Geometric interpretation of quasi modular forms[1]

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Abstract

In the present text we give a geometric interpretation of quasi-modular forms using moduli of elliptic curves with marked elements in their de Rham cohomology. Meanwhile, we prove that the problem of finding differential and polynomial equations for modular forms is equivalent to the problem of constructing such moduli of elliptic curves and calculating its Gauss-Manin connection.

1 Introduction

In an algebraic geometric context modular forms are interpreted in two ways: First, a modular form is a section of tensor products of the canonical bundle of moduli spaces of elliptic curves. Second, one can interpret a modular form as a function from the pairs (F, ω), where F is an elliptic curve and ω is a regular differential form on F, to the base field which has a functional property with respect to the multiplication of ω by a constant. Quasi modular forms arises in different topics (see [1] and the references within there), but still they are analytic functions on the Poincaré upper half plane and no geometric interpretation is available in the literature. It seems to me that there is no way to generalize the first interpretation of modular forms to the context of quasi modular forms, however, the second interpretation generalizes well to the context of quasi modular forms. The objective of the present text is to explain this generalization. For simplicity, we work with modular groups, $\Gamma(N), \Gamma_0(N)$ and $\Gamma_1(N)$.

Let k be any algebraically closed field of characteristic 0. [2] An enhanced elliptic curve for $\Gamma_0(N)$ is a 4-tuple $(F, C, \omega_1, \omega_2,)$, where F is an elliptic curve over k, C is a cyclic subgroup of $F(\mathsf{k})$ of order N and ω_1 and ω_2 are two elements in the algebraic de Rham cohomology of F, namely $H^1_{\mathrm{dR}}(F)$, such that

1. ω_1 is a differential form of the first kind,
2. ω_1, ω_2 form a basis of the k-vector space $H^1_{\mathrm{dR}}(F)$,
3. $$\mathrm{Tr}_{\mathrm{dR}}(\omega_1 \cup \omega_2) = 1,$$

(see [2] for the definition of algebraic de Rham cohomology and $\mathrm{Tr}_{\mathrm{dR}}$). An enhanced elliptic curve for $\Gamma_1(N)$ is a 4-tuple $(F, Q, \omega_1, \omega_2)$, where F, ω_1 and ω_2 are as before and Q is a point of $F(\mathsf{k})$ of order N. An enhanced elliptic curve for $\Gamma(N)$ is a 4-tuple $(F, (P, Q), \omega_1, \omega_2)$, where F, ω_1 and ω_2 are as before and P and Q are a pair of points of $F(\mathsf{k})$ that generates

[1] The text is under construction. Any comment is wellcome.

[2] Most probably, the arguments of the present text can be addapted to the characteristic $\neq 2, 3$

the N-torsion subgroup $F[N]$ with Weil pairing $e(P,Q)$ that is a primitive root of unity of order N.

Let Γ be one of the $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$ and T_Γ be the set of enhanced elliptic curves for Γ modulo canonical isomorphisms. The algebraic group

$$G = \left\{ \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \mid \ k' \in \mathsf{k}, k \in \mathsf{k} - \{0\} \right\}$$

acts in a canonical way on T_Γ:

$$(*, *, \omega_1, \omega_2) \bullet g = (*, *, k\omega_1, k'\omega_1 + k^{-1}\omega_2), \ g \in G, \ (*, *, \omega_1, \omega_2) \in T_\Gamma.$$

A quasi modular form f of weight m and differential order n for Γ is a function $T_\Gamma \to \mathsf{k}$ with the following properties: there are functions $f_i : T_\Gamma \to \mathsf{k}, \ i = 0, 1, 2, \dots, n$ such that

$$f \bullet g = k^{-m} \sum_{i=0}^{n} \binom{n}{i} k'^i k^i f_i, \ \forall g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G. \tag{1}$$

It satisfy also another condition which has to do with the degeneration of elliptic curves and we describe it after giving an algebraic structure to T_Γ. For $n = 0$ we recover the definition of modular forms of weight m.

Now assume that k is a subfield of $\mathbb{C}$. In the present text we prove that T_Γ has a canocial structure of an affine variety such that the action of G becomes algebraic. We construct another affine variety A_Γ and the action of G on A_Γ such that $A_\Gamma - T_\Gamma$ is an invariant divisor under G and T_Γ/G is the modular curve for Γ and A_Γ/G is its compactification. We further construct a vector field R_Γ on A_Γ which sends quasi modular forms of weight m and order n to quasi modular forms of weight $m+2$ and order $n+1$. The second part in the definition of a quasi modular form is that it extends to a regular function on A_Γ. For $\mathsf{k} = \mathbb{C}$ we prove that the function filed of A_Γ and its differential algebra structure given by the vector field R_Γ is isomorphic to the classical differential algebra of quasi modular forms availble in the the literature.

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2 Moduli spaces of elliptic curves I

Recall the notations introduced in Introduction. In this section we consider $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. For simplicity we drop the subscript Γ from our data associated to Γ.

Let T be the moduli space of the triples (F, ω_1, ω_2), where F is an elliptic curve over k and $\omega_1, \ \omega_2 \in H^1_{\mathrm{dR}}(F)$ are as described in Introduction. Let also $\pi : E \to T$ be the corresponding universal family of elliptic curves, where

$$E = \cup_{t=(F,\omega_1,\omega_2)\in T} F$$

and

$$A := \mathrm{Spec}(\mathsf{k}[t_1, t_2, t_3]).$$

Many of the following statements are classical. They can be found in [3]. We have

$$T \cong \mathrm{Spec}(\mathsf{k}[t_1, t_2, t_3, \frac{1}{\Delta}]) \subset A, \ \Delta := 27t_3^2 - t_2^3,$$

and under this isomorphy the action of the algebraic group G is given by

$$t \bullet g := (t_1 k^{-2} + k' k^{-1}, t_2 k^{-4}, t_3 k^{-6}), t = (t_1, t_2, t_3), g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G.$$

Let $\tilde{E}$ be the affine subvariety of $\mathrm{Spec}(\mathsf{k}[x, y, t_1, t_2, t_3])$ given by:

$$\tilde{E} : y^2 - 4(x - t_1)^3 + t_2(x - t_1) + t_3 = 0, \tag{2}$$

One can realize E in an affine coordinates (x, y, t_1, t_2, t_3) as a variety given by $\tilde{E} - \{\Delta \neq 0\}$. In this way $\pi : E \to T$ is the projection in (t_1, t_2, t_3) and associated to $t \in T$ one has the triple $(\pi^{-1}(t),\ [\frac{dx}{y}],\ [\frac{xdx}{y}])$.

In A we consider the Ramanujan vector field:[3]

$$\mathrm{R} = (t_1^2 - \frac{1}{12} t_2)\frac{\partial}{\partial t_1} + (4t_1 t_2 - 6t_3)\frac{\partial}{\partial t_2} + (6t_1 t_3 - \frac{1}{3} t_2^2)\frac{\partial}{\partial t_1} \tag{3}$$

It is characterized uniquely and up to a multiplication by a constant in k by the proprty

$$\nabla_{\mathrm{R}}(\omega_2) = 0,$$

where ∇ is the Gauss-Manin connection of the family $\pi : E \to T$. For an explicit expression of ∇ in the coordinates t_1, t_2, t_3 and associated to the the basis $[\frac{dx}{y}]$, $[\frac{xdx}{y}]$ of the de Rham cohomology $H^1_{\mathrm{dR}}(E/T)$ see [3], (29).

3 Moduli of elliptic curves II

In this section we assume that $\mathsf{k} \subset \mathbb{C}$. Recall also that k is an algebraically closed field. Let us fix $b \in T(\mathsf{k})$ and $\delta_1, \delta_2 \in H_1(E_b, \mathbb{Z})$ such that $\langle \delta_1, \delta_2 \rangle = 1$. We obtain a morphism of groups

$$\mathsf{Mon} : \pi_1(T(\mathbb{C}), b) \to \mathrm{SL}(2, \mathbb{Z})$$

which corresponds to the monodromy group of the family of elliptic curves $\pi : E \to T$, see [3].

Proposition 1. *Let Γ be a normal subgroup of finite rank of $\mathrm{SL}(2,\mathbb{Z})$ and k be a subfield of $\mathbb{C}$. There is an affine variety A_Γ, an action of G from the right on A_Γ, a morphisim $\beta : A_\Gamma \to A$, a vector field R_Γ in A_Γ, all of them defined over k, and a k-point 0_Γ of A_Γ such that*

1. *The induced map $T_\Gamma \to T$, where $T_\Gamma := \beta^{-1}(T)$, is étale.*
2. *The image of the composition*

$$\pi_1(T_\Gamma(\mathbb{C}), \tilde{b}) \hookrightarrow \pi_1(T(\mathbb{C}), b) \stackrel{\mathsf{Mon}}{\to} \mathrm{SL}(2, \mathbb{Z})$$

is Γ, where $\tilde{b}$ is any point in $\beta^{-1}(b)$.

[3] After an affine transformation $(t_1, t_2, t_3) \mapsto (\frac{1}{12}t_1, \frac{1}{12}t_2, \frac{2}{3(12)^2}t_3)$ one gets a vector field for which the corresponding ordinary differential equation is the Ramanujan relation between Eisenstein series, see [5], p. 4.

3. *There is an integer n depending only on Γ such that*

$$\beta(x) \bullet g^n = \beta(x \bullet g),\ \ g \in G, x \in A_\Gamma,$$

4. *We have $\beta(0_\Gamma) = 0 \in A(\mathsf{k})$, the stablizer of 0_Γ is G, and $(A_\Gamma - \{0_\Gamma\})/G$ is a complete variety.*

5. *β maps R_Γ to R,*

Sketch of the proof. Let M be the universal covering of $T(\mathbb{C})$. Define

$$\tilde{M} := M/\sim,$$

where

$$a, b \in M,\ a \sim b \Leftrightarrow\ a \text{ and } b \text{ have the same end point and } \mathsf{Mon}(ab^{-1}) \in \Gamma.$$

We have a canonical finite covering map $\tilde{M} \to T(\mathbb{C})$. This map can be realized as morphism of algebraic varieties over k, i.e there is an algebraic variety T_Γ and a morphisim $T_\Gamma \to T$, both defined over k, such that $T_\Gamma(\mathbb{C})$ is biholomorphic to $\tilde{M}$ and under this biholomorphism the induced map $M_\Gamma(\mathbb{C}) \to T(\mathbb{C})$ is our finite covering map, see [**?**].

In the definition of Mon if we choose another basis $\tilde{\delta}_1, \tilde{\delta}_2 \in H_1(E_b, \mathbb{Z})$ with $\langle \tilde{\delta}_1, \tilde{\delta}_2 \rangle = 1$ then the new monodromy map is $A \cdot \mathsf{Mon} \cdot A^{-1}$, where

$$\begin{pmatrix} \tilde{\delta}_1 \\ \tilde{\delta}_2 \end{pmatrix} = A \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix},\ A \in \mathrm{SL}(2, \mathbb{Z}).$$

Since Γ is a normal subgroup of $\mathrm{SL}(2, \mathbb{Z})$, we conclude that the definition of $\tilde{M}$ does not depend on the choice of $\delta_1,\ \delta_2$. Moreover, different choices of the base point b yields to biholomorphic complex manifolds $\tilde{M}$. The fact that Γ is normal in $\mathrm{SL}(2, \mathbb{Z})$ implies that the property in item 2 does not depend on the choice of $\tilde{b}$.

Now, we want to construct the following commutative diagram of affine varieties:

$$\begin{array}{ccccccc} E & \to & T & \hookrightarrow & A & \leftarrow & \tilde{E} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ F & \to & T_\Gamma & \hookrightarrow & A_\Gamma & \leftarrow & \tilde{F} \end{array}$$

Since all the varieties we want to construct are affine, it is enough to work in the ring level. For an affine variety V let $\mathcal{O}_V$ be the ring of regular functions on V, i.e $V = \mathrm{Spec}(\mathcal{O}_V)$. Therefore we want to construct

$$\begin{array}{ccccccc} \mathcal{O}_E & \hookleftarrow & \mathcal{O}_T & \hookleftarrow & \mathcal{O}_A & \hookrightarrow & \mathcal{O}_{\tilde{E}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_F & \hookleftarrow & \mathcal{O}_{T_\Gamma} & \hookleftarrow & \mathcal{O}_{A_\Gamma} & \hookrightarrow & \mathcal{O}_{\tilde{F}} \end{array}$$

We have $\mathcal{O}_A = \mathsf{k}[t_1, t_2, t_3]$ and $\mathcal{O}_T = \mathsf{k}[t_1, t_2, t_3, \frac{1}{\Delta}]$ and we define

$$\mathcal{O}_F := \mathcal{O}_E \otimes_{\mathcal{O}_T} \mathcal{O}_{T_\Gamma}$$

We also define $\mathcal{O}_{A_\Gamma}$ to be:

$$\mathcal{O}_{A_\Gamma} := \{ f \in \mathcal{O}_{T_\Gamma} \mid f \text{ satisfy a monic polynomial with coefficient in } \mathcal{O}_A \}.$$

Note that $\mathcal{O}_{T_\Gamma}$ is an integral extension of $\mathcal{O}_T$, i.e. any element of $\mathcal{O}_{T_\Gamma}$ satisfy a monic polynomial with coefficients in $\mathcal{O}_T$. Finaly we define

$$\mathcal{O}_{\tilde{F}} := \mathcal{O}_{\tilde{E}} \otimes_{\mathcal{O}_A} \mathcal{O}_{A_\Gamma}$$

All the morphisims in the above diagram are canonicals. The items 1,2,3 follows from the above construction. Since $\mathcal{O}_{T_\Gamma}$ is an integral extention of $\mathcal{O}_T$, the item 4 follows from [**?**], and hence by definition G acts on $\mathcal{O}_{A_\Gamma}$.

Since the stablizer of 0 is the whole G and β commutes with β, G acts on the finite set $\beta^{-1}(0)$. Since there is no non trivial algebraic subgroup H of G such that H/G is finite, we conclude that $\beta^{-1}(0)$ consists of one point, namely 0_Γ.

A vector field on an affine variety V is a map $v : \mathcal{O}_V \to \mathcal{O}_V$ which satisfies the additivity $v(a+b) = v(a) + (b)$ and the Leibniz rule, $v(ab) = v(a)b + av(b)$ for all $a, b \in \mathcal{O}_V$. v is called also a derivation. Item 5 follows from the following well-known facts: Let R be an integral extension of a ring S. Any derivation in R extends in a unique way to a derivation on S, see [**?**].

□

Proposition 2. *Let Γ be one of $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$. The affine variety T_Γ is the moduli of enhanced elliptic curves for Γ.*

For a proof see [**?**].

4 Quasi modular form

In this section we recall the definition of quasi/differential modular forms. For more details see [1, 3]. We use the notations $A = \begin{pmatrix} a_A & b_A \\ c_A & d_A \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

When there is no confusion we will simply write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote by $\mathbb{H}$ the Poincaré upper half plane and

$$\mathrm{j}(A, z) := c_A z + d_A.$$

For $A \in \mathrm{GL}(2, \mathbb{R})$ and $m \in \mathbb{Z}$ we use the slash operator

$$f|_m A = (\det A)^{m-1} \mathrm{j}(A, z)^{-m} f(Az).$$

Let Γ be a subgroup of $\mathrm{SL}(2, \mathbb{Z})$. We define the notion of an $M_m^n(\Gamma)$-function, a differential modular form of weight m and differential order n for Γ. For simplicity we write $M_m^n(\Gamma) = M_m^n$. For $n = 0$ an M_m^0-function is a classical modular form of weight m on $\mathbb{H}$ (see bellow). A holomorphic function f on $\mathbb{H}$ is called M_m^n if the following two conditions are satisfied:

1. There are holomorphic functions $f_i, \ i = 0, 1, \ldots, n$ on $\mathbb{H}$ such that

$$f|_m A = \sum_{i=0}^{n} \binom{n}{i} c_A^i \mathrm{j}(A, z)^{-i} f_i, \ \forall A \in \Gamma. \tag{4}$$

2. $f_i \mid_m A, i = 0, 1, 2, \dots, n$ have finite growths when $\mathrm{Im}(z)$ tends to $+\infty$ for all $A \in \mathrm{SL}(2,\mathbb{Z})$.

We will also denote by M_m^n the set of M_m^n-functions and we set

$$M := \sum_{m\in\mathbb{Z}, n\in\mathbb{N}_0} M_m^n$$

For an $f \in M_m^n$ we have $f|_m I = f_0$ and so $f_0 = f$. Note that for an M_m^n-function f the associated functions f_i are unique. If f is M_m^n-function with the associated functions f_i then f_i is an M_{m-2i}^{n-i}-function with the associated functions $f_{ij} := f_{i+j}$. The set M is a bigraded differential $\mathbb{C}$-algebra:

$$\frac{d}{dz} : M_m^n \to M_{m+2}^{n+1}$$

If $n \le n'$ then $M_m^n \subset M_m^{n'}$ and

$$M_m^n M_{m'}^{n'} \subset M_{m+m'}^{n+n'},\ M_m^n + M_m^{n'} = M_m^{n'}$$

It is useful to define

$$(5) \qquad f||_m A := (\det A)^{m-n-1} \sum_{i=0}^{n} \binom{n}{i} c_{A^{-1}}^{i} \mathrm{j}(A,z)^{i-m} f_i(Az),\ A \in \mathrm{GL}(2,\mathbb{R}),\ f \in M_m^n.$$

The equality (4) is written in the form

$$(6) \qquad f = f||_m A, \forall A \in \Gamma$$

One can prove that

$$f||_m A = f||_m (BA),\ \forall A \in \mathrm{GL}(2,\mathbb{R}),\ B \in \Gamma,\ f \in M_m^n.$$

Using this one can prove that the growth at infinity condition on f is a finite number of conditions for $f||_m \alpha,\ \alpha \in \Gamma\backslash \mathrm{SL}(2,\mathbb{Z})$.

The relation of $||_m$ with $\frac{d}{dz}$ is given by:

$$(7) \qquad \frac{d(f||_m A)}{dz} = \frac{df}{dz}||_{m+2} A,\ \forall A \in \mathrm{GL}(2,\mathbb{R}).$$

Let $A \in \mathrm{SL}(2,\mathbb{Z})$. If $f \in M_m^n(\Gamma)$ with the associated functions f_i then $f||_m A \in M_m^n(A^{-1}\Gamma A)$ with the associated functions $f_i||_m A \in M_{m-2i}^{n-i}(A^{-1}\Gamma A)$.

From now on we assume that there is $h \in \mathbb{N}$ such that

$$T_h := \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$$

Take h the smallest one. Recall that Γ is a normal subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For an $f \in M_m^n(\Gamma)$ and $A \in \mathrm{SL}(2,\mathbb{Z})$ with $[A] = \alpha \in \Gamma\backslash \mathrm{SL}(2,\mathbb{Z})$ we have $(f||_m A)|_m T_h = f$ and so we can write the Fourier expansion of $f||_m A$ at α

$$(f||_m A) = \sum_{n=0}^{+\infty} a_n q_h^n,\ a_n \in \mathbb{C},\quad q_h := e^{2\pi i h z}.$$

We have used the growth condition on f to see that the above function in q_h is holomorphic at 0.

5 Main theorem

Let $\mathcal{O}_{A_\Gamma}$ be the ring of regular functions on A_Γ. Our main theorem in this article is:

Theorem 1. *The differential algebras* $(M, \frac{d}{dz})$ *and* $(\mathcal{O}_{A_\Gamma(\mathbb{C})}, \mathrm{R}_\Gamma)$ *are isomorphic.*

In order to prove this theorem we need the notion of period domain and period map.

6 Period domain

In this section k is the field of complex numbers. Differential modular forms are best viewed as holomorphic functions on the period domain

$$\mathcal{P} := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \mid \mathrm{Im}(x_1\overline{x_3}) > 0 \right\}, \tag{8}$$

We let the group $\mathrm{SL}(2,\mathbb{Z})$ (resp. G) act from the left (resp. right) by usual multiplication of matrices. The Poincaré upper half plane $\mathbb{H}$ is embedded in $\mathcal{P}$ in the following way:

$$z \to \tilde{z} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}.$$

We denote by $\tilde{\mathbb{H}}$ the image of $\mathbb{H}$ under this map. For $\alpha \in \mathrm{SL}(2,\mathbb{Z})$ we also define $\tilde{\mathbb{H}}_\alpha$ to be the image of $\tilde{\mathbb{H}}$ under the action of α from the left on $\mathcal{P}$.

A differential modular form $f \in M_m^n$ is in a one to one correspondance with a holomorphic function $F = \phi(f) : \mathcal{P} \to \mathbb{C}$ with the following properties:

1. The function F is Γ-invariant.
2. There are holomorphic functions $F_i : \mathcal{P} \to \mathbb{C},\ i = 0, 1, \ldots, n$ such that

$$F(x \cdot g) = k^{-m} \sum_{i=0}^{n} \binom{n}{i} k'^i k^i F_i(x),\ \forall x \in \mathcal{P},\ g \in G, \tag{9}$$

3. For all $\alpha \in \mathrm{SL}(2,\mathbb{Z})$ the restriction of F_i to $\tilde{\mathbb{H}}_\alpha$ has finite growth at infinity..

In fact we have $F_i = \phi(f_i)$.

It is a mere calculation to see that the vector field

$$X := x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3} \tag{10}$$

is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ and hence it induces a vector field in the quotient $\Gamma\backslash\mathcal{P}$. Now, viewing differential modular forms as functions on $\Gamma\backslash\mathcal{P}$, the differential operator is given by the vector field X.

7 Period map

The period map is defined by

$$\mathsf{pm} : T_\Gamma(\mathbb{C}) \to \Gamma\backslash\mathcal{P},\ t \mapsto \left[\frac{1}{\sqrt{2\pi i}} \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix} \right].$$

where δ_1, δ_2 is a basis of the $\mathbb{Z}$-module $H_1(E_{\beta(t)}, \mathbb{Z})$ with $\langle \delta_1, \delta_2 \rangle = 1$. It is well-defined and is a biholomorphic map. Further it satisfies

$$\mathsf{pm}(t \bullet g) = \mathsf{pm}(t) \cdot g,\ t \in T_\Gamma(\mathbb{C}),\ g \in G. \tag{11}$$

and

$$d\mathsf{pm}(t)(\mathrm{R}_\Gamma) = X$$

where X is the vector field given by (10) in the quotient $\Gamma\backslash\mathcal{P}$.

8 Proof of Theorem 1

The period map which is a biholomorphism gives us the desired isomorphism of algebras. Under the period map an M^n_m-function corresponds to a regular function $p \in \mathcal{O}_{A_\Gamma(\mathbb{C})}$ with the following property: There are $f_i \in \mathcal{O}_{A_\Gamma(\mathbb{C})}$ such that (1) in Introduction is valid.

9 Fundamental differential/functional equations for modular forms

Let us take a basis $s_1, s_2, \ldots, s_m$ of the k-algebra $\mathcal{O}_{A_\Gamma}$. This gives an embedding of A in $\mathbb{A}^m_{\mathsf{k}}$. Let also

$$I = \ker(\mathsf{k}[x_1, x_2, \ldots, x_m] \to \mathcal{O}_{A_\Gamma}),\ x_i \mapsto s_i$$

and take a set of generators $\{p_1, p_2, \ldots, p_n\}$ of the ideal I.

For $\mathsf{k} = \mathbb{C}$ Theorem 1 implies that there is a finite set $f_1, f_2, \ldots, f_m$ of generators of the $\mathbb{C}$-algebra of quasi modular forms such that they satisfy the functional relations

$$p_i(f_1, f_2, \ldots, f_m) = 0,\ i = 1, 2, \ldots, n$$

and the ordinary differential equation

$$\frac{dF}{dz}(z) = \mathrm{R}_\Gamma(F(z)),\ F = (f_1, f_2, \ldots, f_m) : \mathbb{H} \to \mathbb{C}^m$$

Example 1. $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. This case is discussed in detail in [3]. The algebra of quasi modular forms for Γ is freely generated by the Eisenstein series

$$g_k(z) = a_k\Big(1 + (-1)^k \frac{4k}{B_k} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i z n}\Big),\quad k = 1, 2, 3,\ z \in \mathbb{H}, \tag{12}$$

where B_k is the k-th Bernoulli number ($B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ \ldots$), $\sigma_i(n) := \sum_{d|n} d^i$,

$$(a_1, a_2, a_3) = (\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3), \tag{13}$$

The differential equation between g_i's are given by the Ramanujan relations:

$$\frac{dg_1}{dz} = g_1^2 - \frac{1}{12} g_2,\ \frac{dg_2}{dz} = 4g_1 g_2 - 6g_3,\ \frac{dg_3}{dz} = 6g_1 g_3 - \frac{1}{3} g_2^2 \tag{14}$$

Example 2. $\Gamma = \Gamma(2)$. The affine variety A_Γ in this case is $\mathrm{Spec}(\mathsf{k}[t_1, t_2, t_3])$ and the discriminant variety is given by $(t_1 - t_2)(t_2 - t_3)(t_3 - t_1) = 0$. The differential equation corresponding to the vector field R_Γ is:

$$\mathrm{H}: \begin{cases} \dot{t}_1 = t_1(t_2 + t_3) - t_2 t_3 \\ \dot{t}_2 = t_2(t_1 + t_3) - t_1 t_3 \\ \dot{t}_3 = t_3(t_2 + t_3) - t_1 t_2 \end{cases} \tag{15}$$

which is called also the Halphen equation, becuase he expressed a solution of the system (15) in terms of the logarithmic derivatives of the null theta functions; namely,

$$u_1 = \frac{1}{2}(\ln \theta_4(0|z))', u_2 = \frac{1}{2}(\ln \theta_2(0|z))', u_3 = \frac{1}{2}(\ln \theta_3(0|z))'$$

where

$$\begin{cases} \theta_2(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \\ \theta_3(0|z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \\ \theta_4(0|z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \end{cases}, \ q = e^{2\pi i z}, \ z \in \mathbb{H}.$$

u_1, u_2, u_3 form a basis of the $\mathbb{C}$-algebra of quasi-modular forms for Γ.

Example 3. $\Gamma = \Gamma(3)$. Let

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \ q = e^{2\pi i z}$$

be the Dedekind's η-function. In [4] Y. Ohyama has found that

$$W = (3 \log \eta(\frac{z}{3}) - \log \eta(z))' \tag{16}$$

$$X = (3 \log \eta(3z) - \log \eta(z))' \tag{17}$$

$$Y = (3 \log \eta(\frac{z+2}{3}) - \log \eta(z))' \tag{18}$$

$$Z = (3 \log \eta(\frac{z+1}{3}) - \log \eta(z))' \tag{19}$$

satisfy the equations:

$$\begin{cases} t_1' + t_2' + t_3' = t_1 t_2 + t_2 t_3 + t_3 t_1 \\ t_1' + t_3' + t_4' = t_1 t_3 + t_3 t_4 + t_4 t_1 \\ t_1' + t_2' + t_4' = t_1 t_2 + t_2 t_4 + t_4 t_1 \\ t_2' + t_3' + t_4' = t_2 t_3 + t_3 t_4 + t_4 t_2 \\ \zeta_3^2(t_2 t_4 + t_3 t_1) + \zeta_3(t_2 t_1 + t_3 t_4) + (t_2 t_3 + t_4 t_1) = 0. \end{cases}$$

where $\zeta_3 = e^{\frac{2\pi i}{3}}$. We write the first four lines of the above equation as a solution to a vector field V in $\mathbb{C}^4$ and let $F(t_1, t_2, t_3, t_4)$ be the polynomial in the fifth line. Using a computer, or by hand if one has a good patience for calculations, one can verify the equality $dF(V) = 0$ and so V is tangent to $F = 0$. This discussion suggest that A_Γ is given by $F = 0$ and $\mathrm{R}_\Gamma = V \mid_{\{F=0\}}$.

References

[1] F. Martin and E. Royer. Formes modulaires et périodes, Formes modulaires et transcendance, Sémin. Congr., Soc. Math. France, Vol. 12, 1–117, 2005.

[2] Milne, James S., Étale cohomology. Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980.

[3] H. Movasati. On Differential modular forms and some analytic relations between Eisenstein series, to appear in Ramanujan Journal.

[4] Yousuke Ohyama. Differential equations for modular forms of level three. *Funkcial. Ekvac.*, 44(3):377–389, 2001.

[5] Y.V. Nesterenko and P. Philippon (Editors). *Introduction to algebraic independence theory*, volume 1752 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.