha_i$ is exactly of dimension 4 and so up to multiplication by a rational function in $\mathbb{Q}(t)$ there is a unique vector field $Ra$ which satisfies

$$\alpha_i(Ra) = 0, \ i = 1, 2, 3, 4$$

or equivalently $\nabla_{Ra} \omega = 0$. We calculate this vector field and get the following expression:

$$Ra = \left(\frac{6}{5} t_0^5 + \frac{1}{3125} t_0 t_3 - \frac{1}{5} t_4 \right) \frac{\partial}{\partial t_0} + \left(-125 t_0^6 + t_0^4 t_1 + 125 t_0 t_4 + \frac{1}{3125} t_1 t_3 \right) \frac{\partial}{\partial t_1}$$

$$+ \left(-1875 t_0^7 - \frac{1}{5} t_0^5 t_2 + 2 t_0^4 t_2 + 1875 t_0^2 t_4 + \frac{1}{5} t_1 t_4 + \frac{2}{3125} t_2 t_3 \right) \frac{\partial}{\partial t_2} +$$

$$+ \left(-3125 t_0^8 - \frac{1}{5} t_0^5 t_3 + 3 t_0^4 t_3 + 3125 t_0^2 t_4 + \frac{1}{5} t_2 t_4 + \frac{3}{3125} t_3 \right) \frac{\partial}{\partial t_3} + (5 t_1 t_4 + \frac{1}{625} t_3 t_4) \frac{\partial}{\partial t_4}.$$ 

This appears in the first five lines of the ordinary differential equation (1). The other pieces of this differential equation has to do with the fact that the choice of $Ra$ is not unique. Let

$$\alpha := \frac{t_0 dt_4 - 5 t_4 dt_0}{(t_4 - t_0^5) t_4}.$$ 

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The vector field $R_a$ turns to be unique after putting the condition

$$\alpha (R_a) = 1$$

We have calculated $R_a$ from (15) and (14). The choice of $\alpha$ up to multiplication by a rational function is canonical (see below). However, choosing such a rational function does not seem to be canonical.

**Proposition 2.** There is a unique basis $\tilde{\omega}_i$, $i = 1, 2, 3, 4$ of $H^3_{dR}(W_t)$, $t \in T$ such that

1. It is compatible with the Hodge filtration, i.e. $\tilde{\omega}_i \in F^{4-i} \setminus F^{5-i}$.
2. $\tilde{\omega}_4 = \omega$ and $\langle \tilde{\omega}_1, \tilde{\omega}_4 \rangle = 1$.
3. The Gauss-Manin connection matrix $A$ of the family $W \to T$ in the mentioned basis is of the form

$$A = \begin{pmatrix}
* & \alpha & 0 & 0 \\
* & * & \alpha & 0 \\
* & * & * & b_4 \alpha \\
* & * & * & *
\end{pmatrix}$$

and

$$A(R_a) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & b_2 & b_3 & b_4 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

where $b_2, b_3, b_4 \in \mathbb{C}[t]$.

Our proof of the above proposition is algorithmic and in fact we calculate $b_i$'s

$$b_2 = -\frac{72}{5} t_0^8 t_3 - \frac{24}{3125} t_0^4 t_3 - \frac{3}{5} t_0^3 t_4 - \frac{2}{1953125} t_3^2,$$

$$b_3 = 12 t_0^4 + \frac{2}{625} t_3, \quad b_4 = -\frac{1}{57} (t_0^5 - t_4)^2$$

and $\tilde{\omega}_i$'s:

$$\tilde{\omega}_1 = \omega_1, \quad \tilde{\omega}_2 = (t_0^4 - \frac{1}{3125} t_3) \omega_1 + \left(\frac{t_0^5}{5} - \frac{1}{5} t_4\right) \omega_2, \quad \tilde{\omega}_4 = \omega$$

$$\tilde{\omega}_3 := \left(-\frac{14}{5} t_0^8 + \frac{1}{15625} t_0^4 t_3 - \frac{1}{625} t_0^4 t_3 - \frac{1}{5} t_0^3 t_4 - \frac{1}{15625} t_2 t_4 - \frac{2}{9765625} t_3^2\right) \omega_1 +$$

$$\left(\frac{3}{5} t_0^9 + \frac{2}{15625} t_0^5 t_3 - \frac{3}{5} t_0^4 t_4 - \frac{2}{15625} t_3 t_4\right) \omega_2 + \left(\frac{1}{25} t_0^4 - \frac{2}{25} t_0^4 t_4 + \frac{1}{25} t_2^2\right) \omega_3.$$

The polynomials $b_2$ and $b_3$ appear in the last line of the ordinary differential equation (1).

**Proof.** The equalities in the second item and $\tilde{\omega}_1 \in F^3$ determine both $\tilde{\omega}_1 = \omega_1, \tilde{\omega}_4 = \omega$ uniquely. We first take the 3-forms $\tilde{\omega}_i = \omega_i$, $i = 2, 3$ as in the previous section and write the Gauss-Manin connection of the five parameter family of Calabi-Yau varieties $W_t$, $t \in T$ in the basis $\tilde{\omega}_i$, $i = 1, 2, 3, 4$:

$$\nabla [\tilde{\omega}_i]_{4 \times 1} = [\alpha_{ij}]_{4 \times 4} [\tilde{\omega}_l]_{4 \times 1}.$$

We explain how to modify $\tilde{\omega}_2$ and $\tilde{\omega}_3$ and get the basis in the announcement of the proposition. Let $R$ be the $\mathbb{Q}(t)$ vector space generated by $\alpha_{4,i}$, $i = 1, 2, \ldots, 4$. It does not
It is also convenient to use the basis \( \tilde{\omega}_1 \) and we already mentioned that it is of dimension 4. If we replace \( \tilde{\omega}_2 \) by \( a\tilde{\omega}_1 + a\tilde{\omega}_1 \) then \( \alpha_{11} \) is replaced by \( \alpha_{11} - \alpha_{12} \). Modulo \( R \) the space of differential forms on \( T \) is one dimensional and since \( \alpha_{12} \not\in R \), we choose \( a \) in such a way that \( \alpha_{11} - \alpha_{12} \in R \). We do this and so we can assume that \( \alpha_{11} \in R \). The result of our calculations shows that \( \alpha_{12} \) is a multiple of \( t_0 dt_4 - 5t_4 dt_0 \). We replace \( \omega_2 \) by \( r \tilde{\omega}_2 \) with some \( r \in \mathbb{Q}(t) \) and get the desired form for \( \alpha_{12} \). We repeat the same procedure for \( \tilde{\omega}_3 \). In this step we replace \( \tilde{\omega}_3 \) by \( r_3 \tilde{\omega}_3 + r_2 \tilde{\omega}_2 + r_1 \tilde{\omega}_1 \) with some \( r_1, r_2, r_3 \in \mathbb{Q}(t) \).

8 Polynomial Relations between periods

We take a basis \( \delta_1, \delta_2, \delta_3, \delta_4 \in H_3(W_{t_0, t_4}, \mathbb{Q}) \) such that the intersection form in this basis is given by:

\[
\Psi := \begin{pmatrix}
\langle \delta_1, \delta_j \rangle \\
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & -\frac{6}{5} \\
0 & 0 & \frac{2}{5} & 0 \\
0 & -\frac{2}{5} & 0 & 2 \\
\frac{6}{5} & 0 & -2 & 0
\end{pmatrix}.
\]

It is also convenient to use the basis \( [\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3, \tilde{\delta}_4] = [\delta_1, \delta_2, \delta_3, \delta_4] \Psi^{-1} \). In this basis the intersection form is \( \langle [\delta_1, \delta_j] \rangle = \Psi^{-t} \). Let \( \omega_i, i = 1, 2, 3, 4 \) be the basis of the de Rham cohomology \( H^3_{\text{dR}}(W_{t_0, t_4}) \) constructed in §4 and let \( \tilde{\delta}_i^p \in H^3(W_{t_0, t_4}, \mathbb{Q}) \) be the Poincaré dual of \( \delta_i \), that is, it is defined by the property \( \int_\delta \tilde{\delta}_i^p = \langle \delta, \delta_i \rangle \) for all \( \delta \in H_3(W_{t_0, t_4}, \mathbb{Q}) \). If we write \( \omega_i \) in terms of \( \tilde{\delta}_i^p \) what we get is:

\[
[\omega_1, \omega_2, \omega_3, \omega_4] = [\tilde{\delta}_1^p, \tilde{\delta}_2^p, \tilde{\delta}_3^p, \tilde{\delta}_4^p] [\int_\delta \omega_j]
\]

that is, the coefficients of the base change matrix are the periods of \( \omega_i \)'s over \( \delta_i \)'s and not \( \tilde{\delta}_i \)'s. The matrix \( [\int_\delta \omega_j] \) is called the period matrix associated to the basis \( \omega_i \) of \( H^3_{\text{dR}}(W_{t_0, t_4}) \) and the basis \( \delta_i \) of \( H_3(W, \mathbb{Q}) \). We have

\[
\langle \omega_i, \omega_j \rangle = [\int_\delta \omega_j]^t \Psi^{-t} [\int_\delta \omega_j].
\]

Taking the determinant of this equality we can calculate \( \det([\int_\delta \omega_j]) \) up to sign:

\[
\det(pm) = \frac{12}{5^{10}} \frac{1}{(t_4 - t_0^5)^2}.
\]

There is another effective way to calculate this determinant without the sign ambiguity. For simplicity, we use the restricted parameters \( t_4 = 1 \) and \( t_0 = \psi \) and the notation \( x_{ij} := \int_\delta \omega_j \) as in the Introduction. Proposition 1 and the equality (17) gives us 6 non
trivial relations between $x_{ij}$'s:

\begin{align*}
0 &= -\frac{25}{6}x_{12}x_{21} + \frac{25}{6}x_{11}x_{22} + \frac{5}{2}x_{22}x_{31} - \frac{5}{2}x_{21}x_{32} - \frac{5}{6}x_{12}x_{41} + \frac{5}{6}x_{11}x_{42} \\
0 &= -\frac{25}{6}x_{13}x_{21} + \frac{25}{6}x_{11}x_{23} + \frac{5}{2}x_{23}x_{31} - \frac{5}{2}x_{21}x_{33} - \frac{5}{6}x_{13}x_{41} + \frac{5}{6}x_{11}x_{43} \\
0 &= -\frac{25}{6}x_{14}x_{21} + \frac{25}{6}x_{11}x_{24} + \frac{5}{2}x_{24}x_{31} - \frac{5}{2}x_{21}x_{34} - \frac{5}{6}x_{14}x_{41} + \frac{5}{6}x_{11}x_{44} - \frac{1}{625(\psi^5 - 1)} \\
0 &= -\frac{25}{6}x_{13}x_{22} + \frac{25}{6}x_{12}x_{23} + \frac{5}{2}x_{22}x_{32} - \frac{5}{2}x_{22}x_{33} - \frac{5}{6}x_{13}x_{42} + \frac{5}{6}x_{12}x_{43} + \frac{1}{625(\psi^5 - 1)} \\
0 &= -\frac{25}{6}x_{14}x_{22} + \frac{25}{6}x_{12}x_{24} + \frac{5}{2}x_{24}x_{32} - \frac{5}{2}x_{22}x_{34} - \frac{5}{6}x_{14}x_{42} + \frac{5}{6}x_{12}x_{44} - \frac{\psi^4}{125(\psi^5 - 1)^2} \\
0 &= -\frac{25}{6}x_{14}x_{23} + \frac{25}{6}x_{13}x_{24} + \frac{5}{2}x_{23}x_{33} - \frac{5}{2}x_{23}x_{34} - \frac{5}{6}x_{14}x_{43} + \frac{5}{6}x_{13}x_{44} + \frac{\psi^3}{125(\psi^5 - 1)^2}.
\end{align*}

These equalities correspond to the entries (1, 2), (1, 3), (1, 4), (2, 3), (2, 4) and (3, 4) of (17). In the ideal of $\mathbb{Q}(\psi)[y_{ij}, i, j = 1, 2, 3, 4]$ generated by the polynomials $f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}$ in the right hand side of the above equalities the polynomial $\det([x_{ij}])$ is reduced to the right hand side of (18). For instance, Singular check this immediately (see [14]). Let $y_{ij}$ be indeterminate variables, $R = \mathbb{C}(\psi)[y_{ij}, i, j = 1, 2, 3, 4]$ and

$I := \{f \in R \mid f(x_{ij}) = 0\}$

**Proposition 3.** The ideal $I$ is generated by $f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}$.

**Proof.** Let $E$ be the differential field over $F = \mathbb{C}(\psi)$ generated by $x_{ij}$'s. Note that the matrix $[x_{ij}]$ is the fundamental system of the linear differential equation:

$$
\frac{\partial}{\partial \psi} [x_{ij}] = [x_{ij}]B(\psi)^t,
$$

where $B(\psi)$ is obtained from the matrix (10) by putting $dt_0 = 1$, $dt_4 = 0$, $t_0 = \psi$, $t_4 = 1$. The homology group $H_3(W_\psi, \mathbb{Q})$ has a symplectic basis and hence the monodromy group of $W_\psi$ is a subgroup of $\text{Sp}(4, \mathbb{Z})$. Consequently, the differential Galois group $G(E/F)$ is an algebraic subgroup of $\text{Sp}(4, \mathbb{C})$ and it contains a maximal unipotent matrix which is the monodromy around $z = 0$. By a result of Saxl and Seitz, see [29], we have $G(E/F) = \text{Sp}(4, \mathbb{C})$. Therefore, $\dim G(E/F) = 10$ which is the transcendental degree of the field $E$ over $F$ (see [30]). \qed

### 9 A leaf of $Ra$

The solutions of the the vector field $Ra$ in the moduli space $T$ are the locus of parameters such that all the periods of $\omega$ are constant. We want to choose a solution of $Ra$ and write it in an explicit form. We proceed as follows:

Let $\tilde{\delta}_i, \tilde{\delta}_j$, $i = 1, 2, 3, 4$ be two basis of $H_3(W_t, \mathbb{Q})$ as in §8 and let $C_{4 \times 1} = [c_1, c_2, c_3, c_4]^t$ be given by the equality

$$
[\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle]C = [1, 0, 0, 0]^t.
$$

and so $C = [0, 0, 0, -\frac{6}{5}]^t$. We are interested on the loci $L$ of parameters $s \in T$ such that

$$
\int_{\tilde{\delta}_i} \omega = c_i, \quad i = 1, 2, 3, 4.
$$

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We will write each coordinate of $s$ in terms of periods: first we note that, on this locus we have
\[
\int_{\delta_1} \omega_1 = 1
\]
because
\[
1 = \langle \omega_1, \omega \rangle = \sum_{i,j} \langle \tilde{\delta}_i, \tilde{\delta}_j \rangle \int_{\delta_i} \omega_1 \int_{\delta_j} \omega = \left[ \int_{\delta_1} \omega_1, \ldots, \int_{\delta_4} \omega_1 \right] \left[ (\tilde{\delta}_i, \tilde{\delta}_j) \right] C = \int_{\delta_1} \omega_1.
\]
By our choice $\omega_1$ does not depend on $t_1, t_2$ and $t_3$. Therefore, the locus of parameters $s$ in $T$ such that $\int_{\delta_1} \omega_1 = 1$ is given by
\[
(20) \quad (s_0, s_4) = (t_0, t_4) \cdot \int_{\delta_1} = (t_0 \int_{\delta_1} \omega_1, t_4 \int_{\delta_1} \omega_1)^5
\]
with arbitrary $s_1, s_2, s_3$. This is because for $k = (\int_{\delta_1} \omega_1)^{-1}$, we have $\int_{\delta_1} k \omega_1 = 1$ and under the identification $(t_0, t_4) \mapsto (W_{t_0, t_4}, \omega_1)$, the pair $(t_0, t_4) \cdot k^{-1}$ is mapped to $(W_{t_0, t_4}, k \omega_1)$.

To find $s_1, s_2, s_3$ parameters we proceed as follows: we know that $\omega = s_1 \omega_1 + s_2 \omega_2 + s_3 \omega_3 + \frac{\omega_4}{(\omega_1, \omega_4)}$. This together with (19) and (12) imply that
\[
[\int_{\delta_i} \omega_j]_{4 \times 4} [s_1, s_2, s_3, 625(s_4 - s_0^5)]^t = C
\]
which gives formulas for $s_1, s_2, s_3$ in terms of periods. Let us write all these in terms of the periods of the one parameter family $W_{\psi}$. Recall the notation $x_{ij}$ in the Introduction.

We have $s_0 = \psi x_{11}, s_1 = x_{11}^5$ and
\[
\int_{\delta_i} \omega_j = x_{11}^{-j} x_{ij}.
\]
Note that in the above equality the cycle $\delta_i$ lives in $W_{\psi x_{11}, x_{11}^5}$. We restrict $s_i$’s to $t_0 = \psi, t_4 = 1$, we use the equality (18) and we get:
\[
s_k = -6 \frac{(-1)^{4+k} \det [x_{11}^{-j} x_{ij}]_{i,j=1,2,3,4, i \neq 4, j \neq k}}{5 \det [x_{11}^{-j} x_{ij}]}
= -6 \frac{5^{10}}{12} (1 - \psi^5)^2 (-1)^{4+k} x_{11}^k \det [x_{ij}]_{i,j=1,2,3,4, i \neq 4, j \neq k} \cdot k = 1, 2, 3,
\]
Modulo the ideal $I$ in §8 the expressions for $s_i$’s can be reduced to to the shorter expressions in the right hand side of the equalities in Theorem 3. In the left hand side we have written $t_i$ instead of $s_i$. We also get the relation
\[
625 x_{11}^5 (1 - \psi^5) = -6 \frac{5^{10}}{12} (1 - \psi^5)^2 x_{11}^4 \det [x_{ij}]_{i,j=1,2,3}.
\]
The function $\psi \rightarrow s(\psi) := (s_0(\psi), s_1(\psi), \ldots, s_4(\psi))$ is tangent to the vector field $R\psi$ but it is not its solution. In order to get a solution, one has to make a change of variable in $\psi$. 

10 The parametrization

Let \( \tilde{\omega}_i, i = 1, 2, 3, 4 \) be the basis of the de Rham cohomology of \( W_t, t \in T \) constructed in Proposition 2. We consider the period map:

\[
\text{pm} : T \to \text{Mat}(4), \; t \mapsto \left[ \int_{\delta_t} \tilde{\omega}_j \right]_{4 \times 4},
\]

where \( \text{Mat}(4) \) is the set of \( 4 \times 4 \) matrices. By our construction of \( \tilde{\omega}_i \), its image is of dimension 5 and so it is an embedding in some open neighborhood \( U \) of a point \( p \in L \) in \( T \). We restrict its inverse \( s = (s_0, s_1, s_2, s_3, s_4) \) to \( \text{pm}(L) \), where \( L \) is defined in \( \S 9 \). Note that a point in \( \text{pm}(L) \) is of the form:

\[
P = \begin{pmatrix}
1 & p_{12} & p_{13} & 0 \\
p_\tau & p_{22} & p_{23} & 0 \\
p_{31} & p_{32} & p_{33} & 0 \\
p_{41} & p_{42} & p_{43} & -\frac{6}{5}
\end{pmatrix}.
\]

We consider \( s_0, s_1, s_2, s_3, s_4 \) and all the quantities \( p_{ij} \) as functions of \( \tau \) and set \( \dot{a} = \frac{\partial a}{\partial \tau} \). This is our derivation in (1). Note that \( \tau \) as a function in \( \psi \) is given by:

\[
\tau = \frac{\int_{\delta_2} \Omega}{\int_{\delta_1} \Omega}.
\]

We have \( \dot{s}(\tau) = x(\tau) \cdot \text{Ra}(s(\tau)) \) for some holomorphic function \( x \) in \( U \cap L \), because \( \text{Ra} \) is tangent to the locus \( L \) and \( s \) is a local parametrization of \( L \). Let \( A \) be the Gauss-Manin connection matrix of the family \( W_t, t \in T \) in the basis \( \tilde{\omega}_i, i = 1, 2, 3, 4 \). We have \( d(\text{pm}) = \text{pm} \cdot A^t \), from which it follows

\[
\begin{pmatrix}
0 & \dot{p}_{12} & \dot{p}_{13} & 0 \\
1 & \dot{p}_{22} & \dot{p}_{23} & 0 \\
\dot{p}_{31} & \dot{p}_{32} & \dot{p}_{33} & 0 \\
\dot{p}_{41} & \dot{p}_{42} & \dot{p}_{43} & 0
\end{pmatrix} = \begin{pmatrix}
1 & p_{12} & p_{13} & 0 \\
p_\tau & p_{22} & p_{23} & 0 \\
p_{31} & p_{32} & p_{33} & 0 \\
p_{41} & p_{42} & p_{43} & -\frac{6}{5}
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & 0 \\
x & 0 & x \cdot b_2(s) & 0 \\
0 & x & x \cdot b_3(s) & 0 \\
0 & 0 & x \cdot b_4(s) & 0
\end{pmatrix}.
\]

Here we have used the particular form of \( A \) in Proposition 2. The equalities corresponding to the entries \((1, i), i \geq 2\) together with the fact that \( x \neq 0, b_i(s) \neq 0 \) imply that \( p_{12} = p_{13} = 0 \). The equality for the entry \((2, 1)\) implies that \( x = \frac{1}{p_{22}} \). Using these, we have

\[
(21) \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & \dot{p}_{22} & \dot{p}_{23} & 0 \\
\dot{p}_{31} & \dot{p}_{32} & \dot{p}_{33} & 0 \\
\dot{p}_{41} & \dot{p}_{42} & \dot{p}_{43} & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\tau & p_{22} & p_{23} & 0 \\
p_{31} & p_{32} & p_{33} & 0 \\
p_{41} & p_{42} & p_{43} & -\frac{6}{5}
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & b_2(s) & 0 \\
0 & \frac{1}{p_{22}} & b_3(s) & 0 \\
0 & 0 & b_4(s) & 0
\end{pmatrix}.
\]

11 Periods

Four linearly independent solutions of (8) are given by \( \psi_0, \psi_1, \psi_2, \psi_3 \), where

\[
(22) \sum_{i=0}^{3} \psi_i(\tilde{z}) \epsilon^i + O(\epsilon^4) = \sum_{n=0}^{\infty} \frac{(1 + 5\epsilon)(2 + 5\epsilon) \cdots (5n + 5\epsilon)}{((1 + \epsilon)(2 + \epsilon) \cdots (n + \epsilon))^5} \tilde{z}^{n+\epsilon}, \; \tilde{z} = \frac{z}{5^5},
\]

15
see for instance [21]. In fact, there are four topological cycles \( \delta_1, \delta_2, \delta_3, \delta_4 \in H_3(W_z, \mathbb{Q}) \) such that

\[
\int_{\delta_i} \eta = \frac{(2\pi i)^{1-i}}{5^i} (i-1)! \psi_{i-1}.
\]

Performing the monodromy of (22) around \( z = 0 \), we get the same expression multiplied with \( e^{2\pi i \epsilon} \). Therefore, the monodromy \( \tilde{\psi}_i \) of \( \psi_i \) is given according to the equalities:

\[
\tilde{\psi}_0 = \psi_0, \quad \tilde{\psi}_1 = (2\pi i) \psi_0 + \psi_1, \quad \tilde{\psi}_2 = \frac{(2\pi i)^2}{2!} \psi_0 + (2\pi i) \psi_1 + \psi_2,
\]

\[
\tilde{\psi}_3 = \frac{(2\pi i)^3}{3!} \psi_0 + \frac{(2\pi i)^2}{2!} \psi_1 + (2\pi i) \psi_2 + \psi_3.
\]

This implies that the topological monodromy, which acts on \( H_3(W_{1,z}, \mathbb{Q}) \), in the basis \( \delta_i, \ i = 1, 2, 3, 4 \) is given by

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
1 & 3 & 3 & 1
\end{pmatrix}.
\]

Further, the intersection form in this basis is \( \Psi \) in (16), and the monodromy around the other singularity is

\[
\begin{pmatrix}
1 & -\frac{25}{6} & 0 & -\frac{5}{6} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

see for instance [31], page 5. In fact in [31] the authors have considered the basis \( C[\delta_1, \delta_2, \delta_3, \delta_4]^t \), where

\[
C = \begin{pmatrix}
0 & \frac{25}{6} & 0 & \frac{5}{6} \\
\frac{25}{6} & 0 & \frac{5}{2} & 0 \\
0 & 5 & 0 & 0 \\
5 & 0 & 0 & 0
\end{pmatrix}.
\]

Note that in the mentioned reference when the authors say that with respect to a basis \( \delta_1, \delta_2, \delta_3, \delta_4 \) of a vector space, a linear map is given by the matrix \( T \) then the action of the linear map on \( \delta_i \) is the \( i \)-th coordinate of \( [\delta_1, \delta_2, \delta_3, \delta_4]^t \) and not \( T[\delta_1, \delta_2, \delta_3, \delta_4]^t \). Define

\[
Z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\tau & 1 & 0 & 0 \\
\tau^2 & 2\tau & 2 & 0 \\
\tau^3 & 3\tau^2 & 6\tau & 6
\end{pmatrix}.
\]

Note that

\[
D = Z^{-1} \dot{Z} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

and under the monodromy \( M \), \( \tau \) goes to \( \tau + 1 \) and \( Z \) goes to \( MZ \). Therefore

\[
Q = Z^{-1} P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} (p_{31} - \tau^2) & p_{22} & p_{23} \\
\frac{1}{3} \tau^3 - \frac{1}{2} \tau p_{31} + \frac{1}{6} p_{41} & \frac{1}{2} \tau^2 p_{22} - \frac{1}{2} \tau p_{32} + \frac{1}{6} p_{42} & \frac{1}{2} \tau^2 p_{23} - \frac{1}{2} \tau p_{33} + \frac{1}{6} p_{43} & \frac{1}{5}
\end{pmatrix}
\]
is invariant under the monodromy around 0. The differential equation of \( P \) is given in (21) which we write it in the form \( \dot{P} = \frac{1}{p_{22}} P \cdot A(Ra)^t \). From this we calculate the differential equation of \( Q \):

\[
\dot{Q} = -Z^{-1} \dot{Z} Z^{-1} P + Z^{-1} \dot{P} = -DQ + \frac{1}{q_{22}} QA(Ra)^t = \frac{1}{q_{22}} \begin{pmatrix} 0 & 0 & q_{23} & 0 & 0 \\ q_{32} & -q_{22}^2 + q_{33} & q_{23}b_2 + q_{23}b_3 & 0 & 0 \\ -q_{22}q_{31} + q_{42} & -q_{22}q_{32} + q_{43} & q_{42}b_2 + q_{43}b_3 - \frac{1}{5} b_4 - q_{22}q_{33} & 0 & 0 \end{pmatrix}.
\]

Let us use the new notation \( s_5 = q_{22} \) and \( s_6 = q_{23} \). The first five lines of our differential equation (1) is just \( \dot{s} = \frac{1}{s_5} \mathcal{R}a(s) \) and the next two lines correspond to the equalities of (2,2) and (2,3) entries of the above matrices. Note that in (1) we have used the notation \( t_i \) instead of \( s_i \).

### 12 Calculating \( q \)-expansions

All the quantities \( s_i \) are invariant under the monodromy \( M \) around \( z = 0 \). This implies that they are invariant under the transformation \( \tau \rightarrow \tau + 1 \). Therefore, all \( s_i \)'s can be written in terms of the new variable \( q = e^{2\pi i \tau} \). In order to calculate all these \( q \)-expansions, it is enough to restrict to the case \( t_0 = 1, t_1 = t_2 = t_3 = 0, t_4 = z \). We want to write

\[
s_0 = \int_{s_1} \eta, \quad s_4 = z(\int_{s_1} \eta)^5 \]

in terms of \( q \). Calculating \( \psi_0 \) and \( \psi_1 \) from the formula (22) we get:

\[
\psi_0 = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \tilde{z}^m,
\]

\[
\psi_1 = \ln(\tilde{z})\psi_0(\tilde{z}) + 5\tilde{\psi}_1(\tilde{z}), \quad \tilde{\psi}_1 := \sum_{m=1}^{\infty} \frac{(5m)!}{(m!)^5} \left( \sum_{k=m+1}^{5m} \frac{1}{k} \right) \tilde{z}^m
\]

and so

\[
q = e^{-2\pi i \int_{s_1} \eta / \phi_0} = e^{\int_{s_1} \tilde{\psi}_1(\tilde{z}) / \phi_0(\tilde{z})}.
\]

By comparing few coefficients of \( \tilde{z}^i \) and we get

(24) \[
 s_0 = \int_{s_1} \eta = \frac{1}{5}(\frac{2\pi i}{5})^3 \psi_0 = \frac{1}{5}(\frac{2\pi i}{5})^3(1 + 5!q + 21000q^2 + \cdots)
\]

(25) \[
 s_4 = z(\int_{s_1} \eta)^5 = 5^5(\frac{1}{5}(\frac{2\pi i}{5})^3)^5 \tilde{z}\psi_0^5 = (\frac{2\pi i}{5})^{15}(0 + q - 170q^2 + \cdots).
\]

In the differential equation (1), we consider the weights

(26) \[
\deg(t_i) = 3(i + 1), \quad i = 0, 1, \ldots, 4, \quad \deg(t_5) = 11, \quad \deg(t_6) = 23.
\]
In this way in its right hand side we have homogeneous rational functions of degree 4, 7, 10, 13, 16, 24 which is compatible with the left hand side if we assume that the derivation increases the degree by one. We have $\partial/\partial \tau = \left(\frac{2\pi i}{5}\right)^5 q \partial/\partial q$ and so $(2\pi i)^{-\deg(t)}s_i$, $i = 0, 1, \ldots, 6$ is the solution presented in the Introduction. The initial values (2) in the Introduction are taken from the equalities (24) and (25). In the literature, see for instance [21, 28], we find also the equalities:

$$q_{31} = \frac{1}{2}(p_{31} - \tau^2) = \frac{1}{2} \left(\int_{\delta_3} \eta - \left(\int_{\delta_2} \eta\right)^2\right) = \frac{1}{(2\pi i)^2} \left(\sum_{n=1}^{\infty} \left(\sum_{d|n} n_d q^d \right) \frac{q^n}{n^2}\right),$$

$$q_{14} = \frac{1}{3} \tau^3 - \frac{1}{2} \tau p_{31} + \frac{1}{6} p_{41} = \frac{1}{(2\pi i)^3} \left(\frac{1}{3} \left(\frac{\psi_1}{\psi_0}\right)^3 - \psi_1 \psi_2 + \psi_3\right) = \frac{2}{5} \left(\sum_{n=1}^{\infty} \left(\sum_{d|n} n_d q^d \right) \frac{q^n}{n^3}\right),$$

where $n_d$ are as explained in the Introduction.

### 13 Proof of Theorem 3

The proof of the equalities for $t_0, t_1, t_3, t_4$ is done in §9. In §7 we have calculated $\tilde{\omega}_2, \tilde{\omega}_3$ in terms of $\omega_2$ and $\omega_3$. In §10 and §11 we have defined

$$s_5 = p_{22} = q_{22} = \int_{\delta_2} \tilde{\omega}_2, \quad s_6 = p_{23} = q_{23} = \int_{\delta_2} \tilde{\omega}_3.$$

Using $\int_{\delta_i} \omega_j = x_{ij}^i$ we get the expressions for $s_5, s_6$ in Theorem 3. Note that for simplicity in Theorem 3 we have again used the notation $t_i$ instead of $s_i a^{-\deg(t_i)}$, where $a = \frac{2\pi i}{5}$ and $\deg(t_i)$ is defined in (26).

### 14 Proof of Theorem 1

The Yukawa coupling $k_{\tau \tau \tau}$ is a quantity attached to the family of Calabi-Yau varieties $W_{1,z}$. It can be written in terms of periods:

$$k_{\tau \tau \tau} = -\frac{5^{-4} a^6}{(z \partial_z)^3(z - 1)(\int_{\delta_1} \eta)^2},$$

where $\tau = \int_{\delta_2} \eta$ and $a = \frac{2\pi i}{5}$, see for instance [24] page 258. In [6] the authors have calculated the $q$-expansion of the Yukawa coupling and they have reached to spectacular predictions presented in Introduction. Let us calculate the Yukawa coupling in terms of
our auxiliary quantities \( s_i \). We use the notation \( t_i = s_i a^{-\deg(t_i)} \).

\[
k_{\tau \tau \tau} = \frac{-5^{-4}a^6}{(t_0^4)^3 \left( \frac{\partial (t_0^4)}{\partial \tau} \right)^{-3}(t_0^4 - 1)(a^3t_0)^2} = \frac{-5^{-4} \left( \frac{t_4}{t_0^5} \right)^3}{(t_0^4)^3(t_0^4 - 1)t_0^2} = \frac{-5^{-4}(t_0 t_4 - 5 \bar{t}_0 t_4)^3 t_0^{12}}{t_4^3(t_4 - t_0)}
\]

Theorem 1 is proved.

15 Proof of Theorem 2

First, we note that if there is a polynomial relation with coefficients in \( \mathbb{C} \) between \( t_i, i = 0, 1, \ldots, 6 \) (as power series in \( q = e^{2\pi i \tau} \) and hence as functions in \( \tau \)) then the same is true if we change the variable \( \tau \) by some function in another variable. In particular, we put \( \tau = \frac{t_0}{t_0^{11}} \) and obtain \( t_i \)'s in terms of periods. Now, it is enough to prove that the period expressions in Theorem 3 are algebraically independent over \( \mathbb{C} \). Using Proposition 3, it is enough to prove that the variety induced by the ideal \( \hat{I} = \langle t_i - k_i, i = 0, 1, \ldots, 6 \rangle + I \subset k[y_{ij}, i, j = 1, 2, 3, 4] \) is of dimension \( 16 - 6 - 7 = 3 \). Here \( k_i \)'s are arbitrary parameters, \( I \) is the ideal in \( \mathbb{C}[k_i, i = 0, 1, \ldots, 6] \) and in the expressions of \( t_i \) we have written \( y_{ij} \) instead of \( x_{ij} \). This can be done by any software in commutative algebra (see for instance [14]).

16 Moduli space, III

In this section we introduce moduli interpretation for \( t_5 \) and \( t_6 \). Let \( \hat{\mathbf{a}} \) be the vector field in \( \mathbb{C}^7 \) corresponding to (1) and let \( \hat{\omega}_i, i = 1, 2, 3, 4 \) be the differential forms calculated in Proposition 2. Consider \( t_i, i = 0, 1, 2, \ldots, 6 \) as unknown parameters. We define a new basis \( \hat{\omega}_i, i = 1, 2, 3, 4 \) of \( H^3_{\text{IR}}(W_{t_0, t_4}) \):

\[
\hat{\omega}_1 = \omega_1, \ \hat{\omega}_2 = \frac{1}{t_5} \hat{\omega}_2, \ \hat{\omega}_3 = \frac{5^7}{(t_4 - t_0^5)^2}(-t_6 \hat{\omega}_2 + t_5 \hat{\omega}_3), \ \hat{\omega}_4 = \hat{\omega}_4.
\]

The intersection form in the basis \( \hat{\omega}_i, i = 1, 2, 3, 4 \) is a constant matrix and in fact it is:

\[
(27) \ \ \Phi := \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]
The Gauss-Manin connection composed with $\tilde{R}a$ has also the form:

$$\nabla_{\tilde{R}a} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & (4t-3)^2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is interesting that the Yukawa coupling appears as the only non constant term in the above matrix. Let $X$ be the moduli of pairs $(W,[\alpha_1,\alpha_2,\alpha_3,\alpha_4])$, where $W$ is a Calabi-Yau variety as before, $\alpha_i \in F^{4-i}\backslash F^{5-i}$, $F \subset H^3_{dR}(W)$ is the $i$-th piece of the Hodge filtration, $\alpha_i$'s form a basis of $H^3_{dR}(W)$ and the intersection form in $\alpha_i$'s is given by the matrix (27). We have the isomorphism

$$\{ t \in \mathbb{C}^3 \mid t_5 t_4 (t_4-t_0^2) \neq 0 \} \cong X$$

$$t \mapsto (W_{t_0,t_4}, [\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \hat{\omega}_4])$$

which gives the full moduli interpretation of all $t_i$'s.

17 A conjecture and concluding remarks

A conjecture: We have calculated the first eleven coefficients of

$$\frac{1}{24t_0} - \frac{24}{750}t_1 - 50t_2 - \frac{1}{5}t_3 - t_4, 25t_5, 15625t_6$$

in the differential equation (1).

<table>
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<th>$q^7$</th>
<th>$q^8$</th>
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<td>0</td>
<td>-15</td>
<td>26249</td>
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</tr>
</tbody>
</table>

We have also calculated the Yukawa coupling $\frac{-(t_4-t_0^2)^2}{625t_2^2}$. The numbers $n_d$ in the Introduction are given by:

5, 2875, 609250, 317206375, 242467530000, 229305888887625, 24824974211802200, 29509105070845659250, 375623160937476603550000, 503840510416985243645106250, 704288164978454686113488249750

Based on these calculations we may conjecture:

Conjecture 1. All $q$-expansions of

$$\frac{1}{24} t_0 - \frac{1}{120} t_1 - \frac{1}{30} t_2 - \frac{1}{50} t_3 - \frac{7}{10} t_4, 25t_5 + \frac{1}{125}, 15625t_6$$

have positive integer coefficients.
We have verified the conjecture for the coefficients of \( q^i, i \leq 50 \) (see the author’s web page\(^3\)). The rational numbers which appear in (29) are chosen in such a way that the coefficients \( t_{i,n}, n = 1,2,\ldots,10 \) become positive integers and for each fixed \( i \) they are relatively prime. Writing the series \( t_i \) as Lambert series 
\[
 a_0 + \sum_{d=1}^{\infty} a_d \frac{q^d}{1-q^d}
\]
do not help for understanding the structure of \( t_{i,n} \). It is not possible to factor out some potential of \( d \) from \( a_d \)’s for each \( t_i \). One should probably take out a polynomial in \( q \) from \( t_i \) and then try to understand the nature of the sequences. The on-line encyclopedia of integer sequences does not recognize the integer sequences of \( t_0, t_1, \ldots, t_6 \). This supports the fact that the general formula for \( t_i \)’s or any interpretation of them is not yet known.

Is there a Calabi-Yau monster?: The parameter \( j_* = z^{-1} = \frac{t_5}{t_4} \) classifies the Calabi-Yau varieties of type (3), that is, each such Calabi-Yau variety is represented exactly by one value of \( j_* \) and two such Calabi-Yau varieties are isomorphic if and only if the corresponding \( j_* \) values are equal. This is similar to the case of elliptic curves which are classified by the classical \( j \)-function (see §2). We have calculated also the \( q \)-expansions of \( j_* \):

\[
3125 \cdot j_* = \frac{1}{q} + 770 + 421375q + 274007500q^2 + 236982309375q^3 + 251719793608904q^4 + 304471126588125q^5 + 4014316741748714500q^6 + 562487442070502650877500q^7 + 824572505123979141773850000q^8 + 10134728591533847753384775272872409691q^9 + O(q^{10})
\]
The coefficient 3125 is chosen in such a way that all the coefficients of \( q^i, i \leq 9 \) in \( 3125 \cdot j_* \) are integer and all together are relatively prime. Note that the moduli parameter \( j_* \) in our case has two cusps \( \infty \) and 1, that is, for these values of \( j_* \) we have singular fibers. Our \( q \)-expansion is written around the cusp \( \infty \).

All the beautiful history behind the interpretation of the coefficients of the classical \( j \)-function of elliptic curves, monster group, monstrous moonshine conjecture and Borcherds proof, may indicate us another fascinating mathematics behind the \( q \)-expansion of the \( j_* \)-function of the varieties (3).

Symplectic basis: The basis \( \hat{\delta}_i, i = 1,2,3,4 \) given by \( [\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4]^t = C[\delta_1, \delta_2, \delta_3, \delta_4]^t \), where

\[
 C := \begin{pmatrix}
 0 & -1 & 0 & 0 \\
 -1 & 0 & 0 & 0 \\
 0 & -\tfrac{5}{6} & -\tfrac{5}{6} & 0 \\
 0 & \tfrac{5}{6} & 0 & \tfrac{5}{6}
\end{pmatrix},
\]
is the symplectic basis of \( H_3(W_\psi, \mathbb{Q}) \), that is, the intersection matrix in this basis is given by:

\[
[\langle \hat{\delta}_i, \hat{\delta}_j \rangle]_{4 \times 4} = \begin{pmatrix}
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \\
 -1 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0
\end{pmatrix}.
\]

\(^3\)http://w3.impa.br/~hossein/manyfiles/calculation-haftarticle.txt
In this basis the monodromy group $\Gamma$ of the family $W_\psi$ is generated by

$$\hat{T} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix}, \quad \hat{S} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

In a private communication, Prof. Duco van Straten informed me of the following conjecture: the group $\Gamma$ has infinite index in $\text{Sp}(4, \mathbb{Z})$.

**Modular properties of $t_i$’s:** Recall the notation in §16. There is an algebraic group which acts on the right hand side of the isomorphism (28). It corresponds to the base change in $\hat{\omega}_i$, $i = 1, 2, 3, 4$ such that the new basis is still compatible with the Hodge filtration and we have still the intersection matrix (27):

$$G := \{ g = [g_{ij}]_{4 \times 4} \in \text{GL}(4, \mathbb{C}) \mid g_{ij} = 0, \text{ for } j < i \text{ and } g^t \Phi g = \Phi \},$$

$$(W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4]) \cdot g = (W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4] g).$$

Therefore, we have the action of $G$ on the $t$-space, where $t = (t_0, t_1, \ldots, t_6)$, from the right which we denote it again by $\cdot$. The period map $t \mapsto [\int_{\hat{\delta}_i} \hat{\omega}_j]$ written in the symplectic basis gives us the modular properties of $t_i$ with respect to the monodromy group $\Gamma = \langle \hat{T}, \hat{S} \rangle$ as follows. We regard $t_i$’s as the coordinates of the inverse of the period map restricted to the matrices:

$$(30) \quad P(\tau) := \begin{pmatrix} \tau & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 \\ x_{41} & -\tau x_{32} + x_{31} & -\tau & 1 \end{pmatrix},$$

where

$$x_{31} = \frac{1}{2} (5(\tau + \tau^2) + \frac{1}{(2\pi i)^2} \sum_{d|n} n d^3 (e^{2\pi i r n} - 2n^2)),$$

$$x_{32} = x_{31}', \quad x_{41}' = x_{31} - \tau x_{31}'.$$

(recall that in this paragraph we are using the symplectic basis $\hat{\delta}_i$). For an element $A$ in the monodromy group $\Gamma$ and $\tau$ fixed, there is a unique element $g(\tau, A)$ of $G$ and a matrix $P(\tau)$ of the form (30) such that

$$A \cdot P(\tau) = P(\tau) \cdot g(\tau, A)$$

and so

$$t(\tau) = t(A \cdot P(\tau)) = t(P(\tau)) \cdot g(\tau, A).$$

This is the modular functional equation of $t$. The explicit calculation of these modular properties and the full definition of a modular or a quasi-modular form will be done in the forthcoming articles. In the case of elliptic curves, the same procedure as above gives us the modularity of the Eisenstein series $g_2$ and $g_3$ and the functional equation of $g_1$ with an anomalous shift under $\text{SL}(2, \mathbb{Z})$ (see [25]). In our context, two different phenomena appear: first we do not have a functional equation for individual $t_i$, but we have functional equations involving all $t_i$’s. Second, we have functional equations involving $x_{31}$ and its derivatives.
References


