Around Hilbert 16-th Problem

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Introduction

The present text is the lecture notes of a mini-course given at Santiago-Chile, 17-25 July. Its objective is to introduce the reader with some problems arising from the centennial Hilbert 16-th problem, H16 for short. Our aim is not to collect all the developments and theorems in direction of H16 (for this see for instance [9]), but to present a way of breaking the problem in many pieces and observing the fact that even such partial problems are extremely difficult to treat. Our point of view is algebraic and we want to point out that the both real and complex algebraic geometry would be indispensable for a systematic approach to the H16. Here is Hilbert's announcement of the problem:

16. Problem der Topologie algebraischer Curven und Flächen

Die Maximalzahl der geschlossenen und getrennt liegenden Züge, welche eine ebene algebraische Curve n ter Ordnung haben kann, ist von Harnack Mathematische Annalen, Bd. 10 bestimmt worden; es entsteht die weitere Frage nach der gegenseitigen Lage der Curvenzüge in der Ebene. Was die Curven 6ter Ordnung angeht, so habe ich mich - freilich auf einem recht umständlichen Wege - davon überzeugt, daß die 11 Züge, die sie nach Harnack haben kann, keinesfalls sämtlich außerhalb von einander verlaufen dürfen, sondern daß ein Zug existiren muß, in dessen Innerem ein Zug und in dessen Aeußerem neun Züge verlaufen oder umgekehrt. Eine gründliche Untersuchung der gegenseitigen Lage bei der Maximalzahl von getrennten Zügen scheint mir ebenso sehr von Interesse zu sein, wie die entsprechende Untersuchung über die Anzahl, Gestalt und Lage der Mäntel einer algebraischen Fläche im Raume - ist doch bisher noch nicht einmal bekannt, wieviel Mäntel eine Fläche 4ter Ordnung des dreidimensionalen Raumes im Maximum wirklich besitzt. Vgl. Rohn, Flächen vierter Ordnung, Preisschriften der Fürstlich Jablonowskischen Gesellschaft, Leipzig 1886

Im Anschlußan dieses rein algebraische Problem möchte ich eine Frage aufwerfen die sich, wie mir scheint, mittelst der nämlichen Methode der continuirlichen Coefficientenänderung in Angriff nehmen läßt, und deren Beantwortung für die Topologie der durch Differentialgleichungen definirten Curvenschaaren von entsprechender Bedeutung ist - nämlich die Frage nach der Maximalzahl und Lage der Poincaréschen Grenzcykeln (cycles limites) für eine Differentialgleichung erster Ordnung und ersten Grades von der Form:

$$dy/dx = Y/X,$$

wo X, Y *ganze rationale Funktionen nten Grades in* x, y *sind, oder in homogener Schreibweise*

$$X(ydz/dt - zdy/dt) + Y(zdx/dt - xdz/dt) + Z(xdy/dt - ydx/dt) = 0$$

wo X, Y, Z *ganze rationale homogene Functionen nten Grades von* x, y, z *bedeuten und diese als Funktionen des Parameters* t *zu bestimmen sind.*

16. Problem of the topology of algebraic curves and surfaces

The maximum number of closed and separate branches which a plane algebraic curve of the n-th order can have has been determined by Harnack. There arises the further question as *to the relative position of the branches in the plane. As to curves of the 6-th order, I have satisfied myself-by a complicated process, it is true-that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the maxi mum number of sheets which a surface of the 4-th order in three dimensional space can really have.36*

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincar's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form

$$dy/dx = Y/X,$$

where X and Y are rational integral functions of the n-th degree in x and y. Written homogeneously, this is

$$X(ydz/dt - zdy/dt) + Y(zdx/dt - xdz/dt) + Z(xdy/dt - ydx/dt) = 0$$

where X, Y, and Z are rational integral homogeneous functions of the n-th degree in x, y, z, and the latter are to be determined as functions of the parameter t.

Preliminaries

Here some notations that we use:

- For a topological space X and $x \in X$ we denote by (X, x) an small neighborhood of x in X.

- We will use the fields $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $\mathbb{K}[x, y]$ the ring of polynomials in x, y with coefficients in $\mathbb{K}$.

- The set of polynomial differential 1-forms

$$\Omega^1_{\mathbb{K}^2} := \{Pdy - Qdx \mid P, Q \in \mathbb{K}[x, y]\},$$

  and differential two forms

$$\Omega^2_{\mathbb{K}^2} = \{Pdx \wedge dy \mid P \in \mathbb{K}[x, y]\}.$$

  One usually defines:

$$\Omega^0_{\mathbb{K}^2} := \mathbb{K}[x, y].$$

- The wedge product is defined in the following way:

$$(P_1 dx + Q_1 dy) \wedge (P_2 dx + Q_2 dy) = (P_1 Q_2 - P_2 Q_1) dx \wedge dy.$$

  Exercise 0.1. Verify that for all $\omega_1, \omega_2 \in \Omega^1_{\mathbb{K}^2}$ we have $\omega_1 \wedge \omega_1 = 0$ and $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$.

- We have the differential maps:

$$d_0 : \Omega^0_{\mathbb{K}^2} \to \Omega^1_{\mathbb{K}^2}, \quad d_0(P) = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy.$$

$$d_1 : \Omega^1_{\mathbb{K}^2} \to \Omega^2_{\mathbb{K}^2}, \quad d_1(Pdx + Qdy) = dP \wedge dx + dQ \wedge dy.$$

  Exercise 0.2. Show that $d_1 \circ d_0 = 0$.

  We will usually drop the sub index 0 and 1 and simply write $d = d_0, d = d_1$.

  Exercise 0.3. If $d\omega = 0$ for some $\omega \in \Omega^1_{\mathbb{K}^2}$ then there is a $f \in \Omega^0_{\mathbb{K}^2}$ such that $\omega = df$.

- Stokes formula. Let δ be a closed anti-clockwise oriented path in $\mathbb{R}^2$ which does not intersect itself. Let also Δ be the region in $\mathbb{R}^2$ which δ encloses. Then

$$\int_\delta \omega = \int_\Delta d\omega.$$

  Exercise 0.4. Give a proof of Stokes formula using the classical books in calculus.

- Let $\gamma = (x(t), y(t)) : (\mathbb{K}, 0) \to \mathbb{K}^2$ be an analytic map and $\omega = Pdx + Qdy \in \Omega^1_{\mathbb{K}^2}$. The pull-back of ω by γ is defined to be

$$\gamma^*\omega := (P(x(t), y(t))\frac{\partial x(t)}{\partial t} + Q(x(t), y(t))\frac{\partial y(t)}{\partial t})dt$$

Exercise 0.5. Show that $\gamma^*\omega = 0$ is independent of the parametrization t, i.e if $a : (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ is an analytic map and $\gamma^*\omega = 0$ then $(\gamma \circ a)^*\omega = 0$.

If $\gamma^*\omega = 0$ then we say that ω restricted to the image of γ is zero.

- We denote by

$$\mathbb{K}(x, y) := \{\frac{P}{Q} \mid P, Q \in \mathbb{K}[x, y]\}$$

the field of meromorphic functions in $\mathbb{K}^2$. The set of meromorphic differential i-forms is denoted by $\Omega^i_{\mathbb{K}^2}(*)$(instead of $\mathbb{K}[x, y]$ we have used $\mathbb{K}(x, y)$).

Exercise 0.6. Show that if for $\omega_1, \omega_2 \in \Omega^1_{\mathbb{K}^2}(*)$ we have $\omega_1 \wedge \omega_2$ then $\omega_2 = R\omega_1$ for some $R \in \mathbb{K}(x, y)$. Formulate the problem using polynomial forms and prove it.

- For $\Omega \in \Omega^2_{\mathbb{K}^2}$ and $\omega \in \Omega^1_{\mathbb{K}^2}$ we denote by $\frac{\Omega}{\omega}$ any meromorphic differential 1-form α such that

$$\Omega = \omega \wedge \alpha.$$

Exercise 0.7. Show that such an α exists and is defined up to addition by an element in $K(x, y)\omega$.

- We will use $X_{\mathbb{R}}$ and $X_{\mathbb{C}}$ to distinguish between real and complex objects.

Chapter 1

Polynomial differential equation

In this chapter we introduce limit cycles of polynomial differential equations in $\mathbb{R}^2$ and state the Hilbert 16-th problem.

1.1 Real foliations

What we want to study is the following ordinary differential equation:

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}, \tag{1.1}$$

where P, Q are two polynomials in x and y with coefficients in $\mathbb{R}$ and $\dot{x} = \frac{dx}{dt}$. We will assume that P and Q do not have common factors. Its solutions are the trajectories of the vector field:

$$X := P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

(we will also write $X = (P, Q)$). Let us first recall the first basic theorem of ordinary differential equations.

Theorem 1.1. *For $A \in \mathbb{R}^2$ if $X(A) \neq 0$ then there is a unique analytic function*

$$\gamma : (\mathbb{R}, 0) \to \mathbb{R}^2$$

such that

$$\gamma(0) = A, \ \dot{\gamma} = X(\gamma(t))$$

Proof. Let us write formally

$$\gamma = \sum_{i=0}^{\infty} \gamma_i t^i, \ \gamma_i \in \mathbb{R}^2, \ \gamma_0 := A$$

and substitute it in $\dot{\gamma} = X(\gamma)$. It turns out that if $X(A) \neq 0$ then γ_i can be written in a unique way in terms of of $\gamma_j, \ j < i$. This guaranties the existence of a unique formal γ. □

Exercise 1.1. Recover the proof of convergence of γ from classical books on ordinary differential equations.

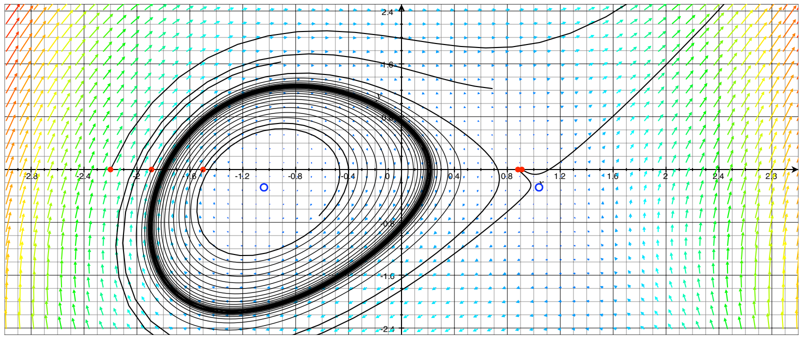

Figure 1.1: A limit cycle crossing $(x, y) \sim (-1.79, 0)$

Exercise 1.2. Describe the trajectories of the following differential equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}, \quad \begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}, \quad \begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases}$$

Example 1.1. The trajectories of the differential equation

$$\begin{cases} \dot{x} = 2y + \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + 0.9y \end{cases} \tag{1.2}$$

are depicted in Figure (1.1).

The collection of the images of the solutions of (1.1) gives us us an analytic singular foliation $\mathcal{F} = \mathcal{F}(X)_{\mathbb{R}} = \mathcal{F}(X) = \mathcal{F}_{\mathbb{R}}$ in $\mathbb{R}^2$. Therefore, when we are talking about a foliation we are not interested in the parametrization of its leaves(trajectories). It is left to the reader to verify that:

Exercise 1.3. For a polynomial $R \in \mathbb{R}[x, y]$ the foliation associated to X and RX in $\mathbb{R}^2 \backslash \{R = 0\}$ are the same.

For this reason from the beginning we have assumed that P and Q have no common factors. Being interested only on the foliation $\mathcal{F}(X)$, we may write (1.1) in the form

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)},$$

$$\omega = 0, \quad \text{where } \omega = Pdy - Qdx \in \Omega^1_{\mathbb{R}^2}.$$

In the second case we use the notation $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{R}} = \mathcal{F}(\omega)$. In this case the foliation $\mathcal{F}$ is characterized by the fact that ω restricted to the leaves of $\mathcal{F}$ is identically zero.

Definition 1.1. The singular set of the foliation $\mathcal{F}(Pdy - Qdx)$ is defined in the following way:

$$\text{Sing}(\mathcal{F}) = \text{Sing}(\mathcal{F})_{\mathbb{R}} := \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = Q(x, y) = 0\}.$$

By our assumption $\text{Sing}(\mathcal{F})$ is a finite set of points. The leaves of $\mathcal{F}$ near a point $A \in \text{Sing}(\mathcal{F})$ may be complicated.

Exercise 1.4. Using a software which draws the trajectories of vector fields, describe the solutions of (1.2) near its singularities.

By Bezout theorem we have

$$\#\mathrm{Sing}(\mathcal{F}) \le \deg(P)\deg(Q)$$

The upper bound can be reached, for instance by the differential equation $\mathcal{F}(Pdy - Qdx)$, where

$$P = (x-1)(x-2)\cdots(x-d),\ Q = (y-1)(y-2)\cdots(y-d').$$

1.2 Poincaré first return map

From topological point of view a leaf L of $\mathcal{F} = \mathcal{F}(\omega)$ is either homeomorphic to $\mathbb{R}$ or to the circle $S^1 := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$. In the second case L is called a closed solution of $\mathcal{F}$ (but not yet a limit cycle).

Exercise 1.5. For a foliation $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{R}}$ the curve $\{R = 0\}$, where $d\omega = Rdx \wedge dy$, intersects all closed leaves of $\mathcal{F}$.

We consider a point $p \in L$ and a transversal section Σ to $\mathcal{F}$ at p. For any point q in Σ near enough to p, we can follow the leaf of $\mathcal{F}$ in the anti-clockwise direction and since L is closed we will encounter a new point $h(q) \in \Sigma$. We have obtained an analytic function

$$h : \Sigma \to \Sigma,$$

which is called the Poincaré first return map. Later in the context of holomorphic foliations we will call it the holonomy map. Usually we take a coordinate system z in Σ with $z(p) = 0$ and write the power series of h at 0:

$$h(z) = \sum_{i=0}^{\infty} \frac{h^{(n)}(0)}{n!} z^n$$

Definition 1.2. $h'(0)$ is called the multiplier of the closed solution L. If the multiplier is 1 then we say that h is tangent to the identity. In this case the tangency order is n if

$$h^{(i)}(0) = 0,\ h^{(n)}(0) \neq 0.$$

A closed solution L of $\mathcal{F}$ is called a limit cycle if its Poincaré first return map is not identity. In case the Poincaré first return map is identity then the leaves of $\mathcal{F}$ near L are also closed. In this case we can talk about the continuous family of cycles $\delta_z,\ z \in \Sigma$, where δ_z is the leaf of $\mathcal{F}$ through z.

Exercise 1.6. Prove that the multiplier and order of tangency do not depend on the coordinate system z in Σ.

Proposition 1.1. *In the above situation, we have*

$$h'(0) = \exp(-\int_{\delta} \frac{d\omega}{\omega}).$$

1.3 Hilbert 16-th problem

It is natural to ask whether a foliation $\mathcal{F}(Pdy - Qdx)$ has a finite number of limit cycles. This is in fact the first part of Hilbert 16-th problem:

Theorem 1.2. *Each polynomial foliation $\mathcal{F}(Pdy - Qdx)$ has a finite number of limit cycles.*

The above theorem was proved by Yu. Ilyashenko and J. Ecalle independently around 80's. We have associated to each foliations $\mathcal{F}$ the number $N(\mathcal{F})$ of its limit cycles of $\mathcal{F}$. It is natural to ask how $N(\mathcal{F})$ depends on the ingredient polynomial P and Q of $\mathcal{F}$.

Problem 1.1. (Hilbert 16'th problem) Fix a natural number $n \in \mathbb{N}$. Is there some natural number $N(n) \in \mathbb{N}$ such that each foliation $\mathcal{F}(Pdx - Qdy)$ with $\deg(P), \deg(Q) \leq n$ has at most $N(n)$ limit cycles.

Of course, it would be of interest to give an explicit description of $N(n)$ and more strongly determine the nature of

$$N(n) := \max\{N(\mathcal{F}(\omega)) \mid \omega = Pdy - Qdx, \ \deg(P), \deg(Q) \leq n\}.$$

One of the objective of the present text is to explain the fact that Hilbert 16'th problem is a combination of many unsolved difficult problems. We note that even the case $n = 2$ is open.

1.4 Stable and unstable limit cycle

For a foliation $\mathcal{F} = \mathcal{F}(Pdy - Qdx)$ let us define the affine degree

$$\deg(\mathcal{F}) = \max\{\deg(P), \deg(Q)\}.$$

We denote by $\mathcal{F}(d)$ the space of degree d foliations. As we remarked before, for a constant number $c \in \mathbb{R}$ the foliations $\mathcal{F}(c\omega)$ and $\mathcal{F}(\omega)$ are the same. Therefore, $\mathcal{F}(d)$ is some open set in the projectivization of the coefficients space of P and Q.

We take a point $\mathcal{F} \in \mathcal{F}(d)$ and ask what happens to the limit cycles of $\mathcal{F}$ if perturb the coefficients of $\mathcal{F}$.

Definition 1.3. A limit cycle δ of $\mathcal{F} \in \mathcal{F}(d)$ is called to be stable if any perturbation of $\mathcal{F}' \in \mathcal{F}(d)$ of $\mathcal{F}$ has a limit cycle near to δ. Otherwise, it is called unstable.

Let us take a perturbation $\mathcal{F}_t \in \mathcal{F}(d), \ t \in (\mathbb{R}^n, 0), \ \mathcal{F}_0 = \mathcal{F}$. Here, t can be chosen from the coefficient space of P and Q.

Example 1.2. For our example (1.2) we will use the following perturbation:

$$\mathcal{F}_t : \begin{cases} \dot{x} = 2y + \epsilon\frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon s y \end{cases}, t := (\epsilon, s - 1) \in (\mathbb{R}^2, 0). \tag{1.3}$$

We take again a transversal section Σ to $\mathcal{F}$ in $p \in \delta$. It turns out that Σ is also transversal to the perturbed foliations $\mathcal{F}_t, \ t \in (\mathbb{R}^n, 0)$. Therefore, we can follow a leaf of $\mathcal{F}_t$ through $z \in \Sigma$ in the anti-clockwise direction and then return to Σ in a new point. In this way, we obtain the perturbed Poincaré return map

$$h : \Sigma \times (\mathbb{R}^n, 0) \to \Sigma \times (\mathbb{R}^n, 0), \ h(z, t) = (h_t(z), t).$$

Proposition 1.2. *For a limit cycle δ if*

1. *the multiplier δ of $\mathcal{F}$ is not equal to one or;*

2. *the multiplier δ of $\mathcal{F}$ is equal to one and its tangency order is odd,*

then δ is stable.

Exercise 1.7. It seems to be classical that the inverse of the above proposition is also true. This is equivalent to say that if the multiplier of a limit cycle δ of $\mathcal{F}$ is equal to one and its tangency order is even then there is a perturbation $\mathcal{F}_t \in \mathcal{F}(d)$ of $\mathcal{F}$ in which δ disappears. In general there must be a perturbation $\mathcal{F}_t \in \mathcal{F}(d)$ of $\mathcal{F}$ with m limit cycles near δ. Prove or disprove all these.

Hilbert 16th problem makes sense only in the polynomial context (in general in the analytic context with compact ambient space). In the C^∞ context one can easily construct differential equations with infinite number of limit cycles: The function

$$f(x,y) = \sin(\frac{1}{x^2+y^2})\exp(-\frac{1}{x^2+y^2})$$

is C^∞ at $0 \in \mathbb{R}^2$ and analytic elsewhere. The ordinary differential equation

$$\begin{cases} \dot{x} = -y + xf(x,y) \\ \dot{y} = x + yf(x,y) \end{cases}$$

has the limit cycles $x^2+y^2 = \frac{1}{n\pi}$, $n \in \mathbb{N}$. Those limit cycles accumulate in $0 \in \mathbb{R}^2$. If we use $\sin(x^2+y^2)$ instead of $\sin(\frac{1}{x^2+y^2})$ in the definition of f we obtain a differential equation which has an infinite number of limit cycles accumulating at infinity.

Exercise 1.8. Verify the details of the above discussion.

Chapter 2

Holomorphic foliations

In this chapter we will do two main things. First, we will consider the foliation $\mathcal{F}(\omega)$ in $\mathbb{C}^2$ instead of $\mathbb{R}^2$. Second, we will compactify the ambient space $\mathbb{C}^2$ into $\mathbb{P}^2$. This will be the beginning of the theory of holomorphic foliations on complex manifolds.

2.1 Complexification

The basic theorem of ordinary differential equations in the complex context can be written in the following way:

Theorem 2.1. *Let $P, Q \in \mathbb{C}[x, y]$ and $X = (P, Q)$. For $A \in \mathbb{C}^2$ if $X(A) \neq 0$ then there is a unique holomorphic function*

$$\lambda : (\mathbb{C}, 0) \to \mathbb{C}^2$$

such that

$$\gamma(0) = A,\ \dot{\gamma} = X(\gamma(t))$$

The proof is similar to the real case. The images of the complex solutions of the vector field X give us a holomorphic foliation $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{C}} = \mathcal{F}_{\mathbb{C}}$ in $\mathbb{C}^2$. The leaves of $\mathcal{F}_{\mathbb{C}}$ are two dimensional real manifolds embedded in a real four dimensional space. If $P, Q \in \mathbb{R}[x, y]$ we will denote by $\mathcal{F}_{\mathbb{R}} = \mathcal{F}(\omega)_{\mathbb{R}}$ the corresponding real foliation in $\mathbb{R}^2$. Note that $\mathbb{R}^2 \subset \mathbb{C}^2$ and

$$\mathcal{F}_{\mathbb{R}} = \mathbb{R}^2 \cap \mathcal{F}_{\mathbb{C}}$$

i.e. the intersection of a leaf of $\mathcal{F}_{\mathbb{C}}$ with $\mathbb{R}^2$ is a union of leaves of $\mathcal{F}_{\mathbb{R}}$. Note that $\mathcal{F}_{\mathbb{C}}$ may has more singularities.

Example 2.1. The holomorphic foliation $\mathcal{F}_d$ defined in $\mathbb{C}^2$ by the 1-form

$$\omega := (y^d - x^{d+1})dy - (1 - x^d y)dx$$

is called the Jouanolou foliation of degree d. Consider the group

$$G := \{\epsilon \in \mathbb{C} \mid \epsilon^{d^2+d+1} = 1\}.$$

It acts on $\mathbb{C}^2$ discontinuously in the following way:

$$(\epsilon, (x, y)) \to (\epsilon^{d+1}x, \epsilon y)\ \epsilon \in G,\ (x, y) \in \mathbb{C}^2$$

It has a fixed point $p_1 = (0,0)$ at $\mathbb{C}^2$ (and two other fixed points $p_2 = [0:1:0], p_3 = [1:0:0]$ at infinity). For each $\epsilon \in G$ we have $\epsilon^*(\omega) = \epsilon^{d+1}\omega$ and so G leaves $\mathcal{F}_d$ invariant. We have

$$\mathrm{Sing}(\mathcal{F}_d)_{\mathbb{C}} = \{(\epsilon, \epsilon^{-d}) \mid \epsilon \in G\}$$

(there is no singularity at infinity) and G acts on $\mathrm{Sing}(\mathcal{F}_d)$ transitively.

2.2 Compactification

In differential and algebraic geometry many theorems are stated for compact varieties and so it would be of interest to compactify the ambient space of our foliations.

The projective space of dimension n is defined as follows:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\sim$$

where

$$a, b \in \mathbb{C}^{n+1} - \{0\},\ a \sim b \Leftrightarrow a = kb, \quad \text{for some } k \in \mathbb{C} - \{0\}.$$

It turns out that $\mathbb{P}^n$ is a complex manifold. We will mainly use $\mathbb{P}^1$ and $\mathbb{P}^2$. The projective space of dimension one $\mathbb{P}^1$ is covered by two charts x, x' biholomorphic to $\mathbb{C}$ and the transition map is given by

$$x' = \frac{1}{x}.$$

The projective space of dimension two $\mathbb{P}^2$ is covered by three charts $(x,y), (u,v), (u',v')$ biholomorphic to $\mathbb{C}^2$ and the transition maps are given by

$$v = \frac{y}{x},\ u = \frac{1}{x}, \qquad v' = \frac{x}{y},\ u' = \frac{1}{y}.$$

Considering the chart $(\mathbb{C}^2, (x,y))$, $\mathbb{P}^2$ becomes a compactification of $\mathbb{C}^2$. A foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$ extends to a holomorphic foliation in $\mathbb{P}^2$ in a unique way. For instance, in the chart (u,v) we have

$$\omega = P(\frac{1}{u}, \frac{v}{u})d(\frac{v}{u}) - Q(\frac{1}{u}, \frac{v}{u})d(\frac{1}{u}) = \frac{\tilde{P}(u,v)dv - \tilde{Q}(u,v)du}{u^{d+2}},\ \tilde{P}, \tilde{Q} \in \mathbb{C}[x,y].$$

Another compactification of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ is $\mathbb{P}^1 \times \mathbb{P}^1$ which is useful for studying the Ricatti foliations given by:

$$\omega = q(x)dy - (p_0(x) + p_1(x)y + p_2(x)y^2)dx,\ p_0, p_1, p_2, q \in \mathbb{C}[x].$$

Substituting $y = \frac{1}{y'}$ we have

$$\omega = \frac{1}{y'^2}(-q(x)dy - (p_0(x)y^2 + p_1(x)y + p_2(x))dx)$$

and so all the projective lines $\{a \in \mathbb{C} \mid q(a) \neq 0\} \times \mathbb{P}^1$ are transversal to the foliation. This will be later used to define the global holonomy of Ricatti foliations.

2.3 Holonomy

A leaf L of a holomorphic foliation $\mathcal{F}$ in $\mathbb{P}^2$ is a Riemann surface which in principle may has infinite genus, i.e the homotopy group $\pi_1(L, p),\ p \in L$ may not be finitely generated. We take a transversal to $\mathcal{F}$ section Σ at p. This is the image of an embedding of $(\mathbb{C}, 0)$ to $\mathbb{P}^2$ such that it maps 0 to p and is transverse to the leaves of $\mathcal{F}$. For instance one can take Σ a pieces of line which crosses p and is not tangent to L. For $\gamma \in \pi_1(L, p)$ we can define the holonomy

$$h_\gamma : \Sigma \to \Sigma$$

of $\mathcal{F}$ along γ in the same way that we have done it for the Poincaré first return map.

2.4 Projective degree

Working with foliations in $\mathbb{P}^2$ it is useful to use the projective degree. It can be proved that a line in $\mathbb{P}^2$ which does not cross any singularity of $\mathcal{F}$ has a fixed number d (counted with multiplicity) of tangency points with the foliation $\mathcal{F}$. In particular, for a generic line we have exactly d simple tangency points. The number d is called the the projective degree of $\mathcal{F}$. A foliation of projective degree d in the affine coordinate $\mathbb{C}^2 \subset \mathbb{P}^2$ is given by the differential form:

$$Pdx + Qdy + g(xdy - ydx)$$

where either g is a non-zero homogeneous polynomial of degree d and $\deg(P),\ \deg(Q) \leq d$ or g is zero and $\max\{\deg(P), \deg(Q)\} = d$. In the first case the line at infinity is not invariant by $\mathcal{F}$ and in the second case it is invariant by $\mathcal{F}$.

We may redefine $\mathcal{F}(d)$ to be the set of holomorphic foliations of projective degree d in $\mathbb{P}^2$. The spaces $\mathcal{F}(d)$ corresponding to two different definitions of the degree have different aspects. For instance, a generic foliation of projective degree d does not have an algebraic solution and a generic foliation of (affine) degree leaves the line at infinity invariant.

2.5 Algebraic solutions and algebraic limit cycles

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For a reduced polynomial $f \in \mathbb{K}[x, y]$ we consider the situation in which the curve

$$L_{f,\mathbb{K}} = \{f = 0\} := \{(x, y) \in \mathbb{K}^2 \mid f(x, y) = 0\}$$

is invariant under a foliation $\mathcal{F}(\omega)$. We will use the notation $L_f := L_{f,\mathbb{C}}$ and $\mathcal{F} = \mathcal{F}_\mathbb{C}$ and etc..

Example 2.2. For various values of $t \in \mathbb{R}$ the set

$$L_{f-t,\mathbb{R}},\ f =: \frac{y^2}{2} + \frac{(x^2-1)^2}{4}$$

is depicted in 3.1. All of these curves are invariant by $\mathcal{F}(df)_\mathbb{R}$.

Let us find an algebraic statement for the mentioned fact. For a smooth point $a \in L_f$, ω restricted to the tangent space of L_f at a is zero. Since df has also the same property, we conclude that $\omega \wedge df$ restricted to the curve L_f is zero. But $\omega \wedge df = Pdx \wedge dy$ for some polynomial $P \in \mathbb{C}[x,y]$ and so P restricted to L_f is zero. Since f is reduced we conclude that

$$\omega \wedge df = f\theta,\ \ \theta \in \Omega^2_{\mathbb{C}^2}.$$

Conversely, if the above equality holds the curve L_f is invariant under $\mathcal{F}(\omega)$. We call L_f an algebraic leaf of $\mathcal{F}(\omega)$.

Definition 2.1. A holomorphic foliations $\mathcal{F}(\omega)$ has a (meromorphic) first integral if there is a rational function $\frac{F}{G},\ F, G \in \mathbb{C}[x,y]$ such that $\mathcal{F} = \mathcal{F}(GdF - FdG)$. $\frac{F}{G}$ is called the first integral of $\mathcal{F}$. All the algebraic curves $\frac{F}{G} = t, t \in \mathbb{C}$ are invariant by $\mathcal{F}$.

Theorem 2.2. *(Darboux) A holomorphic foliation $\mathcal{F}(\omega)$ has either a finite number of algebraic leaves or a meromorphic first integral.*

Proof. Let us assume that $\mathcal{F}(\omega)$ has an infinite number of algebraic irreducible leaves $\{f_i = 0\},\ i \in \mathbb{N},\ f_i \in \mathbb{C}[x,y]$. Then

$$\omega \wedge df_i = f_i P_i dx \wedge dy,\ \ P_i \in \mathbb{C}[x,y]$$

and so

$$\deg(P_i) + \deg(f_i) \leq \deg(f_i) - 1 + \deg(\omega) \text{ or equiv.ly } \deg(P_i) \leq \deg(\omega) - 1.$$

Since the space of polynomials of degree less that $\deg(\omega) - 1$ is finite dimensional, we can assume without loss of generality that $P_2, \ldots, P_h$ form a basis for the $\mathbb{C}$ vector space generated by $P_i,\ i = 1, 2, \ldots$. Therefore, there are $\lambda_i \in \mathbb{C},\ i = 1, \ldots, h$ such that $\sum_{i=1}^{h} \lambda_i P_i = 0$ and so

$$\omega \wedge \eta_1 = 0,\ \ \eta_1 := \sum_{i=1}^{h} \lambda_i \frac{df_i}{f_i}.$$

In the same way there are $\mu_i \in \mathbb{C},\ i = 2, \ldots, h+1$ such that $\sum_{i=2}^{h+1} \mu_i P_i = 0$ and so

$$\omega \wedge \eta_2 = 0,\ \ \eta_2 := \sum_{i=2}^{h+1} \mu_i \frac{df_i}{f_i}.$$

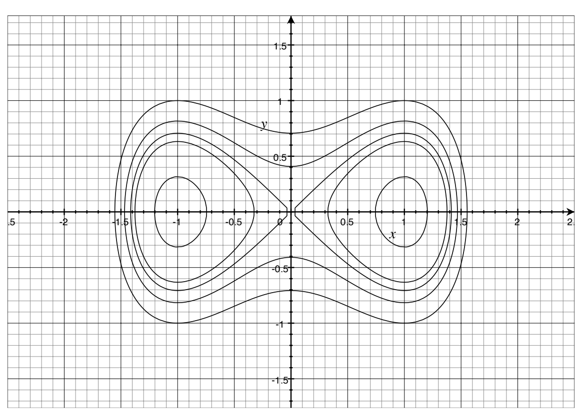

Figure 2.1: Eight: $\frac{y^2}{2} + \frac{(x^2-1)^2}{4} = t,\ t = \frac{1}{20}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$

Let $G_i \in \mathbb{C}[x,y],\ i=1,2$ be such that $\eta_i = G_i \frac{\omega}{f_1 f_2 \cdots f_{h+1}}$. We have

$$0 = d\eta_i = dG_i \wedge \frac{\omega}{f_1 f_2 \cdots f_{h+1}} + G_i d(\frac{\omega}{f_1 f_2 \cdots f_{h+1}}).$$

It follows that $\frac{G_1}{G_2}$ is not a constant and it is a meromorphic first integral of $\mathcal{F}(\omega)$:

$$d(\frac{G_1}{G_2}) \wedge \omega = \frac{G_2 dG_1 \wedge \omega - G_2 dG_1 \wedge \omega}{G_2^2} = 0.$$

□

The above proof shows that for a foliation $\mathcal{F}(\omega)$ either all the leaves are algebraic or the number of its irreducible algebraic leaves is less than

$$\frac{d(d+1)}{2} + 2$$

where $d := \deg(\omega)$.

Problem 2.1. The above upper bound seems to be far from the real one. Find an $N(d)$ such that every $\mathcal{F}$ with $\deg(\mathcal{F}) \leq d$ has at most $N(d)$ irreducible algebraic leaf and there is an $\mathcal{F}$ with $\deg(\mathcal{F}) \leq d$ and such that $\mathcal{F}$ has exactly $N(d)$ algebraic leaves.

Now let us come back to the real context, $f \in \mathbb{R}[x,y],\ \omega \in \Omega^1_{\mathbb{R}^2}$. The real curve $L_{f,\mathbb{R}}$ may have many connected components and in particular it may have many ovals. Take one of them an call it δ. If $\mathcal{F}$ leaves L_f invariant then δ is a leaf of $\mathcal{F}_\mathbb{R}$ and so it has a holonomy:

Definition 2.2. If the holonomy of δ is not the identity map then it is called an algebraic limit cycle.

By Darboux theorem and Harnack theorem[1] it follows that a foliation $\mathcal{F}(\omega)_\mathbb{R}$ has a finite number of algebraic limit cycles. However, the second part of the Hilbert 16-th problem for algebraic limit cycles is not yet solved.

Problem 2.2. (Hilbert 16'th problem for algebraic limit cycles) Fix a natural number $n \in \mathbb{N}$. Is there some natural number $N(n) \in \mathbb{N}$ such that each foliation $\mathcal{F}(Pdx - Qdy)_\mathbb{R},\ P,Q \in \mathbb{R}[x,y]$ with $\deg(P), \deg(Q) \leq n$ has at most $N(n)$ algebraic limit cycles.

We note that the above problem even for the case $n = 2$ is open.

Example 2.3. The foliation

$$\mathcal{F}(nxdy + mydx),\ n,m \in \mathbb{N}$$

has the first integral $x^m y^n$. This example shows that the degree of a foliation can be small but the degree of its algebraic solutions can be big.

[1]A real curve of degree d in $\mathbb{R}P^2$ has at most $\frac{(d-1)(d-2)}{2} + 1$ ovals.

2.6 Lins-Neto's examples

It is a classical fact in real algebraic geometry that a real algebraic curve of genus g has at most $g+1$ ovals in the real plane $\mathbb{R}^2$. In order to prove the algebraic Hilbert 16-th problem we may try to prove that the genus of algebraic leaves of holomorphic foliations of degree is uniformly bounded. Unfortunately, this affirmation is wrong. The first examples may be derived from

Exercise 2.1. There is a Hurwitz-Zeuthen genus formula for calculating the genus of singular algebraic curves. Apply this formula for

$$x^n y^m (x+y-1)^k = t, \ n, m, k \in \mathbb{N}, \ t \in \mathbb{C}$$

and conclude that its genus can go to the infinity.

The pencils $\mathcal{F}_{i,t} = \mathcal{F}(\omega_i + t\eta_i), \ i = 1, 2, \ t \in \mathbb{P}^1$, where

$$\omega_1 = (4x - 9x^2 + y^2)dy - 6y(1-2x)dx, \eta_1 = 2y(1-2x)dy - 3(x^2 - y^2)dx,$$

$$\omega_2 = y(x^2-y^2)dy - 2x(y^2-1)dx, \eta_2 = (4x - x^3 - x^2y - 3xy^2 + y^3)dy + 2(x+y)(y^2-1)dx$$

are studied by A. Lins Neto in [10]. They satisfy

$$d\omega_i = \alpha_i \wedge \omega_i, \ i = 1, 2,$$

where

$$\alpha_i := \lambda_i \frac{dQ_i}{Q_i}, \ \lambda_1 = \frac{5}{6}, \ \lambda_2 = \frac{3}{4},$$

$$Q_1 = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4, \ Q_2 = (y^2-1)(x+2+y^2-2x)(x^2+y^2+2x).$$

Theorem 2.3. *(Lins Neto) The set*

$$E_i = \{t \in \mathbb{P}^1 \mid \mathcal{F}_{i,t} \text{ has a meromorphic first integral}\}$$

is $\mathbb{Q} + \mathbb{Q}e^{2\pi i/3}$ *for* $i = 1$ *and is* $\mathbb{Q} + i\mathbb{Q}$ *for* $i = 2$. *Moreover, for any natural number* n *the set of points* t *in* E_i *such that the algebraic solutions of* $\mathcal{F}(\omega_i + t\eta_i)$ *are of degree (resp. genus) less than* n, *is finite.*

Problem 2.3. Find an algorithm whit the input $\omega_i + t\eta_i, \ t \in E_i$ and the output which is the meromorphic first integral of $\mathcal{F}_i(\omega_i + t\eta_i)$.

Problem 2.4. Using Lins Neto's examples construct explicitly a family $\mathcal{F}_n, \ n \in \mathbb{N}$ of holomorphic foliation in $\mathbb{P}^2_{\mathbb{R}}$ with a fixed degree d and all of them with a meromorphic first integral such that each $\mathcal{F}_n$ has a leaf with N_n ovals and N_n tends to infinity when $n \to \infty$. Discuss the problem in the case $d = 2, 3, 4$.

Problem 2.5. Prove or disprove the following statement: A foliation with a meromorphic first integral has a finite number of families of ovals depending only on the degree of the foliation and not the degree of the first integral. If this statement is true then in the situation of the previous problem there is a natural number k independent of n such that all the ovals of $\mathcal{F}_n$ are nested in k families.

2.7 Minimal set

For a holomorphic foliation in $\mathbb{P}^2$ one may formulate many problems related to the accumulation of its leaves. The most simples one which is still open is the following:

Problem 2.6. Is there a foliation $\mathcal{F}$ in $\mathbb{P}^2$ with a leaf L which does not accumulate in the singularities of $\mathcal{F}$.

For instance the above problem for Jouanolou foliation is proved numerically for $d \leq 4$ and it is still open for general d.

Let us suppose that such an $\mathcal{F}$ and L exist and set $M := \bar{L}$, where the closure is taken in $\mathbb{P}^2$. It follows that M is a union of leaves of $\mathcal{F}$. We may suppose that M does not contain a proper $\mathcal{F}$-invariant subset. In this case we call M a minimal set.

Proposition 2.1. *A foliation in $\mathbb{P}^2$ with algebraic leaf has not a minimal set.*

For many other useful statement on minimal sets see [1]. For local theory of holomorphic foliations see [2].

Chapter 3

Abelian integrals

The objective of this section is to introduce abelian integrals in the context of holomorphic foliations. The main problem in this direction is Arnold-Hilbert infinitesimal problem.

3.1 Continuous family of cycles

Let us be given a polynomial $f \in \mathbb{R}[x, y]$ of degree d. We consider f as a function from $\mathbb{R}^2$ to $\mathbb{R}$. The set of critical points and critical values of f are defined to be

$$P_{\mathbb{R}} = \{(a, b) \in \mathbb{R}^2 \mid f_x(a, b) = f_y(a, b) = 0\},\ C_{\mathbb{R}} := f(P_{\mathbb{R}}).$$

Let us take two consecutive elements of C, namely $c_1 < c_2$, and assume that the real affine variety

$$L_{t,\mathbb{R}} := f^{-1}(t),\ c_1 \leq t \leq c_2$$

contains a connected component δ_t, namely an oval, such that it is a closed cycle in $\mathbb{R}^2$, varies continuously with respect to the parameter t and for $t = c_1, c_2$ it is either a point or a closed polygon of paths.

Example 3.1. To carry an example in mind, take the polynomial $f = y^2 - x^3 + 3x$ in two variables x and y. We have $C = \{-2, 2\}$. For t a real number between 2 and -2 the level surface of f in the real plane $\mathbb{R}^2$ has two connected pieces which one of them is an oval and we can take it as δ_t. In this example as t moves from -2 to 2, δ_t is born from the critical point $(-1, 0)$ of f and end up in the α-shaped piece of the fiber $f^{-1}(2) \cap \mathbb{R}^2$ (See Figure 3.1).

Example 3.2. Another good example is

$$f = \frac{y^2}{2} + \frac{(x^2 - 1)^2}{4}$$

The set of critical values of f is $C = \{0, \frac{1}{4}\}$ and we can distinguish three family of ovals.

3.2 Abelian integrals

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial mapping and $\delta_t \cong \mathbb{S}^1,\ t \in (\mathbb{R}, 0)$ be a continuous family of ovals in the fibers of f. The level surfaces of f are the images of the solutions of the

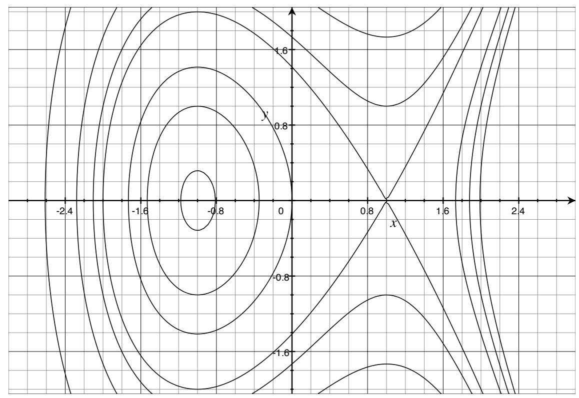

Figure 3.1: Elliptic curves: $y^2 - x^3 + 3x - t,\ t = -1.9, -1, 0, 2, 3, 5, 10$

Hamiltonian ordinary differential equation

$$\mathcal{F}_0 : \begin{cases} \dot{x} = f_y \\ \dot{y} = -f_x \end{cases} . \tag{3.1}$$

We make a perturbation of $\mathcal{F}_0$

$$\mathcal{F}_\epsilon : \begin{cases} \dot{x} = f_y + \epsilon P(x,y) \\ \dot{y} = -f_x + \epsilon Q(x,y) \end{cases} , \epsilon \in (\mathbb{R}, 0), \tag{3.2}$$

where P and Q are two polynomials with real coefficients.

Usually one expects that in the new ordinary differential equation the cycle δ_0 breaks and accumulates, in positive or negative time, in some part of the real plane or infinity.

Let us take a transversal section Σ to $\mathcal{F}_0$. We assume that Σ is parameterized by the image t of f. Let $h_\epsilon(t) : \Sigma \to \Sigma$ be the deformed holonomy along δ_0. Note that $h_0(t) \equiv t$. We write the Taylor series if $h_\epsilon(t)$ in ϵ:

$$h_\epsilon(t) := t + \epsilon M_1(t) + \epsilon^2 M_2(t) + \cdots$$

Proposition 3.1. *If M_1 is not identically zero and $M_1(t_0) = 0$ for some $t_0 \in \Sigma$ then for ϵ small enough there is a limit cycle of $\mathcal{F}_\epsilon$ near δ_{t_0}.*

It is natural to calculate M_1 in terms of the polynomial ingredients of $\mathcal{F}_\epsilon$.

Proposition 3.2. *We have*

$$M_1(t) = -\int_{\delta_t} (Pdy - Qdx)$$

Proof. The deformed foliation (3.2) cab be written in the form

$$df + \epsilon\omega = 0,\ \omega := Pdy - Qdx. \tag{3.3}$$

Let $\delta_{t,h_\epsilon(t)}$ be a path in the leaf of $\mathcal{F}_\epsilon$ through t which connects t to $h_\epsilon(t)$ along the path δ. Since Σ is parameterized by $t = f\mid_\Sigma$, by integrating the 1-form (3.3) over the path $\delta_{t,h_\epsilon(t)}$ we have

$$h_\epsilon(t) - t + \epsilon(\int_{\delta_t} \omega) + O(\epsilon^2) = 0$$

The coefficient of ϵ in the above equality gives us

$$M_1(t) = -\int_{\delta_t} \omega.$$

□

We conclude that

Proposition 3.3. *If the Abelian integral*

$$M_1(t) = \int_{\delta_t} (Pdy - Qdx)$$

is zero for $t = 0$ but not identically zero then for any small ϵ there will be a limit cycle of $\mathcal{F}_\epsilon$ near enough to δ_0.

If the Abelian integral is identically zero (for instance if δ_t is homotopic to zero in the complex fiber of f) then the birth of limit cycles is controlled by iterated integrals (see for instance [6, 15]). In our main example take the ordinary differential equation

$$\mathcal{F}_\epsilon : \begin{cases} \dot{x} = 2y + \epsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon s y \end{cases}, \epsilon \in (\mathbb{R}, 0). \tag{3.4}$$

If $\int_{\delta_0} (\frac{x^2}{2} dy - sydx) = 0$ or equivalently

$$s := \frac{-\int_{\Delta_0} xdx \wedge dy}{\int_{\Delta_0} dx \wedge dy} = \frac{\Gamma(\frac{5}{12})\Gamma(\frac{13}{12})}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})} \sim 1.2636$$

where Δ_0 is the bounded open set in $\mathbb{R}^2$ with the boundary δ_0, then for ϵ near to 0, $\mathcal{F}_\epsilon$ have a limit cycle near δ_0. In fact for $\epsilon = 1$ and $s = 0.9$ such a limit cycle still exists and it is depicted in Figure (1.1).

Exercise 3.1. Formulate the material of this section for foliations with meromorphic first integrals.

3.3 Arnold-Hilbert Problem

A weaker version of H16, known as the infinitesimal Arnold-Hilbert problem asks for a reasonable bound for the number of zeros of real Abelian integrals when the degrees of f and P, Q are bounded.

Problem 3.1. (Infinitesimal Arnold-Hilbert problem) Determine the number

$$Z(m,n) := \max \#\{\int_{\delta_t} \omega = 0 \mid f \in \mathbb{R}[x,y] \ \omega \in \Omega^1_{\mathbb{R}^2}, \ \deg(f) \leq m, \ \deg(\omega) \leq n\},$$

where δ_t is a continuous family of ovals in the real level surfaces of f.

There are some partial solutions to this problem but it is still open in its generality (see [9, 5]). Using the the theory of fewnomials Khovanski has proved that $Z(m,n)$ is a finite number. It is proved by Ilyasheno, Yakovenko and others that $Z(m,n)$ has an exponential growth.

Theorem 3.1. *(Gavrilov-Horozov-Iliev, [5])*

$$N(3,2) = 2.$$

3.4 Arnold-Hilbert problem in dimension zero

Even the zero dimensional version of Arnold-Hilber problem, in which Abelian integrals are algebraic functions, is not completely solved(see [7]).

Let us be given a degree d polynomial $f(x) \in \mathbb{R}[x]$ in one variable x. Let c_1 and c_2 be two real critical values of f such that between c_1 and c_2 there is no more critical value of f and $f^{-1}(t)$ for t between c_1 and c_2 contains two real points x_1 and x_2 such that for t near c_1 they collapse into each other in a critical point p_1 with $f(p_1) = c_1$. In another word we have a zero dimensional cycle which vanishes at p_1.

Definition 3.1. The zero dimensional abelian integral is defined to be

$$\int_{\delta_t} \omega := \omega(x_2) - \omega(x_1),\ \delta_t := [x_2] - [x_1],\ \omega \in \mathbb{R}[x].$$

We define $N_0(m, n)$ in a similar way as we have defined in the previous section.

Proposition 3.4. *We have*

$$n - 1 - [\frac{n}{m}] \leq Z(m, n) \leq \frac{(m-1)(n-1)}{2}. \tag{3.5}$$

The lower bound in this inequality is given by the dimension of the vector space of Abelian integrals

$$V_n = \{\int_{\delta_t} \omega,\ \deg \omega \leq n\}$$

where f is a fixed general polynomial of degree m, while the upper bound is a reformulation of Bezout's theorem. When $m = 3$ we get $Z(m, m-1) = 1$. The space of Abelian integrals V_n is Chebishev, possibly with some accuracy.

Definition 3.2. Recall that V_n is said to be Chebishev with accuracy c if every $I \in V_n$ has at most $\dim V_n - 1 + c$ zeros in the domain D.

Problem 3.2. Give an exact description of the number $N_0(d, d-1)$.

If we do not put any restriction on x_1 and x_2 the sharp upper bound is $(d-1)(d-2)/2$. For instance take $f(x) = (x-1)(x-2)...(x-d)$ and $g = (x-1)(x-2)...(x-d+1)$. In the image of f we can find $d-1$ intervals with mentioned property and the main problem is: How many of those zeros can be grouped in one of such intervals.

Chapter 4

Center problem

In this chapter we consider the space of holomorphic foliation in $\mathbb{P}^2$ which have at least one center singularity (consequently we will use the projective degree). It turns out that such a space is algebraic. The classification of all irreducible components of such an algebraic variety is known as the center (or center-focus) problem in the literature. The main source for the content of this chapter is [13]

4.1 Center singularity

Let $\mathcal{F}$ be a germ of singular foliation at $(\mathbb{C}^2, 0)$. We say that $0 \in \mathbb{C}^2$ is a center singularity of $\mathcal{F}$ or simply a center of $\mathcal{F}$, if there exists a germ of holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ which has non-degenerate critical point at $0 \in \mathbb{C}^2$, and the leaves of $\mathcal{F}$ near 0 are given by $f = const.$. The point 0 is also called a Morse singularity of f. Morse lemma in the complex case implies that there exists a local coordinate system (x, y) in $(\mathbb{C}^2, 0)$ with $x(0) = 0,\ y(0) = 0$ and such that $f(x, y) = x^2 + y^2$. Near the center the leaves of $\mathcal{F}$ are homeomorphic to a cylinder, therefore each leaf has a nontrivial closed cycle that will be called the Lefschetz vanishing cycle.

4.2 Center variety

Let $\mathcal{F}(d)$ be the space of degree d holomorphic foliations in $\mathbb{P}^2$ and $\mathcal{M}(d)$ the closure of the set of foliations of degree d and with at least one center in $\mathcal{F}(d)$. The following example gives us a huge number of these foliations:

Example 4.1. Let $\tau : \mathbb{P}^2 \to M$ be a holomorphic map between $\mathbb{P}^2$ and a complex compact manifold M with $dim(M) \geq 3$. Let also $\mathcal{F}$ be a holomorphic codimension one singular foliation in M. We say that τ has a tangency point $a \in \mathbb{P}^2$ of order two with $\mathcal{F}$ if $\tau(a)$ is a regular point of the foliation $\mathcal{F}$, $\mathcal{F}$ in a coordinate $(x, y) \in (\mathbb{C}^{n-1}, 0) \times (\mathbb{C}, 0)$ around a is given by $y = const.$ and a is a non-degenerate critical point of $y \circ \tau$. This says that the pullback foliation $\tau^*(\mathcal{F})$ has a center at the point a.

I have learned the statement and proof of the following proposition from A. Lins Neto.

Proposition 4.1. *$\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$.*

Proof. Let $\mathcal{M}_0(d)$ be the set of all foliations in $\mathcal{M}(d)$ with a center at the origin $(0,0) \in \mathbb{C}^2 \subset \mathbb{P}^2$ and with a local first integral of the type

$$f = xy + f_3 + f_4 + \cdots + f_n + h.o.t. \tag{4.1}$$

in a neighborhood of $(0,0)$. Let us prove that $\mathcal{M}_0(d)$ is an algebraic subset of $\mathcal{F}(d)$.
Let $\mathcal{F}(\omega) \in \mathcal{M}_0(d)$ and $\omega = \omega_1 + \omega_2 + \omega_3 + \ldots + \omega_{d+1}$ be the homogeneous decomposition of ω, then in a neighborhood around $(0,0)$ in $\mathbb{C}^2$, we have

$$\omega \wedge df = 0 \Rightarrow (\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}) \wedge (d(xy) + df_3 + df_4 + \cdots) = 0$$

Putting the homogeneous parts of the above equation equal to zero, we obtain

$$\begin{cases} \omega_1 \wedge d(xy) = 0 \Rightarrow \omega_1 = k.d(xy), \ k \text{ is constant} \\ \omega_1 \wedge df_3 = -\omega_2 \wedge d(xy) \\ \ldots \\ \omega_1 \wedge df_n = -\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy) \\ \ldots \end{cases} \tag{4.2}$$

Dividing the 1-form ω by k, we can assume that $k = 1$. Let $\mathcal{P}_n$ denote the set of homogeneous polynomials of degree n. Define the operator :

$$S_n : \mathcal{P}_n \to (\mathcal{P}_n dx \wedge dy)$$

$$S_n(g) = \omega_1 \wedge d(g)$$

We have

$$S_{i+j}(x^i y^j) = d(xy) \wedge d(x^i y^j) = (xdy + ydx) \wedge (x^{i-1} y^{j-1}(jxdy + iydx))$$

$$= (j - i)x^i y^j dx \wedge dy$$

This implies that when n is odd S_n is bijective and so in (4.2), f_n is uniquely defined by the terms f_m, ω_m's $m < n$, and when n is even

$$Im(S_n) = A_n dx \wedge dy$$

where A_n is the subspace generated by the monomials $x^i y^j, \ i \neq j$. When n is even the existence of f_n implies that the coefficient of $(xy)^{\frac{n}{2}}$ in

$$-\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy)$$

which is a polynomial, say P_n, with variables

$$\text{coefficients of } \omega_2 \ldots \omega_{n-1}, f_2, \ldots, f_{n-1}$$

is zero. The coefficients of $f_i, i \leq n-1$ is recursively given as polynomials in coefficients of $\omega_i, i \leq n-1$ and so the algebraic set

$$X : P_4 = 0 \ \& \ P_6 = 0 \ \& \ldots \& \ P_n = 0 \ \ldots$$

consists of all foliations $\mathcal{F}$ in $\mathcal{F}(d)$ which have a formal first integral of the type 4.1 at $(0,0)$. From a result of Mattei and Moussu (theorem A, [11]), it follows that $\mathcal{F}$ has a holomorphic first integral of the type (4.1). This implies that $\mathcal{M}_0(d) = X$ is algebraic. Note that by Hilbert zeroes theorem, a finite number of P_i's defines $\mathcal{M}_0(d)$. The set $\mathcal{M}(d)$ is obtained by the action of the group of automorphisms of $\mathbb{P}^2$ on $\mathcal{M}_0(d)$. Since this group is compact we conclude that $\mathcal{M}(d)$ is also algebraic. □

4.3 Components of holomorphic foliations with a first integral

Let $\mathcal{P}_{d+1}$ be the set of polynomials of maximum degree $d+1$ in $\mathbb{C}^2$ and $f \in \mathcal{P}_{d+1}$. The leaves of the foliation $\mathcal{F}(df)$ are contained in the level surfaces of f. Let $\mathcal{I}(d)$ be the set of the mentioned holomorphic foliations in $\mathcal{F}(d)$.

Theorem 4.1. *([8]) $\mathcal{I}(d),\ d \geq 2$ is an irreducible component of $\mathcal{M}(d)$.*

We can restate the above result as follows: Let $\mathcal{F} \in \mathcal{I}(d)$, p one of the center singularities of $\mathcal{F}$ and $\mathcal{F}_t$ a holomorphic deformation of $\mathcal{F}$ in $\mathcal{F}(d)$ such that its unique singularity p_t near p is still a center.

Theorem 4.2. *In the above situation, there exists an open dense subset U of $\mathcal{I}(d)$, such that for all $\mathcal{F}(df) \in U$, there exists polynomial $f_t \in \mathcal{P}_{d+1}$ such that $\mathcal{F}_t = \mathcal{F}(df_t)$.*

This theorem also says that the persistence of one center implies the persistence of all other centers.

Let $\mathcal{F}$ be a foliation in $\mathbb{C}^2$ given by the polynomial 1-form

$$\omega(f,\lambda) = \omega(f_1,\ldots,f_r,\lambda_1,\ldots,\lambda_r) = f_1\cdots f_r \sum_{i=1}^{r} \lambda_i \frac{df_i}{f_i} \tag{4.3}$$

where the f_i's are irreducible polynomials in $\mathbb{C}^2$ and $d_i = deg(f_i)$. $\mathcal{F}$ is called a logarithmic foliation and it has the multi-valued first integral $f = f_1^{\lambda_1}\cdots f_r^{\lambda_r}$ in $U = \mathbb{C}^2 \backslash (\cup_{i=1}^{r}\{f_i = 0\})$. We can prove that generically, the degree of $\mathcal{F}$ is $d = \sum_{i=1}^{r} d_i - 1$.

Let $\mathcal{L}(d_1, d_2, \ldots, d_r)$ be the set of all logarithmic foliations of the above type.

Theorem 4.3. *([14]) The set $\mathcal{L}(d_1, d_2, \ldots, d_r)$ is an irreducible component of $\mathcal{M}(d)$, where $d = \sum_{i=1}^{r} d_i - 1$.*

The classification of degree two polynomial differential equations was done by Dulac in [4]. Going to the language of holomorphic foliations in $\mathbb{P}^2$, instead of using the language of polynomial differential equations, this classification was completed in [3] for degree two holomorphic foliation in $\mathbb{P}^2$. Deformation of real Hamiltonian equations with a center singularity, generating limit cycles from the Lefschetz vanishing cycles of the center, has been one of the methods of approach to Hilbert sixteen problem, for this see Roussarie's book [16] and its reference. Yu.S. Ilyashenko in [8] shows that the persistence of a center after deformation of a generic Hamiltonian equation implies that the deformed equation is also Hamiltonian. He uses this fact to get a certain number of limit cycles after deformation of Hamiltonian equations.

4.4 Some results using the projective degree

In this section we use the projective degree and redefine $\mathcal{M}(d)$ and $\mathcal{F}(d)$. Let $\mathcal{P}_d$ be the set of polynomials of maximum degree d in $\mathbb{C}^2$ and

$$(F,G) \in \mathcal{P}_{a+1} \times \mathcal{P}_{b+1},\ \frac{a+1}{b+1} = \frac{q}{p},\ g.c.d.(p,q) = 1,\ \mathcal{R} := \{F = 0, G = 0\} \subset \mathbb{P}^2$$

The foliation $\mathcal{F} = \mathcal{F}(pGdF - qFdG)$ has the first integral:

$$f : \mathbb{P}^2 \backslash \mathcal{R} \to \mathcal{S},\ f(x,y) = \frac{F(x,y)^p}{G(x,y)^q}$$

i.e., the leaves of the foliation $\mathcal{F}$ are contained in the level surfaces of f. Let $\mathcal{I}(a,b)$ be the closure of the set of the mentioned holomorphic foliations in $\mathcal{F}(d)$.

Theorem 4.4. *([13, 12]) $\mathcal{I}(a,b), a+b > 2$ is an irreducible component of $\mathcal{M}(d)$, where $d = a+b$.*

We can restate our main theorem as follows: Let $\mathcal{F} \in \mathcal{I}(a,b)$, p one of the center singularities of $\mathcal{F}$ and $\mathcal{F}_t$ a holomorphic deformation of $\mathcal{F}$ in $\mathcal{F}(d)$, where $d = a+b$, such that its unique singularity p_t near p is still a center.

Theorem 4.5. *In the above situation, if $a+b > 2$ then there exists an open dense subset U of $\mathcal{I}(a,b)$, such that for all $\mathcal{F}(pGdF - qFdG) \in U$, $\mathcal{F}_t$ admits a meromorphic first integral. More precisely, there exist polynomials F_t and G_t such that $\mathcal{F}_t = \mathcal{F}(pG_t dF_t - qF_t dG_t)$, where F_t and G_t are holomorphic in t and $F_0 = F$ and $G_0 = G$.*

This theorem also says that the persistence of one center implies the persistence of all other centers and dicritical singularities (the points of $\{F = 0\} \cap \{G = 0\}$).

4.5 The center variety and Limit cycles

Let X be an irreducible component of $\mathcal{M}(d)$, $\mathcal{F} \in X - sing(\mathcal{M}(d))$ be a real foliation, i.e. its equation has real coefficients, p be a real center singularity and $\delta_t, t \in (\mathbb{R}, 0)$ be a family of real vanishing cycles around p. The cyclicity of δ_0 in a deformation of $\mathcal{F}$ inside $\mathcal{F}(d)$ is greater than $codim_{\mathcal{F}(d)}(X) - 1$. Roughly speaking, the cyclicity of δ_0 is the maximum number of limit cycles appearing near δ_0 after a deformation of $\mathcal{F}$ in $\mathcal{F}(d)$. The proof of this fact and the exact definition of cyclicity can be found in [16]. We have $codim_{\mathcal{F}(d)}(\mathcal{L}(d+1)) - 1 = \frac{(d+2)(d+1)}{2}$ and this is the number obtained by Yu. Ilyashenko in [8]. One can state similar results for the components obtained in Theorem 4.4 and 4.3.

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