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# MEROMORPHIC CONNECTIONS ON $\mathbb{P}^1$ AND THE MULTIPLICITY OF ABELIAN INTEGRALS

H. MOVASATI

*Dedicated to Professor César Camacho on his 60th birthday*

ABSTRACT. In this paper, we introduce the notion of Abelian integrals in differential equations for an arbitrary vector bundle on $\mathbb{P}^1$ with a meromorphic connection. In this general context, we give an upper bound for the maximum multiplicity of Abelian integrals.

## 1. INTRODUCTION

After a century, Hilbert sixteenth problem in differential equations is not solved. Challenges to solve it have developed and develop new areas of mathematics, such as holomorphic foliations, real algebraic geometry, and Abelian integrals with their own problems which in many cases are far from the original one. One of the problems arising in this direction is the Arnold–Hilbert problem which asks for the maximum number of zeros of certain Abelian integrals. Very large upper bounds are found, but they do not seem to be realistic (see, e.g., the expository article [7]). Another particular problem which can be considered is the maximum multiplicity of zeros of Abelian integrals. P. Mardešić in [15] gives an upper bound for the desired number in the case of a generic Hamiltonian differential equation. This upper bound is a polynomial of degree 4 in $n$, where $n$ is the degree of the Hamiltonian system and 1-form of the integral. But again it is not believed that so many limit cycles can be obtained by perturbing the Hamiltonian system around a cycle with trivial holonomy. Since then many authors have tried to give an effective answer to this problem. Here we mention the works of Khovanski, Gabrielov, Bolibruch, Moura, and others. In particular, the works [3, 17] have obtained the same upper bound for a

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linear system of differential equations using quite different methods. In [16], the author approaches the problem in the case of hyperelliptic integrals.

The aim of this article is to put this problem in the context of meromorphic connections on vector bundles on $\mathbb{P}^1$ and give an upper bound for the multiplicity of zeros of Abelian integrals. Our upper bound (see Theorems 1 and 3) generalizes the result of Mardešić, Bolibruch, and Moura. In the particular case of a trivial vector bundle together with its trivializing section, our upper bound is a polynomial of degree two in the rank of the vector bundle. This kind of upper bounds has been already obtained by Bolibruch and Moura. Note that we have not yet a direct relation in all cases between the usual notion of Abelian integrals and the notion of Abelian integrals in our context. Such relations are developed only for Lefschetz pencils with some generic properties. In this new context, one can use the machinery of meromorphic connections for studying Abelian integrals.

In Sec. 1, we recall the notion of meromorphic connections on vector bundles on $\mathbb{P}^1$ and, consequently, the notion of Abelian integrals in this context. Theorems 1 and 3 give an upper bound for the maximum multiplicity of Abelian integrals in our context. Section 2 is devoted to the study of linear equations of our Abelian integrals. Finally, in Sec. 3, we give the connection of our notion of Abelian integrals with the usual one in the case of a Lefschetz pencil.

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## 2. Meromorphic connections

Let $V$ be a locally free sheaf (vector bundle) of rank $\alpha$ on $\mathbb{P}^1$ and $D = \sum_{i=1}^{r} m_i c_i$ be a positive divisor in $\mathbb{P}^1$, i.e., all $m_i$ are positive. We denote by $C \subset \mathbb{P}^1$ the set of $c_i$. A meromorphic connection $\nabla$ on $V$ with the pole divisor $D$ is a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla : V \to \Omega^1_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} V$$

satisfying the Leibniz identity

$$\nabla(f\omega) = df \otimes \omega + f\nabla\omega, \quad f \in \mathcal{O}_{\mathbb{P}^1}, \quad \omega \in V,$$

where $\Omega^1_{\mathbb{P}^1}(D)$ is the sheaf of meromorphic 1-forms in $\mathbb{P}^1$ with poles on $D$ (the pole order of a section of $\Omega^1_{\mathbb{P}^1}(D)$ at $c_i$ is less than $m_i$). For any two meromorphic connections $\nabla_1$ and $\nabla_2$ with the same pole divisor $D$, $\nabla_1 - \nabla_2$ is a $\mathcal{O}_{\mathbb{P}^1}$-linear mapping.

Let $t$ be the affine coordinate of $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$, where $\infty$ is the point at infinity in $\mathbb{P}^1$. By the Leibniz rule and by composing $\nabla$ with the holomorphic vector field $\partial/\partial t$ we can define:

$$\nabla_{\partial/\partial t} : H^0(\mathbb{P}^1, V(*\infty)), \to H^0(\mathbb{P}^1, V(D + *\infty)),$$

where $H^0(\mathbb{P}^1, V(*\infty))$ (respectively, $H^0(\mathbb{P}^1, V(D+*\infty))$) is the set of meromorphic global sections of $V$ with arbitrary poles at $\infty$ (respectively, and poles of maximum multiplicity $m_i$ at $c_i$). Since $\partial/\partial t$ is a holomorphic vector field in $\mathbb{P}^1$ with a zero of multiplicity two at $\infty$, if $\omega \in H^0(\mathbb{P}^1, V(*\infty))$ has a pole (respectively, zero) of order $m$ at $\infty$, then $\nabla_{\partial/\partial t}\omega$ has a pole (respectively, zero) of order $m - 1$ (respectively, $\max\{2, m + 1\}$) at $\infty$. If there is no confusion, we write $\nabla = \nabla_{\partial/\partial t}$.

For any point $b \in \mathbb{P}^1\backslash C$, we can find a frame $\{e_1, e_2, \ldots, e_\alpha\}$ of holomorphic sections of $V$ in a neighborhood of $b$ such that $\nabla e_i = 0$ for all $i$ and any other solution of $\nabla\omega = 0$ is a linear combination of $e_i$. Analytic continuations of this frame in $\mathbb{P}^1\backslash C$ define the monodromy operator

$$T : \pi_1(\mathbb{P}^1\backslash C, b) \to GL(V_b).$$

We say that $\nabla$ is irreducible if the action of the monodromy on a nonzero element of $V_b$ generates the whole $V_b$.

Let $V^*$ be the dual vector bundle of $V$. A natural dual connection

$$\nabla^* : V^* \to \Omega^1_{\mathbb{P}^1}(D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} V^*$$

is defined on $V^*$ as follows:

$$\langle \nabla^*\delta, \omega\rangle = d\,\langle \delta, \omega\rangle - \langle \delta, \nabla\omega\rangle\,, \quad \delta \in V^*, \omega \in V.$$

If $\{e_1, e_2, \ldots, e_\alpha\}$ is a base of flat sections in a neighborhood of $b$, then we can define its dual as follows:

$$\langle \delta_i, e_j\rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

We can easily verify that $\delta_i$ are flat sections. The associated monodromy for $\nabla^*$ with respect to this basis is just $T^*$, where $T^*$ is the composition of $T$ with the transpose operator. We can also define a natural connection on

$$\bigwedge^k V = \{\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \mid \omega_i \in V\}$$

with the pole divisor $D$ as follows:

$$\nabla(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} \omega_1 \wedge \omega_2 \wedge \cdots \widehat{\omega_i, \nabla\omega_i} \cdots \wedge \omega_k,$$

where $\widehat{\omega_i, \nabla\omega_i}$ means that we replace $\omega_i$ by $\nabla\omega_i$.

Every line bundle $L$ in $\mathbb{P}^1$ is of the form $L_{a\infty}$, where $a$ is an integer and $L_{a\infty}$ is the line bundle associated with the divisor $a\infty$. We define $c(L) = a$ (Chern class). According to the Grothendieck decomposition theorem, every vector bundle $V$ on $\mathbb{P}^1$ can be written as $V = \bigoplus_{i=1}^{\alpha} L_i$, where $L_i$ are line bundles. We define $c(V) = \sum_{i=1}^{\alpha} c(L_i)$. In view of Theorem 1, the following definition is natural.

**Definition 1.** For any meromorphic global section of $V$ with poles at $C \cup \{\infty\}$ define its degree as the sum of its pole orders. For any natural number $n$, let $H_\nabla(n)$ be the smallest number such that for all $\omega$ of degree $n$ the set $\{\nabla^i\omega \mid i = 0, 1, 2, \ldots, H_\nabla(n) - 1\}$ generates each fiber $V_b$, $b \in \mathbb{P}^1 \backslash C \cup \{\infty\}$.

Of course, we have

$$H_\nabla(n) \geq \alpha.$$

Let $V$ be a line bundle. In this case $H_\nabla(n)$ is the maximum multiplicity of a zero of $\omega$ of degree $n$ minus one and, therefore, $H_\nabla(n) = n + c(V) + 1$. In the general case, we can only give an upper bound for $H_\nabla(n)$. One of the main results of this article is item 3 of the following theorem.

**Theorem 1.** *If the connection $\nabla$ over $V$ is irreducible, then for any global meromorphic nonzero section of $V$ with poles at $C \cup \{\infty\}$, say $\omega$, the following assertions hold*:

1. $\{\nabla^i\omega \mid i = 0, 1, 2, \ldots\}$ *generates each fiber $V_b$, $b \in \mathbb{P}^1 \backslash C \cup \{\infty\}$*;
2. $\{\nabla^i\omega \mid i = 0, 1, 2, \ldots, \alpha - 1\}$ *generates a generic fiber $V_b$*;
3. *finally, we have*

$$H_\nabla(n) \leq \frac{\alpha(\alpha-1)}{2} \sum m_i + \alpha(n+1) - \frac{\alpha(\alpha-1)}{2} + c(V).$$

*Proof.* Let us prove item 1. If there exists $\omega \in H^0(\mathbb{P}^1, V(*\infty))$ such that $\{\nabla^i\omega \mid i = 0, 1, 2, \ldots\}$ does not generate $V_b$, then there exists nonzero $\delta_b \in V_b^*$ such that

$$\left\langle \delta_b, \nabla^i\omega \right\rangle = 0, \quad i = 0, 1, 2, \ldots.$$

Consider the flat section $\delta$ passing through $\delta_b$. Since

$$\left.\frac{\partial^i \langle \delta, \omega \rangle}{\partial^i t}\right|_b = \left\langle \delta_b, \nabla^i\omega \right\rangle = 0,$$

we conclude that $\langle \delta, \omega \rangle$ is identically zero. Since $\nabla$ is irreducible, we conclude that $\omega$ is the zero section.

Now let us prove item 2. Let $k$ be the largest number such that for all nonzero $\omega \in H^0(\mathbb{P}^1, V(*\infty))$, $A = \omega \wedge \nabla\omega \wedge \cdots \wedge \nabla^k\omega$ is not identically zero. We prove that $k = \alpha - 1$. Fix a nonzero $\omega$ with the property $A \wedge \nabla^{k+1}\omega = 0$. Let $B = \mathbb{P}^1 - C \cup \{\infty\} \cup \mathrm{zero}(A)$ and $V'$ be the vector bundle over $B$ generated by $\omega, \nabla\omega, \cdots, \nabla^k\omega$. Since $A \wedge \nabla^{k+1}\omega = 0$, $\nabla$ induces on $V'$ a well-defined holomorphic connection. But this means that $V_b'$ is invariant with respect to the monodromy. $\nabla$ is irreducible and, therefore, $k + 1 = \dim(V_b') = \alpha$.

The proof of item 3 is as follows. For any global meromorphic section $\omega$ of $V$ with poles at $C \cup \{\infty\}$, we define

$$A = \omega \wedge \nabla\omega \wedge \cdots \wedge \nabla^{\alpha-1}\omega.$$

According to item 2, $A$ is a nonzero global meromorphic section of $\wedge^\alpha V$. Let $m$ be the order of the pole of $\omega$ at $\infty$. Then the sum of poles of $\omega$ at $C$ is at most $n-m$. Since $\nabla$ increases the pole order from $a_i$ to $a_i + m_i$ at each $c_i$, we conclude that the sum of pole orders of $A$ at $C$ is at most $\dfrac{\alpha(\alpha-1)}{2}\sum m_i + \alpha(n-m)$. Each $\nabla^i\alpha$ has a pole (respectively, zero if $m-i$ is positive) of order $m-i$ at $\infty$ and, therefore, $A$ has order

$$m + m - 1 + m - 2 + \cdots + m - (\alpha - 1) = m\alpha - \frac{\alpha(\alpha-1)}{2}$$

at infinity. We conclude that the multiplicity of a zero of $A$ in $\mathbb{P}^1 \backslash C \cup \{\infty\}$ is less than

$$\frac{\alpha(\alpha-1)}{2}\sum m_i + \alpha(n-m) + m\alpha - \frac{\alpha(\alpha-1)}{2} + c(V).$$

If $b \in \mathbb{P}^1 \backslash C \cup \{\infty\}$ is a point such that $\omega$, $\nabla\omega$, $\ldots$, $\nabla^i\omega$, $i \geq \alpha - 1$, do not generate $V_b$, then $A$ has a zero of multiplicity $i - (\alpha - 2)$. The theorem is proved. $\square$

The author does not think that this upper bound is the best one. More precisely, for any vector bundle $V$ and divisor $D$ on $\mathbb{P}^1$, can we find a meromorphic connection $\nabla$ on $V$ with pole divisor $D$ such that $H_\nabla(n)$ is the above number?

Let $\delta$ be a flat section of $V^*$ in a small open set $U$ around $b$ and $\omega$ be a global meromorphic section of $V$ with poles at $C \cup \{\infty\}$. From now on, we use the notation $\int_\delta \omega$ instead of $\langle \delta, \omega \rangle$. Later we will see the justification of this notation by usual Abelian integrals.

We fix the number $n$ and suppose that the degree of $\omega$ is less than $n$. What is the maximum multiplicity of $\int_\delta \omega$ at $t \in U$, say $H_\nabla(n, \delta)$? Let $S(n)$ be the vector space of meromorphic sections of $V$ with poles at $C \cup \{\infty\}$ and degree less than $n$. Since $S(n)$ is a finite-dimensional vector space, $H_\nabla(n, \delta_t)$ is a finite number.

**Theorem 2.** *If $\nabla$ is irreducible, then for all $t \in U$*

$$H_\nabla(n, \delta) \geq \dim_{\mathbb{C}} S(n) - 1.$$

*The equality occurs except for a finite number of points in $U$.*

*Proof.* Let $\omega_1, \omega_2, \ldots, \omega_b$ be a basis for the vector space $S(n)$. Consider the determinant

$$W_b(t) = \det\left[\frac{\partial^i}{\partial t^i}\int_\delta \omega_j\right]_{b\times b}.$$

It suffices to prove that $W_b(t)$ is not identically zero. Let $a \leq b$ be the smallest number such that $W_a(t)$ is identically zero. There exist holomorphic functions $p_i$, $i = 1, 2, \ldots, a$, in $U$ such that

$$A_a - \sum_{i=1}^{a-1} A_i \frac{p_i}{p_a} = 0,$$

where $A_i$ is the $i$th column of the matrix

$$\left[\frac{\partial^i}{\partial t^i} \int_\delta \omega_j\right]_{a\times a}.$$

We have $a$ equalities. If we apply $\partial/\partial t$ to the $i$th equation and subtract the $(i+1)$th equation, we conclude that

$$\left[\frac{\partial^i}{\partial t^i} \int_\delta \omega_j\right]_{b-1\times b-1} \times \left[\frac{\partial(p_i/p_b)}{\partial t}\right]_{b-1\times 1} = 0.$$

By the hypothesis, this implies that

$$\left[\frac{\partial(p_i/p_b)}{\partial t}\right]_{b-1\times 1} \equiv 0$$

or, equivalently,

$$\sum_{i=1}^{a} c_i \int_\delta \omega_i = 0,$$

where $c_i$ are constant. Since $\nabla^*$ is irreducible, we have

$$\sum_{i=1}^{a} c_i\omega_i = 0,$$

is a contradiction. $\square$

The following is complementary to our main theorem.

**Theorem 3.** *If $\nabla$ is irreducible, then*

$$H_\nabla(n) - 1 = \sup\{H_\nabla(n,\delta)\},$$

*where $\delta$ runs through all flat sections of $V^*$ in $\mathbb{P}^1\backslash C \cup \{\infty\}$.*

*Proof.* If there exists a degree $n$ section $\omega$ such that

$$\{\nabla^i\omega \mid i = 0, 1, 2, \ldots, p-1\}$$

does not generate $V_b$, then there exists $\delta_b \in V_b^*$ such that

$$\int_{\delta_b} \nabla^i\omega = 0, \quad i = 0, 1, 2, \ldots, p-1.$$

Consider the flat section $\delta$ passing through $\delta_b$. We conclude that $\int_\delta \omega$ has the multiplicity $p$ at $b$ and hence $H_\nabla(n,\delta_b) < H_\nabla(n)$. The proof of the other part is similar. $\square$

## 3. Regular connections and linear equations

Consider the connection $\nabla^*$ on $V^*$ as before and fix a trivialization mapping for $V^*$ around a singular point $c_i$. $\nabla^*$ is called regular at $c_i$ if each flat section of $V^*$ in a sector with the vertex $c_i$ has at most a polynomial growth near $c_i$ (see [12, p. 36] or [1, p. 8]). $\nabla$ is called regular if it is regular at all $c_i$.

Let $\omega_1, \omega_2, \ldots, \omega_\alpha$ be global meromorphic sections of $V$ with poles at $C \cup \{\infty\}$. Also, let $\delta_1, \ldots, \delta_\alpha$ be a base of flat sections of $V^*$ in a neighborhood of $b$. The Wronskian function is defined as follows:

$$W(t) = W(\omega_1, \ldots, \omega_\alpha)(t) = \det \left[ \int_{\delta_j} \omega_i \right]_{\alpha\times\alpha}.$$

The quotient of two such functions is a meromorphic function in $\mathbb{P}^1 \backslash C$ and by regularity of $\nabla^*$, we conclude that it is extended meromorphically to the whole $\mathbb{P}^1$. Fix $\omega$. By a similar argument as stated in Theorem 2 and by irreducibility of $\nabla$, we know that $W(\omega, \nabla\omega, \ldots, \nabla^{\alpha-1}\omega)$ is not identically zero. The set

$$\left\{ \int_\delta \omega \mid \delta \text{ is a flat section of } V^* \right\}$$

is a base for the space of solutions of the following linear equation:

$$\psi : \begin{vmatrix} Y & \int_\delta \omega \\ Y' & \int_\delta \nabla\omega \\ \vdots & \vdots \\ Y^{(\alpha)} & \int_\delta \nabla^\alpha \omega \end{vmatrix} = 0, \tag{1}$$

which can be written in the other form

$$\psi : Y^{(\alpha)} + \sum_{i=1}^{\alpha} (-1)^i P_i Y^{(i)} = 0, \tag{2}$$

where

$$P_i = \frac{W(\omega, \nabla\omega, \ldots, \widehat{\nabla^{\alpha-i}\omega}, \ldots, \nabla^\alpha\omega)}{W}, \quad W = W(\omega, \nabla\omega, \ldots, \nabla^{\alpha-1}\omega).$$

Since $\int_{\delta_i} \tilde{\omega}$ has a polynomial growth at the points of $C$, $\psi$ is regular and, therefore, it must be Fuchsian, i.e., $P_i$ has poles of order at most $i$ (see [1]). The union of poles of $P_i$ is the singular set of the Picard–Fuchs equation $\psi$. It has three types of singularities:

1. $C$. At $c_i \in C$, the solutions of (2) branch;
2. $Z$, the zeros of $W$. In these singularities, like regular points, we have a space of solutions of dimension $\alpha$. Note that

$$P_1 = \frac{\partial W / \partial t}{W}.$$

For this reason, in [1] these are called apparent singularities. For a zero $b$ of $W$ we can find a flat section $\delta$ of $V^*$ such that $\int_\delta \omega$ has a multiplicity greater than $\alpha$ at $b$;

3. $\infty$. Let $m$ be the order of the pole of $\omega$ at $\infty$. The solutions of (2) in a neighborhood of $\infty$ are meromorphic functions with poles of order at most $m$ at $\infty$.

Since $P_1 = \dfrac{\partial W/\partial t}{W}$, we have

$$\mathrm{Res}(P_1 dt, t=c) = \mathrm{mul}(W, t=c), \quad c \in C \cup Z.$$

Now we consider a regular linear equation $\psi$ with singularities at $C \cup Z \cup \{\infty\}$ and suppose that it has apparent singularities in $Z$ and a singularity of type 3 at $\infty$. Furthermore, assume that $\psi$ has the same monodromy representation as $\nabla^*$.

**Proposition 1.** *$\psi$ is obtained by a meromorphic global section of $V$ with poles at $C \cup \{\infty\}$.*

*Proof.* In a neighborhood of $b$, we consider a base of flat sections $\delta_i$ of $V^*$ and a base $e_i$, $i = 1, 2, \ldots, \alpha$, for solutions of $\psi$ such that the monodromy representation of the both $\nabla^*$ and $\psi$ with respect to these bases is the same. Define a section of $V = (V^*)^*$ as follows:

$$\omega(\delta_t) = e_i(t).$$

This is a single-valued holomorphic section of $V$ in $\mathbb{P}^1 \backslash C \cup \{\infty\}$. Since $\psi$ and $\nabla^*$ are regular, $\omega$ can be extended meromorphically to $C$. □

Let $\phi_b$ be the maximum multiplicity of solutions of $\psi$ at $b$. If $b$ is a regular point of $\psi$, then $\phi_b = \alpha - 1$ and if it is an apparent singularity of $\psi$, then $\phi_b \geq \alpha$. In the last case, by the definition of $W$, we can see that $W$ has a zero of order at least $\phi_b - (\alpha - 1)$ at $b$ and by $P_1 = \dfrac{\partial W/\partial t}{W}$, we have

$$\phi_b \leq \mathrm{Res}(P_1\, dt,\, t = b) + (\alpha - 1).$$

*Remark 1.* Let us choose a trivialization of $V$ in a small disk $D$ around a singular point $c_i$ of the connection $\nabla$, $V|_D \cong D \times \mathbb{C}^\alpha$, and a coordinate $z$ in $D$. In this coordinate, we can write

$$\nabla v = \frac{\partial v}{\partial z} + \sum_{j=1}^{m_i} \frac{C_j}{z^j} v + A(z) v,$$

where $v$ is a holomorphic vector in $D$, $C_j$, $1 \leq j \leq m_i$ (respectively, $A(z)$) is a constant (respectively, holomorphic in $z$) matrix. $C_1$ is called the residue of the connection at $c_i$. Now we can apply the Levelt theory (see [1, Sec. 1, 2.2]) to understand the local theory of this connection.

## 4. The Lefschetz pencil

Let $M$ be a projective compact complex manifold of dimension two, $\{M_t\}_{t\in\mathbb{P}^1}$ be a pencil of hyperplane sections of $M$, and $f$ be the meromorphic function on $M$ whose level sets are $M_t$ (see [13]). Let $\mathcal{R}$ be the set of indeterminacy points of $f$, $L_t = M_t - \mathcal{R}$, $C = \{c_1, c_2, c_3, \dots, c_r\}$ be the set of critical values of $f$, $\beta = \dim(H^1(L_t,\mathbb{C}))$ for $t \in \mathbb{P}^1 - C$, and $\mathbb{C}[t]$ be the ring of polynomials in $t$. Since $f|_{M-\mathcal{R}}$ is a $C^\infty$-fibration over $\mathbb{P}^1 - C$ (see [13]), $\beta$ is independent of $t$. We assume the following:

1. the axis of the pencil intersects $M$ transversally. This is equivalent to the fact that in a coordinate system $(x, y)$ around each indeterminacy point of $f$, we can write $f = x/y$;
2. the critical points of $f$ are isolated;
3. the pole divisor $D = M_\infty$ of $f$ is a regular fiber, i.e., $\infty \notin C$.

We define $\Omega^i(*D)$ to be the set of meromorphic $i$-forms in $M$ with poles of arbitrary order along $D$. The set $\tilde{H} = \bigcup_{t\in B} H^1(L_t,\mathbb{C})$, where $B = \mathbb{P}^1 - C$, has a natural structure of a complex manifold and the natural projection $\tilde{H} \to B$ is a holomorphic vector bundle which is called the cohomology vector bundle. The sheaf of holomorphic sections of $\tilde{H}$ is also denoted by $\tilde{H}$. In what follows, when we consider $f$ as a holomorphic function, we mean its restriction to $M - \mathcal{R}$. Let $\mathbb{C}_{M-\mathcal{R}}$ be the sheaf of constant functions in $M - \mathcal{R}$ and $R^1 f_*\mathbb{C}_{M-\mathcal{R}}$ be the first direct image of the sheaf $\mathbb{C}_{M-\mathcal{R}}$ (see [6]). Any element of $R^1 f_*\mathbb{C}_{M-\mathcal{R}}(U)$, where $U$ is an open set in $B$, is a holomorphic section of the cohomology fiber bundle mapping. It is easy to verify that

$$\tilde{H} \cong R^1 f_*\mathbb{C}_{M-\mathcal{R}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \quad \text{in } B.$$

Now we introduce the Gauss–Manin connection on $\tilde{H}$. Consider a holomorphic coordinate $(t, 0)$ in a small open disk $U$ in $\mathbb{P}^1$. The Gauss–Manin connection is defined as follows:

$$\nabla : \tilde{H}(U) \to \Omega^1_{\mathbb{P}^1} \otimes_{\mathcal{O}_U} \tilde{H}(U),$$
$$\nabla(g \otimes c) = dg \otimes c, \quad c \in R^1 f_*\mathbb{C}_{M-\mathcal{R}}(U), \quad g \in \mathcal{O}_{\mathbb{P}^1}(U).$$

The sheaf of flat sections of $\nabla$ is $R^1 f_*\mathbb{C}_{M-\mathcal{R}}$. Let $\partial/\partial t$ be a vector field in $U$. We write

$$\nabla_{\partial/\partial t} = \frac{\partial}{\partial t} \circ \nabla, \quad \nabla_{\partial/\partial t}(g \otimes c) = \frac{\partial g}{\partial t} \otimes c.$$

In the same way, we can define the cohomology fiber bundle $\tilde{H}_c$ of compact fibers $M_t$. Since $\tilde{H}_c$ is a $\nabla$-invariant vector subbundle of $\tilde{H}$, we have the restriction of $\nabla$ to $\tilde{H}_c$ which we denote again by $\nabla$.

Let $\omega$ be a meromorphic 1-form in $M$ with poles along some fibers of $f$. Also, let $\{\delta_t\}_{t\in\mathbb{P}^1-C}$, $\delta_t \subset L_t$, be a continuous family of cycles. The Abelian integral $\int_{\delta_t}\omega$ appears in the deformation $df+\epsilon\omega$ of $df$ inside holomorphic foliations (differential equations) and it is related to the number of limit cycles born from the cycles $\delta_t$ (see [10]). The pair $(\tilde{H},\nabla)$ is defined in $\mathbb{P}^1-C$ and in order to be in the context of this paper, we prove the following proposition.

**Proposition 2.** *Under assumptions* 1–3*, there exist a vector bundle $V$, a vector subbundle $\bar{V}\subset V$, and a meromorphic connection on $V$ with poles in $C$*

$$\nabla : V \to \Omega^1_{\mathbb{P}^1}(D)\otimes_{\mathcal{O}_{\mathbb{P}^1}} V, \quad D=\sum m_i c_i,$$

*such that*:

1. *$\bar{V}$ is $\nabla$-invariant*;
2. $(V,\nabla)$ (*respectively,* $(\bar{V},\nabla)$) *coincides with* $(\tilde{H},\nabla)$ (*respectively, with* $(\tilde{H}_c,\nabla)$) *in* $\mathbb{P}^1-C$;
3. *the Brieskorn lattice* (*Petrov module in the context of differential equations*) *of $f$*
$$H'=\frac{\Omega^1(*D)}{df\wedge\Omega^0(*D)+d\Omega^0(*D)}$$
*is a $\mathbb{C}[t]$-isomorphism to the module of global sections of $V$ with poles of arbitrary order at $\infty$.*

This is a task which is done in detail in [11]. If the singularities of $f$ are nondegenerate, i.e., in a holomorphic coordinate $(x,y)$ around a singularity $p_i$ we can write $f=f(p_i)+x^2+y^2$, then all $m_i$ are equal to one. In other words, $\nabla$ is logarithmic.

The pair $(V,\nabla)$ is not irreducible but if $H^1(M,\mathbb{C})=0$ and $f$ satisfies conditions 1–3 and has nondegenerate singularities with distinct images then $(\bar{V},\nabla)$ is irreducible (see [13, 7.3]). The following proposition justifies the use of $(\bar{V},\nabla)$ instead of $(V,\nabla)$.

**Proposition 3.** *For a meromorphic* 1*-form $\omega$ in $M$ with poles of order at most $n$ along $D$, the integral $\int_{\delta_t}\omega$ is a polynomial of degree $n$. $\nabla^i_{\partial/\partial t}\omega$, $i>n$, restricted to each fiber has no residues in $\mathcal{R}$ and hence is a meromorphic section of $\bar{V}$.*

*Proof.* We have
$$p(t):=\int_{\delta_t}\omega=t^n\int_{\delta_t}\frac{\omega}{f^n}.$$
Since the 1-form $\omega/f^n$ has no poles along $D$, $p(t)/t^n$ has finite growth at $t=\infty$. Since $p(t)$ is holomorphic in $\mathbb{C}$ (even in the points of $C$), we conclude that $p(t)$ is a polynomial of degree at most $n$. The second part is a direct consequence of the first one and the fact that
$$\frac{\partial}{\partial t}\int_{\delta_t}\omega=\int_{\delta_t}\nabla_{\partial/\partial t}\omega.$$

The proposition is proved. $\square$

I tried to study the maximum multiplicity of Abelian integrals in the context of meromorphic connections. My motives were the paper [15] and also a paper of mine, where the extension of cohomology vector bundles and their connections to the critical values of a meromorphic function is discussed. The upper bound obtained in Theorem 1 seems to be far from the best one (at least for Gauss–Manin connections). Some works in differential equations (see [9]) suggest that the number $H_\nabla(n)$ must be very sensitive with respect to $\nabla$.

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Author's address:
IPM, P.O.Box: 19395-5746,Tehran-Iran
E-mail: movasati@mathematik.tu-darmastadt.de