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A Course in Hodge Theory

With Emphasis on Multiple Integrals

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باز آن جان چونک محو عشق گشت

یعجب الزراع آمد بعد کشت

If seeds are planted firmly in the ground,
Wheat will eventually grow all around;
Then in the mill they grind it to make bread-
Its value soars now with it men are fed;
Next by men's teeth the bread is ground again,
Life, wisdom, and intelligence they gain,
And when in love that life becomes effaced
Farmers rejoice the seed's not gone to waste!

Jalal al-Din Muhammad Balkhi, Masnavi, Book 1, Section 148. Translation: Jawid Mojaddedi, Oxford World's Classic. Calligraphy: nastaliqonline.ir.

This book is dedicated to the memory of my parents Rogayeh and Ali, and to my family Sara and Omid

Preface

The main objective of the present book is to give an introduction to Hodge theory and its main conjecture, the so-called Hodge conjecture. We aim to explore the origins of Hodge theory much before the introduction of Hodge decomposition of the de Rham cohomology of smooth projective varieties. This is namely the study of elliptic, abelian and multiple integrals originated from the works of Cauchy, Abel, Jacobi, Riemann, Poincaré, Picard and Lefschetz among many others. Therefore, the reader is warned that he or she will find in this book a partial presentation of the modern Hodge theory. The present book is intended to be an incomplete resuscitation of Picard and Simart's treatise *Théorie des fonctions algébriques de deux variables indépendantes* after almost a century, keeping in mind that the main object of study is the multiple integral itself and not other by-products. A complete analysis of this treatise and other contributions need a historian in mathematics. This is beyond the scope of this book. Another main emphasis of this book is on the computational aspects of the theory such as computing homologies by means of vanishing cycles, de Rham cohomologies, Gauss-Manin connections, Hodge cycles, etc. The development of Hodge theory during the last decades has put it far from its origin and the introduction of mirror symmetry by string theorists and the period manipulations of the B-model Calabi-Yau varieties, have risen the need for a text in Hodge theory with more emphasis on periods and multiple integrals. We aim to present materials which are not covered in J. Lewis's book *A survey of the Hodge conjecture* and C. Voisin's books *Hodge theory and complex algebraic geometry, I, II*, therefore, the reader will not find in this book some of the fundamental theorems in modern Hodge theory. We have tried to keep the text self-sufficient, however, a basic undergraduate knowledge of Complex Analysis, Differential Equations, Algebraic Topology and Algebraic Geometry will make the reading of the text smoother. The text is mainly written for two primary target audiences: graduate students who want to learn Hodge theory and get a flavor of why the Hodge conjecture is hard to deal with, and mathematicians who use periods and multiple integrals in their research and would like to put them in a Hodge theoretic framework. We hope that our text, together with those mentioned above, makes Hodge theory more accessible to a broader public.

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Frequently used notations

$\mathsf{k}, \bar{\mathsf{k}}$	A field of characteristic zero and its algebraic closure.
$\bar{\mathbb{Q}}$	The field of algebraic numbers.
R	A finitely generated ring over the field k.
$\mathsf{T} := \mathrm{Spec}(\mathsf{R})$	A parameter space.
Θ_{T}	The set of vector fields in T.
$t \in \mathsf{T}$	A point in the parameter space.
$\mathrm{Mat}(n \times m, \mathsf{R})$	The set of $n \times m$ matrices with entries in R.
$\mathrm{Mat}(n, \mathsf{R})$	The set of $n \times n$ matrices with entries in R.
$V^\vee$	The dual of an R-module V. We always write a basis of a free R-module of rank r as a $r \times 1$ matrix. For a basis δ of V and α of $V^\vee$ we denote by $$[\delta, \alpha^{\mathsf{tr}}] := [\alpha_j(\delta_i)]_{i,j}$$ the corresponding $r \times r$ matrix.
M^{tr}	The transpose of a matrix M. We also write $M = [M_{ij}]$, where M_{ij} is the (i, j) entry of M. The indices i and j always count the rows and columns, respectively.
d	The differential operator or a natural number which is the degree of a tame polynomial.
$\mathbb{P}^{n+1}$	The projective space.
$(x_0, x_1, \ldots, x_{n+1})$	Homogeneous coordinates of $\mathbb{P}^{n+1}$
$x = (x_1, x_2, \ldots, x_{n+1})$	Affine coordinates of $\mathbb{C}^{n+1}$, for $n = 1, 2$ we use the classical notations (x, y) and (x, y, z), respectively.
f	A tame polynomial in $\mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$.
s	The parameter in the tame polynomial $f - s$.
x^β, x^α	Monomials.
$\nu_1, \nu_2, \ldots$	The weights of the variables $x_1, x_2, \ldots$.
U	An open set in the usual topology or an affine variety.
L_t	A fiber of a tame polynomial.
X	A smooth hypersurface in $\mathbb{P}^{n+1}$.

ρ, ρ_0	The Picard number of a surface X and $\rho_0 := b_2 - \rho$, where b_2 is the second Betti number of X.
$h^{i,j}$	Hodge numbers of the projective variety X.
r	Dimension of the moduli space of hypersurfaces.
Y	A subvariety of X of codimension 1. It is usually the hyperplane section of X.
Z, Z_i	Algebraic subvarieties/cycles of X.
Z_∞	The algebraic cycle obtained by intersection of X with a linear $\mathbb{P}^{\frac{n}{2}+1}$.
$\check{Z}$	A primitive algebraic cycle, that is, $\check{Z} \cdot Z_\infty = 0$.
n, m	The dimension of X and any number between 0 and $2n$, respectively.
$H^m_{\mathrm{dR}}(X)$, $H^m_{\mathrm{dR}}(U)$	Algebraic de Rham cohomology
$\omega, \eta, \omega_\beta, \eta_\beta$	Differential forms in U or elements of $H^m_{\mathrm{dR}}(X)$ or $H^m_{\mathrm{dR}}(U)$ etc..
Δ	The discriminant in R of a tame polynomial or a simplex.
Δ^n	The n-dimensional simplex.
$\hat{\Delta}$	A divisible element in R in order to get tameness for f.
$\check{\Delta}$	The double discriminant which is an element in R.
$H_m(X,\mathbb{Z}), H_m(U,\mathbb{Z})$	The singular homology with coefficients in $\mathbb{Z}$.
$H_m(X,\mathbb{Z})_0, H^m_{\mathrm{dR}}(X)_0$	The primitive (co)homology.
$H_m(U,\mathbb{Z})_\infty,\ H_m(U,\mathbb{Z})_{\not\infty}$	The $\mathbb{Z}$-module of cycles at infinity.
u	Polarization which is an element in $H^2_{\mathrm{dR}}(X)$ obtained by $X \subset \mathbb{P}^{n+1}$.
∂	The boundary map.
δ	A homology class.
λ	A path in a topological space.
σ, τ	Maps derived from the Leray-Thom-Gysin isomorphism.
$\delta = \{\delta_t\}_{t \in U}$	A continuous family of cycles.
$G_{\mathrm{tors}}, G_{\mathrm{free}}$	The torsion and free subgroups of an abelian group G, respectively.
$\cup$	Cup product in singular or de Rham cohomology.
$\cap$	Cap product in singular (co)homologies.
$\mathbb{S}^n$	The n-dimensional sphere.
$\mathbb{B}^n$	The n-dimensional ball.
$a \cdot b,\ \langle a, b \rangle$	Intersection of topological or algebraic cycles a and b.
$\mathrm{Hodge}_n(X,\mathbb{Z})$	The $\mathbb{Z}$-module of Hodge cycles.
A_β	Rational numbers which are responsible for distinguishing between differential forms.
$\Gamma(a),\ B(a_1, a_2, \cdots)$	The Γ and B-function, respectively.
B_β	B-factors of the periods of the Fermat variety.
C	The set of critical values of a map.
I, x^I	A basis of the Milnor module.
X^d_n	The Fermat variety.
$\mathsf{H}, \mathsf{H}', \mathsf{H}''$	Brieskorn modules.

F^i	Hodge filtration.
W	Weight filtration.
$\frac{\omega}{df}$	Gelfand-Leray form.
M	Gauss-Manin system of the tame polynomial f.
$\mathrm{jacob}(f)$	Jacobian ideal of the polynomial f.
μ	The Milnor number of the tame polynomial f.
μ_d	The group of d-th roots of unity.
∇	Gauss-Manin connection.
∇_v	Connection along a vector field v.
$\mathbf{P}$	Period matrix, period map.
ξ	An invariant of Hodge cycles.
V_δ	Hodge locus corresponding to the Hodge cycle δ.
$(x)_y$	Pochhammer symbol.
$\langle x\rangle_y := (x-y+1)_{y-1}$	A modified Pochhammer symbol.
$\{x\}$	Fractional part of x.

Contents

Chapter 1
Introduction

Los niños habían de recordar por el resto de su vida la augusta solemnidad con que su padre se sentó a la cabecera de la mesa, temblando de fiebre, devastado por la prolongada vigilia y por el encono de su imaginación, y les reveló su descubrimiento: La tierra es redonda como una naranja, (Gabriel García Márquez, Cien años de soledad).

1.1 What is Hodge Theory?

What is now called *Hodge theory* is a culmination of a great amount of effort starting from the works of Augustin Louis Cauchy (1789-1857), Niels Henrik Abel (1802-1829), Carl Gustav Jacob Jacobi (1804-1851) and Georg Friedrich Bernhard Riemann (1826-1866) on elliptic and abelian integrals, Jules Henri Poincaré's (1854-1912) *Analysis Situs*, Charles Émile Picard's (1856-1941) intensive study of multiple integrals, Solomon Lefschetz's (1884-1972) treatise on the topology of smooth projective varieties, William Vallance Douglas Hodge's (1903-1975) description of the de Rham cohomology of projective varieties, Phillip Augustus Griffiths' (1938-) breakthrough applications of it in algebraic geometry and Pierre René Deligne's (1944-) vast generalization of it into mixed Hodge structures; just to mention the name of some of the main contributers. Officially, it took this name after Hodge in his treatise *The theory and applications of harmonic integrals* in 1941, proved the so-called *Hodge decomposition* of the de Rham cohomology of Kähler manifolds, and in particular complex smooth projective varieties. Among many others, Griffiths and Deligne took it out of the context of complex analysis, give to it a more algebraic framework, and made it as a new branch of algebraic geometry. Nowadays, the term Hodge theory stands for the study of Hodge structures, their variations, period domains etc. which is basically the Hodge theory after 1950's. In the present book we would like to recover many aspects of Hodge theory before 1950's, this is the study of multiple integrals as it was started by Picard and Poincaré almost a century ago. Our main objective is to introduce and study *Hodge theory before Hodge*, using

all the machineries produced after Hodge. Throughout the text we will never need the Hodge decomposition, despite the fact that the rigorous proofs for few important theorems in Hodge theory, such as hard Lefschetz theorem, are first done using harmonic forms and Hodge decompositions.

The study of n-dimensional multiple integrals, that is, the integration

$$\int_{\delta} \omega \tag{1.1}$$

of algebraic differential n-forms ω over topological cycles δ of dimension n and lying in affine algebraic varieties, goes back to 19th century. Cauchy, Abel, Jacobi, Riemann were among many mathematicians who studied the one dimensional case, nowadays known as elliptic and abelian integrals. Poincaré and Picard were the pioneers in the study of two dimensional integrals. Picard jointly with Simart, wrote a two-volume treatise on this subject *Theorie des fonctions algébriques de deux variables indépendentes*. These two volumes must be considered as the founding stone of Hodge theory. The study of where we are integrating, that is δ, and what we are integrating, that is ω, took different paths in the history of mathematics:

The tale of δ: Between 1911 and 1924, Lefschetz motivated by Picard-Simart's treatise and with *Analysis Situs* of Poincaré in hand, started an intensive investigation of the topology of algebraic varieties, in his own words, he wanted "to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry" (S. Lefschetz in [Lef68] page 854). It is a remarkable fact that at the time Lefschetz's work was being done, while the study of algebraic topology was getting under way, the topology tools available were still primitive, for instance singular homology was not yet defined, despite the fact that the computation of its rank (Betti number) was a major preoccupation in mathematics. His treatise *L'analysis situs et la géométrie algébrique* appeared in 1924 and it paved the road for a further deep understanding of the topology and arithmetic of algebraic varieties. The singular homology reached its perfection when Eilenberg and Steenrod in their book *Foundations of algebraic topology* in 1952 took a bunch of proved theorems and properties of the singular homology and made them into a set of axioms. Finally, it was known where δ lives! It is remarkable to note that it took some decades for the precise formulations and proofs of Lefschetz topological ideas. Rigorous proofs were given by using harmonic integrals and sheaf theory, which is the history of ω in the next paragraph. However, these methods only obtain the homology groups with complex or real coefficients, whereas the direct method of Lefschetz enables us to use integer coefficient.

The tale of ω: In the first appearances of the integrand, that is ω, its type was mainly distinguished by terms like *differential forms of the first kind, second kind etc*. This terminology arose from the study of abelian integrals (one dimensional case), and its generalization into two dimensions was satisfactory enough so that that Lefschetz was able to state and prove his, nowadays called *Lefschetz (1,1)-theorem*. The things were not going so smoothly in dimensions greater than 2. It was after the invention of the de Rham cohomology and its Hodge decomposition that Hodge in 1941 was able to distinguish differential forms from each other so

that the Hodge conjecture was formulated. This was done in the category of C^∞ and harmonic forms, however, the original ω as in the works of Poincaré and Picard was always algebraic, that is, it was always constructed by means of algebraic operations like, sum, multiplication, square root etc. The comeback to algebraic differential forms was done after the work of Atiyah and Hodge in 1955 which led Grothendieck to define the algebraic de Rham cohomology, and soon after, the Hodge filtration by Griffiths and Deligne. Many other cohomology theories, such as étale cohomology, were invented later. The interest on multiple integrals was slowly lost in time; it was substituted with Hodge structures, their variations and applications of all these in algebraic geometry. The students were learning about Picard group and number, without knowing their original definitions by Picard in terms of multiple integrals.

We will mainly use the term multiple integral instead of another commonly used term *period*. According to Poincaré in [Poi87] page 323, Picard has given the name period to multiple integrals. This has to do with the one dimensional case in which abelian integrals appear as periods of Riemann's theta series, for further details see §3.2. This name does not seem to be adequate and even Poincaré in the mentioned article complains about this. In general there is no generalization of Riemann's theta series for double and higher dimensional integrals and so they cannot be interpreted as periods of some periodic functions.

1.2 The main purposes of the present book

There are few motivations for writing this book. Below we list some of them.

Study of multiple integrals: As it was mentioned in the previous section, the interest in multiple integrals was slowly lost during a century of activities in Hodge theory. Therefore, one of the main aims of the present book is to recover the study of multiple integrals. Of course, we are going to use all the related machineries developed until recently. The interest on this mainly arose after Physicists, and in particular string theorists, started to use multiple integrals on Calabi-Yau threefolds, in order to make amazing predictions in enumerative algebraic geometry. There have been many complains that modern Hodge theory is not adequate to their need, for instance, many times they prefer to work directly with integrals on cycles with boundaries, instead of trying to talk about Hodge structures and use the available results. We hope that this book will serve as the state-of-the-art realization of Picard's project in a language as similar as possible to his in [PS06], and in this way, accessible to other areas of science.

Hodge conjecture before Hodge: This conjecture arose from a wish to classify the homology classes of algebraic cycles. A concrete formulation and realization of this wish was done in two dimensions by Lefschetz using integrals of the first kind. Going to higher dimensions the things were ill-defined." As might be expected, one of the major difficulties has been to frame a definition of integrals of the second kind, or rather of the differential forms which are their integrands, which is suitable

for stack-theoretic methods.[i] A preliminary study of certain special cases suggested such a definition... but it is not obvious that this definition coincides with that of Picard and Lefschetz in all cases." (W.V.D Hodge and M. F. Atiyah in [HA55] page 56). Despite the fact that it was possible to formulate the Hodge conjecture using only closed (p,q)-forms, and so leaving the realm of algebraic geometry, it was only formulated after the discovery of the Hodge decomposition for Kähler manifolds by Hodge in his treatise *The Theory and Applications of Harmonic Integrals* in 1941. Since this was done in the framework of smooth and harmonic forms, the original algebraic formulation was still a desire as we see in the above quotation of Atiyah-Hodge article which is written 14 years after Hodge's own treatise. Another objective of the present text is to collect recent and old developments on the Hodge conjecture, with emphasis on its formulation using multiple integrals.

Computational Hodge conjecture: As far as many texts in Hodge theory are inspired by the trueness of the Hodge conjecture, the present text is mainly inspired by its falsity. In order to have some concrete evidences for this, the author believes that any counterexample would only be possible using a computer search for non-algebraic Hodge cycles. Therefore, in the present text we aim to cover computational aspects of Hodge theory, from computing the homology of projective varieties by means of Lefschetz's vanishing cycles to computing the Gauss-Manin connection and Hodge cycles of special varieties, such as the Fermat variety. From this point of view, our focus has been mainly on hypersurfaces and we give an exposition of many results of Griffiths for this class of varieties. We would like to make the Hodge conjecture a down-to-earth conjecture so that an amateur mathematician could think about it and verify it for special varieties. In this way the *computational Hodge conjecture* that we deal in this book, is not at all trivial for varieties such that the Hodge conjecture is well-known (for instance surfaces).

History of Hodge theory: The mathematics is growing fast and it is quite difficult to keep the track of both new developments and the historical origin of many concepts and ideas; specially for young people who must start publishing in order to find academic jobs. At first contact with Hodge theory, it is frustrating to know that in modern Hodge theory multiple integrals are not the main object of study. They just appear in the introduction of many books concerned with Hodge theory in order to give a historical justification of their contents, and then, they disappear. "... students who have sat through courses on differential and algebraic geometry (read by respected mathematicians) turned out to be acquainted neither with the Riemann surface of an elliptic curve $y^2 = x^3 + ax + b$ nor, in fact, with the topological classification of surfaces (not even mentioning elliptic integrals of first kind and the group property of an elliptic curve, that is, the Euler-Abel addition theorem). They were only taught Hodge structures and Jacobi varieties!" (V. I. Arnold, On teaching mathematics, Palais de Découverte in Paris, 7 March 1997). For this reason, one of the aims of the present text is to analyze the historical account of Hodge theory, something which might be called *Hodge theory before Hodge*. Since the author is not a historian and his writing abilities might not be enough efficient in order to

[i] Here, the name stack is used instead of sheaf.

give a complete account of this, the strategy of using many quotations from old and new texts, from the works of many respected mathematicians who had major impacts on the theory, is adopted. Moreover, if a mathematical object in this book carries names of mathematicians, we have tried to explore this naming by studying the original articles and making proper citations. We hope this will pave the road for further historical study of the subject.

The author's own story: There are few objects in mathematics which show to a great extent its unity. Multiple integrals must be considered a class of such objects. For this the author would like to tell his own story with abelian and multiple integrals. This mainly explores many other fascinating aspects of integrals that might be far from the main objectives of the present book. This includes the appearance of abelian integrals in differential equations and holomorphic foliations, special values of special functions, relations with modular and automorphic forms and generalization of this into Calabi-Yau modular forms, Darboux, Halphen and Ramanujan differential equations etc. We only cite the proper references for the sake of completeness and do not enter into the details of these topics.

"Descartes wrote: "La géométrie est l'art de raisonner juste sur des figures fausses (Geometry is the art of correct reasoning on false figures)." "Figures" is plural: it is very important to have various perspectives and to know in which way each is wrong, (P. Deligne in [RS14] page 179)". Following this advice, I have tried to put many hand drawn figures in this book in order to make the understanding of arguments smoother.

1.3 Prerequisites and how to read the book?

We assume that the reader is familiar with basic undergraduate courses in mathematics, such as Linear algebra, complex analysis in one variable, algebra (group, rings, modules), etc. A basic algebraic geometry, commutative algebra and differential topology will make the reading smoother. However, these can be learned in parallel with reading this book. In a similar way, we do not assume that the reader is familiar with singular homology and cohomology, instead, we use the axiomatic approach of Eilenberg and Steenrod. Hopefully, this will work as Lefschetz himself developed his ideas without having a concrete notion of singular homology. This can be even generalized to many string theorists who use mathematical objects with less care. From this point of view we mainly use the strategy that getting familiar with a mathematical object is more important than its rigorous construction.

As for the author's own courses which resulted in this book, it is convenient to use many old texts, such as Picard-Simart book [PS06], for the presentation of the material. It is highly recommended that a course based on this book to be enriched by reading activities of many texts of 19th century. This will help the students to understand the mentality of mathematicians at that time, and more importantly, to keep their own ideas in a down-to-earth level.

We have tried to keep the content of each chapter independent from the others. Therefore, the general reader with a good background in mathematics might start having a look at chapters and sections depending on his interests. Repetition of formulas and statements have been allowed. Hopefully, this will make the reading of each chapter smoother. At the beginning of each chapter we have explained its dependence on previous chapters. In this way, the reader can start from a random chapter which is of his interest and try to read it with a minimum amount of effort spent on other chapters.

There are exercises in this book for which even the author is not aware of any solution, for instance, a solution to Exercise 2.11 might lead to counterexamples to the Hodge conjecture. These exercises are mainly labeled with a double star. Those exercises are for the sake of thinking and discussing the particular cases, and not necessarily solving them. In many cases a computer assisted approach can be used in order to get an insight on an exercise. Exercises with a star may need more advanced techniques than those presented in the corresponding chapter.

1.4 Synopsis of the contents of this book

In this book we aim to give a modern approach to the place where the integration is performed and the integrands. The first one is namely the study of the singular homology and the topology of algebraic varieties, Hodge cycles, cycles at infinity etc. and the second one is the algebraic de Rham cohomology of affine varieties, Brieskorn modules, Hodge filtrations, etc. We focus on hypersurfaces and generalization of these using tame polynomials.

Chapters 2 and 3 aim to describe the origins of Hodge conjecture. This is namely the Lefschetz's Puzzle on Picard's ρ_0-formula in [PS06]. We do not give rigorous definitions and theorems in this chapter, instead we want that the reader get a flavor of Hodge theory done almost a century ago, and how it was rich despite the lack of a proper language and notations. Many skipped details will be given throughout the book. These chapters, and in particular Chapter 2, serves as a second Introduction to the content of the present book.

Chapter 4 is devoted to the axiomatic approach of Eilenberg-Steenrod to singular homology. The basic idea is to familiarize the reader with the notation $H_n(X,\mathbb{Z})$ in order to get a weak feeling of what it is. Hopefully, a strong feeling will be produced in the next chapters. If the reader is not comfortable with the presentation of singular homology in Chapter 4, he must keep reading Chapters 5, 6, 7. The main ideas of these chapters are formulated by Poincaré, Picard and Lefschetz. This was when there was no concrete definition of singular homology.

Chapters 5, 6 and 7 are fully dedicated to the study of the topology of algebraic varieties. Chapter 5 is devoted to Lefschetz theorems on the topology of smooth projective varieties. The whole chapter is based on a main theorem which is proved using Picard-Lefschetz theory in Chapter 6. The main focus of the present book is on the class of smooth hypersurfaces and generalization of these as fibers of tame

polynomials. For this reason, in Chapter 7 we study the topology of an affine variety associated to a tame polynomial. A good source for the topic of this chapter is the book [AGZV88]. Since this book is mainly concerned with the local theory of fibrations with applications to singularities, we have collected and proved some theorems on the topology of tame polynomials which have some fresh ideas. In particular, our approach to the calculation of the intersection matrices of tame polynomials and joint cycles has a slightly new feature.

Chapter 8 is devoted to the announcement of the Hodge conjecture using only (p,q)-forms. The Hodge decomposition is not necessary for this. This is not yet our final version of the Hodge conjecture as it uses the integration of C^∞-forms, whereas we would like to present it using integration of algebraic differential forms. In this way we are near to our final goal which is the study of multiple integrals.

Chapter 9 is dedicated to the Lefschetz $(1,1)$ theorem. This is the Hodge conjecture in dimension two and it is the strongest evidence that the Hodge conjecture must be true in general. We follow the same line of thought as in the Lefschetz's original book. This is namely by using the integrals of the first kind.

Chapters 10 and 11 are dedicated to the study of integrands and the distinction between them under the name *mixed Hodge structures*. For a tame polynomial f in $n+1$ variables and with coefficients in a ring, we introduce the Brieskorn module H of f which is a finer version of the algebraic de Rham cohomology of the affine variety $f=0$. We find a canonical basis of the Brieskorn module H and explain algorithms for writing any other element in terms of this basis.

In Chapter 12 we introduce the Gauss-Manin system associated to f. The difference between differential forms, which can be formulated in precise words using the notion of mixed Hodge structure, is more transparent using the Gauss-Manin system.

Chapter 13 is intended to combine the content of Chapters 6, 7, which are of topological nature, with the content of Chapters 10, 11, 12, which are of algebraic nature, in order to study multiple integrals.

Chapter 14 is dedicated to a celebrated theorem of M. Noether which states that a generic surface X of degree $d \geq 4$ in a projective space of dimension three has no curves apart from a hypersurface section of X.

In Chapter 15 we aim to explore the Hodge cycles of the Fermat variety. We compute the periods of the Fermat variety and give algorithms to compute a basis of the $\mathbb{Z}$-module of Hodge cycles and its rank. For smooth surfaces in $\mathbb{P}^3$ the $\mathbb{Z}$-module of Hodge cycles is known as the Picard or Neron-Severi group, and its rank is known as the Picard number. A table of Picard numbers for Fermat surfaces is given in this chapter. There is a relation between the dimension of the space of Hodge cycles and the rank of elliptic curves over function fields. In this chapter we explain this and we recover Shioda's record on the construction of an elliptic curve of rank 68.

Chapter 16 is devoted to the study of periods of Hodge cycles of the Fermat variety and an invariant ξ of such Hodge cycles. We explain the relation of such periods with algebraic relations between the values of the Γ-function on rational points. The invariant ξ originally comes from the study of codimensions of components of the

Hodge locus, however, it can be defined independently of this. This invariant serves as a tool to distinguish Hodge cycles from each other.

Chapter 17 is fully dedicated to the study of algebraic cycles of the Fermat variety and their intersection numbers with each others. These are namely the linear cycles and Aoki-Shioda cycles. We aim to introduce all the machinery in order to verify the Hodge conjecture using these cycles and for special values of the degree and the dimension.

Chapters 18 and 19 were originally written as follow-up articles for the first version of the present book. In Chapter 18 we show how the data of integrals of algebraic differential forms over algebraic cycles can be used in order to prove that algebraic and Hodge cycle deformations of a given algebraic cycle are equivalent. We take a difference of two linear cycles inside the Fermat variety with intersection of codimension two in both cycles, and gather evidences that the Hodge locus corresponding to this is smooth and reduced. This implies the existence of new algebraic cycles in the Fermat variety whose existence is predicted by the Hodge conjecture for all hypersurfaces, but not the Fermat variety itself. The main goals of Chapter 18 are pursued in Chapter 19 in which we focus on cubic hypersurfaces.

In Chapter 20 we explain the implementation of many algorithms appearing throughout the book, in `Singular`, a computer programming language for polynomial computations. This includes computation of Gauss-Manin connections, Hodge cycles, etc.

In Chapter 21 we have collected a bunch of problems which are elementary to state, in the sense that we only need a high school mathematics or at most a basic knowledge in linear algebra to understand them. They have arisen throughout the book and the curiosity to explore their origin might motivate undergraduate students to learn more advanced mathematics.

1.5 Further reading

In the first complete version of the present book there were many missing references in modern Hodge theory. This is mainly due to its narrow point of view and might have been resulted because of emphasis on integrals and computations. In order to give a correct view of Hodge theory for novices and students, we give few references to some classical books and developments in the topic. Griffiths and Harris' book [GH94] contains the foundational topics such as Hodge structures. For advanced topics Voisin's books [Voi02b, Voi03] and Lewis' lectures notes [Lew99] mentioned in the preface are now classical text books. The texts [GMV94, CEZGuT14, KK98, Ara12] contain many lecture notes in various topics of Hodge theory and are alternatives for further reading. The development of Hodge theory in terms of mixed Hodge structures is nicely covered in [PS08, Kul98], see also [EZ91]. In terms of Hodge structures, their degenerations and period domains the book [CMSP03] gives a full discussion of the topic. For this topic Griffiths' article [Gri70] is still fresh and contains a summary of main results. A development in

Hodge theory, and specially after the works of Steenbrink, is in singularity theory for which we refer to [Ste77a, SZ85, SS85, Sab99, Sab07, Sab08, Her02, Sai01, Sai82, Sai89, Dim87, Dim92, NS12, Ta07, Sie74]. For our purpose we have mainly used [AGZV88] and the references therein. A completely different topic originated from analogies between de Rham cohomology with its Hodge decomposition and Étale cohomology with its Galois action, see for instance [Del71a], is p-adic Hodge theory for which we refer to [Fal88]. High precision approximation of periods, which results in computing the Picard rank of surfaces, are done in [Ser19, LS18], and it gives us a new horizon in computational Hodge theory, see also [Sim08] for the discussion of algebraic cycles from an algorithmic computational point of view.

Despite the fact that period manipulations and Hodge theory of Calabi-Yau varieties, and connection of this to mirror symmetry in Physics, has been one of our motivations for writing the present book, we do not touch this topic and a huge amount of literature produced so far. A mathematical analysis of Hodge theory of physicists has been done in [Voi99, Del97, Mor93, AMSY16] and the author's book [Mov17a]. We also refer to [LY96b, LY96a, Bv95, HLY96, SZ10, SXZ13, HLTY18] and the reference therein for a vast literature on usage of Hodge theory and periods for Calabi-Yau manifolds. Another important development is the study of tautological systems initiated in the articles [LY13, LSY13, HLZ16] which itself deserves another book.

Chapter 2
Origins of Hodge theory

The French are much more reserved with strangers than the Germans. It is extremely difficult to gain their intimacy, and I do not dare to urge my pretensions as far as that; finally every beginner had a great deal of difficulty getting noticed here. I have just finished an extensive treatise on a certain class of transcendental functions to present it to the Institute which will be done next Monday. I showed it to Mr Cauchy, but he scarcely deigned to glance at it, (In a letter by Niels Henrik Abel, see [OR16]).

2.1 Origins of singular homology

In order to trace back the origins of Hodge theory, one must go back to the study of elliptic and abelian integrals. This certainly explains the origin of singular homology and de Rham cohomology. It paves the road for higher dimensional integrals and the birth of the Hodge conjecture. The first manifestation related to singular homology and de Rham cohomology is surely the foundational work of Cauchy on residues and integration in the complex domain, see Exercise 2.2.

We start with an elliptic integral of the form

$$\int_a^b \frac{dx}{\sqrt{p(x)}}, \tag{2.1}$$

where $p(x)$ is a polynomial of degree 3 and with three distinct real roots, and a, b are two consecutive elements among the roots of p and $\pm\infty$. If $p(x)$ has repeated roots one can compute it easily, see Exercise 2.4. It is an easy exercise to show that all the above integrals can be calculated in terms of only two of them, see Exercise 2.5. In many calculus books we find tables of integrals and there we never find a formula for elliptic integrals. Already in the 19th century, it was known that if we choose p randomly (in other words for generic p) such integrals cannot be calculated in terms of until then well-known functions. For particular examples of p we have

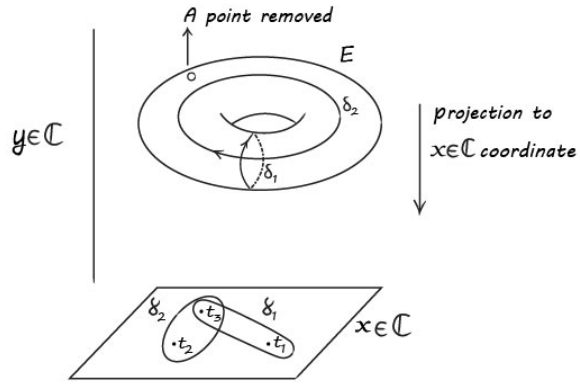

Fig. 2.1 The elliptic curve $y^2 = p(x)$ in the four dimensional space $\mathbb{C}^2$.

some formulas calculating elliptic integrals in terms of the values of the Gamma function on rational numbers, see Exercise 2.6.

The fact that we only need two of the integrals in (2.1) in order to calculate the others, can be easily seen by considering the integration in the complex domain $x \in \mathbb{C}$, in which we may discard the assumption that p has only real roots. The integration is done over a path γ in the $x \in \mathbb{C}$ domain which connects two roots of p, and avoids other roots except at its start and end points. "C'est à Cauchy que revient la gloire d'avoire fondé la theorie des intégrales prises entre des limites imaginaire" (H. Poincaré in [Poi87]). An amazing fact that we learn in a complex analysis course is that if the path γ moves smoothly, without violating the properties as before, then the value of the integral does not change. This is certainly the origin of homotopy theory, or at least one of them. The next step in the study of elliptic integrals is the invention of the y variable which is basically the square root of $p(x)$. One must look its origin in the works of Cauchy, Abel, Jacobi and in particular Riemann. Up to multiplication by a constant which can be computed explicitly, the integral (2.1) can be written as

$$\int_\delta \frac{dx}{y}, \ \delta \in H_1(E, \mathbb{Z}),$$

where

$$E := \left\{ (x,y) \in \mathbb{C}^2 \ \mid \ y^2 = p(x) \ \right\}. \tag{2.2}$$

This is called an elliptic curve in Weierstrass form. We may define $H_1(E, \mathbb{Z})$ as the abelization of the fundamental group of E, that is, the quotient of the fundamental group of E by its subgroup generated by commutators. In our case, in order to get explicit examples of δ, we take a closed path in the $x \in \mathbb{C}$ domain which only turns around two roots of p and we know that this path lifts up to a closed path in E which is not zero in $H_1(E, \mathbb{Z})$, see Figure 2.1. However, in general we use the so-called

Pochhammer cycles, see Exercise 4.7. After a great amount of work in the 19th century in order to classify surfaces, orientable or not, together with the concept of the compactification of E, we know the topology of E; it is a punctured torus and so $H_1(E,\mathbb{Z})$ has only two linearly independent generators. Generalization of all these to arbitrary projective varieties and tame polynomials is the main focus of Chapters 5, 6 and 7.

2.2 Origins of de Rham cohomology

Let us now consider the integrals $\int_\delta \frac{P(x)dx}{y}$, where $P(x) \in \mathbb{C}[x]$ is a polynomial in x and δ is a closed path in the elliptic curve E in (2.2). Since the $\mathbb{C}$-vector space of polynomials $\mathbb{C}[x]$ is generated by monomials x^n, $n \in \mathbb{N}_0$, in order to compute these integrals, it is enough to compute:

$$\int_\delta \frac{x^n dx}{y},\ \delta \in H_1(E,\mathbb{Z}),\ \ n \in \mathbb{N}_0. \tag{2.3}$$

In a similar way as in the previous section, it turns out that we need only two integrals in (2.3) in order to compute all others. For examples of this see Exercise 2.8. In the mathematics of 20th century $\frac{x^n dx}{y}$ could stand alone and it was given the name *differential 1-form*. Using them one defined the de Rham cohomology $H^1_{\mathrm{dR}}(E)$ and the old fact about integrals took new interpretation, namely, the first de Rham cohomology of the elliptic curve E is a $\mathbb{C}$-vector space of dimension two. It is generated by two elements represented by $\frac{dx}{y}$ and $\frac{xdx}{y}$. A generalization of this fact in the framework of tame polynomials is done in Chapter 10. The definition of de Rham cohomology for arbitrary oriented manifolds took a long path, going first to the category of C^∞ manifolds and differential forms, therefore, getting far from the original algebraic context of abelian and elliptic integrals. This might be the reason why in many classical books on the topology of manifolds, one learns the integration of C^∞-forms over topological cycles, without a single reference to elliptic or abelian integrals. The comeback to the integration of algebraic differential forms on topological cycles was done many decades later, when Grothendieck in [Gro66] inspired by the work of Atiyah and Hodge in [HA55] defined the algebraic de Rham cohomology.

2.3 The Legendre relation: a first manifestation of algebraic cycles

We already know that all the integrals of the elliptic curve E are reduced to 4 elliptic integrals:

$$\int_{\delta_i} \frac{x^j dx}{y}, \quad i = 1,2, j = 0,1,$$

where δ_1, δ_2 is a basis of the $\mathbb{Z}$-module $H_1(E,\mathbb{Z})$. The Legendre relation between elliptic integrals is the following equality:

$$\int_{\delta_1} \frac{dx}{y} \int_{\delta_2} \frac{xdx}{y} - \int_{\delta_2} \frac{dx}{y} \int_{\delta_1} \frac{xdx}{y} = 2\pi\sqrt{-1}. \tag{2.4}$$

where we have assumed that the intersection number of δ_1 and δ_2 is $+1$. We have also assumed that the elliptic curve E is of the form $y^2 = 4x^3 + t_2x + t_3$ for some $t_2, t_3 \in \mathbb{C}$ with $27t_3^2 - t_2^3 \neq 0$. This relation is due to the only non-zero dimensional algebraic cycle of the elliptic curve E. This is namely the elliptic curve E itself. We can interpret (2.4) as the following integration:

$$\frac{1}{2\pi\sqrt{-1}} \int_E \frac{dx}{y} \cup \frac{xdx}{y} = 1 \tag{2.5}$$

Here, we have to define the cup product $\cup$ in the de Rham cohomology. Even though, the above equality is not explained, we learn the following lessons:

1. The presence of an algebraic cycle in an algebraic variety X implies algebraic relations between the periods of X.
2. For explicit computations of such relations, we need to compute the intersection numbers of topological cycles like δ_1 and δ_2.
3. An integral over algebraic cycles like E splits into a transcendental piece $2\pi\sqrt{-1}$ and the algebraic piece which is the number 1.

The last item indicates that one can compute integrals like (2.5) by means of algebraic methods. This has been one of the key observations to go beyond classical singular homology and de Rham cohomology. One can make the equality (2.4) even more explicit by computing the periods. For instance, for the family of elliptic curves $E_z : y^2 - 4x^3 + 12x - 4(2-4z) = 0$, we get

$$F\left(\frac{1}{6}, \frac{5}{6}, 1\middle|z\right) F\left(\frac{-1}{6}, \frac{7}{6}, 1\middle|1-z\right) + F\left(\frac{1}{6}, \frac{5}{6}, 1\middle|1-z\right) F\left(\frac{-1}{6}, \frac{7}{6}, 1\middle|z\right) = \frac{6}{\pi}, \tag{2.6}$$

where

$$F(a,b,c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \ c \notin \{0,-1,-2,-3,\ldots\}, \tag{2.7}$$

is the Gauss hypergeometric function and $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol. For the details of the computations see [Mov12] §8.6.

2.4 Multiple integrals

As we saw in the previous sections, the study of linear independent one dimensional integrals leads naturally to the study of the topology of curves. Smooth curves are two dimensional real manifolds, and so we have a good visualization of them; we can draw them on a blackboard. However, for higher dimensional integrals, and in particular two dimensional ones, the visualization of the place of integration fails because we are dealing with spaces of dimension bigger than or equal to four. We have to find an alternative algebraic method which helps us to understand the topology of algebraic varieties. This line of thought in the first half of 20th century lead to the development of algebraic topology. This captures the topology of higher dimensional spaces by means of algebraic objects. In particular, a precise definition of the homologies:

$$X \text{ a smooth variety of dimension } n \ \to H_q(X,\mathbb{Z}), \ \ q = 0,1,\ldots,n$$

were formulated in the mid-twentieth. The foundation of algebraic topology was started by Poincaré in his treatise *Analysis Situs* [Poi95] with a series of five supplements. What was really studied in these works were the rank (Betti numbers) and torsion elements of a homology and not the homology itself. The reader can also think of an element $\delta \in H_q(X,\mathbb{Z})$ in the same style of the end 19th century, that is, it is the image of a smooth generically one to one map from a compact, oriented q-dimensional manifold, such as a q-dimensional sphere, to X, or rather a collection of such maps. All the necessary material on homologies are gathered in Chapter 4.

In the third supplement [Poi02] to Analysis Situs, Poincaré applies his theory to the affine variety

$$U := \{(x,y,z) \in \mathbb{C}^3 \mid z^2 = p(x,y)\}, \tag{2.8}$$

where $p(x,y)$ is a polynomial in two variables and $p(x,y) = 0$ is a smooth curve in $\mathbb{C}^2$. The main motivation for Poincaré was the investigation of the double integrals

$$\iint \frac{x^i y^j dxdy}{\sqrt{p(x,y)}}, \ i,j \in \mathbb{N}_0. \tag{2.9}$$

Émile Picard is the main responsible for a systematic study of double integrals. "During the period 1882-1906 Picard developed almost single handedly the foundations of this theory" (S. Lefschetz in [Lef68] page 866). He together with Simart, in 1897 and 1906 published two books on the subject. These two books contain many earlier results of Picard. The main tools in these books are the algebraic geometry of Noether, Severi, Castelnuevo and others, and the basics of algebraic topology after Riemann, Betti and Poincaré. Lefschetz after reading these two books felt the need for a systematic study of the topology of algebraic varieties and after eleven years of labor and isolation he published his treatise in 1924. After Lefschetz the study of multiple integrals were almost forgotten. It was mainly replaced with Hodge structures, many cohomology theories etc. This was until string theorists who entered the

scene around eighties. They started to use triple integrals and produce miraculous predictions on the number of curves on Calabi-Yau threefolds.

We will briefly describe the Hodge theory of double integrals (2.9) in §2.5. This will hopefully give a sketch of some of the main topics of the present text. However, the origin of the Hodge conjecture lies in the study of another class of double integrals which we are going to explain it in Chapter 3.

2.5 Tame polynomials and Hodge cycles

One of the main difficulties for Picard and Pioncaré in order to study the double integrals of algebraic surfaces is the absence of a good knowledge of the topology of surfaces, something which was completely understood fifty years later and after the breakthrough works of Lefschetz in [Lef24b] and Hodge in [Hod41]. This is one of the main reasons why the class of hyperelliptic surfaces (abelian surfaces) was a favorite example in Picard-Simart book [PS06], for more details see Chapter 3. One knows their topology and this makes their study smoother. Actually, even now abelian varieties are favorite examples for checking the Hodge conjecture. An algebraic surface of [PS06] and most of the works in algebraic geometry of 19th century was essentially given by a polynomial in $\mathbb{C}^3$, even those without this property like a hyperelliptic surface, were projected in $\mathbb{C}^3$, loosing the smoothness property.

Surfaces in $\mathbb{C}^3$, and in general, hypersurfaces in $\mathbb{C}^{n+1}$ may have complicated singularities and this may cause many difficulties to study the corresponding integrals. In the present text we will be mainly concerned with smooth hypersurfaces. This will also open the way for the case of hypersurfaces with isolated singularities. One of the branches of mathematics which took the works of Picard and Lefschetz in a more or less original format was *Singularity Theory*. By this we mainly mean the content of the book [AGZV88] and the references therein. In order to make a systematic link between a local context as in [AGZV88] and the global context as in Picard and Lefschetz, we introduce tame polynomials. In order to have a down-to-earth glance at the content of the present text let us pick a very explicit example of a tame polynomial, this is namely

$$f(x,y,z) := z^2 + p(x,y) = z^2 + g(x,y) + \cdots, \tag{2.10}$$

where g is a homogeneous polynomial of degree d in x,y such that $g(x,1)$ has d different roots, and $\cdots$ means a linear combinations of monomials in x,y of degree $\leq d-1$. This is a class of polynomials which appears frequently in Picard-Simart book [PS06] and it is the main topic of Poincaré's article [Poi87]. The full topological and algebraic treatment of tame polynomials is done in Chapters 7 and 10, respectively. We define the affine variety U as in (2.8). Our condition on g implies that U has at most isolated singularities. The usage of double integral sign will be abandoned and instead we will write the following:

$$\int_\delta \frac{x^i y^j dx \wedge dy}{z}, \quad \delta \in H_2(U,\mathbb{Z}), \tag{2.11}$$

which is essentially (2.9). Note that instead of $dxdy$ we have written $dx \wedge dy$ which is a differential 2-form. In Picard-Simart book the integrands as differential forms are always under integral sign and never stand alone. This is despite the fact that differential forms were originated from the works of Johann Friedrich Pfaff and Élie Cartan on partial differential equations and the later was contemporary and compatriot to Picard. Only after the works of Georges de Rham the integrand stood as an independent object under the name of a differential form.

From now on assume that U is smooth. We prove that $H_2(U,\mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank and we compute its rank

$$\mu := \operatorname{rank}(H_2(U,\mathbb{Z})) = (d-1)^2 \tag{2.12}$$

and give a basis of this using *vanishing cycles*. We also give an explicit basis of the $\mathbb{C}$-vector space $H^2_{\mathrm{dR}}(U)$. For instance, if f is a small deformation of $z^2 + g(x,y) - 1$ then $H^2_{\mathrm{dR}}(U)$ is generated by

$$\frac{x^i y^j dx \wedge dy}{z}, \quad 0 \le i,j \le d-2. \tag{2.13}$$

Both these results imply that for the surface U we have in total $\mu \cdot \mu$ non-trivial integrals. There are two phenomena which produces algebraic relations between these integrals. The first one is basically of the same nature as in §2.3. The second one is due to the presence of algebraic curves in U. We can compactify U inside a smooth projective variety X such that $Y := X - U$ is a union of smooth curves Y_i, $i = 1,2,\ldots,s$. Let C be any curve in X. We have the topological cycles $[Y_i]$, $[C] \in H_2(X,\mathbb{Z})$, and any $\mathbb{Z}$-linear combination of these cycles is an example of an algebraic cycle. Recall that this is in the same way that we associate to any real one dimensional curve in a Riemann surface a homological class. We can find a linear commbination δ of these cycles such that its intersection with all Y_i's is zero, and hence, its support is in U, for more details see Section 9.3. For simplicity, we write $\delta \in H_2(U,\mathbb{Z})$. The cycle δ is a very special one. It follows that

$$\int_\delta \frac{x^i y^j dx \wedge dy}{z} = 0, \tag{2.14}$$

$$\forall i,j \in \mathbb{N}_0, \quad 0 \le i,j \le d-2, \quad \frac{1}{2} + \frac{i+1}{d} + \frac{j+1}{d} < 1.$$

Following the terminology in higher dimensions, we call $\delta \in H_2(U,\mathbb{Z})$ with the property (2.14) a Hodge cycle, see Chapters 8 and 11. Note that for $d = 2,3,4$ any $\delta \in H_2(U,\mathbb{Z})$ is a Hodge cycle. Lefschetz $(1,1)$ theorem implies any Hodge cycle δ is a $\mathbb{Z}$-linear combination of cycles constructed from algebraic cycles as above, that is, any Hodge cycle is an algebraic cycle, see Chapter 9. In higher dimensions a similar affirmation is known as the *Hodge conjecture* which is one of the millennium

problems of the Clay Institute for Mathematics. In this book we are interested in Hodge cycles because of their relations with integrals. For instance, a consequence of δ being an algebraic cycle is that if $f = z^2 + g(x,y) - 1$ has coefficients in the field $\bar{\mathbb{Q}}$ of algebraic numbers then

$$\int_\delta \frac{x^i y^j dx \wedge dy}{z} \in \bar{\mathbb{Q}} \cdot \pi \tag{2.15}$$

for all $i, j \in \mathbb{N}_0$ such that $\frac{1}{2} + \frac{i+1}{d} + \frac{j+1}{d} \notin \mathbb{N}$. This follows from a theorem of Deligne, see [DMOS82] Proposition 1.5, and the description of the de Rham cohomology of hypersurfaces in weighted projective spaces, see Chapter 11. For a more explicit version of (2.15) see [MR06], Theorem 1.1. These properties show the importance of algebraic cycles in the study of multiple integrals.

2.6 Exercises

2.1. The two curves $x^2 + y^2 = 1$ and $xy = 1$ are different in the real plane $\mathbb{R}^2$ but isomorphic in the complex plane $\mathbb{C}^2$. Which of the pictures in Figure 2.2 is a correct intuition of these curves? Justify your answer.

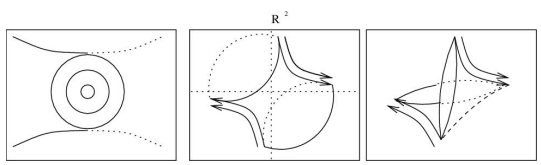

Fig. 2.2 A cylinder in the complex plane

2.2. Let $p \in \mathbb{C}[x]$ be a polynomial of degree d and with no double roots. Show that integrals of the form

$$\int_\delta \frac{q(x)}{p(x)} dx, \quad q \in \mathbb{C}[x],$$

where δ is a closed path in the $x \in \mathbb{C}$ plane minus the roots of p, is reduced to d^2 integrals $\int_{\delta_j} \frac{dx}{x-t_i}$, where t_i's are roots of p and δ_j is a small closed path in the $x \in \mathbb{C}$ plane encircling t_j anticlockwise, see Figure 2.3. Note that d of these integrals are $2\pi\sqrt{-1}$ and the rest are zero. In a modern language, $H^1_{\mathrm{dR}}(\mathbb{C} - \{p = 0\})$ is generated by the differential 1-forms $\frac{dx}{x-t_i}$ and the singular homology $H_1(\mathbb{C} - \{p = 0\}, \mathbb{Z})$ is generated by δ_j's. Formulate and prove the exercise assuming that $p, q \in \mathbb{Q}[x]$. You are only allowed to choose differential forms with rational coefficients.

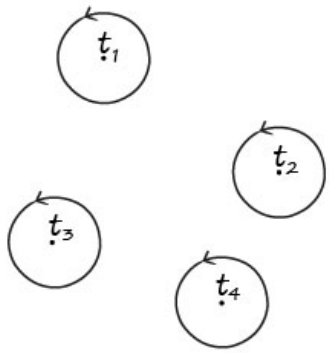

Fig. 2.3 Homology of punctured plane.

2.3. Show that the definite integral $\int_a^b \frac{dx}{\sqrt{p(x)}}$ converges, where $p(x)$ is a polynomial of degree d and with distinct real roots, and a, b are two consecutive elements among the roots of p and $\pm\infty$.

2.4. Compute the indefinite integral

$$\int \frac{dx}{\sqrt{p(x)}}, \tag{2.16}$$

where p is a polynomial of degree 1 and 2. Compute it also when p is of degree 3 but it has double roots. These integrals are computable because $y^2 = p(x)$ is a rational curve!

2.5. Let p be a polynomial of degree 3 and with three real roots $t_1 < t_2 < t_3$. Show that two of the four integrals

$$\int_{-\infty}^{t_1} \frac{dx}{\sqrt{p(x)}}, \quad \int_{t_1}^{t_2} \frac{dx}{\sqrt{p(x)}}, \quad \int_{t_2}^{t_3} \frac{dx}{\sqrt{p(x)}}, \quad \int_{t_3}^{+\infty} \frac{dx}{\sqrt{p(x)}},$$

can be computed in terms of the other two. Formulate and prove a similar statement for a polynomial p of arbitrary degree.

2.6. * For particular examples of polynomials p of degree 3, there are some formulas for elliptic integrals 2.16 in terms of the values of the Gamma function on rational numbers. For instance, verify the equality

$$\int_7^{+\infty} \frac{dx}{\sqrt{x^3 - 35x - 98}} = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{2\pi i\sqrt{-7}}. \tag{2.17}$$

In [Wal06] page 439 we find also the formulas

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\Gamma(\frac{1}{3})^2}{2^{\frac{4}{3}}3^{\frac{1}{2}}\pi},$$
$$\int_0^1 \frac{dx}{\sqrt{x-x^3}} = \frac{\Gamma(\frac{1}{4})^2}{2^{\frac{3}{2}}\pi^{\frac{1}{2}}}.$$

These formulas can be also derived using the software Mathematica. The Chowla-Selberg theorem, see for instance Gross's articles [Gro78, Gro79], describes this phenomenon in a complete way. The right hand side of (2.17) can be written in terms of the Beta function which is more natural when one deals with the periods of algebraic differential forms.

2.7. Show that $\int_\delta \frac{dx}{\sqrt{p(x)}}$, where δ is an interval between two consecutive real roots of p or $\pm\infty$, and $\deg(p) = 4$, can be calculated in terms of elliptic integrals, that is those with $\deg(p) = 3$. Try first the particular cases $p(x) = x^4 \pm 1$. Try to use any integral calculator, for instance Mathematica, to see whether you get a result. In modern terms, both $y^2 = p(x)$, $\deg(p) = 3,4$ are curves of genus one.

2.8. Let $p(x) := 4(x-t_1)^3 - t_2(x-t_1) - t_3$, where t_1, t_2, t_3 are three parameters. Let also δ be a path in the $x \in \mathbb{C}$ plane which connects two roots of p and $y := \sqrt{p(x)}$. Show that

$$\int_\delta \frac{x^2dx}{y} = (2t_1)\int_\delta \frac{xdx}{y} + (-t_1^2 + \frac{1}{12}t_2)\int_\delta \frac{dx}{y},$$
$$\int_\delta \frac{x^3dx}{y} = (3t_1^2 + \frac{3}{20}t_2)\int_\delta \frac{xdx}{y} + (-2t_1^3 + \frac{1}{10}t_1t_2 + \frac{1}{10}t_3)\int_\delta \frac{dx}{y},$$
$$\int_\delta \frac{x^4dx}{y} = (4t_1^3 + \frac{3}{5}t_1t_2 + \frac{1}{7}t_3)\int_\delta \frac{xdx}{y} + (-3t_1^4 - \frac{1}{10}t_1^2t_2 + \frac{9}{35}t_1t_3 + \frac{5}{336}t_2^2)\int_\delta \frac{dx}{y},$$
$$\int_\delta \frac{x^5dx}{y} = (5t_1^4 + \frac{3}{2}t_1^2t_2 + \frac{5}{7}t_1t_3 + \frac{7}{240}t_2^2)\int_\delta \frac{xdx}{y} +$$
$$(-4t_1^5 - \frac{2}{3}t_1^3t_2 + \frac{2}{7}t_1^2t_3 + \frac{19}{420}t_1t_2^2 + \frac{1}{30}t_2t_3)\int_\delta \frac{dx}{y}.$$

Explain the general algorithm for $\int_\delta \frac{x^ndx}{y}$ with $n \in \mathbb{N}$. The parameter t_1 is the origin of the theory of geometric quasi-modular forms developed in [Mov12], even though it seems to be useless. For the purpose of this exercise one can put $t_1 = 0$. In a modern language one says that $\frac{x^ndx}{y}$ induces an element in the $\mathbb{C}$-vector space $H^1_{\mathrm{dR}}(E)$ and $\frac{x^ndx}{y}, n = 0,1$ form a basis of this vector space, where E is the curve given by $y^2 - p(x) = 0$.

2.9. Show that for $m_1, m_2, \cdots, m_{n+1}$ positive even numbers, the following

$$\delta := \left\{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1\right\} \tag{2.18}$$

is a bounded subset of $\mathbb{R}^{n+1}$. It is actually diffeomorphic to the n-dimensional sphere $\mathbb{S}^n$. For $n=2$ and for examples of m_1, m_2, m_3 use any graphic software to draw δ and see this fact. The manifold δ is the boundary of the following closed set in $\mathbb{R}^{n+1}$:

$$\Delta := \left\{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} \leq 1\right\}.$$

Compute the volume of Δ. You may prove the general formula:

$$\int_\Delta x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1} =$$

$$\frac{1}{\left(\sum_{i=1}^{n+1} \frac{\beta_i+1}{m_i}\right) \cdot \prod_{i=1}^{n+1} m_i} \frac{\Gamma(\frac{\beta_1+1}{m_1}) \cdot \Gamma(\frac{\beta_2+1}{m_2}) \cdots \Gamma(\frac{\beta_{n+1}+1}{m_{n+1}})}{\Gamma(\sum_{i=1}^{n+1} \frac{\beta_i+1}{m_i})} \prod_{i=1}^{n+1} (1-(-1)^{\beta_i+1}).$$

This generalizes the volume of the $(n+1)$-dimensional ball:

$$\mathrm{Vol}(\mathbb{B}^{n+1}) = \begin{cases} \frac{\pi^{\frac{n+1}{2}}}{\frac{n+1}{2}!}, & n \text{ is odd}, \\ \frac{\pi^{\frac{n}{2}}}{(\frac{n+1}{2} \frac{n-1}{2} \cdots \frac{1}{2})}, & n \text{ is even}. \end{cases}$$

2.10. * Let d be an even number and p_1 be a polynomial in x, y with rational coefficients and of degree strictly less than d. Define

$$\Delta := \{(x,y,z) \in \mathbb{R}^3 \quad | \quad z^2 + x^d + y^d + p_1(x,y) - 1 < 0\}. \tag{2.19}$$

If we take the coefficients of the monomials in p_1 very small, then the above set is an open subset of $\mathbb{R}^3$ isomorphic to the three dimensional ball $\mathbb{B}^3$.

1. For $d = 2, 4$ and a homegenous polynomial p_1 compute the integral

$$\int_\Delta x^i y^j z^j dx \wedge dy \wedge dz$$

   for $\frac{1}{2} + \frac{i+1}{d} + \frac{j+1}{d} \notin \mathbb{N}$, and show that it is a product of an algebraic number with π. This is a consequence of the fact that for the affine variety $U \subset \mathbb{C}^3$ defined by $z^2 + x^d + y^d + p_1(x,y) - 1 = 0$ all the cycles in $H_2(U, \mathbb{Z})$, and in particular the boundary of Δ above, are Hodge.
2. Let δ be the boundary of Δ. For $d = 6$, δ is called Hodge if $\int_\delta \frac{dx \wedge dy}{z} = 0$. Is there any example of p such that δ is Hodge. For a more elaborated version of this see Exercise 2.11.

2.11. ** Let d and n be even numbers and

$$f_t := x_1^d + x_2^d + \cdots + x_{n+1}^d - s - \sum_{\alpha \in I_d} t_\alpha x^\alpha,$$

where

$$I_d := \left\{\alpha \in \mathbb{Z}^{n+1} \mid 0 \le \alpha_i \le d-2,\ \ 2 \le \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} \le d\right\}$$

and f_t is parameterized by $t = (s, \cdots, t_\alpha, \cdots) \in \mathbb{R}^{N+1}$. One can easily check that

$$N := \#I_d := \binom{d+n+1}{n+1} - (n+1)^2.$$

Later, we will learn that this is the dimension of the moduli of hypersurfaces of degree d and dimension n. For $s \in \mathbb{R}$ let $t_s := (s, 0, \cdots, 0) \in \mathbb{R}^{N+1}$. For $t \in \mathbb{R}^{N+1}$ near t_1, the real algebraic variety $\delta_t := \{x \in \mathbb{R}^{n+1} \mid f_t(x) = 0\}$ is smooth and diffeomorphic to the n-dimensional sphere. We have an open connected subset U of $\mathbb{R}^{N+1}$ whose points enjoy the mentioned property and for a point t in the boundary of U, δ_t is singular. For instance, the half line $t_s,\ s \in \mathbb{R}^+$ is inside U. The point t_0 is in the boundary of U and δ_{t_0} is a point. For $t \in U$ let $\Delta_t := \{x \in \mathbb{R}^{n+1} \mid f_t(x) < 0\}$ be the bounded open set in $\mathbb{R}^{n+1}$ with the boundary δ_t and let $I(t)$ be its volume:

$$I(t) := \int_{\Delta_t} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}. \tag{2.20}$$

We are interested in the following real analytic variety

$$M := \left\{ t \in U \middle| \frac{\partial I}{\partial s}(t) = 0 \right\}. \tag{2.21}$$

Note that $I(t)$ for fixed t_α's and as a function in s is an increasing function, and so, $\frac{\partial I}{\partial s}$ is non-negative in U.

1. Prove that
$$I(t_s) = \frac{2^{n+1}}{(n+1)\cdot d^n} \frac{\Gamma(\frac{1}{d})^{n+1}}{\Gamma(\frac{n+1}{d})} s^{\frac{n+1}{d}}$$
and so for $d < n+1$ we have the point t_0 in the boundary of U with $\frac{\partial I}{\partial s}(t_0) = 0$.
2. For $(n,d) = (2,4)$ a cycle δ_t with $t \in M$ is called a Hodge cycle. Prove or disprove that M is non-empty.
3. For $(n,d) = (4,6)$ and a singular point t of M, δ_t is called a Hodge cycle. This is a point t such that M locally around t is not a smooth real manifold of codimension 1. Prove or disprove that M is non-empty. If M is non-empty prove or disprove that the singular set of M is non-empty.

The origin of this exercise, and in particular the relation between the notion of a Hodge cycle used here with the classical definition, will be clarified in Chapter 11. This exercise is motivated by the so-called *infinitesimal 16th Hilbert problem* on the number of zeros of abelian integrals, see for instance [GM07, Gav01] and the references therein.

Chapter 3
Origins of the Hodge conjecture

As was the case for almost all our scientists of that day my mathematical isolation [form 1911-24] was complete. This circumstance was most valuable in that it enabled me to develop my ideas in complete mathematical calm (Solomon Lefschetz in [Lef68] page 854).

3.1 Lefschetz's puzzle: Picard's ρ_0-formula

The main content of the Hodge conjecture arose from the desire of classifying topological cycles carried by algebraic varieties and this goes back to the early state of both Algebraic Geometry and Algebraic Topology. "From the ρ_0-formula of Picard, applied to a hyperelliptic surface Φ (topologically the product of four circles) I had come to believe that the second Betti number $R_2(\Phi) = 5$, where as clearly $R_2(\Phi) = 6$. What was wrong? After considerable time it dawned upon me that Picard only dealt with finite 2-cycles, the only useful cycles for calculating periods of certain double integrals. Missing link? The cycle at infinity, that is the plane section of the surface at infinity. This drew my attention to cycles carried by an algebraic curve, that is to algebraic cycles, and ... the harpoon was in." (S. Lefschetz in his mathematical autobiography [Lef68] page 854). Lefschetz himself formulated a criterion for cycles carried by an algebraic curve which can be generalized to real codimension two cycles on algebraic varieties and nowadays it is known as Lefschetz $(1,1)$ theorem. He states his result in the following form: "On an algebraic surface U a 2-dimensional homology class contains the carrier cycle of a virtual algebraic curve if and only if all the algebraic double integrals of the first kind have zero periods with respect to it" (W.V.D. Hodge in [Hod57], page 12). It is a pity no modern book in Hodge theory states the Lefschetz $(1,1)$ theorem in its original form, namely, using integrals.

In this section we explain Lefschetz's puzzle and we explain what dawned upon him so that he started to study topological cycles carried by algebraic curves. Our story begins from page 448 of the second volume of Picard and Simart's treatise

[PS06]. Any complete account of what we are going to talk about, would be a book on its own. Therefore, we skip many details and leave them to the reader.

3.2 Hyperelliptic surfaces

In [PS06] there is no distinction between birational surfaces, and so, what Picard and Simart refer as a *hyperelliptic surface* is essentially a two dimensional (polarized) complex torus or its algebraic counterpart which is an abelian surface. "In its early phase (Abel, Riemann, Weierstrass), algebraic geometry was just a chapter in analytic function theory" (S. Lefschetz in [Lef68] page 855). In [PS06] any polarized torus is identified with its (singular) image under a projection to $\mathbb{P}^3$ which is constructed using explicit transcendental theta functions. Nowadays, the term hyperelliptic surface refers to a very small class of surfaces obtained by a quotient of product of two elliptic curves, see [BHPV04] page 148. According to [PS06] Vol, II, page 439, Picard and Humbert started the investigation of hyperelliptic surfaces. Humbert's paper [Hum93] is fully dedicated to the study of such surfaces, and in particular Kummer surfaces. The contribution of Poincaré and Appell is throughly explained in this paper. Around this time there was a lot of activity on *abelian functions*, and so the investigation of equations for abelian surfaces was a natural and accessible topic. The main reason for focusing on such surfaces is of course the topology which is easy to describe; a torus is homeomorphic to a product of cycles. As it was usual for Picard's taste, he needed to have such surfaces, or some affine part of it, as hypersurfaces in $\mathbb{C}^3$. For this purpose, he used *normal theta functions*, which are nowadays called Riemann's theta functions. For a nice historical and modern overview of Riemann's theta functions see Chai's article [Cha14]. A purely algebraic approach to the topic of equations for abelian varieties was introduced by Mumford in a series of articles [Mum66b], which in turn, are based on many works of Baily, Cartier, Igusa, Siegel, Weil, see the introduction of the first article. It does not seem to me that the contribution of Humbert and Picard as founders of the subject is acknowledged in the modern treatment of equations defining abelian varieties. Humbert is mainly acknowledged for his works on Humbert surfaces in the moduli of abelian surfaces and Appell-Humbert theorem on classification of line bundles on a complex torus. Actually the modern meaning of hyperelliptic surface seems to appear first in the work of Bombieri and Mumford in [BM77]. "I believe Enrico and I knew that the word "hyperelliptic" had been used classically as a name for abelian surfaces but we felt that this usage was no longer followed, i.e. after Weil's books, the term "abelian varieties" had taken precedence. So the word "hyperelliptic" seemed to be a reasonable term for this other class of surfaces," (D. Mumford, personal communication, Januay 15, 2016).

Let $a, b \in \mathbb{R}^g$ and $\mathbb{H}_g$ be the genus g Siegel domain. It is the set of $g \times g$ symmetric matrices τ over the complex numbers whose imaginary part is positive definite, for more details see Exercise 8.5. The Riemann theta function $\theta \begin{bmatrix} a \\ b \end{bmatrix}$ with characteristics

(a,b) is the following holomorphic function:

$$\theta\begin{bmatrix}a\\b\end{bmatrix} \ : \ \mathbb{C}^g\times\mathbb{H}_g\to\mathbb{C},$$
$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) := \sum_{n\in\mathbb{Z}^g}\exp\left(\frac{1}{2}(n+a)^{\mathsf{tr}}\cdot\tau\cdot(n+a)+(n+a)^{\mathsf{tr}}\cdot(z+b)\right).$$

Here, we regard a vector like a,b etc. as $g\times 1$ matrix. Such a theta function satisfies the functional equation

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z+\tau m+n,\tau) = \exp\left(a^{\mathsf{tr}}n-b^{\mathsf{tr}}m-\frac{1}{2}m^{\mathsf{tr}}\tau m-m^{\mathsf{tr}}z\right)\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau). \quad (3.1)$$

Let $\Lambda := \{\tau m+n \mid m,n\in\mathbb{Z}^g\}$ and $\mathbb{C}^g/\Lambda$ be the corresponding complex compact torus. In a more geometric language, one says that the exponential factors in (3.1) form a line bundle in $\mathbb{C}^g/\Lambda$ and $\theta\begin{bmatrix}a\\b\end{bmatrix}$ is a holomorphic section of this line bundle.

Let us now consider the genus two case, that is, $g=2$. Let $\theta_i,\quad i=1,2,3,4$ be a collection of linearly independent Riemann's theta functions with the same characteristics a,b. What is considered in [PS06] and [Hum93] is essentially the image X of the following map

$$\mathbb{C}^2/\Lambda\to\mathbb{P}^3,\ z\mapsto[\theta_1;\theta_2;\theta_3;\theta_4]. \quad (3.2)$$

Algebraic Geometry of Picard and many others at that time was limited to affine varieties and so instead of the projective geometry notation, one was mainly interested in the surface

$$U := \left\{(x,y,z)\in\mathbb{C}^3 \mid f(x,y,z)=0\right\}\subset X, \quad (3.3)$$

where f is the polynomial relation between the following quotient of theta series

$$(x,y,z) = (\frac{\theta_2(z,\tau)}{\theta_1(z,\tau)},\frac{\theta_3(z,\tau)}{\theta_1(z,\tau)},\frac{\theta_4(z,\tau)}{\theta_1(z,\tau)}). \quad (3.4)$$

The affine variety U is called a hyperelliptic surface. Apparently, in 19th century, the domain and image of the map (3.2) were not distinguished from each other. Even worse, the affine variety U and the compact variety X do not seem to be distinguished from each other. Let us redefine X to be the abelian surface $\mathbb{C}^2/\Lambda$ and let Y be the subvariety of X which is the zero locus of θ_1. In the discussion below we assume that Y is a curve of genus two and redefine U to be $U := X\backslash Y$. Once again, it seems to the author that in [PS06] there is no distinction between X,Y and U, and their image under the map (3.2). In order to understand Lefschetz's puzzle and also Picard's computation of ρ_0, we adopt the modern language of writing the long exact sequence of $U\subset X$:

$$\begin{array}{ccccccccccccc} H_3(X) & \to & H_1(Y) & \to & H_2(U) & \to & H_2(X) & \to & H_0(Y) & \to & H_1(U) & \to & H_1(X) \\ 4 & & 4 & & 5 & & 6 & & 1 & & 4 & & 4 \end{array}.$$

The number under a homology is its rank and it is called a Betti number. For shortness, we have removed $\mathbb{Z}$ from our singular homology notation, that is, $H_2(X) := H_2(X,\mathbb{Z})$ etc. This way of writing maps together, means that the image of a map above is the kernel of the next map. A precise description of what we are going to discuss will be presented in Chapter 4, however, some good intuition will be enough to proceed further. The first and fourth maps are obtained by intersecting topological cycles in X with Y. The second and fifth maps are obtained in the following way. For a topological cycle δ in Y, we replace each point p of δ with a circle in U centered at p. Doing this continuously, we get a topological cycle in U. Other maps are obtained by inclusions. The variety X is topologically a product of four circles and so using Künneth formula one can compute its Betti numbers. The curve Y is topologically a compact oriented surface with two handles, that is, it is of genus two. Therefore, the Betti numbers of Y are easy to compute. The non-trivial rank computations in the above sequence are those of $H_i(U),\ i = 1,2$. These computations are done in [PS06], of course, in the context of one dimensional and double integrals. Our focus will be on the computation of Picard's ρ_0 which in our context is

$$\rho_0 := \mathrm{rank} H_2(U) = 5. \tag{3.5}$$

"Tel est le nombre des intégrales doubles distinctes de second espèce pour les surfaces hyperelliptiques non singulières" ([PS06] Vol. II page 448). Let us have a look at the last map which is actually an isomorphism. This fact translated into the spirit of [PS06] implies the following. In order to find the number of linearly independent one dimensional integrals, it does not matter to take which variety, U or X. Therefore, as far as one dimensional integrals are concerned, there is not too much difference between the compact variety X and its affine subset U. The third map which is obtained by inclusion is an injection, but not surjective. The double integrals of Picard and Poincaré were only over topological cycles living in affine varieties. The integration over topological cycles in X seems to be considered in the first half of 20th century and in particular after the invention of de Rham cohomology. Therefore, the dimension of the homology and de Rham cohomology of U and X differs by 1. As we quoted at the beginning of this section, this is caused by the homology class of the curve Y. In [PS06] we find two different methods for computing ρ_0. The first one is of singular homology nature and we do not reproduce it here. The second method is of de Rham cohomology nature and is actually the verification of $\dim H^2_{\mathrm{dR}}(U) = 5$.

3.3 Computing Picard's ρ_0

In [PS06] Vol. II page 448 we find an algebraic method for computing ρ_0 which we explain it in this section. A hyperelliptic curve of genus two is given by the equation

$$S := \left\{ (x,y) \in \mathbb{C}^2 \mid y^2 = p(x) \right\},$$

where p is a degree 5 polynomial and it has not repeated roots. Similar to the case of elliptic curves, one can prove that all integrals on this curve is reduced to the integrals

$$\int_\delta \frac{x^i dx}{y}, \quad i = 0,1,2,3, \tag{3.6}$$

where for δ we have four candidates. In other words, the de Rham cohomology $H^1_{\mathrm{dR}}(S)$ has a basis represented by the differential forms in the integrands of (3.6), and the singular homology $H_1(S,\mathbb{Z})$, where δ lives there, is a free $\mathbb{Z}$-module of rank 4. The curve S as a topological space is obtained by removing a point from a compact real two dimensional surface with two holes.

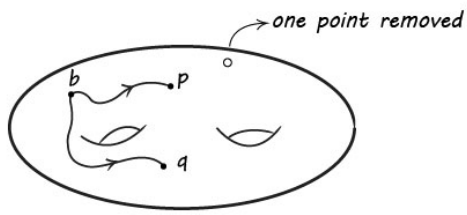

Fig. 3.1 A genus two curve with one point removed.

The first de Rham cohomology and singular homology of S do not differ from the compactification $\bar{S}$ of S. This is obtained by adding the removed point back to the surface S. This situation is particular to the one dimensional case and the fact that only one point is removed. As we will see in the next paragraph, for hyperelliptic surfaces this is no more true and this is the origin of Lefschetz's puzzle. In the following we will need the Jacobian variety of $\bar{S}$ which is the following complex torus:

$$\begin{aligned} J(\bar{S}) &:= \mathbb{C}^2/\Lambda, \\ \Lambda &:= \left\{ \left(\int_\delta \frac{dx}{y}, \int_\delta \frac{xdx}{y} \right) \middle| \delta \in H_1(S,\mathbb{Z}) \right\}. \end{aligned} \tag{3.7}$$

Let U be the set of two points (without order) in S, that is

$$U := \{\{p,q\} \mid p,q \in S\} = S \times S/\sim, \tag{3.8}$$

where $\sim$ is defined by $(p,q) \sim (q,p)$. Adding the point at infinity to S, that is $\bar{S} := S \cup \{\infty\}$, we get a compact surface of genus two and in a similar way we can define X which is topologically the set of two points in $\bar{S}$. The surface X is the same as Φ in Lefschetz's autobiography [Lef68]. For a fixed point $b \in S$, we have the following

map:

$$X \to J(\bar{S}), \tag{3.9}$$
$$\{p,q\} \mapsto \left(\int_b^p \frac{dx}{y}, \int_b^p \frac{xdx}{y}\right) + \left(\int_b^q \frac{dx}{y}, \int_b^q \frac{xdx}{y}\right),$$

which turn out to be a birational map; it is a biholomorphism in some open subset of X and $J(\bar{S})$, for a modern treatment of this see for instance [Mil]. The surjectivity of (3.9) follows from Jacobi's theorem. This map is not injective. By Abel's inversion theorem, a fiber of (3.9) is characterized by the fact that for two points $\{p_i, q_i\},\ i = 1,2$ in it, there is meromorphic function f on $\bar{S}$ such that the zeros of f, respectively poles of f, are $\{p_1, q_1\}$, respectively $\{p_2, q_2\}$. For instance, the zero divisor of $x-a$ for any constant a is mapped to a fixed point in the Jacobian $J(\bar{S})$.

Almost all the surfaces in [PS06] are represented in $\mathbb{C}^3$ and this is also the case of X. We want to write X in coordinate system. The most natural embedding is of course inside $\mathbb{C}^4$ and it is done using symmetric functions:

$$\begin{aligned} &X \hookrightarrow \mathbb{C}^4, \\ &(x_1, y_1, x_2, y_2) \mapsto (x, y, z, w), \\ &\text{where } x := x_1 + x_2,\ y = x_1 x_2,\ z = y_1 + y_2,\ w = y_1 y_2. \end{aligned} \tag{3.10}$$

One can easily describe the image of this map, see Exercise 3.2. Since $y_1 y_2 = \frac{1}{2}(z^2 - f(x_1) - f(x_2))$ and $f(x_1) + f(x_2)$ can be written in terms of x, y, it is enough to project X into $\mathbb{C}^3$ with the coordinate system (x, y, z). Under the map (3.10) we only see the affine part U of X and the curve Y lies at infinity. Now, the next job is to describe double integrals on U. The integrals

$$\iint (x_1^p x_2^q - x_1^q x_2^p) \frac{dx_1 dx_2}{y_1 y_2},\ \ p,q = 0,1,2,3,\ \ p < q \tag{3.11}$$

gives us essentially six double integrals on X of the form

$$\int R(x,y) \frac{dxdy}{w}.$$

This is because the integrand is invariant under $(x_1, y_1) \to (x_2, y_2)$. We have six differential 2-forms (3.11) and it turns out that a linear combination of these 2-forms is exact. This can be easily derived from the identity

$$\frac{\partial}{\partial x_1}\left(\frac{f(x_1)}{(x_2 - x_1)\sqrt{f(x_1)f(x_2)}}\right) - \frac{\partial}{\partial x_2}\left(\frac{f(x_2)}{(x_1 - x_2)\sqrt{f(x_1)f(x_2)}}\right) = \frac{U(x_1, x_2)}{\sqrt{f(x_1)f(x_2)}}, \tag{3.12}$$

where $U(x_1, x_2)$ is a polynomial of degree 3 in both variables x_1 and x_2 and it is defined through the above equality, see Exercise 3.3. This equality in [PS06] Vol.II page 198, 262, 321, 450 is attributed to Weierstrass.

3.4 Exercises

3.1. Show that the set of n-points of a Riemann surface has a natural structure of a complex manifold of dimension n. Note that a complex manifold by definition is smooth. Is this in contradiction with the fact that the image of the map (3.10) is singular?

3.2. Show that the image of the map (3.10) is given by the equations

$$\begin{cases} w^2 = Q(x,y), \\ z^2 - 2w = P(x,y), \end{cases}$$

where $Q(x,y)$ and $P(x,y)$ are the quantities $p(x_1)p(x_2)$ and $p(x_1)+p(x_2)$ written in the new variables $x := x_1 + x_2$ and $y := x_1x_2$. Describe the singularities of the projections of X in (x,y,z), (x,y,w) affine spaces.

3.3. Prove the equality (3.12). Note that

$$\frac{p(x_1)-p(x_2)}{x_1-x_2} - \frac{1}{2}(p'(x_1)+p'(x_2))$$

is an anti-symmetric polynomial in x_1, x_2 and so it is divisible by $x_1 - x_2$.

3.4. A *Kummer surface* is obtained by identifying the points x and $-x$ in the Jacobian $J(\bar{S})$ (defined in (3.7)) of the hyperelliptic curve $\bar{S}$ of genus two. Therefore, it has 16 singularities. Let $i : S \to S,\ \ i(x,y) = (x,-y)$ be the involution of the hyperelliptic curve S. Prove that

1. For two point $p,q \in S$, the sum of the images of $\{p,q\}$ and $\{i(p),i(q)\}$ under the map (3.9) is zero.
2. Conclude that the surface $w^2 = p(x,y)$ in Exercise 3.2 is birational to the Kummer surface and under the isomorphism part of this birationality we can see at least 10 nodal singularities of the Kummer surface.

Chapter 4
Homology theory

Create the homology theory of simplicial complexes basis-free (without incidence matrices) by means of the boundary operator. Instead of Betti numbers and torsion coefficients, focus on the homology groups themselves and the homomorphisms between them, (an advice of E. Noether to P. Alexandroff and H. Hopf, see [Rem95] page 5).

4.1 Early history of topology

We start this chapter with some history of topology. The reader may also consult [Wei99] for a complete account on this. We tell the part of the history which is related to integrals. Topology in its early phase was called *Analysis Situs*. "Cette théorie [Analysis situs] a été fondée par Riemann, qui lui a donné ce nom [...] Indépendamment de Riemann, Betti avait de son côté étudié les divers ordres de connexion dans les espaces à n dimensions....Dans son Mémoire sur les fonctions algébriques de deux vaiables, M. Picard avait montré l'intérêt que présentent des considérations de ce genre dans l'étude des surfaces algébriques. Tout récemment, M. Poincaré a repri d'une manière général cette question de l'Analysis Situs, et, après avoir complété et précisé les résultat obtenus par Betti, a appelé l'attention sur les différences considérables que préssentent ces théorie, suivant qu'il s'agit d'un espace à deux dimensions ou d'un espace à un plus grande nombre de dimensions" (Picard and Simart in [PS06], Vol. I, page 19). This was the prehistory of topology. The next phase of topology started with Solomon Lefschetz."Soon after his doctorate he [S. Lefschetz] began to study intensely the two volume treatise of Picard-Simart, *fonctions algébriques de deux variables*, and he first tried to extend to several variables the treatment of double integrals of the second kind found in the second volume. He was unable to do this directly, and it led him to a recasting of the whole theory, especially the topology [...] The former [Topology book of Lefschetz published in 1930] was widely acclaimed and established the name *Topology* in place of the previously used term *analysis situs*" (P. Griffiths in [GSW92]). One of

the main branches of topology is algebraic topology by which we mainly mean the singular homology and cohomology. "In fact the homology group itself was barely recognized [at the time Lefschetz was engaged in the investigation of the topology of algebraic varieties]. Instead one dealt with chains and cycles, systems of linear relations on cycles given by boundaries, and then passed directly to the Betti numbers and torsion coefficients. During the period 1925-1935 there was a gradual shift of interest from the numerical invariants to the homology groups themselves. This shift was due in part to the influence of E. Noether, and developments in abstract algebra [...] the period 1936-9 saw the development of cohomology and cup products at the hands of Alexander, Čech and Whitney. It was not likely that Lefschetz, who had enjoyed a monopoly on products and intersections for ten years, would have nothing to say on the subject.[...] The term *singular* [homology] was applied by Veblen to emphasize the fact that his cells and chains were continuous images of polyhedral cells and chains, and were not themselves polyhedral" (Norman E. Steenrod in [Ste57] page 33-34-36).

Axiomatization is one of the traditional processes in mathematics. An object may be constructed during decades and by many mathematicians, and finally one finds that such an object satisfies an enumerable number of axioms, previously stated as theorems, and these axioms determine the object uniquely. An outstanding example to this is the axiomatization of homology theory of topological spaces by Samuel Eilenberg and Norman Steenrod in 1952 (see [ES52]). In this chapter we present the axiomatic approach to Homology theory introduced by Eilenberg and Steenrod. Despite the fact that in §4.4 we define singular homology and cohomology, we content ourselves to a list of properties without proof. The reader who does not feel comfortable with our presentation and needs more background in this topic is referred to classical texts such as [ES52, Mas91, Bre93]. In the author's opinion sometimes getting familiar with a mathematical object and using it correctly, even without knowing its precise definition, is more important than its rigorous definition. This can be also applied to our case, in which some of the fundamental theorems by Poincaré and Lefschetz were formulated without rigorous definitions.

For a smooth variety X defined over the field of complex numbers we are going to discuss the construction of (co)homologies which are the $\mathbb{Z}$-modules

$$H_q(X,\mathbb{Z}),\ H^q(X,\mathbb{Z})\ q = 0,1,2,\ldots.$$

Since the invention of singular cohomology, many cohomology theories, such as étale cohomology, have been defined in the framework of algebraic geometry, however, non of these can be considered as a true replacement for singular cohomology. If we use an automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ of the field of complex numbers and define X_σ to be obtained by replacing the coefficients of the defining polynomials of X with the action of σ on them, then there is no canonical isomorphism between $H_q(X,\mathbb{Z})$ and $H_q(X_\sigma,\mathbb{Z})$. In other words, the singular (co)homologies cannot be defined in the framework of Algebraic Geometry.

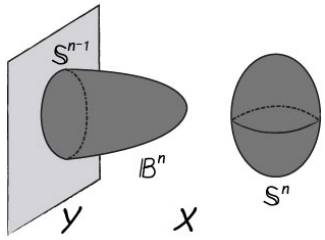

Fig. 4.1 Elements of singular homology

4.2 Before getting into axiomatic approach

This chapter serves as reference for all our need in singular homology and cohomology. It must be learned parallel to Chapters 5, 6 and 7. We are going to talk about topological spaces in general, however, what we have in mind is a very particular class of them. These are namely algebraic varieties which inherit their topology from $\mathbb{C}^n$. For now, what we need is that for a topological space X and $n \in \mathbb{N}_0$, we have $\mathbb{Z}$-modules $H_n(X,\mathbb{Z})$, possibly with torsions. A differentiable map from the n-dimensional sphere $\mathbb{S}^n$ to X induces an element of $H_n(X,\mathbb{Z})$. For the purpose of the present text we may regard an arbitrary element of $H_n(X,\mathbb{Z})$ as a $\mathbb{Z}$-linear sum of such spherical elements. In a similar way, for a pair (X,Y) of topological spaces X and Y with $Y \subset X$ and $n \in \mathbb{N}_0$, we have $\mathbb{Z}$-modules $H_n(X,Y,\mathbb{Z})$. An element of this $\mathbb{Z}$-module is either as before or a differentiable map from the n-dimensional ball $\mathbb{B}^n$ to X such that the boundary $\mathbb{S}^{n-1}$ of $\mathbb{B}^n$ goes to Y, see Figure 4.1. As before any element of $H_n(X,Y,\mathbb{Z})$ can be thought of a $\mathbb{Z}$-linear combination of the mentioned elements. The properties of singular homology and cohomology can be learned along the way we use them for the study of the topology of algebraic varieties.

4.3 Eilenberg-Steenrod axioms of homology

Despite the fact that we are going to deal with only smooth manifolds, this class of topological space is not enough for constructing singular homologies. The name itself gives us a clue, namely, for constructing homologies we need singular objects,

and so, we have to enlarge the category of smooth manifolds. The enlargement that we have in mind is the category of polyhedra or triangulated spaces. The main reason that we use this is Theorem 4.1. The reader who is interested in more details is referred to [ES52] and [Mas91].

An admissible category $\mathscr{A}$ of pairs (X,A) of topological spaces X and A with $A \subset X$ and the maps between them $f : (X,A) \to (Y,B)$ with $f : X \to Y$ and $f(A) \subset B$ satisfies the following conditions:

1. If (X,A) is in $\mathscr{A}$ then all pairs and inclusion maps in the lattice of (X,A)

$$\begin{array}{ccccccc} & & & (X,\emptyset) & & & \\ & & \nearrow & & \searrow & & \\ (\emptyset,\emptyset) & \to & (A,\emptyset) & & & (X,A) & \to (X,X) \\ & & \searrow & & \nearrow & & \\ & & & (A,A) & & & \end{array}$$

   are in $\mathscr{A}$.
2. If $f : (X,A) \to (Y,B)$ is in $\mathscr{A}$ then $(X,A),(Y,B)$ are in $\mathscr{A}$ together with all maps that f defines of members of the lattice of (X,A) into corresponding members of the lattice of (Y,B). If A and B are empty sets then for simplicity we write $f : X \to Y$.
3. If f_1 and f_2 are in $\mathscr{A}$ and their composition is defined then $f_1 \circ f_2 \in \mathscr{A}$.
4. If $I = [0,1]$ is the closed unit interval and $(X,A) \in \mathscr{A}$ then the Cartesian product

$$(X,A) \times I := (X \times I, A \times I)$$

   is in $\mathscr{A}$ and the maps given by

$$g_0, g_1 : (X,A) \to (X,A) \times I,$$

$$g_0(x) = (x,0),\ g_1(x) = (x,1)$$

   are in $\mathscr{A}$.
5. There is in $\mathscr{A}$ a space consisting of a single point. If $X,P \in \mathscr{A}$ and P is a single point space and $f : P \to X$ is any map then $f \in \mathscr{A}$.

The category of all topological pairs and the category of polyhedra are admissible. In this text we will not need these general categories. The reader is referred to [ES52] for examples of admissible categories. What we need in this text is the category of real differentiable manifolds, possibly with boundaries, which is admissible and it is a sub category of the category of polyhedra. A polyhedra is also called a triangulable space.

An axiomatic homology theory is a collection of functions

$$(X,A) \mapsto H_q(X,A),\ q = 0,1,2,3,\ldots$$

from an admissible category of pairs (X,A) of topological spaces X and A with $A \subset X$ to the category of abelian groups such that to each continuous map $f : (X,A) \to$

(Y,B) in the category it is associated a homomorphism:

$$f_q : H_q(X,A) \to H_q(Y,B),\ q = 0,1,2,\ldots$$

We denote by $H_*(X,A)$ the disjoint union of $H_q(X,A)$ and by $f_* : H_*(X,A) \to H_*(Y,B)$ the corresponding map constructed from f_q's. We have the following axioms of Eilenberg and Steenrod.

1. If f is the identity map then f_* is also the identity map.
2. For $(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C)$ we have $(g \circ f)_* = g_* \circ f_*$.
3. There are connecting homomorphisms

   $$\partial : H_q(X,A) \to H_{q-1}(A)$$

   called boundary maps, such that for any $(X,A) \to (Y,B)$ in $\mathscr{A}$ the following diagram commutes:

   $$\begin{array}{ccc} H_q(X,A) & \to & H_q(Y,B) \\ \downarrow & & \downarrow \\ H_{q-1}(A) & \to & H_{q-1}(B) \end{array},\ q = 0,1,2,\ldots$$

   Here A means $(A,\emptyset)$.
4. The exactness axiom: The homology sequence

   $$\cdots \to H_{q+1}(X,A) \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \to \cdots \to H_0(X,A),$$

   where i_* and j_* are induced by inclusions, is exact. This means that the kernel of every homomorphism coincides with the image of the previous one.
5. The homotopy axiom: for homotopic maps $f,g : (X,A) \to (Y,B)$ we have $f_* = g_*$.
6. The excision axiom: If the closure of a subset U[i] in X is contained in the interior of A and the inclusion

   $$(X \backslash U, A \backslash U) \xhookrightarrow{i} (X,A)$$

   belongs to the category, then i_* is an isomorphism.
7. The dimension axiom: For a single point set $X = \{p\}$ we have $H_q(X) = 0$ for $q > 0$.

The coefficient group of a homology theory is defined to be $H_0(X)$ for a single point set X. From the first and second axioms it follows that for any two single point sets X_1 and X_2 we have an isomorphism $H_0(X_1) \cong H_0(X_2)$.

Axiomatic cohomology theories are dually defined, i.e for $(X,A) \xrightarrow{f} (Y,B)$ we have $H^q(Y,B) \xrightarrow{f^*} H^q(X,A)$ and the coboundary maps $\delta : H^{q-1}(A) \to H^q(X,A)$ with similar axioms as listed above. We just change the direction of arrows and instead of subscript q we use superscript q.

---

[i] In [ES52] it is assumed that U is open. However, in the page 200 of the same book, the authors show that for singular homology or cohomology the openness condition is not necessary.

In [Mil62] we find the Milnor's additivity axiom which does not follow from the previous ones if the admissible category of topological spaces has topological sets which are disjoint union of infinite number of other topological sets:

8. Milnor's additivity axiom. If X is a disjoint union of open subsets X_α with inclusion maps $i_\alpha : X_\alpha \hookrightarrow X$, all belonging to the category, the homomorphisims

$$(i_\alpha)_q : H_q(X_\alpha) \to H_q(X)$$

must provide an injective representation of $H_q(X)$ as a direct sum.

The amazing point of the above axioms is the following:

Theorem 4.1 *In the category of polyhedra the homology (cohomology) theory exists and it is unique for a given coefficient group.*

The singular homology (cohomology) is the first explicit example of the homology (cohomology) theory. Its precise construction took more than sixty years in the history of mathematics. For more details, the reader is referred to [Mas91]. The uniqueness is a fascinating observation of Eilenberg and Steenrod. For a proof of uniqueness the reader is referred to [ES52] page 100 and [Mil62].

In order to stress the role of the coefficient group G we sometimes write:

$$H_q(X,A,G) = H_q(X,A) \text{ and so on.}$$

Using the above axiom we can show that

$$H_m(\mathbb{S}^n) \cong \begin{cases} G, & \text{if } m = 0,n \\ 0 & \text{otherwise} \end{cases}, \tag{4.1}$$

$$H_m(\mathbb{B}^n, \mathbb{S}^{n-1}) \cong \begin{cases} G, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}. \tag{4.2}$$

4.4 Singular homology

As we mentioned before, one of the motivations for the development of algebraic topology was a systematic study of multiple integrals. For this reason, the first example of homology theory is the singular homology constructed from simplicial complexes, where our integrations take place.

Fix an abelian group G. Let X be a C^∞ manifold and

$$\Delta^n = \left\{ (t_1, t_2, \cdots, t_{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} t_i = 1 \text{ and } t_i \geq 0 \right\}$$

be the standard n-simplex. A C^∞ map $f : \Delta^n \to X$ is called a singular n-simplex, see Figure 4.2. The map f needs to be neither surjective nor injective. Therefore, its

image may not be so nice as Δ^n. One of the most important examples of a singular n-simplex of the present book is the case in which X is the Fermat variety. This is explained in Exercise 4.2.

Let $C_n(X)$ be the set of all formal and finite sums

$$\sum_i n_i f_i,\ n_i \in G, f_i \text{ a singular } n\text{-simplex}.$$

$C_n(X)$ has a natural structure of an abelian group. For the simplex Δ^n we denote by

$$I_k : \Delta^{n-1} \to \Delta^n, \quad k = 0,1,2,\ldots,n,$$

$$I_k(t_1,t_2,\ldots,t_n) := (t_1,t_2,\ldots,t_k,0,t_{k+1},\cdots,t_n),$$

the canonical inclusion for which the image is the k-th face of Δ^n, where $k = 0,1,2,\ldots,n$. For f a singular n-simplex we define

$$\partial_n f = \sum_{k=0}^{n} (-1)^k f \circ I_k \in C_{n-1}(X)$$

and by linearity we extend it to the homomorphism of abelian groups:

$$\partial = \partial_n : C_n(X) \to C_{n-1}(X)$$

which we call it the boundary map. We get the complex

$$C_\bullet(X): \qquad C_0(X) \leftarrow C_1(X) \leftarrow \cdots \leftarrow C_n(X) \leftarrow \cdots,$$

where the arrows are the boundary maps ∂. The kernel of the boundary map is

$$Z_n(X) = \ker(\partial_n)$$

and it is called the group of singular n-cycles. The image of the boundary map is

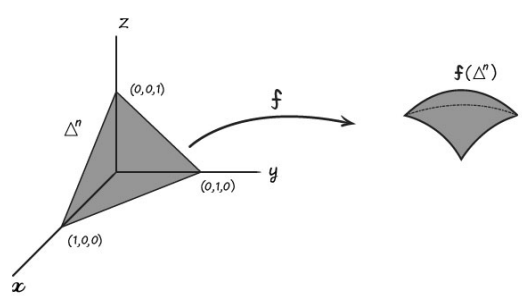

Fig. 4.2 An n-simplex.

$$B_n(X) = \mathrm{Im}(\partial_{n+1})$$

and it is called the group of singular n-boundaries. It is an easy exercise to show that $\partial_n \circ \partial_{n+1} = 0$ and so $B_n(X) \subset Z_n(X)$. The n-th homology group of X with coefficients in G is defined to be

$$H_n(X,G) := H_n(C_\bullet(X), \partial) = \frac{Z_n(X)}{B_n(X)}.$$

The elements of $H_n(X,G)$ are called homology classes with coefficients in G. The boundary map

$$\delta : H_n(X,A,G) \to H_{n-1}(A,G)$$

is given by the boundary map ∂_n (prove that it is well-defined).

In order to construct relative homologies we proceed as follows: For the pair (X,A) of topological spaces with $A \subset X$ we define

$$C_n(X,A) := \frac{C_\bullet(X)}{C_\bullet(A)}$$

and we have the complex:

$$C_\bullet(X,A): \qquad C_0(X,A) \leftarrow C_1(X,A) \leftarrow \cdots \leftarrow C_n(X,A) \leftarrow \cdots,$$

where the arrows are induced by the boundary maps ∂. In a similar way, we define $Z_n(X,A)$ and $B_n(X,A)$. The following definition turns out to be natural

$$H_n(X,A,G) := H_n(C_\bullet(X,A)) = \frac{Z_n(X,A)}{B_n(X,A)}.$$

Singular homology and cohomology is the beginning of a branch of mathematics called Homological Algebra. For a historical account on this see Weibel's article [Wei99].

Theorem 4.2 *The singular homology (resp. cohomology) satisfies all the axioms of homolology theory (resp. cohomology theory)*

For cohomology theory there are different constructions. First, it was constructed singular cohomology and then it appeared Cech cohomology of constant sheaves (see [BT82]). The introduction of the de Rham cohomology was a significant step toward the formulation of Hodge theory.

4.5 Some consequences of the axioms

In theory, it is possible to prove all the properties of the homology and cohomology theory from the axioms and without using singular homology and cohomology. However, in practice one first proves Theorem 4.1 and then one proves that singular

homology and cohomology satisfy all the axioms. In this way, one is allowed to use all the geometric intuition of simplexes. In this section we give a list of such properties. For proofs see [ES52].

1. Universal coefficient theorem for homology: for a polyhedra X there is a natural short exact sequence

$$0 \to H_q(X,\mathbb{Z}) \otimes G \to H_q(X,G) \to \mathrm{Tor}(H_{q-1}(X,\mathbb{Z}),G) \to 0.$$

For two abelian groups A and B, $\mathrm{Tor}(A,B) := \mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ is the Tor functor. It satisfies the following properties. The first item serves as its definition.

- Let $0 \to F_1 \xrightarrow{h} F_0 \xrightarrow{k} A \to 0$ be a short exact sequence with F_0 a free abelian group (it follows that F_1 is free too). Then there is an exact sequence as follows:
$$0 \to \mathrm{Tor}(A,B) \to F_1 \otimes B \xrightarrow{h\otimes 1} F_0 \otimes B \xrightarrow{k\otimes 1} A \otimes B \to 0.$$
One can use this property to define or calculate $\mathrm{Tor}(A,B)$. It is recommended to students to prove the below properties using only this one.
- $\mathrm{Tor}(A,B)$ and $\mathrm{Tor}(B,A)$ are isomorphic.
- If either A or B is torsion free then $\mathrm{Tor}(A,B) = 0$.
- For $n \in \mathbb{N}$ we have
$$\mathrm{Tor}(\frac{\mathbb{Z}}{n\mathbb{Z}},A) \cong \{x \in A \mid nx = 0\}$$
and so $\mathrm{Tor}(\frac{\mathbb{Z}}{n\mathbb{Z}},\frac{\mathbb{Z}}{m\mathbb{Z}}) = \frac{\mathbb{Z}}{gcd(n,m)\mathbb{Z}}$.

For further properties of Tor see [Mas91], p. 270.

2. Universal coefficient theorem for cohomology: For a polyhedra X there is a natural short exact sequence:

$$0 \to \mathrm{Ext}(H_{q-1}(X,\mathbb{Z}),G) \to H^q(X,G) \to \mathrm{Hom}(H_q(X,\mathbb{Z}),G) \to 0.$$

For two abelian groups A and B, $\mathrm{Ext}(A,B)$ is the Ext functor. It satisfies the following properties. The first item serves as its definition.

- Let $0 \to F_1 \xrightarrow{h} F_0 \xrightarrow{k} A \to 0$ be a short exact sequence with F_0 a free abelian group (it follows that F_1 is free too). Then there is an exact sequence as follows:
$$0 \leftarrow \mathrm{Ext}(A,B) \leftarrow \mathrm{hom}(F_1,B) \xleftarrow{\mathrm{hom}(h,1)} \mathrm{hom}(F_0,B) \xleftarrow{\mathrm{hom}(k,1)} \mathrm{hom}(A,B) \leftarrow 0.$$
One can use this property to define or calculate $\mathrm{Ext}(A,B)$. It is recommended to students to prove the below properties using only this one.
- If A is a free abelian group then $\mathrm{Ext}(A,B) = 0$ for any abelian group B.
- If B is a divisible group then $\mathrm{Ext}(A,B) = 0$ for any abelian group A.
- For $n \in \mathbb{N}$ we have

$$\mathrm{Ext}(\frac{\mathbb{Z}}{n\mathbb{Z}}, B) \cong \frac{B}{nB}.$$

For further properties of Ext see [Mas91], p. 313. Note that the canonical map

$$H^q(X,G) \to \mathrm{Hom}(H_q(X,\mathbb{Z}),G)$$

gives us a pairing

$$H^q(X,G) \times H_q(X,\mathbb{Z}) \to G,\ (\alpha,\beta) \mapsto \int_\beta \alpha. \tag{4.3}$$

The usage of integral sign here is purely symbolic. It will be justified when later we introduce the de Rham cohomology of smooth manifolds.

3. For a triple $Z \subset Y \subset X$ of polyhedras we have the long exact sequence:

$$\cdots \to H_q(Y,Z,G) \to H_q(X,Z,G) \to H_q(X,Y,G) \to H_{q-1}(Y,Z,G) \to \cdots.$$

4. For X and Y two polyhedra we have a cross product maps

$$H^p(X,G_1) \times H^q(Y,G_2) \to H^{p+q}(X \times Y, G_1 \otimes G_2),\ \ (\omega_1,\omega_2) \mapsto \omega_1 \times \omega_2.$$

For a list of its properties see [Mas91], 324.

5. Künneth theorem for homology: Let X and Y be two polyhedra. Then we have a natural exact sequence:

$$0 \to \bigoplus_{i+j=q} H_i(X,G) \otimes_G H_j(Y,G) \to H_q(X \times Y, G) \to \bigoplus_{i+j=q-1} \mathrm{Tor}(H_i(X,G), H_j(Y,G)) \to 0.$$

6. Künneth theorem for cohomology: Let X and Y be two polyhedra. Let us assume that all the cohomologies of X with coefficients in $\mathbb{Z}$ are finitely generated and at least one of the two spaces X and Y has all cohomology groups torsion free. Then we have a canonical isomorphism

$$\bigoplus_{i+j=q} H^i(X,\mathbb{Z}) \otimes H^j(Y,\mathbb{Z}) \cong H^q(X \times Y, \mathbb{Z})$$

given by the cross product. see [Mas91], p. 344.

7. For X,Y as in the previous item and $\delta_1 \in H_i(X,\mathbb{Z}), \delta_2 \in H_j(Y,\mathbb{Z}),\ \omega_1 \in H^i(X,\mathbb{Z}),\ \omega_2 \in H^j(X,\mathbb{Z})$ we have

$$\int_{\delta_1 \otimes \delta_2} \omega_1 \times \omega_2 = \int_{\delta_1} \omega_1 \int_{\delta_2} \omega_2.$$

8. There are natural cup and cap products:

$$H^p(X,G_1) \times H^q(X,G_2) \to H^{p+q}(X, G_1 \otimes_{\mathbb{Z}} G_2),\ (\alpha,\beta) \mapsto \alpha \cup \beta$$

$$H^p(X,G_1)\times H_q(X,G_2)\to H_{q-p}(X,G_1\otimes_{\mathbb{Z}} G_2),(\alpha,\beta)\mapsto \alpha\cap\beta.$$

For some properties which $\cup$ and $\cap$ satisfy see [Mas91], p. 329, for instance we have

$$\alpha\cap(\beta\cap\gamma)=(\alpha\cup\beta)\cap\gamma,\ \alpha\in H^p(X,G_1),\ \ \beta\in H^q(X,G_2),\ \gamma\in H_r(X,G_3).$$

The cup product is defined using the cross product. Let $d: X\to X\times X,\ d(x)=(x,x)$ be the diagonal map. We define

$$\omega_1\cup\omega_2=d^*(\omega_1\times\omega_2).$$

9. The cap product for $p=q$, $G_1=G$, $G_2=\mathbb{Z}$ and X a connected space generalizes the integration map (4.3):

$$H^q(X,G_1)\times H_q(X,G_2)\to G_1\otimes G_2,\ (\alpha,\beta)\mapsto \int_\beta \alpha:=\alpha\cap\beta. \tag{4.4}$$

10. Top (co)homology: Let X be a compact connected oriented manifold of dimension n. We have $H^n(X,\mathbb{Z})\cong\mathbb{Z},\ H_n(X,\mathbb{Z})\cong\mathbb{Z}$. The choice of a generator of $H_n(X,\mathbb{Z})$ or $H^n(X,\mathbb{Z})$ corresponds to the choice of an orientation and so we sometimes refer to it as a choice of an orientation for X. We denote by $[X]$ a generator of $H_n(X,\mathbb{Z})$ and write

$$\int_X \alpha:=\int_{[X]}\alpha,\ \alpha\in H^n(X,\mathbb{Z}).$$

Let Y be a compact connected oriented manifold of dimension m. For a C^∞ map $f: X\to Y$ we have the map $f_*: H_n(X,\mathbb{Z})\to H_n(Y,\mathbb{Z})$. If there is no danger of confusion then the image of $[X]$ in $H_n(Y,\mathbb{Z})$ is denoted again by $[X]$. In many cases X is a submanifold of Y and f is the inclusion.

11. Intersection map: Let X be an oriented manifold. There is a natural intersection map

$$H_p(X,\mathbb{Z})\times H_q(X,\mathbb{Z})\to H_{p+q-n}(X,\mathbb{Z}),(\alpha,\beta)\mapsto\alpha\cdot\beta.$$

If X is connected for $q=n-p$ this gives us

$$H_p(X,\mathbb{Z})\times H_{n-p}(X,\mathbb{Z})\to\mathbb{Z}. \tag{4.5}$$

12. Poincaré duality theorem: Let X be a compact oriented manifold of dimension n. Poincaré duality says that

$$P: H^q(X,\mathbb{Z})\to H_{n-q}(X,\mathbb{Z}),\ \alpha\mapsto\alpha\cap[X]$$

is an isomorphism. This implies that the intersection map (4.5) is unimodular. This means that any linear function $H_{n-q}(X,\mathbb{Z})\to\mathbb{Z}$ is expressible as intersection with some element in $H_q(X,\mathbb{Z})$ and any class in $H_q(X,\mathbb{Z})$ having intersection number zero with all classes in $H_{n-q}(X,\mathbb{Z})$ is a torsion class. Note that P is

an isomorphism in the level of torsions, and this cannot be deduced from the mentioned consequence. For $\alpha \in H^q(X,\mathbb{Z})$ we say that α and $P(\alpha)$ are Poincaré duals. We have the equality

$$\int_{P(\alpha)} \omega = \int_X \omega \cup \alpha,\ \omega \in H^{n-q}(X,\mathbb{Z}),\ \alpha \in H^q(X,\mathbb{Z}).$$

By Poincaré duality the intersection map in homology is dual to cup product map and in particular (4.5) is dual to

$$H^{n-q}(X,\mathbb{Z}) \times H^q(X,\mathbb{Z}) \to \mathbb{Z},\ (\omega_1,\omega_2) \mapsto \int_X \omega_1 \cup \omega_2.$$

4.6 Leray-Thom-Gysin isomorphism

In this section we are going to study an isomorphism which is mainly attributed to three mathematicians J. Leary, R. Thom, W. Gysin. Our main source for this is Cheniot's article [Ché91] page 392. This gives us the contributions of Thom and Leray. In the literature one mainly finds an immediate consequence of this isomorphism in cohomology which is known under the names *Gysin sequence* or *Gysin map*.

Let us be given a compact oriented submanifold Y of real codimension c in an oriented manifold X. One can define a map

$$H_{m-c}(Y,\mathbb{Z}) \to H_m(X,X\backslash Y,\mathbb{Z}) \tag{4.6}$$

for any m, with the convention that $H_m(Y) = 0$ for $m < 0$, in the following way: Let us be given a cycle δ in $H_{m-c}(Y)$. Its image by this map is obtained by thickening a cycle representing δ, each point of it growing into a closed c-disk transverse to Y in X. In lower dimensions, Figure 4.3 and Figure 4.4 may help the reader to have a better understanding of the map (4.6).

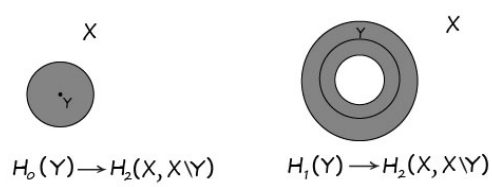

Fig. 4.3 Leray-Thom-Gysin isomorphism I

Theorem 4.3 (Leray-Thom-Gysin isomorphism) *The map (4.6) is an isomorphism.*

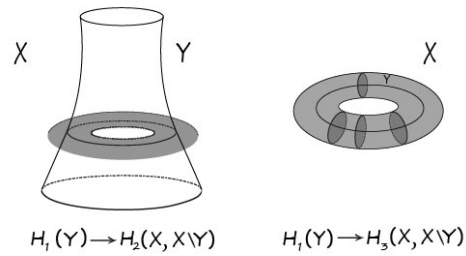

Fig. 4.4 Leray-Thom-Gysin isomorphism II

Note that by definition of the map (4.6), its inverse is given by the intersection of topological cycles in $H_m(X, X\setminus Y, \mathbb{Z})$ with Y. Recall that a tubular neighborhood of Y in X is a C^∞ embedding $f : E \to X$, where E is the normal bundle of Y in X, such that

1. f restricted to the zero section of E induces the identity map in Y.
2. $f(E)$ is an open neighborhood of Y in X.

We know that a tubular neighborhood of Y in X exists ([Hir76], Theorem 5.2). Now using excision property, it is enough to prove the theorem for X, a vector bundle and Y its zero section. The rest of the proof can be found in [Ché91] page 392.

Let X and Y as above. Writing the long exact sequence of the pair $(X, X\setminus Y)$ and using (4.6) we obtain:

$$\cdots \to H_m(X,\mathbb{Z}) \xrightarrow{\tau} H_{m-c}(Y,\mathbb{Z}) \xrightarrow{\sigma} H_{m-1}(X\setminus Y,\mathbb{Z}) \xrightarrow{i} H_{m-1}(X,\mathbb{Z}) \to \cdots \tag{4.7}$$

The map τ is the intersection with Y, see Figure 4.5 The map σ is the composition of the boundary operator with (4.6). The image of $\delta \in H_{m-c}(Y,\mathbb{Z})$ by this map is obtained in the following way: each point of δ is growing into a $(c-1)$-dimensional sphere which is the boundary of a closed c-disk transverse to Y in X, see Figure 4.6.

4.7 Exercises

4.1. Show that the boundary maps in the complex $C_\bullet(X)$ satisfy

$$\partial_n \circ \partial_{n+1} = 0.$$

Show also that the map $\delta : H_n(X, A, G) \to H_{n-1}(A, G)$ induced by the boundary map ∂_n is well-defined.

4.2. Let $m_1, m_2, \ldots, m_{n+1}$ be positive integers bigger than one. Consider the following affine variety in $\mathbb{C}^{n+1}$:

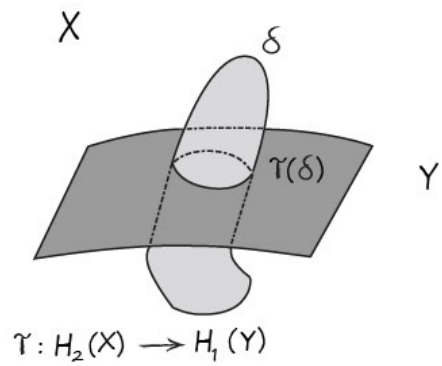

Fig. 4.5 Intersection map

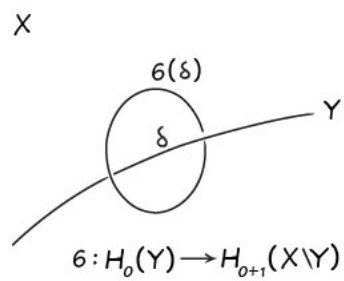

Fig. 4.6 Spheres around the points of a cycle

$$U := \left\{ x \in \mathbb{C}^{n+1} \mid x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1 \right\}.$$

Let

$$\Delta^n := \left\{ (t_1, t_2, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1 \right\}$$

be the standard n-simplex, $I_{m_i} := \{0, 1, 2, \ldots, m_i - 2\}$ and let $\zeta_{m_i} = e^{\frac{2\pi\sqrt{-1}}{m_i}}$ be an m_i-th primitive root of unity. For $\beta \in I := I_1 \times I_2 \times \cdots \times I_{n+1}$ and $a \in \{0,1\}^{n+1}$ let

$$\Delta_{\beta+a} : \Delta^n \to U$$

$$\Delta_{\beta+a}(t) = \left(t_1^{\frac{1}{m_1}} \zeta_{m_1}^{\beta_1+a_1},\ t_2^{\frac{1}{m_2}} \zeta_{m_2}^{\beta_2+a_2},\ \ldots, t_{n+1}^{\frac{1}{m_{n+1}}} \zeta_{m_{n+1}}^{\beta_{n+1}+a_{n+1}} \right),$$

where for a positive number r and a natural number s, $r^{\frac{1}{s}}$ is the unique positive s-th root of r. Prove that for

$$\delta_\beta := \sum_a (-1)^{\sum_{i=1}^{n+1}(1-a_i)} \Delta_{\beta+a}$$

we have $\partial\delta_\beta = 0$, and so, it induces a non-zero element in $H_n(U,\mathbb{Z})$ which we denote it by the same letter. In fact, they from a basis of the $\mathbb{Z}$-module $H_n(U,\mathbb{Z})$. The details of this will be explained in §7.9, see also Exercise 15.3.

Let us assume that $m_1, m_2, \cdots, m_{n+1}$ are even numbers. In this case we have

$$\delta := \left\{ (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1 \right\} \subset U. \tag{4.8}$$

Write δ as a sum of singular n-simplexes and conclude that it induces an element $\delta \in H_n(U,\mathbb{Z})$. Write δ as a linear combination of $\delta_\beta,\ \ \beta \in I$.

4.3. Prove that for a manifold X, the first homology group $H_1(X,\mathbb{Z})$ is the abelization of the homotopy group $\pi_1(X,b),\ b \in X$ of X, that is,

$$H_1(X,\mathbb{Z}) = \pi_1(X,b)/[\pi_1(X,b), \pi_1(X,b)],$$

where for a group G, $[G,G]$ is the subgroup of G generated by the commutators $aba^{-1}b^{-1},\ \ a,b \in G$.

4.4. Show that any point in $\mathbb{R}^n$ is a deformation retract of $\mathbb{R}^n$.

4.5. How one can hang a picture on a wall by looping a string over two nails in such a way that if either one of the nails is removed, the picture falls? (Source Wikipedia, Pochhammer counter). Can you generalize this puzzle to three or four nails? How about arbitrary number of nails?

4.6. Discuss in more details the content of Exercise 4.2 in the one dimensional case $n = 1$. In this case U is a Riemann surface with some removed points. How many? Determine the genus of U. Can you determine a $\mathbb{Z}$-linear combination of δ_β's which are homologous to cycles around the removed points? Can you do it at least for the particular cases $m_1 = m_2 = 3,4,5$?

4.7. Let $y = A(x)$ be a multi-valued function in x, for instance take $y = p(x)^a$, where p is a polynomial in x and a is a complex number. Let also $E \subset \mathbb{C}^2$ be the Riemann surface which is the graph of y. The first concrete examples of a closed cycle δ in E is obtained using Pochhammer cycles. The Pochhammer cycle associated to the points t_1, t_2 in the $x \in \mathbb{C}$ plane and a path γ connecting t_1 to t_2, is the commutator

$$[\gamma_2, \gamma_1] = \gamma_2^{-1} \cdot \gamma_1^{-1} \cdot \gamma_2 \cdot \gamma_1,$$

where γ_2 is a loop along γ starting and ending at some point t_2 in the middle of γ, which encircles t_2 once anticlockwise, and γ_1 is a similar loop with respect to t_1, see Figure 4.7. In our case t_1 and t_2 are two roots of p.

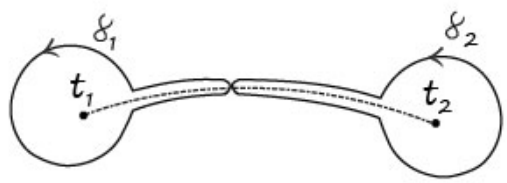

Fig. 4.7 Paths around two removed points.

1. Show that $[\gamma_2, \gamma_1]$ lifts to a closed path δ in E which is non-zero in $H_1(E, \mathbb{Z})$.
2. For $a = \frac{1}{n}$, $n \in \mathbb{N}$ describe a basis of $H_1(E, \mathbb{Z})$ and the corresponding intersection matrix. Recall that $H_1(E, \mathbb{Z})$ is by definition the fundamental group of E modulo its subgroup generated by commutators.
3. Show that the Pochhammer cycle is homotopic to the one depicted in Figure 4.8 and 4.9.

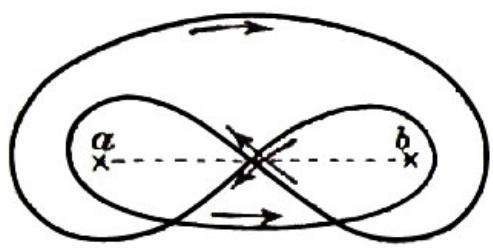

Fig. 4.8 The picture of a Pochhammer cycle taken from [Rie53] page 87.

4.8. The discussion in §2.1 can be carried out for an arbitrary polynomial f of degree d and with distinct roots. The corresponding integrals are usually called abelian integrals. Prove that

1. The curve E is obtained by removing $d - 2[\frac{d-1}{2}]$ points from a Riemann surface of genus $g := [\frac{d-1}{2}]$.
2. Therefore, $H_1(E, \mathbb{Z})$ is of rank $d - 1$ and so we have $d - 1$ linearly independent integrals.

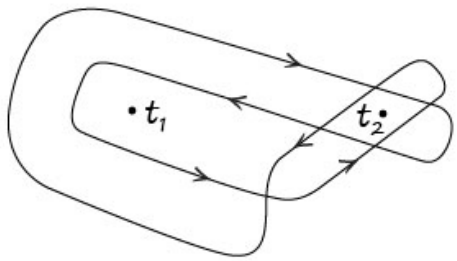

Fig. 4.9 Another way of drawing a Pochhammer cycle.

4.9. One of Poincaré's favorite double integrals in [Poi02] is

$$\iint \frac{R(x,y)dx \wedge dy}{P(x,y)Q(x,y)}$$

where $P,Q,R \in \mathbb{C}[x,y]$ are three polynomials. He attributes the following equality to an unpublished work of Stieltjes:

$$\frac{1}{(2\pi\sqrt{-1})^2}\int_\delta \frac{R(x,y)dx \wedge dy}{P(x,y)Q(x,y)} = \frac{R(a)}{\frac{\partial P}{\partial x}(a)\frac{\partial Q}{\partial y}(a) - \frac{\partial P}{\partial y}(a)\frac{\partial Q}{\partial x}(a)}, \tag{4.9}$$

where we have assumed that the curve $\{P=0\}$ intersects $\{Q=0\}$ transversely at a. This is the same as to say that the denominator of the right hand side of (4.9) is not zero. The integration takes place over a two dimensional topological cycles of $\mathbb{C}^2 - (\{P=0\} \cup \{Q=0\})$ constructed near a. Give a precise description of the cycle δ such that the above equality holds. Hint: Use the change of coordinates $(x,y) \mapsto (P(x,y), Q(x,y))$.

4.10. Let us be given a closed submanifold Y of real codimension c in a manifold X. Using Leray-Thom-Gysin isomorphism, in §4.6 we have defined the intersection map $H_q(X) \to H_{q-c}(Y)$. Let $\tilde{X}$ is another submanifold of X which intersects Y transversely. Construct the relative intersection map

$$H_q(X,\tilde{X}) \to H_{q-c}(Y, Y \cap \tilde{X}). \tag{4.10}$$

4.11. Prove the affirmations (4.1) and (4.2) using the Eilenberg-Steenrod axioms or by explicit definition of singular homology. Calculate the homology and cohomology groups of the projective spaces $\mathbb{R}P^n(n), \mathbb{C}P(n)$.

4.12. List all the axioms of a cohomology theory.

4.13. Show that the Milnor's additivity axiom for a finite disjoint union of topological spaces follows from the axioms 1 till 7.

4.14. Let us be given a homology theory $H_q(X,\mathbb{Z})$ with $\mathbb{Z}$-coefficients. Is

$$X \rightsquigarrow \mathrm{Hom}(H_q(X,\mathbb{Z}),\mathbb{Z})$$

a cohomology theory? If no, which axiom fails?

4.15. Try to prove some of the consequences of homology theory by yourself. For this purpose you can consult [ES52].

4.16. Prove all the properties of Tor mentioned in this text using just the first property in the list.

Chapter 5
Lefschetz theorems

Here's to Lefschetz (Solomon L.)
Who's as argumentative as hell,
When he's at last beneath the sod,
Then he'll start to heckle God (W. V. D. Hodge in [Hod86] page 30).

5.1 Introduction

In 1924 Lefschetz published his treatise [Lef24b] on the topology of algebraic varieties. "As I see it at last it was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry" (S. Lefschetz in [Lef68] page 854). When it was written the knowledge of topology was still primitive and Lefschetz "made use most uncritically of early topology à la Poincaré and even of his own later developments" (S. Lefschetz in [Lef68] page 854). "One of the great lessons to be learned form a study of Lefschetz's work on algebraic varieties is that before proceeding to the investigation of transcendental properties it is necessary first to acquire a through understanding of the topological properties of a variety" (W. V. D. Hodge in [Hod57] page 4). Lefschetz succeeded to formulate his theorems, however, he did not give rigorous proofs for most of them. " When I was a graduate student at Princeton, it was frequently said that Lefschetz never stated a false theorem nor gave a correct proof" (P. Griffiths in [GSW92] page 289). "As a condensed account of the theory of algebraic manifolds, this little volume [Lefschetz's book [Lef24b]] is so successful that it would seem beside the point to criticise the somewhat hasty manner in which the author disposes of several rather delicate details" (Review of [Lef24b] by J. W. Alexander in Bull. Amer. Math. Soc. 31 (1925), 558-559).

Later, Lefschetz theorems were proved using harmonic forms, Morse theory, sheaf theory and spectral sequences. "But none of these very elegant methods yields Lefschetz's full geometric insight, e.g. they do not show us the famous vanishing cycles" (K. Lamotke in [Lam81] page 15). Two temptations to give precise proofs for

Lefschetz theorems using the same topological methods of Lefschetz are due to A. Wallace 1958 and K. Lamotke 1981, see [Lam81]. In this chapter we use the later source and we present Lefschetz theorems on hyperplane sections. Unfortunately, up to the time of writing the present text there is no topological proof for the so called *hard Lefschetz theorem*.

In this chapter if the coefficients ring of the homology or cohomology is not mentioned then it is supposed to be the ring of integers $\mathbb{Z}$. Most of the theorems are stated for homologies and it is left to the reader to formulate similar theorems for cohomologies. So far, we have not stated the Hodge conjecture for the cohomology groups $H^q(X,\mathbb{Q})$, where X is a smooth projective variety of dimension n and $0 \leq q \leq 2n$. However, in this section we have stated many of its consequences for the algebraic cycles in the homology group $H_{2n-q}(X,\mathbb{Q})$. Note that the Poincaré dual of such cycles lie in $H^q(X,\mathbb{Q})$.

5.2 Main theorem

Let X be a smooth projective variety of dimension n. By definition X is embedded in some projective space $\mathbb{P}^N$ and it is the zero set of a finite collection of homogeneous polynomials. Let also Y,Z be two codimension one transversal hyperplane sections of X, and so they are smooth, see Figure 5.1 and Figure 5.2 A modern

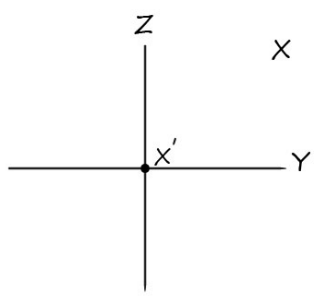

Fig. 5.1 Hyperplance sections

terminology is to say that Y,Z are two smooth very ample divisors of X. We assume that Y and Z intersect each other transversely at $X' := Y \cap Z$. In order to get familiar with the notion of transversality see Exercise 5.1. Using the *Veronese embedding* $\mathbb{P}^N \hookrightarrow \mathbb{P}^M$, $M := \binom{N+d}{d} - 1$ we can reduce the study of the intersection of X with hypersurfaces of degree d to the study of hyperplane sections.

We do not assume that Y and Z are hyperplane sections associated to the same embedding $X \subset \mathbb{P}^N$. However, we assume that for some $k \in \mathbb{N}$ the divisor $Y - kZ$

is principal, i.e. for some rational function $f = \frac{F}{G^k}$ on X, $Y = \{F = 0\}$ is the zero divisor of order one of f and $Z = \{G = 0\}$ is the pole divisor of order k of f. Here, F and G are two homogeneous polynomials in $(x_0, x_1, \cdots, x_N)$ with $\deg(F) = k \cdot \deg(G)$. The following theorem is the main ingredient of Lefschetz's theorems on hyperplane sections.

Theorem 5.1 *We have*

$$H_q(X \backslash Z, Y \backslash X', \mathbb{Z}) = \begin{cases} 0 & \text{if } q \neq n \\ \text{free } \mathbb{Z}\text{-module of finite rank} & \text{if } q = n \end{cases}, \quad n := \dim(X).$$

Since all the homologies in Theorem 5.1 are torsion free, the same theorem is valid for cohomologies with $\mathbb{Z}$-coefficients. Later, we will see how to calculate the rank of $H_q(X \backslash Z, Y \backslash X')$ by means of algebraic methods. A proof and further generalizations of Theorem 5.1 will be presented in Chapter 6 in which we develop the Picard-Lefschetz theory. A reader who wants to get a feeling of this theorem using a basic Algebraic Topology is invited to think on Exercise 4.2. The idea of using hyperplane sections is the main tool in Lefschetz's book [Lef24b] in order to study the topology of algebraic varieties. He took the embryo of this idea from Picard-Simart book [PS06] and Poincaré's articles [Poi02, Poi87].

Remark 5.1 In our proof of Theorem 5.1, we will realize that the assumption that Z is a transversal hyperplane section, can be replaced with a weaker one. We only need that the pencil formed by Y and Z has isolated critical points in $X \backslash Z$ and its fibers, except Z itself, intersect each other transversely at the axis of the pencil $X \cap Z$. This follows from our hypothesis on Z, for further details see Chapter 6 and Theorem 6.4.

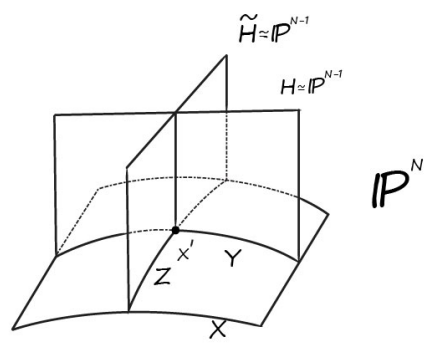

Fig. 5.2 Hyperplane sections

5.3 Some consequences of the main theorem

Let us state some consequences of Theorem 5.1. Let

$$U := X\backslash Z,\ V := Y\backslash X'.$$

U is an affine variety and V is an affine subvariety U of codimension one.

Corollary 5.1 *Let X be smooth projective space and Z be a smooth hyperplane section of X. We have*

$$H_q(U,\mathbb{Z}) = 0,\ \textit{for } q > \dim U,\ U := X\backslash Z.$$

The homology group $H_n(U,\mathbb{Z})$ is free of finite rank.

Proof. We write the long exact sequence of the pair $V \subset U$ and we get the five term exact sequence

$$0 \to H_n(V) \to H_n(U) \to H_n(U,V) \to H_{n-1}(V) \to H_{n-1}(U) \to 0 \tag{5.1}$$

and the isomorphisms

$$H_q(U) \cong H_q(V),\ q \neq n, n-1. \tag{5.2}$$

Now, our result follows by induction on n. For $n = 1$ it it trivial because X is a two dimensional oriented compact real manifold and U is obtained by removing a non-empty finite set of points from X, and hence, it is not compact. Let us assume that it is true in dimension $n-1$. We know that X' is also a smooth hyperplane section of Y and $\dim(Y) = n-1$. Therefore, $H_{n-1}(V)$ is free and $H_q(V) = 0$ for $q > n-1$ and in particular $H_n(V) = 0$. This together with (5.2) implies that $H_q(U) = 0,\ q > n$. The five term exact sequence mentioned above reduces to the four term exact sequence:

$$0 \to H_n(U) \to H_n(U,V) \to H_{n-1}(V) \to H_{n-1}(U) \to 0. \tag{5.3}$$

This implies that $H_n(U)$ is a subset of $H_n(U,V)$ and so by theorem 5.1 it is free. □

Remark 5.2 In the exact sequence (5.3), $H_n(U), H_n(U,V)$ and $H_{n-1}(V)$ are free $\mathbb{Z}$-modules and $H_{n-1}(U)$ may have torsions. Later, we will see that for complete intersection affine varieties $H_{n-1}(U) = 0$ and so (5.3) reduces to three terms which are all free $\mathbb{Z}$-modules.

Corollary 5.2 *The intersection map with Y*

$$H_{n+q}(X) \to H_{n+q-2}(Y),\ q = 2,3,\ldots \tag{5.4}$$

is an isomorphism

Proof. We write the long exact sequence of the pair $X\backslash Y \subset X$ and use the Leray-Thom-Gysin isomorphism and obtain

$$\cdots \to H_{n+q}(X\backslash Y) \to H_{n+q}(X) \to H_{n+q-2}(Y) \to H_{n+q-1}(X\backslash Y) \to \cdots \tag{5.5}$$

Now, our statement follows from Corollary 5.1. □

Later we will use the dual of the intersection map in Corollary 5.2

$$H^{n+q-2}(Y) \to H^{n+q}(X).$$

This is an isomorphism of Hodge structures of weight $(1,1)$, that is, if we use the de Rham cohomology instead of singular cohomology then it sends (p,q)-forms to $(p+1,q+1)$-forms. Despite the fact that so far we have not discussed these topics, it is possible to derive a consequence of the Hodge conjecture:

Conjecture 5.1 *Let us assume that $n+q,\ q \geq 2$ is an even number. For any algebraic cycle $Z = \sum_{i=1}^{r} n_i Z_i,\ n_i \in \mathbb{Z}, \dim(Z_i) = \frac{n+q}{2} - 1$ in Y, there is some $m \in \mathbb{N}$, such that mZ is homologous to another algebraic cycle $\tilde{Z}$ in Y with the property that $\tilde{Z}$ is the intersection of an algebraic cycle of dimension $\frac{n+q}{2}$ in X with Y.*

The above conjecture follows from the rational Hodge conjecture for the $(n-q)$-th cohomology of X. Note that we have used Poincaré duality and the dual of (5.4) is the map $H_{n-q}(Y) \to H_{n-q}(X)$ induced by inclusion. At this point, we cannot expect that Z itself is obtained by intersection of some algebraic cycle with X. Let us consider two non-trivial cases of Conjecture 5.1. For $n = 4$ and $q = 2$, it is a theorem and it follows from Lefschetz $(1,1)$ theorem for X which is valid with cohomologies with integer coefficients, and hence, we can put $m = 1$. For $n = 6$ and $q = 2$ Conjecture 5.1 is still open.

5.4 Lefschetz theorem on hyperplane sections

In this section we are going to prove the following theorem:

Theorem 5.2 (Lefschetz hyperplane section theorem) *Let X be a smooth projective variety of dimension n and $Y \subset X$ be a smooth hyperplane section. Then*

$$H_q(X,Y,\mathbb{Z}) = 0,\ 0 \leq q \leq n-1,$$

In other words, the inclusion $Y \hookrightarrow X$ induces isomorphisms of the homology groups in all dimensions strictly less than $n-1$ and a surjective map in H_{n-1}.

Proof. The proof is essentially based on the long exact sequence of triples

$$V \subset U \subset X,$$

$$V \subset Y \subset X,$$

the Leray-Thom-Gysin isomorphisms for the pairs $V := Y\backslash X' \subset Y$ and $U := X\backslash Z \subset X$ and Theorem 5.1. The long exact sequence of the first triple together with Theorem 5.1 gives us isomorphisms

$$H_q(X,V) \cong H_q(X,U),\ q \neq n, n+1$$

induced by inclusions, and the five term exact sequence:

$$0 \to H_{n+1}(X,V) \to H_{n+1}(X,U) \to H_n(U,V) \to H_n(X,V) \to H_n(X,U) \to 0.$$

Now, we use Leray-Thom-Gysin isomorphism for the pair (U,X) and obtain $H_q(X,U) \cong H_{q-2}(Z)$. Combining these two isomorphisms we get

$$H_q(X,V) \cong H_{q-2}(Z), q \neq n, n+1. \tag{5.6}$$

We write the long exact sequence of the second triple and the pair $X' \subset Z$ in the following way:

$$\begin{array}{ccccccccc}
\cdots \to & H_q(Y,V) & \to & H_q(X,V) & \to & H_q(X,Y) & \to & H_{q-1}(Y,V) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_{q-2}(X') & \to & H_{q-2}(Z) & \to & H_{q-2}(Z,X') & \to & H_{q-3}(X') & \to \cdots
\end{array}$$

Some words must be said about the down arrows: The first and fourth down arrows are the Leray-Thom-Gysin isomorphism for the pair (V,Y). The second down arrow is the isomorphism (5.6) which is the intersection with Z map, see Figure 5.3.

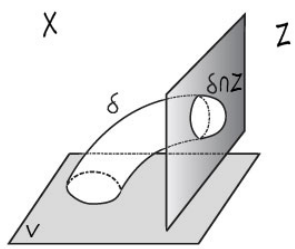

Fig. 5.3 Intersection of a cycle with boundary

The third morphism is obtained by intersecting the cycles with Z, see Exercise 4.10. The above diagram commutes and this follows from the fact that all the down arrow maps are obtained by intersection of topological cycles with the variety used in the image homology. By five lemma, there is an isomorphism

$$H_q(X,Y) \cong H_{q-2}(Z,X'),\ q \neq n, n+1, n+2.$$

Now, the theorem is proved by induction on n. $\square$

Again, we can introduce a consequence of the Hodge conjecture, even though, we have not stated it so far.

Conjecture 5.2 *Let q be an even natural number. For any algebraic cycle $Z = \sum_{i=1}^{r} n_i Z_i,\ n_i \in \mathbb{Z}, \dim(Z_i) = \frac{q}{2},\ \ q < n-1$ in X there are $m \in \mathbb{N}$ and an algebraic cycle $\tilde{Z}$ in Y such that mZ is homologous to $\tilde{Z}$.*

This follows from the Hodge conjecture for the $(2n-q)$-the cohomology of Y with rational coefficients.

5.5 Topology of smooth complete intersections

We start this section by studying the topology of projective spaces:

Proposition 5.1 *For $i \in \mathbb{N}_0$ we have*

$$H_i(\mathbb{P}^n) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \langle [\mathbb{P}^{\frac{i}{2}}] \rangle & \text{if } i \text{ is even} \end{cases}$$

where $\mathbb{P}^{\frac{i}{2}}$ is any linear projective subspace of $\mathbb{P}^n$.

Proof. We take a linear subspace $\mathbb{P}^{n-1} \subset \mathbb{P}^n$, write the long exact sequence of $\mathbb{P}^n \backslash \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and use the equalities

$$H_i(\mathbb{P}^n \backslash \mathbb{P}^{n-1}) = 0,\ i \neq 0.$$

Note that $\mathbb{P}^n \backslash \mathbb{P}^{n-1} \cong \mathbb{C}^n$ can be retracted to a point. We conclude that the maps

$$H_i(\mathbb{P}^n) \to H_{i-2}(\mathbb{P}^{n-1})$$

induced by intersection with $\mathbb{P}^{n-1}$, are isomorphism and $H_1(\mathbb{P}^n) = 0$. Now the proposition follows by induction on n. $\square$

Iterate the sequence $X \supset Y \supset X'$ to

$$X = X_0 \supset X_1 = Y \supset X_2 = X' \supset X_3 \supset \cdots \supset X_n \supset X_{n+1} = \emptyset \tag{5.7}$$

so that X_q is a smooth hyperplane section of X_{q-1} and hence $\dim X_q = n-q$.

Proposition 5.2 *We have*

$$H_q(X, X_i) = 0,\ q \leq \dim(X_i) = n-i.$$

Proof. From theorem 5.2 it follows that

$$H_q(X_i, X_{i+1}) = 0,\ i \leq \dim(X_{i+1}) = n-(i+1). \tag{5.8}$$

Now, we prove the proposition by induction on i. For $i = 1$ it is Theorem 5.2. Assume that $H_q(X, X_i) = 0,\ q \leq \dim(X_i) = n - i$. We write the long exact sequence of the triple $X_{i+1} \subset X_i \subset X$ and use the induction hypothesis and (5.8) to obtain the proposition for $i + 1$. □

Proposition 5.3 *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension n. Then*

$$H_q(\mathbb{P}^{n+1}, X) = 0,\ q \leq n.$$

In particular, we have

$$H_q(X) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ \mathbb{Z} & \text{if } q \text{ is even} \end{cases} \quad \text{for } \ q \leq n-1.$$

Proof. We can use the Veronese embedding of $\mathbb{P}^{n+1}$ such that X becomes a smooth hyperplane section of $\mathbb{P}^{n+1}$. □

We have seen that a hypersurface in $\mathbb{P}^N$ is a smooth hyperplane section.

Definition 5.1 A smooth projective variety $X \subset \mathbb{P}^N$ of dimension n is called a (smooth) complete intersection of type $(d_1, d_2, \cdots, d_{N-n})$ if it is given by $N - n$ homogeneous polynomials $f_1, f_2, \ldots, f_{N-n} \in \mathbb{C}[x_0, x_1, \ldots, x_N]$, $\deg(f_i) = d_i$ such that the matrix

$$[\frac{\partial f_i}{\partial x_j}]_{i=1,2,\ldots,N-n, j=0,1,\ldots,N}$$

has the maximum rank $N - n$ for all points $x \in X$.

Proposition 5.4 *If $X \subset \mathbb{P}^N$ is a smooth complete intersection of dimension n. We have*

$$H_q(\mathbb{P}^N, X) = 0,\ q \leq n.$$

In particular,

$$H_q(X) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ \mathbb{Z} & \text{if } q \text{ is even} \end{cases} \quad \text{for } \ q \leq n-1.$$

Proof. There is a sequence of projective varieties

$$\mathbb{P}^N = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_{N-n} = X \tag{5.9}$$

such that X_i is a hyperplane section (after a proper Veronese embedding) of X_{i-1} for $i = 1, 2 \ldots, N - n$. In fact for $i = 1, 2, \ldots, N - n$, X_i is induced by the zero set of $f_1, f_2, \ldots, f_i$. Now, our assertion follows from Proposition 5.2. □

Let X be a smooth complete intersection in $\mathbb{P}^N$ and Z be a smooth hyperplane section corresponding to $X \subset \mathbb{P}^N$. We call $U := X \backslash Z$ an affine smooth complete intersection.

Proposition 5.5 *Let U be an affine smooth complete intersection variety as above. We have*

1. *For $q \leq n-1$, $H_q(U) = 0$ and so by Corollary 5.1 $H_q(U) = 0$ for all q except for $q = n$.*
2. *The four term exact sequence (5.3) reduces to three term exact sequence of free finitely generated $\mathbb{Z}$-modules.*

Proposition 5.5 is not true for arbitrary affine variety U, see Exercise 5.6.

Proof. Let H be the hyperplane in $\mathbb{P}^N$ such that $Z = H \cap X$. We have a sequence $U = U_{N-n} \subset U_{N-n-1} \subset \cdots \subset U_1 \subset U_0 = \mathbb{C}^N$ which is obtained from (5.9) by removing H from X_i's. We use the isomorphism in (5.2) to each consecutive inclusions of U_i's, we compose them and we get the isomorphism $H_q(U) \cong H_q(\mathbb{C}^N)$ for $q \neq n, n+1, \cdots, N$. In particular, for $0 < q \leq n-1$ we have $H_q(U) = 0$. This is similar to the proof of Corollary 5.2. The second part follows from the first part. □

Proposition 5.6 *Let X be a smooth complete intersection in $\mathbb{P}^N$ of dimension n and let Y be a smooth hyperplane section of X. The intersection map*

$$H_q(X) \to H_{q-2}(Y),\ q \neq n, n+1$$

is an isomorphism.

Proof. The proof follows from the first part of Proposition 5.5, the long exact sequence of (X,U) and the Leray-Thom-Gysin isomorphism. □

Theorem 5.3 *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension n. Then*

$$H_q(X) = \begin{cases} 0 & \text{if } q \text{ is odd, } q \neq n \\ \mathbb{Z} & \text{if } q \text{ is even, } q \neq n \end{cases}$$

and $H_n(X)$ is free.

Proof. For $q < n$ this is Proposition 5.3. For $q > n$ we use Poincaré duality $H^{2n-q}(X) \cong H_q(X)$ and Lefschetz theorem on hyperplane section in cohomology $H^{2n-q}(X) \cong H^{2n-q}(\mathbb{P}^{n+1})$. In order to prove that $H_n(X)$ is free, we consider X as a hyperplane section of another hypersurface Y of dimension $n+1$ (do not confuse this X and Y with those used earlier). We write the long exact sequence of the pair $Y \backslash X \subset Y$ and we have

$$\cdots \to H_{n+2}(Y \backslash X) \to H_{n+2}(Y) \to H_n(X) \to H_{n+1}(Y \backslash X) \to H_{n+1}(Y) \to \cdots.$$

Corollary 5.1 implies that $H_{n+2}(Y \backslash X) = 0$ and $H_{n+1}(Y \backslash X)$ is free. This together with $H_{n+2}(Y) = 0$ or $\mathbb{Z}$ implies the desired statement. □

5.6 Hard Lefschetz theorem

Let us consider the sequence $X_i,\ i = 0, \ldots, n+1$ in (5.7), where X_{i+1} is a transversal hyperplane section of X_i. We do not assume that all of these hyperplane sections are

relative to the same embedding $X \subset \mathbb{P}^N$. We are going to study the hard Lefschetz theorem which is an easy exercise in the case of hypersurfaces (see Exercise 5.5) and a deep non-trivial statement in general.

Theorem 5.4 (Hard Lefschetz theorem) *For every $q = 1, 2, \ldots, n$ the intersection with X_q*

$$H_{n+q}(X, \mathbb{Q}) \to H_{n-q}(X, \mathbb{Q}),\ x \mapsto x \cdot [X_q]$$

is an isomorphism.

First of all note that the hard Lefschetz theorem is stated with rational coefficients. It is false with $\mathbb{Z}$-coefficients for trivial reasons. Take for instance X a smooth curve and Y a hyperplane section which consists of m points with $m > 1$. We use canonical isomorphisms $H_2(X, \mathbb{Z}) \cong \mathbb{Z},\ H_0(X, \mathbb{Z}) \cong \mathbb{Z}$ and obtain the map $\mathbb{Z} \to \mathbb{Z},\ x \mapsto mx$ which is not surjective. For the moment I do not have any counterexample with the non injective map in Theorem 5.4 with $\mathbb{Z}$ coefficients. Since Theorem 5.4 is valid with $\mathbb{Q}$ coefficients, we have to look a counterexample among torsion elements of $H_{n+q}(X, \mathbb{Z})$. For instance, we have to find a three dimensional projective variety X and a torsion $\alpha \in H_4(X, \mathbb{Z})$ with zero intersection with Y (a two dimensional X does not work, see Proposition 5.8). "...a multiple of a very ample class is again very ample and a multiple of an isomorphism will not (usually) be an isomorphism," (P. Deligne, personal communication, January 31, 2016).

There is no topological proof of hard Lefschetz theorem in the literature. This is in some sense natural because it is a theorem with coefficients in $\mathbb{Q}$. The first precise proof available in the literature is due to Hodge by means of harmonic integrals. According to P. Griffiths in [GSW92] this is a case in which Lefschetz's intuition was right on target, but the direct, geometric approach was insufficient to give a complete proof. "...there is a proof of hard Lefschetz theorem using not Hodge theory but rather l-adic cohomology and a semi-simplicity result coming with the Weil conjecture plus Lefschetz theorem [vanishing cycles are conjugated by monodromy]" (P. Deligne, personal communication, January 31, 2016).

We remark that it is enough to prove Theorem 5.4 for $q = 1$. For an arbitrary q the theorem follows from the case $q = 1$, the diagram

$$\begin{array}{ccc} H_{n+q}(X, \mathbb{Q}) & \xrightarrow{\cdot [X_q]} & H_{n-q}(X, \mathbb{Q}) \\ \downarrow & & \uparrow \\ H_{(n-1)+q-1}(X_1, \mathbb{Q}) & \xrightarrow{\cdot [X_q]} & H_{(n-1)-(q-1)}(X_1, \mathbb{Q}) \end{array}$$

and induction on q, where the up arrow is induced by the inclusion and the down arrow is the map of intersection with X_1. Therefore, we use our initial notation $Y = X_1$ and we have to prove that the intersection with the hyperplane section induces an isomorphism $H_{n+1}(X, \mathbb{Q}) \to H_{n-1}(X, \mathbb{Q})$. We write the long exact sequence of the pairs $X \backslash Y \subset X$ and $Y \subset X$ and we have

$$\begin{array}{ccccccccc}
 & & H_n(X,\mathbb{Z}) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & H_n(X,Y,\mathbb{Z}) & & & & & & \\
 & & \downarrow & & & & & & \\
0 \to H_{n+1}(X,\mathbb{Z}) & \xrightarrow{\tau} & H_{n-1}(Y,\mathbb{Z}) & \to & H_n(X\backslash Y,\mathbb{Z}) & \to & H_n(X,\mathbb{Z}) & \to & H_{n-2}(Y,\mathbb{Z}) \to \cdots \\
 & & i\downarrow & & & & & & \\
 & & H_{n-1}(X,\mathbb{Z}) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{5.10}$$

The following proposition is an immediate consequence of the above diagram and the fact that the map $H_{n+1}(X,\mathbb{Q}) \to H_{n-1}(X,\mathbb{Q})$ is the composition $i \circ \tau$.

Proposition 5.7 *The hard Lefschetz theorem and the following statements are equivalent:*

1.

$$H_{n-1}(Y,\mathbb{Q}) = \mathrm{Im}(H_{n+1}(X,\mathbb{Q}) \to H_{n-1}(Y,\mathbb{Q})) \oplus \ker(H_{n-1}(Y,\mathbb{Q}) \to H_{n-1}(X,\mathbb{Q})).$$

2.

$$\mathrm{Im}(H_{n+1}(X,\mathbb{Q}) \to H_{n-1}(Y,\mathbb{Q})) \cap \ker(H_{n-1}(Y,\mathbb{Q}) \to H_{n-1}(X,\mathbb{Q})) = \{0\}.$$

Note that by Poincaré duality $H_{n+1}(X,\mathbb{Q})$ and $H_{n-1}(X,\mathbb{Q})$ have the same dimension. For more equivalent versions of the hard Lefschetz theorem see [Lam81].

Proposition 5.8 *Let Y be a smooth hyperplane section of a projective variety X. The intersection with $[Y]$ induces an isomorphism*

$$H_{n+1}(X)_{\mathrm{tors}} \cong H_{n-1}(Y)_{\mathrm{tors}}$$

In particular, if $\dim(X) = 2$ *then $H_3(X)$ is torsion free.*

Proof. This follows from the horizontal line of the diagram (5.10) and the fact that $H_n(X\backslash Y)$ has no torsion, see Corollary 5.1. □

Remark 5.3 Let $n+q$ be even. If the Hodge conjecture is true for the $(n-q)$-th rational cohomology of X (this contains the Poincaré dual of algebraic cycles in $H_{n+q}(X,\mathbb{Q})$) then it is true for for the $(n+q)$-th rational cohomology of X (this contains the Poincaré dual of algebraic cycles in $H_{n-q}(X,\mathbb{Q})$) but not necessarily vice versa. If the Hodge conjecture is true for $(n+q)$-th rational cohomology of X then the following conjecture is valid.

Conjecture 5.3 *Any algebraic cycle in $H_{n-q}(X,\mathbb{Q})$ is homologous to an intersection of an algebraic cycle in $H_{n+q}(X,\mathbb{Q})$ with Y.*

5.7 Lefschetz decomposition

Let us consider the sequence (5.7). We assume that all hyperplane sections in this sequence are associated to the same embedding $X \subset \mathbb{P}^N$. Each X_q gives us a homology class

$$[X_q] \in H_{2n-2q}(X, \mathbb{Z}).$$

It is left to the reader to check the equalities:

$$[X_q] \cdot [X_{q'}] = [X_{q+q'}]. \tag{5.11}$$

Definition 5.2 An element $x \in H_{n+q}(X, \mathbb{Z}),\ q = 0, 1, 2, \ldots, n$ (resp. $x \in H_{n-q}(X, \mathbb{Z})$) is called primitive if

$$[X_{q+1}] \cdot x = 0$$

(resp. $[X_1] \cdot x = 0$).

According to Lamotke in [Lam81] page 32 the term 'primitive' is due to A. Weil (1958). Recall that by hard Lefschetz theorem if $[X_q] \cdot x = 0$ then x is a torsion element of $H_{n+q}(X, \mathbb{Z})$.

Definition 5.3 *The primitive homology groups are defined in the following way*

$$H_{n+q}(X, \mathbb{Q})_0 := \left\{ x \in H_{n+q}(X, \mathbb{Q}) \mid [X_{q+1}] \cdot x = 0 \right\},$$

$$H_{n-q}(X, \mathbb{Q})_0 := \left\{ x \in H_{n-q}(X, \mathbb{Q}) \mid [X_1] \cdot x = 0 \right\}.$$

Theorem 5.5 (Lefschetz decomposition in homology) *Any element $x \in H_{n+q}(X, \mathbb{Q})$ can be uniquely written as a sum*

$$x = x_0 + [X_1] \cdot x_1 + [X_2] \cdot x_2 + \cdots \tag{5.12}$$

and any element $x \in H_{n-q}(X, \mathbb{Q})$ as

$$x = [X_q] \cdot x_0 + [X_{q+1}] \cdot x_1 + [X_{q+2}] \cdot x_2 + \cdots, \tag{5.13}$$

where $x_i \in H_{n+q+2i}(X, \mathbb{Q})$ are primitive and $q \geq 0$.

In other words, we have the decomposition of $H_{n+q}(X, \mathbb{Q})$ and $H_{n-q}(X, \mathbb{Q})$ into primitive parts:

$$H_{n+q}(X, \mathbb{Q}) = H_{n+q}(X, \mathbb{Q})_0 \oplus [X_1] \cdot H_{n+q+2}(X, \mathbb{Q})_0 \oplus \cdots \oplus [X_i] \cdot H_{n+q+2i}(X, \mathbb{Q})_0 \oplus \cdots, \tag{5.14}$$

$$H_{n-q}(X, \mathbb{Q}) = H_{n-q}(X, \mathbb{Q})_0 \oplus [X_{q+1}] \cdot H_{n+q+2}(X, \mathbb{Q})_0 \oplus \cdots \oplus [X_{q+i}] \cdot H_{n+q+2i}(X, \mathbb{Q})_0 \oplus \cdots. \tag{5.15}$$

Here, i runs from 0 to $\left[\frac{n-q}{2}\right]$.

Proof. We use the fact that intersection with $[X_q]$ induces an isomorphism $H_{n+q}(X) \to H_{n-q}(X)$ which transforms (5.12) into (5.13). Therefore, it is enough to prove (5.12).

For this we use decreasing induction on q starting from $q = n, n-1$, where every element is primitive. For the induction step from $n+q+2$ to $n+q$ it suffices to show that every $x \in H_{n+q}(X)$ can be written uniquely as

$$x = x_0 + [X_1] \cdot y,\ x_0 \text{ primitive} \tag{5.16}$$

because the induction hypothesis applied to y yields the decomposition (5.12). In order to prove (5.16) consider $[X_{q+1}] \cdot x$. According to hard Lefschetz theorem (Theorem 5.4) there is exactly one $y \in H_{n+q+2}(X)$ with $[X_{q+2}] \cdot y = [X_{q+1}] \cdot x$ and thus

$$x_0 := x - [X_1] \cdot y$$

is primitive. In order to show the uniqueness assume that $0 = x_0 + [X_1] \cdot y$ with x_0 primitive. Then $[X_{q+1}] \cdot x_0 = 0$, hence $[X_{q+2}] \cdot y = 0$ and Theorem 5.4 implies $y = 0$, and hence $x_0 = 0$. $\square$

5.8 Lefschetz theorems in cohomology

Let $u \in H^2(X, \mathbb{Z})$ denote the Poincaré dual of of the algebraic cycle $[Y] \in H_{2n-2}(X, \mathbb{Z})$, i.e.

$$u \cap [X] = [Y].$$

Let also

$$u^q = \underbrace{u \cup u \cup \cdots \cup u}_{q\text{-times}} \in H^{2q}(X, \mathbb{Z})$$

which is the Poincaré dual of X_q. We have

$$u^q \cap x = [X_q] \cdot x$$

and so Theorem 5.4 says that for every $q = 1, 2, \ldots, n$ the cap product with the q-th power u^q

$$H_{n+q}(X, \mathbb{Q}) \to H_{n-q}(X, \mathbb{Q}),\ \alpha \mapsto u^q \cap \alpha$$

is an isomorphism. We use the Poincaré duality and the hard Lefschetz theorem in cohomology turns out to be the following:

Theorem 5.6 (Hard Lefschetz theorem) *Let $u \in H^2(X, \mathbb{Z})$ be the Poincaré dual of a hyperplane section of a smooth projective variety X. For $q = 1, 2, \ldots, n$ the cup product with the q-th power of u*

$$L_q : H^{n-q}(X, \mathbb{Q}) \to H^{n+q}(X, \mathbb{Q}),$$

$$\alpha \mapsto u^q \cup \alpha.$$

is an isomorphism.

Define the primitive cohomology in the following way:

$$H^{n-q}(X,\mathbb{Q})_0 := \ker\left(L^{q+1} : H^{n-q}(X,\mathbb{Q}) \to H^{n+q+2}(X,\mathbb{Q})\right).$$

The Poincaré dual to the decomposition in Theorem 5.5 is:

Theorem 5.7 (Lefschetz decomposition) *The natural map*

$$\oplus_q L^q : \oplus_q H^{m-2q}(X,\mathbb{Q})_0 \to H^m(X,\mathbb{Q})$$

is an isomorphism.

5.9 Intersection form

A smooth projective variety $X \subset \mathbb{P}^N$ of dimension n as a topological space is compact and oriented, and hence, by Poincaré duality theorem the intersection bilinear form

$$\langle \cdot,\cdot \rangle : \ H_{n+q}(X,\mathbb{Z}) \times H_{n-q}(X,\mathbb{Z}) \to \mathbb{Z} \tag{5.17}$$

is unimodular (non-degenerate).

Proposition 5.9 *The intersection form 5.17 with respect to the decompositions* (5.14) *and* (5.15) *is orthogonal, that is,*

$$\left\langle [X_i]\cdot H_{n+q+2i}(X,\mathbb{Q})_0,\ \ [X_{q+j}]\cdot H_{n+q+2j}(X,\mathbb{Q})_0 \right\rangle = 0, \quad \textit{for } i \neq j.$$

Moreover, the induced bilinear form $H_{n+q}(X,\mathbb{Q})_0 \times H_{n-q}(X,\mathbb{Q})_0 \to \mathbb{Q}$ *is non-degenerate.*

Proof. The intersection of an element $[X_i]\cdot\alpha,\ \alpha \in H_{n+q+2i}(X,\mathbb{Q})_0$ with an element $[X_{q+j}]\cdot\beta,\ \beta \in H_{n+q+2j}(X,\mathbb{Q})_0$ is of the form $[X_{q+i+j}]\cdot\alpha\cdot\beta$. If $i > j$ then $i+j > 2j$ and so by definition of a primitive element $[X_{q+2j+1}]\cdot\beta = 0$, and hence, $[X_{q+i+j}]\cdot\beta = 0$. This proves the first part of the proposition. The second part follows from the fact that (5.17) is non-degenerate. □

5.10 Cycles at infinity and torsions

The main protagonists of the present section are cycles at infinity and torsion elements of homologies. "One of the most impressive features of his [E. Picard] celebrated treatise is the way in which, by such primitive and indirect means, he did succeed in obtaining a deep understanding of the topological nature of an algebraic surface, though naturally, owing to the use of transcendental methods, some of the finer aspects of the topology of an algebraic surface, such as torsion, were lost." (W.V.D Hodge in [Hod57] page 4). Our main interest on torsion cycles arises from the fact that in the literature there is no analogous of the Hodge conjecture for torsion cycles, that is, for a torsion element of the homologies of a smooth projective

variety, we do not have any criterion, even conjectural, which tell us whether it is the support of an algebraic cycle or not.

Let us assume that X is a smooth projective variety in $\mathbb{P}^N$ and let $Y = X \cap \mathbb{P}^{N-1}$ be a smooth hyperplane section of X. We will regard $\mathbb{P}^{N-1}$ as the projective space at infinity and $\mathbb{C}^N = \mathbb{P}^N \backslash \mathbb{P}^{N-1}$. Using Corollary 5.1, the long exact sequence (4.7) turns out to be

$$0 \to H_{n+1}(X,\mathbb{Z}) \to H_{n-1}(Y,\mathbb{Z}) \xrightarrow{\sigma} H_n(X-Y,\mathbb{Z}) \xrightarrow{i} H_n(X,\mathbb{Z}) \to H_{n-2}(Y,\mathbb{Z}) \to \cdots \tag{5.18}$$

Definition 5.4 A cycle $\delta \in H_n(X-Y,\mathbb{Z})$ is called a cycle at infinity if it is in the kernel of the map i in (5.18) or equivalently in the image of σ. We denote by

$$\begin{aligned} H_n(X-Y,\mathbb{Z})_\infty &:= \ker(H_n(X-Y,\mathbb{Z}) \xrightarrow{i} H_n(X,\mathbb{Z})) \\ &= \operatorname{Im}(H_{n-1}(Y,\mathbb{Z}) \xrightarrow{\sigma} H_n(X-Y,\mathbb{Z})) \end{aligned}$$

the $\mathbb{Z}$-modules of cycles at infinity.

For a visualization of a cycle at infinity see Figure 4.6. Recall that by definition of the primitive cohomology $H_n(X,\mathbb{Z})_0$, it contains all elements $\delta \in H_n(X,\mathbb{Z})$ such that the intersection $\delta \cdot [Y] \in H_{n-2}(X,\mathbb{Z})$ is zero. Here, $[Y] \in H_{2n-2}(X,\mathbb{Z})$ is the homology class of Y. By Lefschetz hyperplane theorem the inclusion $Y \subset X$ induces an isomorphism $H_{n-2}(Y,\mathbb{Z}) \to H_{n-2}(X,\mathbb{Z})$, and so, we can define the primitive cohomology in the following way:

$$H_n(X,\mathbb{Z})_0 := \ker(H_n(X,\mathbb{Z}) \to H_{n-2}(Y,\mathbb{Z})).$$

From the long exact sequence (5.18) we derive the natural isomorphism:

$$H_n(X,\mathbb{Z})_0 \cong \frac{H_n(X-Y,\mathbb{Z})}{H_n(X-Y,\mathbb{Z})_\infty}.$$

Proposition 5.10 *Let $\delta \in H_n(X-Y,\mathbb{Z})$. The followings are equivalent:*

1. *There is a non-zero integer m such that $m\delta$ is a cycle at infinity.*
2. $$\langle \delta, \tilde{\delta} \rangle = 0, \quad \forall \tilde{\delta} \in H_n(X-Y,\mathbb{Z}). \tag{5.19}$$
3. $$\int_\delta \omega = 0, \ \ \forall \omega \in \operatorname{Im}\left(H^n(X,\mathbb{Z}) \to H^n(X-Y,\mathbb{Z})\right), \tag{5.20}$$

where inside Im *we have the restriction map.*

Proof. $1 \to 2$. By definition of singular homology, the union of the support of a basis of $H_n(X-Y,\mathbb{Z})$ is compact in $X-Y$, let us call it K, and so, there is a neighborhood U of Y in X such that U and K do not intersect each other. By definition of the map $H_{n-1}(Y,\mathbb{Z}) \xrightarrow{\sigma} H_n(X-Y,\mathbb{Z}))$ in §4.6, the image of a basis of $H_{n-1}(Y,\mathbb{Z})$ has support

in any small neighborhood of Y in X. Therefore, $m\delta$, and hence δ, does not intersect any cycle in $H_n(X-Y,\mathbb{Z})$.

$2 \to 1$. This follows from the fact that the intersection form in $H_n(X,\mathbb{Z})_0$ is non-degenerate, see the second part of Proposition 5.9. If we have a cycle δ with the property (5.19) then under the map induced by the inclusion $X-Y \subset X$, we get a cycle in $H_n(X,\mathbb{Z})_0$ which does not intersect any cycle in $H_n(X,\mathbb{Z})_0$, and so, it is a torsion cycle.

$2 \leftrightarrow 3$. This easily follows from the fact that over rational numbers cohomology is dual to homology. $\square$

Definition 5.5 We define

$$H_n(X-Y,\mathbb{Z})_{\breve{\infty}} := \{\delta \in H_n(X-Y,\mathbb{Z}) \mid \exists m \in \mathbb{Z},\ m\delta \in H_n(X-Y,\mathbb{Z})_\infty\}$$

to be the set of cycles $\delta \in H_n(X-Y,\mathbb{Z})$ satisfying the equivalent conditions in Proposition 5.10.

Proposition 5.11 *We have*

$$(H_n(X,\mathbb{Z})_0)_{\text{tors}} \cong \frac{H_n(X-Y,\mathbb{Z})_{\breve{\infty}}}{H_n(X-Y,\mathbb{Z})_\infty}, \tag{5.21}$$

$$(H_n(X,\mathbb{Z})_0)_{\text{free}} \cong \frac{H_n(X-Y,\mathbb{Z})}{H_n(X-Y,\mathbb{Z})_{\breve{\infty}}}. \tag{5.22}$$

Moreover, in the first equality, we can remove 0 *in the left hand side if* $H_{n-2}(Y,\mathbb{Z})$ *is torsion free, that is,*

$$H_n(X,\mathbb{Z})_{\text{tors}} \cong \frac{H_n(X-Y,\mathbb{Z})_{\breve{\infty}}}{H_n(X-Y,\mathbb{Z})_\infty}. \tag{5.23}$$

Proof. The proposition follows from the exact sequence (5.18).

5.11 Exercises

5.1. For the following exercise the reader might consult §1 of Lamotke's article [Lam81]. Let $U \subset \mathbb{C}^N$ be an affine variety given by the zero set of finitely many polynomials in $x_1, x_2, \cdots, x_N$. A point $p \in U$ is smooth if there is a polynomial map

$$f = (p_1, p_2, \ldots, p_s) : \mathbb{C}^N \to \mathbb{C}^s,$$

where $p_1, p_2, \cdots p_n \in \mathbb{C}[x_1, \cdots, x_N]$ and $s \leq N$, such that in a Zariski open neighborhood of p in $\mathbb{C}^N$ the differential of f has the maximum rank s and U is the inverse image of 0 by the map f. We say that s (resp. $N-s$) is the codimension (resp. dimension) of U near p. A hyperplane $H : g = 0,\ g := \sum_{i=1}^N a_i x_i - a_0,\ a_i \in \mathbb{C}$ intersects U transversally at p if $g(p) = 0$ and the differential of (f,g) has the maximal rank $s-1$. For the projective variety $X \subset \mathbb{P}^N$ we take affine charts and we generalize the

above definitions to this case. Let $\check{\mathbb{P}}^N$ be the projective space of all hyperplanes in $\mathbb{P}^N$.

1. Show that all hyperplanes of $\mathbb{P}^N$ which do not intersect X transversally at some point form a closed irreducible subvariety $\check{X}$ of $\check{\mathbb{P}}^N$.
2. For a hypersurface $X \subset \mathbb{P}^{n+1}$ given by the homogeneous polynomial f in $x_0, x_1, \cdots, x_{n+1}$ show that $\check{X}$ is given by the image of X under the *Gauss map*:

$$\mathbb{P}^N \dashrightarrow \check{\mathbb{P}}^N, x \mapsto [\frac{\partial f}{\partial x_0} : \frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_{n+1}}] \tag{5.24}$$

3. Find generators for the ideal sheaf of $\check{X}$ for the Fermat variety $X = X_n^d$.

5.2. State and prove Corollary 5.1, Corollary 5.2 and Theorem 5.2 for cohomologies with rational coefficients.

5.3. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of type $(d_1, d_2, \ldots, d_{N-n})$. Show that $[X] \in H_{2n}(\mathbb{P}^N)$ is $d_1 \cdot d_2 \cdots d_{N-n}$ times the homology class $[\mathbb{P}^n]$ of the linear projective space $\mathbb{P}^n \subset \mathbb{P}^N$:

$$[X] = d_1 \cdot d_2 \cdots d_{N-n}[\mathbb{P}^n].$$

5.4. Using Proposition 5.1 and Proposition 5.4, we know that for a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ and $q < n$ an even number $H_q(X) \cong H_q(\mathbb{P}^{n+1})$ and $H_q(\mathbb{P}^{n+1})$ is generated by the homology class of a linear projective subspace $\mathbb{P}^{\frac{q}{2}} \subset \mathbb{P}^{n+1}$. However, there might be no such $\mathbb{P}^{\frac{q}{2}}$ inside X. This is the first indication that the integral Hodge conjecture is wrong. A canonical element $[X \cap \mathbb{P}^{\frac{q}{2}+1}] \in H_q(X)$ is obtained by a transverse intersection of a linear space $\mathbb{P}^{\frac{q}{2}+1} \subset \mathbb{P}^{n+1}$ with X.

1. Show that in $H_q(X)$ we have

$$[X \cap \mathbb{P}^{\frac{q}{2}+1}] = d \cdot [\mathbb{P}^{\frac{q}{2}}]$$

2. ** Let us take a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2$. For instance, take the coefficients of the defining equation of X all transcendental numbers algebraically independent from each other. Fix a generator δ of $H_q(X)$, $q < n$. Classify the cycles

$$i \cdot \delta, \ \ i = 1, 2, \ldots, d-1$$

which are not the homology classes of algebraic subvarieties of X of dimension $\frac{q}{2}$, for some results in this direction see [Kol92, Tot13]. For $n = 3$ and $d \geq 6$ Griffiths and Harris in [GH85] conjectured that every curve in a generic hypersurface $X \subset \mathbb{P}^4$ of degree d has degree a multiple of d.
3. Show that for the Fermat variety X, all the cycles in the previous item are homology classes of algebraic subvarieties of X.
4. * A generic quintic threefold $X \subset \mathbb{P}^4$ has 2875 lines $\mathbb{P}^1$. This is the beginning of a very beautiful interaction between Physics and Mathematics, see [CK99].

5.5. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and

$$H_{n+q}(X,\mathbb{Z}) \to H_{n-q}(X,\mathbb{Z}),\ x \mapsto x \cdot [X_q] \tag{5.25}$$

be the map which for a $(n+q)$-dimensional cycle it associates its intersection with X_q. By Lefschetz hyperplane section theorem we know that $H_{n-q}(X,\mathbb{Z}) \cong \mathbb{Z}$. The same theorem in cohomology and Poincaré duality implies that $H_{n+q}(X,\mathbb{Z}) \cong \mathbb{Z}$. Combining these two, (5.25) is $\mathbb{Z} \to \mathbb{Z},\ \ x \mapsto a \cdot x$, for some $a \in \mathbb{Z}$. Compute the number a.

5.6. Show that for the complement of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ and $q \leq n$ we have

$$H_q(\mathbb{P}^{n+1} \backslash X) = \begin{cases} 0 & \text{if } q \text{ is even} \\ \frac{\mathbb{Z}}{d\mathbb{Z}} & \text{if } q \text{ is odd} \end{cases}.$$

This disproves the first part of Proposition 5.5 in the case of varieties which are not affine smooth complete intersections. Hint: Use a Gysin sequence. Why $\mathbb{P}^{n+1} \backslash X$ is not an affine complete intersection?

5.7. For $n,d \geq 2$ prove that the image of the Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$ constructed by monomials of degree d is not a complete intersection. Hint: use Exercise 5.6. How about the case $n = 1$?

5.8. * Let δ_β's be the topological cycles introduced in Exercise 4.2. For $\beta, \beta' \in I$, we have

$$\langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^n \langle \delta_{\beta'}, \delta_\beta \rangle,$$

and if $\beta \neq \beta'$ and $\beta_k \leq \beta'_k \leq \beta_k + 1,\ k = 1,2,\ldots n+1,\ \beta \neq \beta'$ then

$$\langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^{\frac{n(n+1)}{2}} (-1)^{\Sigma_{k=1}^{n+1} \beta'_k - \beta_k}.$$

We have also

$$\langle \delta_\beta, \delta_\beta \rangle = (-1)^{\frac{n(n+1)}{2}} (1 + (-1)^n),\ \beta \in I.$$

In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle \delta_\beta, \delta_{\beta'} \rangle = 0$ (source [Mov11] page 111 and [AGZV88] page 66). A method to compute intersection number of topological cycles will be introduced in Chapter 7.

5.9. * For examples of n and $m_1, m_2, \ldots, m_{n+1}$, describe the $\mathbb{Z}$-modules $H_n(U,\mathbb{Z})_\infty$ and $H_n(U,\mathbb{Z})_{\breve{\infty}}$. In particular, try to find some cases where these two $\mathbb{Z}$-modules are not the same. Try the cases $n = 2$, $m_1 = m_2 = m_3 = 3,4$.

Chapter 6
Picard-Lefschetz theory

...suppose I have an algebraic variety, and hyperplane sections, and I want to understand how they are related, by looking at a pencil of hyperplane sections. The picture is very simple. I draw it in my mind something like a circle in the plane and a moving line that sweeps it. Then I know how this picture is false: the variety is not one-dimensional, but higher-dimensional, and when the hyperplane section degenerates, it is not just two intersection points coming together. The local picture is more complicated, like a conic that becomes a quadratic cone, (P. Deligne in [RS14] page 184).

6.1 Introduction

"In the classical mémoire of Brill-Noether (Math. Ann., 1874), the foundations of "geometry on an algebraic curve" were laid down centered upon the study of linear series cut out by linear systems of curves upon a fixed curve $f(x,y) = 0$.... The next step in the same direction was taken by Castelnuovo (1892) and Enriques (1893). They applied analogous methods to the creation of an entirely new theory of algebraic surfaces. Their basic instrument was the study of linear systems of curves on a surface." (Solomon Lefschetz in [Lef68] page 855). These quotations from the autobiography of Lefschetz contain the early history of pencils in Algebraic Geometry. Lefschetz in his 1924 book [Lef24b] generalized this to a pencil of hyperplanes in general position with respect to a given variety in order to study its topology. He was inspired by the works of Picard and Poincaré [PS06, Poi02] in which they systematically analyzed algebraic surfaces fibered by algebraic curves. In this way he founded the so-called Picard-Lefschetz theory. In this chapter we introduce basic concepts of Picard-Lefschetz theory and we prove Theorem 5.1 in Chapter 5. Concrete examples and computations in this theory will be presented in Chapter 7, and hence, it might be useful to read both the present chapter and Chapter 7 simultaneously.

6.2 Ehresmann's fibration theorem

Many of the Lefschetz intuitive arguments are made precise by appearance of a fiber bundle which is the basic stone of the so-called Picard-Lefschetz theory. Despite the fact that the theorem below is proved two decades after Lefschetz treatise, it is the starting point of Picard-Lefschetz theory.

Theorem 6.1 *(Ehresmann's Fibration Theorem [Ehr47]). Let $f : Y \to B$ be a proper submersion between the C^∞ manifolds Y and B. Then f fibers Y locally trivially i.e., for every point $b \in B$ there are a neighborhood U of b and a C^∞-diffeomorphism $\phi : U \times f^{-1}(b) \to f^{-1}(U)$ such that*

$$f \circ \phi = \pi_1 = \text{ the first projection.}$$

Moreover, if $N \subset Y$ is a closed submanifold such that $f|_N$ is still a submersion then f fibers Y locally trivially over N i.e., the diffeomorphism ϕ above can be chosen to carry $U \times (f^{-1}(b) \cap N)$ onto $f^{-1}(U) \cap N$.

The map ϕ is called the fiber bundle trivialization map. Ehresmann's theorem can be rewritten for manifolds with boundary. For this we modify the second part of the theorem, saying that N is the boundary of Y. One needs this version of Ehresmann's fibration theorem in the local Picard-Lefschetz theory developed for singularities, see [AGZV88] and §6.5. Ehresmann's fibration theorem for stratified spaces is also known as the Thom-Mather theorem.

In Theorem 6.1 assume that f is not a submersion. Because of this, we define C' to be the union of critical values of f and critical values of $f|_N$, and C be the closure of C' in B. By a critical point of the map f we mean the point in which f is not submersion. Now, we can apply the theorem to the function

$$f : Y \backslash f^{-1}(C) \to B \backslash C = B'.$$

For any set $K \subset B$, we use the following notations:

$$Y_K = f^{-1}(K),\ Y'_K = Y_K \cap N,\ L_K = Y_K \backslash Y'_K$$

and for any point $c \in B$, by Y_c we mean the set $Y_{\{c\}}$. By $f : (Y,N) \to B$ we mean the map f together with $f|_N$. It is called the critical fiber bundle map.

Definition 6.1 Let $A \subset R \subset S$ be topological spaces. R is called a strong deformation retract of S over A if there is a continuous map $r : [0,1] \times S \to S$ such that

1. $r(0,\cdot) = \mathrm{id}$,
2. $\forall x \in S,\ r(1,x) \in R$ and $\forall x \in R,\ r(1,x) = x$,
3. $\forall t \in [0,1],\ x \in A,\ r(t,x) = x$.

See Figure 6.1. Here, r is called the contraction map.

We use the following theorem to define a generalized vanishing cycle and also to find relations between the homology groups of $Y \backslash N$ and the generic fiber L_c of f.

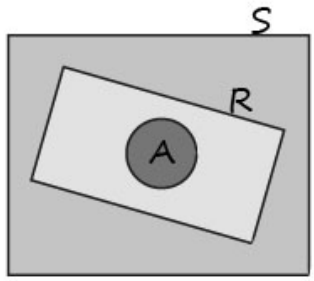

Fig. 6.1 Strong deformation retract

Theorem 6.2 *Let $f : Y \to B$ and C' as before, $A \subset R \subset S \subset B$ and $S \cap C$ be a subset of the interior of A in S. Every retraction from S to R over A can be lifted to a retraction from L_S to L_R over L_A.*

Proof. According to Ehresmann's fibration theorem $f : L_{S\backslash C} \to S\backslash C$ is a C^∞ locally trivial fiber bundle. The homotopy covering theorem, see [Ste51] §11.3, implies that the contraction of $S\backslash C$ to $R\backslash C$ over $A\backslash C$ can be lifted so that $L_{R\backslash C}$ becomes a strong deformation retract of $L_{S\backslash C}$ over $L_{A\backslash C}$. Since $C \cap S$ is a subset of the interior of A in S, the singular fibers can be filled in such a way that L_R is a deformation retract of L_S over L_A. $\square$

Under the hypothesis of theorem 6.2, and for any subset $K \subset A$, we have the isomorphism induced by contraction

$$H_k(L_{S\backslash C}, L_K) \cong H_k(L_{R\backslash C}, L_K), \quad k = 0, 1, 2, \ldots$$

6.3 Monodromy

Let λ be a path in $B' = B\backslash C$ with the initial and end points b_0 and b_1. In the sequel by λ we will mean both the path $\lambda : [0,1] \to B$ and the image of λ; the meaning being clear from the text.

Proposition 6.1 *There is an isotopy*

$$H : L_{b_0} \times [0,1] \to L_\lambda$$

such that for all $x \in L_{b_0}$, $t \in [0,1]$ and $y \in N$

$$H(x,0) = x,\ H(x,t) \in L_{\lambda(t)},\ H(y,t) \in N. \tag{6.1}$$

For every $t \in [0,1]$ the map $h_t = H(.,t)$ is a diffeomorphism between L_{b_0} and $L_{\lambda(t)}$. The different choices of H and paths homotopic to λ would give a class of homotopic maps

$$\{h_\lambda : L_{b_0} \to L_{b_1}\},$$

where $h_\lambda = H(\cdot,1)$.

Proof. The interval $[0,1]$ is compact and the local trivializations of L_λ can be fitted together along γ to yield an isotopy H. □

We can state a stronger version of Proposition 6.1 in which we use fibers (Y_K, Y'_K) instead of $L_K := Y_K \backslash Y'_K$. See Figure 6.2.

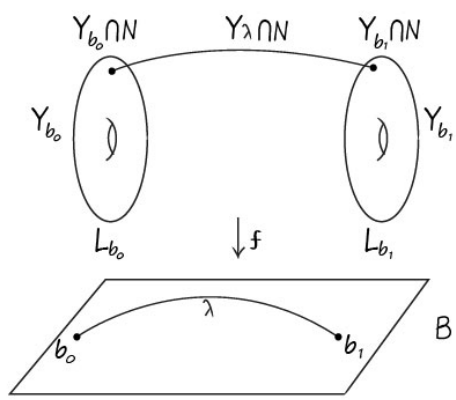

Fig. 6.2 Monodromy along a path

All the choices of isotopies in Proposition 6.1 define collection of homotopic maps $h_\lambda : L_{b_0} \to L_{b_1}$, and so, we have a unique well-defined map

$$h_\lambda : H_*(L_{b_0}) \to H_*(L_{b_1}). \tag{6.2}$$

We will only need to consider the homology class of cycles, however, many of the arguments can be rewritten for their homotopy classes.

Definition 6.2 For any regular value b of f, we can define

$$h : \pi_1(B', b) \times H_*(L_b) \to H_*(L_b),$$

$$h(\lambda, \cdot) = h_\lambda(\cdot).$$

The image of $\pi_1(B',b)$ in $\mathrm{Aut}(H_*(L_b))$ is called the monodromy group and its action h on $H_*(L_b)$ is called the action of monodromy on the homology groups of L_b. See Figure 6.3.

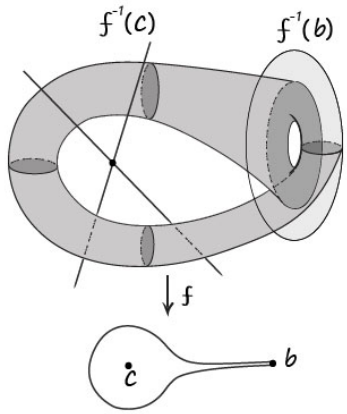

Fig. 6.3 The footprints of a cycle after monodromy

Following the article [Ché96], we give the generalized definition of a vanishing cycle.

Definition 6.3 Let K be a subset of B and b be a point in $K\backslash C$. Any relative k-cycle $\Delta \in H_k(L_K, L_b)$ of L_K with a boundary in L_b is called a k-thimble above (K,b) and its boundary $\delta \in H_{k-1}(L_b)$ is called a vanishing $(k-1)$-cycle above K.

6.4 Vanishing cycles

What we studied in the previous section is developed first in the complex context. Let Y be a complex compact manifold, N be a submanifold of Y of codimension one, $B = \mathbb{P}^1$ and f be a holomorphic function. The set C of critical values of f is finite and so each point in C is isolated in $\mathbb{P}^1$. We write

$$C := \{c_1, c_2, \ldots, c_s\}.$$

Let $c_i \in C$, D_i be a small closed disk around c_i and $\tilde{\lambda}_i$ be a path in $B' := \mathbb{P}^1\backslash C$ which connects $b \in B'$ to b_i in the boundary of D_i. Let also λ_i be the path $\tilde{\lambda}_i$ plus the path which connects b_i to c_i in D_i (see Figure 6.4). Define the set K in the three ways as

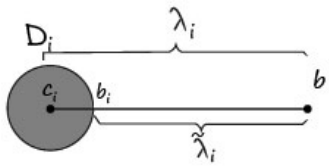

Fig. 6.4 From a regular to a singular value

follows:

$$K^s = \begin{cases} \lambda_i & s=1 \\ \lambda_i \cup D_i & s=2 \\ \tilde{\lambda}_i \cup \partial D_i & s=3 \end{cases}. \tag{6.3}$$

We can now define the vanishing cycle in L_b above K^s for $s = 1,2,3$. Since K^1 and K^3 are subsets of K^2, the vanishing cycle above K^1 or K^3 is also vanishing above K^2. In the case of K^1, we have the intuitional concept of a vanishing cycle. If c_i is a critical point of $f\,|_N$ we can see that the vanishing cycle above K^2 may not be vanishing above K^1, see Exercise 6.1. The case K^3 gives us the vanishing cycles obtained by a monodromy around c_i. In this case we have the Wang isomorphism

$$v : H_{k-1}(L_b) \tilde{\to} H_k(L_{K^3}, L_b)$$

see [Ché91]. Roughly speaking, the image of the cycle α by v is the footprint of α, taking the monodromy around c_i. Let γ_i be the closed path which parametrizes K^3, that is, γ_i starts from b, goes along $\tilde{\lambda}_i$ until b_i, turns around c_i anticlockwise on ∂D_i and finally comes back to b along $\tilde{\lambda}_i^{-1}$. Let also $h_{\gamma_i} : H_k(L_{b_i}) \to H_k(L_{b_i})$ be the monodromy around the critical value c_i. We have

$$\partial \circ v = h_{\lambda_i} - \mathrm{id},$$

where ∂ is the boundary operator. Therefore the cycle α is a vanishing cycle above K^3 if and only if it is in the image of $h_{\lambda_i} - \mathrm{id}$. For more information about the generalized vanishing cycle the reader is referred to [Ché96]. In general by vanishing along the path λ_i we mean vanishing above K^2.

6.5 The case of isolated singularities

First, let us recall some definitions from local theory of vanishing cycles. For all missing proofs the reader is referred to [AGZV88]. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at $0 \in \mathbb{C}^n$. For the proof of

Theorem 5.1 we will only need the simple case:

$$f(x_1, x_2, \cdots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2. \tag{6.4}$$

We can take an small closed disc D with center $c := 0 \in \mathbb{C}$ and a closed ball $\mathbb{B}^{2n}$ with center $0 \in \mathbb{C}^n$ such that

$$f : (f^{-1}(D) \cap \mathbb{B}^{2n}, f^{-1}(D) \cap \partial\mathbb{B}^{2n}) \to D$$

is a C^∞ fiber bundle over $D \backslash \{0\}$. Restricted to the boundary $\partial\mathbb{B}^{2n}$, f is a fiber bundle over D itself. Here, fibers are real manifolds with boundaries which lie in $\partial\mathbb{B}^{2n}$. This is sometimes called Milnor fibration. In what follows we consider the mentioned domain and image for f, see Figure 6.5 The Milnor number of f is defined to be

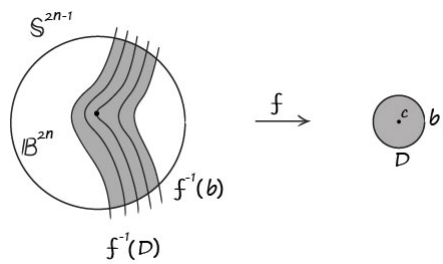

Fig. 6.5 Milnor fibration of a singularity

$$\mu = \mu(f, 0) := \dim\left(\frac{\mathscr{O}_{\mathbb{C}^n,0}}{\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \cdots, \frac{df}{dx_n} \rangle}\right),$$

where $(x_1, x_2, \cdots, x_n)$ is a local coordinate system around 0 and $\mathscr{O}_{\mathbb{C}^n,0}$ is the ring of holomorphic functions in a neighborhood of 0 in $\mathbb{C}^n$.

Proposition 6.2 *For $b \in \partial D$ the relative homology group $H_k(f^{-1}(D), f^{-1}(b))$ is zero for $k \neq n$ and it is a free $\mathbb{Z}$-module of rank μ, where μ is the Milnor number of f. A basis of $H_n(f^{-1}(D), f^{-1}(b))$ is given by hemispherical homology classes.*

By definition a hemispherical homology class is the image of a generator of infinite cyclic group $H_n(\mathbb{B}^n, \mathbb{S}^{n-1}) \cong \mathbb{Z}$, where

$$\mathbb{S}_t^{n-1} := \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum x_j^2 = t\right\},\ 0 \leq t \leq 1,\ \mathbb{S}^{n-1} := \mathbb{S}_1^{n-1}$$

and

$$\mathbb{B}^n := \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum x_j^2 \leq 1\right\} = \cup_{t \in [0,1]} \mathbb{S}_t^{n-1},$$

under the homeomorphism induced by a continuous mapping of the closed n-ball $\mathbb{B}^n$ into $f^{-1}(\lambda)$ which sends the $(n-1)$-sphere $\mathbb{S}_t^{n-1}$, to $L_{\lambda(t)}$. Here $\lambda : [0,1] \to D$ is a straight path which connects 0 to b, see Figure 6.6.

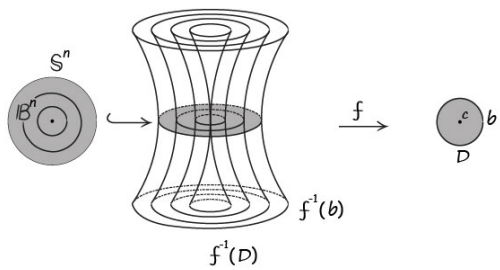

Fig. 6.6 Lefschetz thimble

Definition 6.4 We denote the image of $\mathbb{B}^n$ in $f^{-1}(D)$ by Δ and call it a Lefschetz thimble (of the singularity 0). Its boundary is denoted by δ and it is called a Lefschetz vanishing cycle in the singularity 0.

Remark 6.1 For the non-degenerate singularity (6.4), the Milnor number of f is one and $H_n(f^{-1}(D), f^{-1}(b))$, $b = 1$, is generated by the image of the generator of $H_n(\mathbb{B}^n, \mathbb{S}^{n-1})$ under the canonical inclusion $\mathbb{R}^n \subset \mathbb{C}^n$. In general, we choose a basis in Proposition (6.2) in the following way. There are μ paths λ_i in D with the following properties. The paths λ_i connect 0 to b in D and they do not intersect each other except at their start/end points, and at 0 they intersect each other transversally. Further, $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ near 0 is the clockwise direction. There exists a versal deformation $\tilde{f}$ of f having μ non-degenerate critical points with distinct values, let us say $c_1, c_2, \ldots, c_\mu$, such that the deformed path $\tilde{\lambda}_i$ connects b to c_i, and $\tilde{\lambda}_i$'s do not intersect each other except at b. Along each $\tilde{\lambda}_i$, and hence λ_i, there is a unique (up to sign) vanishing cycle in $f^{-1}(b)$. For an example of this see Exercise 6.5.

6.6 Picard-Lefschetz formula

Recall the notations in §6.4 related to the map $f : Y \to B$, and in particular the notation of Figure 6.4. We further assume that f has isolated singularities and $f|_N$ has no singularities. Let $\check{\lambda}_i$ be a path in B which starts in b, goes along $\tilde{\lambda}_i$ until b_i, turns around c_i anticlockwise and on the boundary of D_i and returns back to b along $\tilde{\lambda}_i$. By monodromy around c_i, we mean the monodromy along $\check{\lambda}_i$.

Theorem 6.3 (Picard-Lefschetz formula) *Let f be as above and assume that the fiber of f over c_i has non-degenerated critical points. The monodromy h around the*

critical value c_i is given by the Picard-Lefschetz formula

$$h(\delta) = \delta + \sum_j (-1)^{\frac{n(n+1)}{2}} \langle \delta, \delta_j \rangle \delta_j, \quad \delta \in H_{n-1}(L_b),$$

where j runs through all the Lefschetz vanishing cycles in the singularities of L_{c_i} and $\langle \cdot, \cdot \rangle$ denotes the intersection number of two cycles in L_b.

See Figure (6.7). In the case of isolated singularities vanishing above K^1 and K^2

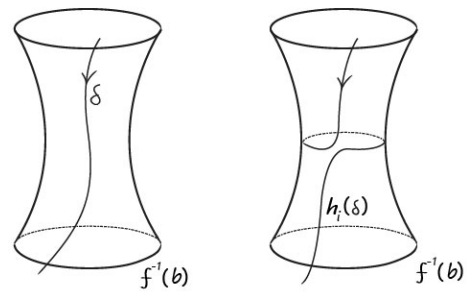

Fig. 6.7 Picard-Lefschetz formula

are the same. Moreover, by the Picard-Lefschetz formula the reader can verify that three types of the definition of a vanishing cycle coincide. For a proof of Theorem 6.3 see [Lam81] page 40. The topological phenomenon behind the Picard-Lefschetz formula is also called Dehn twist.

A monodromy map keeps the intersection form $\langle \cdot, \cdot \rangle$ in $H_{n-1}(L_b)$ invariant. This is already reflected in the Picard-Lefschetz formula:

$$\left\langle \alpha + (-1)^{\frac{n(n+1)}{2}} \langle \alpha, \delta \rangle \delta, \beta + (-1)^{\frac{n(n+1)}{2}} \langle \beta, \delta \rangle \delta \right\rangle = \langle \alpha, \beta \rangle,$$

$$\forall \alpha, \beta \in H_{n-1}(L_b, \mathbb{Z}).$$

Note that $\langle \cdot, \cdot \rangle$ is $(-1)^n$-symmetric. In general, a $\langle \cdot, \cdot \rangle$-preserving map from the homology group $H_{n-1}(L_b, \mathbb{Z})$ to itself is not a composition of some Picard-Lefschetz mappings. A nice example comes from mirror symmetry and the work of Candelas et al in [CdlOGP91]. They compute explicitly the monodromy group of the so-called mirror quintic family of Calabi-Yau threefolds. Later, in [BT14] Thomas and Brav prove that such a monodromy group has an infinite index in $\mathrm{Sp}(4, \mathbb{Z})$. For further details, see Exercise 7.7.

In case the fiber of f over c_i has arbitrary singularies, for instance non-isolated or non-degenerated, a theorem of Landman in [Lan73] says that the monodromy map h satisfies $(h^m - I)^n = 0$ for some $m \in \mathbb{N}$, where I is the identity map and $n-1$ is the

complex dimension of the fibers of f. This implies that all the eignvalues of h are roots of unity. For instance, for a homogeneous tame polynomial g, the origin $0 \in \mathbb{C}$ is the only critical value and the monodromy h around 0 satisfies $h^d = I$, where d is the degree of g. This follows from the computation of the de Rham cohomology and Gauss-Manin connection for g, see §11.4.

6.7 Vanishing cycles as generators

We are now ready to consider a global (along fibers) version of Theorem 6.2.

Proposition 6.3 *Assume that $c_i \in \mathbb{P}^1$ is not a critical point of f restricted to N and f has only isolated critical points $p_1, p_2, \ldots, p_k$ in L_c and these are all the critical points of $f : (Y,N) \to \mathbb{P}^1$ within Y_{c_i}. Let also $K = K^2$. The following statements are true:*

1. *For all $k \neq n$ we have $H_k(L_K, L_b) = 0$. This means that there is no $(k-1)$-vanishing cycle along λ_i for $k \neq n$;*
2. *$H_n(L_K, L_b)$ is freely generated by hemispherical homology classes. It is a free $\mathbb{Z}$-module of rank*

$$\mu_{c_i} := \sum_{j=1}^{k} \mu_{p_j}.$$

Proof. The proof is a modification of a similar argument in [Lam81], paragraph 5.4.1. We mainly use the excision and homotopy axioms. We can assume that $K = D$ and b is a point in the boundary of D. We choose a ball B_i around p_i, $i = 1, 2, \cdots, k$ and a smaller D, if necessary, so that f restricted to the boundary of B_i's is a submersion and so by Ehresmann's theorem, f restricted to $L_D \backslash \mathrm{int}(B)$, say it g, is a fibration over D (including c), where $\mathrm{int}(B)$ is the union of interiors of B_i's. Here, the fibers have boundaries in the boundary of B. We want to cut out from (L_D, L_b) the interior of the pair $(g^{-1}(D), g^{-1}(b))$. The pair (L_D, L_b) is homotopic to $(L_D, L_b \cup g^{-1}(D))$ and this is homotopic to $(L_D \cap B, L_b)$. Here we use the transversality of f on smaller balls in order to perform the homotopy. Now, we use the excision axiom for an open set in L_b and then again the homotopy axiom to get exactly the pair $(L_D \cap B, L_b \cap B)$. The proposition follows from Proposition 6.2. □

Let $\{c_1, c_2, \ldots, c_s\}$ the set of critical values of $f : (Y,N) \to \mathbb{P}^1$, and $b \in \mathbb{P}^1 \backslash C$.

Definition 6.5 Consider a system of s paths $\lambda_1, \ldots, \lambda_s$ starting from b and ending at $c_1, c_2, \ldots, c_s$, respectively, and such that:

1. each path λ_i has no self intersection points,
2. two distinct paths λ_i and λ_j meet only at their common origin $\lambda_i(0) = \lambda_j(0) = b$ (see Figure 6.8).
3. $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ near b is the anticlockwise direction.

This system of paths is called a distinguished set of paths. The set of vanishing cycles along the paths $\lambda_i,\ i=1,\dots,s$ is called a distinguished set of vanishing cycles related to the critical points $c_1,c_2,\dots,c_s$.

Note that in general associated to each c_i we may have many degenerated critical points. We order the corresponding vanishing cycles according to the comments in Remark 6.1. This ordering will be important in §7.10, where we compute the intersection number of vanishing cycles.

We consider the set of discs D_i and paths $\tilde{\lambda}_i$ as in §6.4. Fix a point $\infty \in \mathbb{P}^1$ which may be the critical value of f. Assume that Y is a compact complex manifold, $f : (Y,N) \to \mathbb{P}^1$ restricted to N has no critical values, except probably ∞, and f has only isolated critical points in $Y \backslash Y_\infty$.

Theorem 6.4 *The relative homology group $H_k(L_{\mathbb{P}^1\backslash\infty}, L_b)$ is zero for $k \neq n$ and it is a freely generated $\mathbb{Z}$-module of rank r for $k=n$, where r is the sum of Milnor numbers of all the singularities of f in $L_{\mathbb{P}^1\backslash\infty}$.*

Proof. Let C be the set of critical values of f in $\mathbb{P}^1\backslash\infty$. Our proof is a slight modification of the arguments in [Lam81] Section 5. Let

$$K_i = \lambda_i \cup D_i,\ K = \cup_{i=1}^s K_i.$$

The pair (K,b) is a strong deformation retract of $(\mathbb{P}^1\backslash\infty, b)$. Therefore, by Theorem 6.2 (L_K, L_b) is a strong deformation retract of $(L_{\mathbb{P}^1\backslash\{\infty\}}, L_b)$. The set $\tilde{\lambda} = \cup\tilde{\lambda}_i$ can be retract within itself to the point b and so (L_K, L_b) and $(L_K, L_{\tilde{\lambda}})$ have the same homotopy type. By the excision theorem we conclude that

$$H_k(L_{\mathbb{P}^1\backslash\infty}, L_b) \simeq \oplus_{i=1}^s H_k(L_{K_i}, L_b) \simeq \oplus_{i=1}^s H_k(L_{D_i}, L_{b_i})$$

Proposition 6.3 finishes the proof.

Corollary 6.1 *Suppose that $H_{n-1}(L_{\mathbb{P}^1\backslash\infty}) = 0$ for some $\infty \in \mathbb{P}^1$, which may be a critical value. Then a distinguished set of vanishing $(n-1)$-cycles related to the critical points in the set $C\backslash\{\infty\} = \{c_1,c_2,\dots,c_s\}$ generates $H_{n-1}(L_b)$. Further, if $H_n(L_{\mathbb{P}^1\backslash\infty}) = 0$, they form a basis of $H_{n-1}(L_b)$.*

Proof. Write the long exact sequence of the pair $(L_{\mathbb{P}^1\backslash\infty}, L_b)$:

$$\dots \to H_n(L_{\mathbb{P}^1\backslash\infty}) \to H_n(L_{\mathbb{P}^1\backslash\infty}, L_b) \xrightarrow{\sigma} H_{n-1}(L_b) \to H_{n-1}(L_{\mathbb{P}^1\backslash\infty}) \to \dots \tag{6.5}$$

Knowing this long exact sequence, the assertion follows from the hypothesis.

Figure 6.9 contains a rough idea of the proof of Theorem 6.4. It also shows how vanishing cycles generate a homology group.

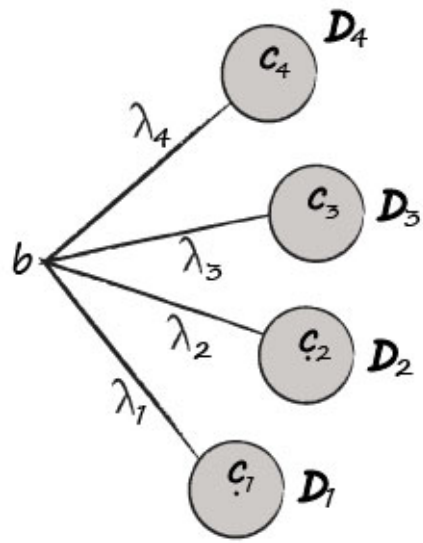

Fig. 6.8 A distinguished set of paths

6.8 Lefschetz pencil

In this section we recall from [Lam81] some basic definitions related to Lefschetz pencils. For the proofs of enumerated statements the reader is referred to the same reference.

Let $\mathbb{P}^N$ be the projective space of dimension N. The hyperplanes of $\mathbb{P}^N$ are the points of the dual projective space $\check{\mathbb{P}}^N$ and we use the notation

$$H_y \subset \mathbb{P}^N,\ y \in \check{\mathbb{P}}^N.$$

Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n. By definition X is a connected complex manifold. The dual variety of X is defined to be:

$$\check{X} := \{y \in \check{\mathbb{P}}^N \mid H_y \text{ is not transverse to } X\}.$$

We know that

1. $\check{X}$ is an irreducible variety of dimension at most $N-1$.

Any line $G \cong \mathbb{P}^1$ in $\check{\mathbb{P}}^N$ (linear projective space of dimension one) gives us in $\mathbb{P}^N$ a pencil of hyperplanes $\{H_t\}_{t\in G}$ which is the collection of all hyperplanes containing a projective subspace $A \cong \mathbb{P}^{N-2}$ of dimension $N-2$ of $\mathbb{P}^N$. The projective space A is

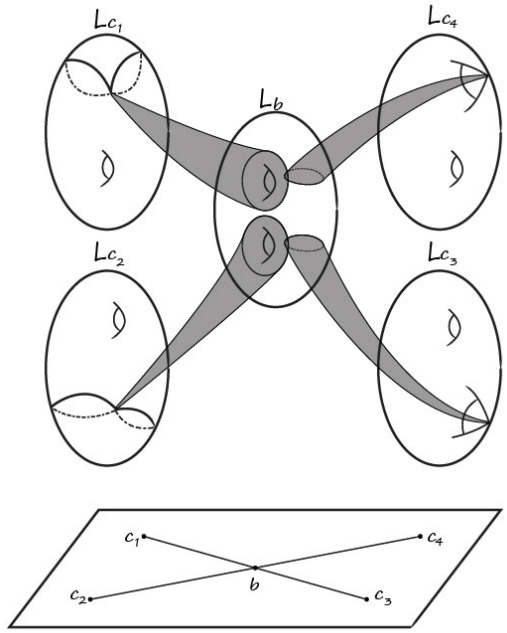

Fig. 6.9 Vanishing cycles generating a homology group

called the axis of the pencil, see Figure 6.10 Let G be a line in $\check{\mathbb{P}}^N$ which intersects

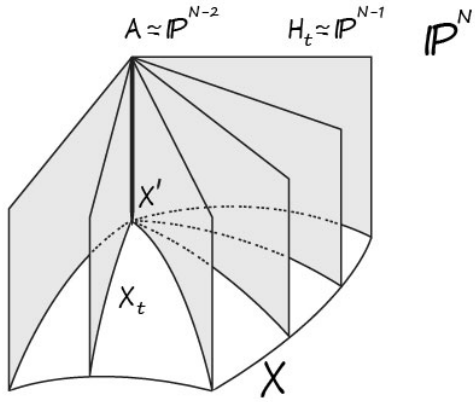

Fig. 6.10 A pencil of hyperplane intersections of X

$\check{X}$ transversely. If $\dim(\check{X}) < N-1$ this means that G does not intersects $\check{X}$ and if $\dim(\check{X}) = N-1$ this means that G intersects $\check{X}$ transversely in smooth points of $\check{X}$. We define

$$X_t := X \cap H_t,\ t \in G,\ X' := X \cap A.$$

2. A intersects X transversely and so X' is a smooth codimension two subvariety of X.
3. For $t \in G \cap \check{X}$ the hyperplane section X_t has a unique singularity which lies in $X_t \backslash X'$ and in a local holomorphic coordinate system is given by:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0.$$

4. Let $\mathbb{P}^{N-1}$ be the space of lines G in $\check{\mathbb{P}^N}$ passing through b. The space of lines $G \in \mathbb{P}^{N-1}$ such that G is not transversal to $\check{X}$ form a proper closed algebraic subset of $\mathbb{P}^{N-1}$ (we will need this in the proof of Theorem 6.5).

We fix a linear isomorphism $G \cong \mathbb{P}^1$ such that $t = 0, \infty \notin G \cap \check{X}$. Let L_0 and L_∞ be the linear polynomials such that $\{L_0 = 0\} = H_0$ and $\{L_\infty = 0\} = H_\infty$. Our pencil of hyperplanes is now given by

$$bL_0 + aL_\infty = 0,\ [a;b] \in \mathbb{P}^1.$$

We define

$$f := \frac{L_0}{L_\infty}\Big|_X$$

which is a meromorphic function on X with the indeterminacy set X'. Let us now see how Ehresmann's theorem applies to a Lefschetz pencil $\{X_t\}_{t \in G}$. Let

$$C := G \cap \check{X}.$$

Proposition 6.4 *The map $f : X \backslash X' \to G$ is a C^∞ fiber bundle over $G \backslash C$.*

Proof. We consider the blow-up of X along the indeterminacy points of f, i.e.

$$Y := \{(x,t) \in X \times G \mid x \in X_t\}.$$

see Figure 6.11. There are two projections

$$X \xleftarrow{p} Y \xrightarrow{g} G.$$

We have

$$Y' := p^{-1}(X') = X' \times G$$

and p maps $Y \backslash Y'$ isomorphically to $X \backslash X'$. Under this map f is identified with g. We use Ehresmann's theorem and we conclude that g is a fiber bundle over $G \backslash C$. Since it is regular restricted to Y', we conclude that g restricted to $Y \backslash Y'$ is a C^∞ fiber bundle over $G \backslash C$.

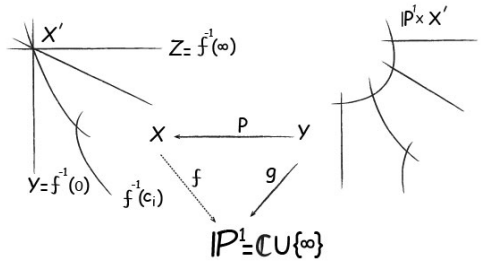

Fig. 6.11 Blow-up along the axis of a Lefschetz pencil

Remark 6.2 In many useful situations, the rational function f has not the properties mentioned above which is equivalent to say that the line G is not transversal to $\check{X}$. Let us recall one of these. We assume that G crosses an arbitrary point $\infty \in \check{X}$, and so, the corresponding hyperplane section $Z := X_\infty$ may have any type of singularities. For any other c in the intersection of $G\backslash\infty$ with $\check{X}$, we assume that the singularities of X_c are all isolated and lies in $X_c\backslash X'$ and the varieties $X_c,\ c \in G\backslash\infty$ intersect each other transversely at X'. In this situation, the variety Y might have singularities in $X' \times \{\infty\} \subset Y$, however, it is smooth outside of this set. The proofs can be recovered along the same line of reasoning as in [Lam81] §2.

6.9 Proof of the main theorem

Theorem 5.1 follows from Theorem 6.4. We have to verify that the hypothesis of Theorem 6.4 is fulfilled. By our hypothesis there is a meromorphic function f on X such that Y is the zero divisor of order one of f and Z is the pole divisor of order k of f. Since Y intersects Z transversely, for any point $p \in X' := X \cap Z$ there is a coordinate system $(x, y, \ldots)$ around p such that the meromorphic function f can be written as

$$f = \frac{x}{y^k}. \tag{6.6}$$

A similar blow-up argument along the variety X' (as in Proposition 6.4) implies that f is a C^∞ fiber bundle map over $\mathbb{C}\backslash C$, where C is the set of critical values of f restricted to $X\backslash Z$. This also implies that f in $X\backslash Z$ has isolated singularities. If this is not the case then we take an irreducible component S of the locus of singularities of f restricted to $X\backslash Z$ which is of dimension bigger than one. The variety S necessarily intersects Y in some point p. The point p does not lie in $Y\backslash Z$ because $Y\backslash Z$ is smooth. It does not lie in X' because Y intersects Z transversely.

6.10 Vanishing cycles are all conjugate by monodromy

In this section we reproduce a beautiful argument due to Lefschetz showing that, for a generic pencil of hyperplane sections, the vanishing cycles are all conjugate by monodromy. The main ingredient of the proof is the irreducibility of the discriminant locus. Our source for the following theorem is [Lam81] 7.3.5.

Theorem 6.5 *Let $f : X \dashrightarrow \mathbb{P}^1$ be the meromorphic function of a generic pencil of hyperplane sections of X as it is explained in §6.8. Let also $b \in \mathbb{P}^1$ be a regular value of f. We consider two cycles $\delta_1, \delta_2 \in H_{n-1}(L_b, \mathbb{Z})$ which vanish along some paths in some critical points of f. There is a closed path λ in $\mathbb{P}^1 \backslash C$ starting from and ending at b such that the monodromy of δ_1 along λ is $\pm\delta_2$.*

Similar theorems are stated in [Mov00] Theorem 2.3.2, Corollary 3.1.2 for generic pencils of type $\frac{F^p}{G^q}$ in $\mathbb{P}^n$ and in [AGZV88] Theorem 3.4 for a versal deformation of a singularity.

Proof. For the proof we use the notation of the line $G \cong \mathbb{P}^1 \subset \check{\mathbb{P}}^N$. The idea of the proof is depicted in Figure 6.12. Let $\lambda_i, \ \ i = 1, 2$ be a path in G which connects b

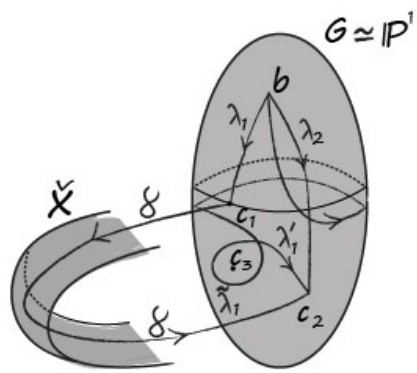

Fig. 6.12 Vanishing cycles are conjugate by monodromy

to the point $c_i \in C$, and such that δ_i vanishes along λ_i. Let D be the locus of points $t \in \check{X}$ such that the line G_t through b and t does not intersect $\check{X}$ transversally. The proof of our theorem is based on two facts 1. $\check{X}$ is an irreducible variety 2. D is a closed proper algebraic subset of $\check{X}$. These facts imply that $\check{X} \backslash D$ is a connected open subset of $\check{X}$. Since $c_1, c_2 \in \check{X} \backslash D$, there is a path γ in $\check{X} \backslash D$ from c_1 to c_2. After a blow up at the point b and using the Ehresmann's theorem, we conclude that there is an isotopy

$$H : [0,1] \times G_{c_1} \to \cup_{t\in[0,1]} G_{\gamma(t)}$$

such that

1. $H(0,\cdot)$ is the identity map;
2. for all $t \in [0,1]$, $H(t,\cdot)$ is a C^∞ isomorphism between G_{c_1} and $G_{\gamma(t)}$ which sends points in $\check{X}$ to $\check{X}$;
3. For all $t \in [0,1]$, $H(t,b) = b$ and $H(t,c_1) = \gamma(t)$.

Let $\lambda_t' = H(t,\lambda_1)$. In each line $G_{\gamma(t)}$ the cycle δ_1 vanishes along the path λ_t' in the unique non-degenerate singular point of $X_{\gamma(t)}$. Therefore, δ_1 vanishes along λ_1' in $c_2 = \gamma(1)$. Consider λ_2 and λ_1' as the paths which start from b and end in a point b_1 near c_2 and put $\lambda = \lambda_2^{-1}\lambda_1'$. By uniqueness (up to sign) of the Lefschetz vanishing cycle along a fixed path we can see that the path λ is the desired path. □

6.11 Global invariant cycle theorem

Topogical cycles invariant by the monodromy group play an important role in Lefschetz's proof of Noether's theorm. We will present this proof in Chapter 14. In this section we first remind a general theorem, without providing the necessary definitions, concerning these cycles and then prove its special case which is mainly due to Lefschetz.

Let $Y \subset X$ be two smooth projective varieties and assume that there is an open Zariski subset U of X such that Y is a fiber over a point, say it 0, of a smooth proper algebraic morphism $\pi : U \to B$. Let also

$$\rho : \pi_1(B,0) \to \mathrm{Aut}(H^m(Y,\mathbb{Q})) \tag{6.7}$$

be the monodromy map.

Theorem 6.6 (Global invariant cycle theorem) *The space of invariant cycles*

$$H^m(Y,\mathbb{Q})^\rho := \{\delta \in H^m(Y,\mathbb{Q}) \ \mid \ \rho(\gamma)(\delta) = \delta, \ \ \forall \gamma \in \pi_1(B,0)\}$$

is equal to the image of the restriction map

$$i^* : H^m(X,\mathbb{Q}) \to H^m(Y,\mathbb{Q})$$

which is a morphism of Hodge structures, where $i : Y \hookrightarrow X$ is the inclusion map. Further, for m even, any Hodge class in $H^m(Y,\mathbb{Q})^\rho$ is in the image of a Hodge class in $H^m(X,\mathbb{Q})$

The above theorem appears in Deligne's article [Del71b], Theorem 4.1.1 under the name "théorèm de la partie fixe", see also [Del68]. It has been restated and used by Voisin in [Voi07, Voi13b].

In this book we will need the following particular case which is written in homology:

Proposition 6.5 *Let Y be a smooth hyperplane section of X. The space of invariant cycles in $H_{n-1}(Y,\mathbb{Z})$ is equal to the image of the intersection map τ in* (5.10)*, that is,*

$$H_{n-1}(Y,\mathbb{Z})^{\rho} = \mathrm{Im}(H_{n+1}(X,\mathbb{Z}) \xrightarrow{\tau} H_{n-1}(Y,\mathbb{Z})). \tag{6.8}$$

Proof. Let $Y_t \subset X, \quad t \in \mathbb{P}^1$ be a pencil with $Y_0 = Y$ and assume that its axis intersects X transversely, and hence, it has at most fibers with isolated singularities. We will freely use the notations related to pencils introduced in this chapter, and in particular §6.8. The argument for $\supset$ is as follows. If $H_{n+1}(X,\mathbb{Z}) \xrightarrow{\tau_t} H_{n-1}(Y_t,\mathbb{Z})$ is the intersection map with Y_t then for a cycle $\delta \in H_{n+1}(X,\mathbb{Z})$, the cycles $\tau_t(\delta)$ for $t \in \mathbb{P}^1 \backslash C$ are mapped to each other under monodromy along any path, and so, $\tau_0(\delta) \in H_{n-1}(Y,\mathbb{Z})^{\rho}$.

The proof of $\subset$ is as follows. Let $\delta_t \in H_{n-1}(Y,\mathbb{Z}), \quad t \in \mathbb{P}^1 \backslash C$ be the track of a monodromy invariant cycle. By Picard Lefschetz formula we know that δ_t does not intersect any vanishing cycle in $H_{n-1}(Y,\mathbb{Z})$. This together with the fact that the pencial has at most isolated singularities, imply that we can assume that the union of $\delta_t, t \in \mathbb{P}^1 \backslash C$ do not intersct small neighborhoods of singularities, and so, we can cut out open balls around the singularities, and still talk about the monodromy of δ_t for all $t \in \mathbb{P}^1$, and in particular for critical values of t, that is, $t \in C$. Here, we have applied Ehresmann's fibration theorem for manifolds with boundaries. Now, the union of $\delta_t, \quad t \in \mathbb{P}^1$ gives us an $(n+1)$-dimensional cycle with the desired property. $\square$

Remark 6.3 There is a version of Picard-Lefschetz theory in which fibers are symplectic manifolds and Lefschetz vanishing cycles are Lagrangian, see [Sei08].

6.12 Exercises

6.1. Following the notations in §6.4, write down an explicit example of a vanishing cycle above K^2 which is not vanishing above K^1. Hint: For this use a critical value of $f|_N$ and Figure 6.13.

6.2. Rewrite the proof of Theorem 6.4 using the fibrations

$$f : \mathbb{C} \to \mathbb{C}, \;\; f(x) = P(x),$$

$$f : \mathbb{C}^2 \to \mathbb{C}, \;\; f(x,y) = y^2 - P(x),$$

where P is a polynomial of degree d and with distinct roots. For instance, take $P(x) = x^d - dx$.

6.3. Prove that the dimension of the n-th primitive cohomology of a hypersurface X of degree d in $\mathbb{P}^{n+1}$ is given by

$$\dim H_n(X,\mathbb{Q})_0 = (d-1)^{n+1} - (d-1)^n + (d-1)^{n-1} - \cdots + (-1)^n(d-1)$$

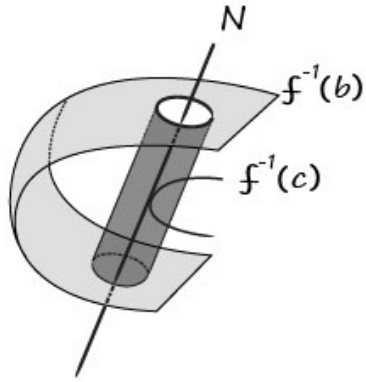

Fig. 6.13 Tangency point and a vanishing cycle

Find a similar formula in the cases of X a complete intersection of type $(d_1, d_2, \ldots, d_s)$.

6.4. Describe the topology of the algebraic curve

$$xy(x+y-1) = \frac{1}{100}$$

over $\mathbb{R}$ and $\mathbb{C}$ and compare them.

6.5. Let $f := \mathbb{C} \to \mathbb{C},\ f(x) := x^d$. The point $0 \in \mathbb{C}$ is the only critical value of f. Let $\lambda(s) = s,\ 0 \le s \le 1$. The set

$$\delta_i := [\zeta_d^{i+1}] - [\zeta_d^i],\ i = 0, \ldots, d-2$$

is a distinguished set of vanishing cycles for $H_0(\{f = 1\}, \mathbb{Z})$. The vanishing takes place along λ.

Chapter 7
Topology of tame polynomials

In contrast to what one might think, if arguments are topological there is a better chance to translate them into abstract algebraic geometry than if they are analytic, such as the proof given by Hodge. Grothendieck asked me to look at the 1924 book L'analysis situs et la géométrie algébrique by Lefschetz. It is a beautiful and very intuitive book, and it contained some of the tools I needed [for the proof of Weil conjectures], (P. Deligne in [RS14] page 181).

7.1 Introduction

A direction in which Lefschetz's topological ideas were developed in the same style as his own, is in singularity theory and the study of topology of isolated hypersurface singularities. By this we mainly mean the content of the book [AGZV88] and the references therein. This study has a purely local nature and so it might get irrelevant to the study of topological classes of algebraic cycles, which have a global nature. However, many nice results in this direction can be transported, with a little effort, to the study of topology of the fibers of tame polynomials. Examples of tame polynomials range from the fibrations attached to elliptic and hyperelliptic integrals to deformation of Fermat varieties. Most of the study in Picard-Simart book is also concentrated on these kind of polynomials in three variables. The importance of these polynomials will hopefully justify the repetition of many arguments which have already appeared in Chapter 6. Another main emphasis of the present chapter is the computation of the intersection of vanishing cycles. This will provide us with a tool in order to study the $\mathbb{Z}$-module of Hodge cycles as a lattice. These lattices in the two dimensional case are known as Mordell-Weil lattices and their classification has been the focus of a good amount of research, see [Shi91] and the references therein.

7.2 Vanishing cycles and orientation

We consider in $\mathbb{C}$ the canonical orientation

$$\frac{1}{-2\sqrt{-1}}dx \wedge d\bar{x} = d(\mathrm{Re}(x)) \wedge d(\mathrm{Im}(x)).$$

This corresponds to the anti-clockwise direction in the complex plane. In this way, every complex manifold carries an orientation obtained by the orientation of $\mathbb{C}$, which we call it the canonical orientation. For a complex manifold of dimension n and a holomorphic nowhere vanishing differential n-form ω on it, the orientation obtained from $\frac{1}{(-2\sqrt{-1})^n}\omega \wedge \overline{\omega}$ differs from the canonical one by $(-1)^{\frac{n(n-1)}{2}}$, see Exercise 7.1 in this chapter. Holomorphic maps between complex manifolds preserve the canonical orientation. For a zero dimensional manifold an orientation is just a map which associates ± 1 to each point of the manifold.

Let $f = x_1^2 + x_2^2 + \cdots + x_{n+1}^2$. For a real positive number t, the n-th homology of the complex manifold

$$L_t := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid f(x) = t\}$$

is generated by the so called vanishing cycle

$$\delta_t = \mathbb{S}^n(t) := L_t \cap \mathbb{R}^{n+1}.$$

It vanishes along the path γ which connects t to 0 in the real line. The (Lefschetz) thimble

$$\Delta_t := \cup_{0 \leq s \leq t} \delta_s = \{x \in \mathbb{R}^{n+1} \mid f(x) \leq t\}$$

is a real $(n+1)$-dimensional manifold which generates the relative $(n+1)$-th homology $H_{n+1}(\mathbb{C}^{n+1}, L_t, \mathbb{Z})$. We consider for $\mathbb{S}^n(t)$ the orientation η such that

$$\eta \wedge \mathrm{Re}(df) = \mathrm{Re}(dx_1) \wedge \mathrm{Re}(dx_2) \wedge \cdots \wedge \mathrm{Re}(dx_{n+1})$$

(up to multiplication with positive-valued functions). This is by definition the orientation of Δ_t.

Proposition 7.1 *The orientation of* $(\mathbb{C}^{n+1}, 0)$ *obtained by the intersection of two thimbles is* $(-1)^{\frac{n(n+1)}{2}}\varepsilon^{n+1}$*, where* $\varepsilon = \pm 1$ *is the orientation of* $(\mathbb{C}, 0)$ *obtained by the intersection of their vanishing paths, see Figure 7.1.*

Proof. Let α be a complex number near to 1 with $|\alpha| = 1$ and

$$h : L_t \to L_{\alpha^2 t},\ x \mapsto \alpha^{-1} \cdot x.$$

The oriented cycle $h_* \delta_t$ is obtained by the monodromy of δ_t along the shortest path which connects t to $\alpha^2 t$. Now the orientation of Δ_t wedge with the orientation of $h_* \Delta_t$ is:

$$
\begin{aligned}
&= \mathrm{Re}(dx_1)\wedge\mathrm{Re}(dx_2)\wedge\cdots\wedge\mathrm{Re}(dx_{n+1})\wedge \\
&\quad \mathrm{Re}(\alpha^{-1}dx_1)\wedge\mathrm{Re}(\alpha^{-1}dx_2)\wedge\cdots\wedge\mathrm{Re}(\alpha^{-1}dx_{n+1}) \\
&= (-1)^{\frac{n^2+n}{2}}\mathrm{Im}(\alpha)^{n+1}\mathrm{Re}(dx_1)\wedge\mathrm{Im}(dx_1)\wedge\mathrm{Re}(dx_2)\wedge\mathrm{Im}(dx_2)\wedge\cdots \\
&\quad \wedge\mathrm{Re}(dx_{n+1})\wedge\mathrm{Im}(dx_{n+1}) \\
&= (-1)^{\frac{n^2+n}{2}}\varepsilon^{n+1} \text{ the canonical orientation of } \mathbb{C}^{n+1}.
\end{aligned}
$$

This does not depend on the orientation η that we chose for δ_t. Note that ε is the sign of $\mathrm{Im}(\alpha)$ and if it is positive then $\mathrm{Re}(dt)\wedge h_*\mathrm{Re}(dt)$ is the canonical orientation of $\mathbb{C}$. $\square$

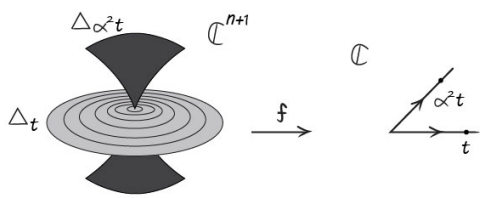

Fig. 7.1 Intersection of thimbles

7.3 Tame polynomials

Let $n \in \mathbb{N}_0$ and

$$\nu := (\nu_1, \nu_2, \ldots, \nu_{n+1}) \in \mathbb{N}^{n+1}.$$

Let also R be a localization of $\mathbb{Q}[t]$, $t = (t_1, t_2, \ldots, t_s)$ a multi parameter, over a multiplicative subgroup of $\mathbb{Q}[t]$. For simplicity, one can take R to be the field of rational functions in t. We regard the entries of t as parameters, that is, we frequently substitute t with a value and work with such a special case. We denote by

$$\mathsf{R}[x] := \mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$$

the polynomial ring with coefficients in R and variables $x_1, x_2, \ldots, x_{n+1}$. It is considered as a graded algebra with

$$\deg(x_i) := \nu_i.$$

For $n = 0$ (resp. $n = 2$ and $n = 3$) we use the notations x (resp. x, y and x, y, z).

Definition 7.1 A polynomial $f \in \mathsf{R}[x]$ is called a homogeneous polynomial of degree d with respect to the grading ν if f is a linear combination of monomials of the type

$$x^{\beta} := x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}},\ \deg(x^{\beta}) = \nu \cdot \beta := \sum_{i=1}^{n+1} \nu_i \beta_i = d.$$

For an arbitrary polynomial $f \in \mathsf{R}[x]$ one can write in a unique way $f = \sum_{i=0}^{d} f_i,\ f_d \neq 0$, where f_i is a homogeneous polynomial of degree i. The number d is called the degree of f.

Definition 7.2 A polynomial $f \in \mathsf{R}[x]$ is called a tame polynomial if there exist natural numbers $\nu_1, \nu_2, \ldots, \nu_{n+1} \in \mathbb{N}$ such that for $g := f_d$, the last homogeneous piece of f in the graded algebra $\mathsf{R}[x],\ \deg(x_i) = \nu_i$, the R-module

$$\mathsf{V}_g := \frac{\mathsf{R}[x]}{\mathrm{jacob}(g)},$$

is finitely generated. Here,

$$\mathrm{jacob}(g) := \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \cdots, \frac{\partial g}{\partial x_{n+1}} \rangle \subset \mathsf{R}[x]$$

is the Jacobian ideal of g. In this way we say that g has an isolated singularity at the origin.

For simplicity, we assume that the coefficients of f are polynomials in t, and hence, we can evaluate f for any $t \in \mathbb{C}^s$. We denote by f_t the polynomial after such an evaluation. The notation $f(x,t)$ will be also used frequently. A detailed algebraic study of tame polynomials, together with many examples, will be discussed in Chapter 10. In particular, we will introduce a polynomial $\Delta(t)$ in parameters t such that $f_t = 0$ is a singular variety if and only if $\Delta(t) = 0$. For the purpose of this chapter we will need the zero set of $\Delta(t)$ and so we define:

$$\{\Delta(t) = 0\} := \{t \in \mathbb{C}^s | \text{the affine variety } f_t = 0 \text{ is singular}\}. \tag{7.1}$$

7.4 Picard-Lefschetz theory of tame polynomials

Let us consider a tame polynomial f in the ring $\mathsf{R}[x]$, where R is a localization of $\mathbb{C}[t]$, $t = (t_1, t_2, \ldots, t_s)$ a multi parameter, over a multiplicative group generated by $\hat{\Delta} \in \mathbb{C}[t]$. Let also

$$\begin{aligned} \mathsf{T} &:= \mathbb{C}^s \backslash \{t \in \mathbb{C}^s \mid \hat{\Delta}(t) = 0\}, \\ L &:= \{(x,t) \in \mathbb{C}^{n+1} \times \mathsf{T} \mid f_t(x) = 0\}, \\ \mathsf{T}_\Delta &:= \mathsf{T} \backslash \{t \in \mathsf{T} \mid \Delta(t) = 0\}, \\ L_t &:= \pi^{-1}(t) = \{x \in \mathbb{C}^{n+1} \mid f_t(x) = 0\}, \end{aligned} \tag{7.2}$$

where f_t is the polynomial obtained by fixing the value of t, Δ is the discriminant of f and $\pi : L \to \mathsf{T}$ is the canonical projection. Let g be the last homogeneous piece of f and $\mathbb{N}_{n+1} = \{1,2,\ldots,n+1\}$, $S = \{i \in \mathbb{N}_{n+1} \mid \nu_i = 1\}$ and $S^c = \mathbb{N}_{n+1} \backslash S$.

Definition 7.3 The homogeneous polynomial g has a strongly isolated singularity at the origin if g has an isolated singularity at the origin and for all $R \subset \{1,2,3\ldots,n+1\}$ with $S \subset R$, g restricted to $\cap_{i \in R}\{x_i = 0\}$ has also an isolated singularity at the origin.

If $\nu_1 = \nu_2 = \cdots = \nu_{n+1} = 1$ then the condition "strongly isolated" is the same as "isolated". The Picard-Lefschetz theory of tame polynomials is based on the following statement:

Theorem 7.1 *If the last homogeneous piece of a tame polynomial f is either independent of any parameter in R or it has a strongly isolated singularity at the origin then the projection $\pi : L \to \mathsf{T}$ is a locally trivial C^∞ fibration over T_Δ.*

Proof. We give only a sketch of the proof. First, assume that the last homogeneous piece of f, namely g, has a strongly isolated singularity at the origin. Let us add the new variable x_0 to $\mathsf{R}[x]$ and consider the homogenization $F(x_0,x) \in \mathsf{R}[x_0,x]$ of f. Let F_t be the specialization of F at $t \in \mathsf{T}$. Define

$$\bar{L} := \{([x_0 : x], t) \in \mathbb{P}^{1,\nu} \times \mathsf{T} \mid F_t(x_0,x) = 0\},$$

where $\mathbb{P}^{1,\nu}$ is the weighted projective space of type

$$(1,\nu) = (1,\nu_1,\nu_2,\ldots,\nu_{n+1}).$$

Let $\bar{\pi} : \bar{L} \to \mathsf{T}$ be the projection in T. If all the weights ν_i are equal to 1 then $D := \bar{L} \backslash L$ is a smooth submanifold of $\bar{L}$ and $\bar{\pi}$ and $\bar{\pi}\mid_D$ are proper regular (i.e. the derivative is surjective). For this case one can use directly Ehresmann's fibration theorem (see Theorem 6.1 in Chapter 6). For arbitrary weights we use the generalization of Ehresmann's theorem for stratified varieties. In $\mathbb{P}^{1,\nu}$ we consider the following stratification

$$(\mathbb{P}^{1,\nu} \backslash \mathbb{P}^\nu) \bigcup (\mathbb{P}^\nu \backslash \mathbb{P}^{S^c}) \bigcup \cup_{I \subset S^c} (\mathbb{P}^I \backslash \mathbb{P}^{<I}),$$

where for a subset I of $\mathbb{N}_{n+1}$, $\mathbb{P}^I$ denotes the sub projective space of the weighted projective space $\mathbb{P}^\nu$ given by $\{x_i = 0 \mid i \in \mathbb{N}_{n+1} \backslash I\}$ and

$$\mathbb{P}^{<I} := \cup_{J \subset I,\, J \neq I} \mathbb{P}^J.$$

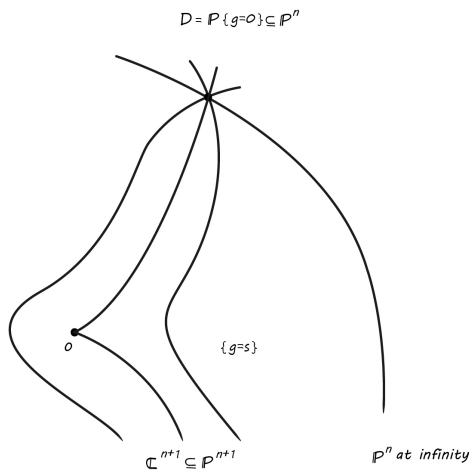

Fig. 7.2 Fibration of a homogeneous tame polynomial

Now in T consider the one piece stratification and in $\mathbb{P}^{1,\nu} \times \mathsf{T}$ the product stratification. This gives us a stratification of $\bar{L}$. The morphism $\bar{\pi}$ is proper and the fact that g has a strongly isolated singularity at the origin implies that $\bar{\pi}$ restricted to each strata is regular. We use Verdier's Theorem ([Ver76], Theorem 4.14, Remark 4.15) and obtain the local trivialization of π on a small neighborhood of $t \in \mathsf{T}$ and compatible with the stratification of $\bar{L}$. This yields to a local trivialization of π around t. If g is independent of any parameter in R then $\bar{L} \backslash L = G \times \mathsf{T}$, where G is the variety induced in $\{g = 0\}$ in $\mathbb{P}^{\nu}$. We choose an arbitrary stratification in G and the product stratification in $G \times \mathsf{T}$ and apply again Verdier's Theorem. $\square$

The hypothesis of Theorem 7.1 is not the best one. For instance, the homogeneous polynomial

$$g := x^3 + tzy + tz^2 \in \mathsf{R}[x,y,z],$$

$$\mathsf{R} := \mathbb{C}[t,s,\frac{1}{t}],\ \deg(x) = 2,\ \deg(y) = \deg(z) = 3$$

depends on the parameter t and $g(x,y,0)$ has not an isolated singularity at the origin. However, for $f := g - s$, π is a C^∞ locally trivial fibration over $\mathsf{T} = \mathbb{C}^2 \backslash \{t = 0\} \cup \{s = 0\}$. I do not know any theorem describing explicitly the atypical values of the morphism π. Such theorems must be based either on a precise desingularization of $\bar{L}$ and Ehresmann's theorem or various types of stratifications depending on the polynomial g. For more information in this direction the reader is referred to the works of J. Mather, R. Thom and J. L. Verdier around 1970 (see [Mat73] and the references therein). Theorem 7.1 (in the general context of morphisms of algebraic varieties) is also known as the second theorem of isotopy (see [Ver76] Remark 4.15).

7.5 Distinguished set of vanishing cycles

Let $f \in \mathbb{C}[x]$ be a tame polynomial and let $C \subset \mathbb{C}$ be the set of critical values of f. We fix a regular value $b \in \mathbb{C}\backslash C$ of f and consider a system of paths $\lambda_i,\ i = 1,2,\dots,\mu$ connecting b to the points of C. We assume that λ_i's do not intersect each other except at their start/end points and at the points of C they intersect each other transversally. In a similar way as in Definition 6.5 we define a distinguished set of vanishing cycles $\delta_i \subset f^{-1}(b),\ i = 1,2,\dots,\mu$ (defined up to homotopy). For each singularity p of f we use a separate versal deformation which is defined in a neighborhood of p. If the completion of f has a non-zero double discriminant then we can deform f and obtain another tame polynomial $\tilde{f}$ with the same Milnor number in such a way that f and $\tilde{f}$ have C^∞ isomorphic regular fibers and $\tilde{f}$ has distinct μ critical values. For the notions of completion and double discriminant see Definition 10.4 and §10.10. In this case we can use $\tilde{f}$ for the definition of a distinguished set of vanishing cycles.

Fix an embedded sphere in $f^{-1}(b)$ representing the vanishing cycle δ_i. For simplicity, we denote it again by δ_i. In the literature the union $\cup_{i=1}^{\mu}\delta_i$ is known as a bouquet of μ spheres.

Theorem 7.2 *For a tame polynomial $f \in \mathbb{C}[x]$ and a regular value b of f, the complex manifold $f^{-1}(b)$ has the homotopy type of $\cup_{i=1}^{\mu}\delta_i$. In particular, a distinguished set of vanishing cycles generates the homology $H_n(f^{-1}(b),\mathbb{Z})$.*

Proof. The proof of this theorem is a well-known argument in Picard-Lefschetz theory, see for instance [Lam81] §5, [Bro88] Theorem 1.2, [Mov00] Theorem 2.2.1 and [DN01]. We have reproduced this argument in the proof of Theorem 7.4 □

Theorem 7.3 *If the tame polynomial $f \in \mathbb{C}[x]$ has μ distinct critical values and the discriminant of its completion is irreducible then for two vanishing cycles δ_0, δ_1 in a regular fiber $f^{-1}(b)$ of f, there is a homotopy class $\lambda \in \pi_1(\mathbb{C}\backslash C, b)$ such that the monodromy of δ_0 along λ is $\pm\delta_1$, where C is the set of critical values of f.*

Proof. The proof is similar to the case of a Lefschetz pencil in Theorem 6.5. Let $F \in \mathsf{R}[x]$ be the completion of f, where R is some localization of $\mathbb{C}[t]$, and $\Delta_0 := \{t \in \mathsf{T} \mid \Delta_F(t) = 0\}$. We consider $f - s,\ s \in \mathbb{C}$ as a line G_{c_0} in T which intersects Δ_0 transversally in μ points. If there is no confusion we denote by b the point in T corresponding to $f - b$. In the proof of Theorem 6.5 we replace $\check{X}$ and $\check{\mathbb{P}^N}$ with Δ_0 and T, respectively. □

Note that in the above theorem we are still talking about the homotopy classes of vanishing cycles. The author believes that the discriminant of a complete tame polynomial is always irreducible. This can be easily checked for $\nu_1 = \nu_2 = \cdots = \nu_{n+1} = 1$ and many particular cases of weights.

7.6 Monodromy of zero dimensional varieties

We consider the one variable tame polynomial $f = f_t = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$ in $\mathsf{R}[x]$, where $\mathsf{R} = \mathbb{C}[t_0, t_1, \ldots, t_{d-1}]$. The homology $H_0(\{f_t = 0\}, \mathbb{Z})$ is the set of all finite sums $\sum_i r_i[x_i]$, where $r_i \in \mathbb{Z}, \sum_i r_i = 0$ and x_i's are the roots of f_t. The monodromy is defined by the continuation of the roots of f along a path in $\pi_1(\mathsf{T}, b)$. To calculate the monodromy we proceed as follows:

Let us consider a polynomial f near $(x-1)(x-2)\cdots(x-d)$ such that it has $\mu := d-1$ distinct real critical values, namely $c_1, c_2, \ldots, c_\mu$. Let b the point in T corresponding to f. We consider f as a function from $\mathbb{C}$ to itself and take a distinguished set of paths λ_i, $i = 1, 2, \ldots, \mu$ in $\mathbb{C}$ which connects 0 to the critical values of f. This means that the paths λ_i do not intersect each other except at 0 and the order $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ around 0 is anticlockwise. The cycle $\delta_i = [i+1] - [i], i = 1, 2, \ldots, \mu$ vanishes along the path λ_i and $\delta = (\delta_1, \delta_2, \ldots, \delta_\mu)$ is called a distinguished set of vanishing cycles in $H_0(L_b, \mathbb{Z})$. Now, the monodromy around the critical value c_i is given by

$$\delta_j \mapsto \begin{cases} \delta_j & j \neq i-1, i, i+1 \\ -\delta_j & j = i \\ \delta_j + \delta_i & j = i-1, i+1 \end{cases}.$$

In $H_0(L_b, \mathbb{Z})$ we have the intersection form induced by

$$\langle x, y \rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad x, y \in L_b.$$

By definition $\langle \cdot, \cdot \rangle$ is a symmetric form in $H_0(L_b, \mathbb{Z})$, i.e. for all $\delta_1, \delta_2 \in H_0(L_b, \mathbb{Z})$ we have $\langle \delta_1, \delta_2 \rangle = \langle \delta_2, \delta_1 \rangle$. Let Ψ_0 be the intersection matrix in the basis δ:

$$\Psi_0 := \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{7.3}$$

The monodromy group keeps the intersection form in $H_0(L_b, \mathbb{Z})$. In other words:

$$\Gamma_{\mathbb{Z}} \subset \{A \in \mathrm{GL}(\mu, \mathbb{Z}) \mid A \Psi_0 A^{\mathsf{tr}} = \Psi_0\}. \tag{7.4}$$

Consider the case $d = 3$. We choose the basis $\delta_1 = [2] - [1]$, $\delta_2 = [3] - [2]$ for $H_0(L_b, \mathbb{Z})$. In this basis the intersection matrix is given by

$$\Psi_0 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

There are two critical points for f for which the monodromy is given by:

$$\delta_1 \mapsto -\delta_1,\ \delta_2 \mapsto \delta_2 + \delta_1,$$

$$\delta_2 \mapsto -\delta_2,\ \delta_1 \mapsto \delta_2 + \delta_1.$$

Let $g_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. The monodromy group satisfies the equalities:

$$\begin{aligned} \Gamma_{\mathbb{Z}} &= \langle g_1, g_2 \mid g_1^2 = g_2^2 = I, g_1 g_2 g_1 = g_2 g_1 g_2 \rangle \\ &= \{I, g_1, g_2, g_1 g_2 g_1, g_2 g_1, g_1 g_2\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \right. \\ &\quad \left. \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \end{aligned}$$

For this example (7.4) turns out to be an equality (one obtains equations like $(a-b)^2 + a^2 + b^2 = 2$ for the entries of the matrix A and the calculation is explicit). See Exercise 7.4 in this chapter.

7.7 Monodromy of families of elliptic curves

Monodromy groups with $\mathbb{Z}$-coefficients are topological objects and cannot be defined in the framework of Algebraic Geometry over arbitrary field. One of the most celebrated of these groups is

$$\mathrm{SL}(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\} \tag{7.5}$$

which appears as the monodromy group of the family of elliptic curves over $\mathbb{C}$. Let

$$E_t : y^2 - 4x^3 + t_2 x + t_3 = 0, \tag{7.6}$$

$$t \in \mathsf{T} := \mathbb{C}^2 \backslash \{(t_2, t_3) \in \mathbb{C}^2 \mid \Delta = 0\},$$

where $\Delta := 27t_3^2 - t_2^3$. The elliptic curve E_t as a topological space is a torus minus a point and hence $H_1(E_t, \mathbb{Z})$ is a free rank two $\mathbb{Z}$-module. We want to compute the monodromy representation

$$\pi_1(\mathsf{T}, b) \to \mathrm{Aut}(H_1(E_b, \mathbb{Z})),$$

where b is a fixed point in T. Fix the parameter $t_2 \neq 0$ and let t_3 vary. Exactly for two values $\tilde{t}_3,\ \check{t}_3 = \pm\sqrt{\frac{t_2^3}{27}}$ of t_3, the curve E_t is singular. In E_b we can take two cycles δ_1 and δ_2 such that $\langle \delta_1, \delta_2 \rangle = 1$ and δ_1 (resp. δ_2) vanishes along a straight line connecting b_3 to $\tilde{t}_3$ (resp. $\check{t}_3$). The corresponding anticlockwise monodromy around the critical value $\tilde{t}_3$ (resp $\check{t}_3$) can be computed using the Picard-Lefschetz formula:

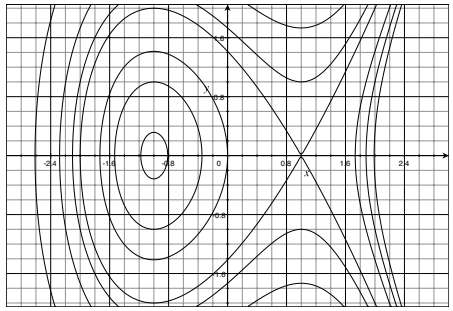

Fig. 7.3 Elliptic curves: $y^2 - x^3 + 12x - 4\psi,\ \psi = -1.9, -1, 0, 2, 3, 5, 10$

$$\delta_1 \mapsto \delta_1,\ \delta_2 \mapsto \delta_2 + \delta_1\ (\text{ resp. } \delta_1 \mapsto \delta_1 - \delta_2,\ \delta_2 \mapsto \delta_2).$$

Therefore, the image of $\pi_1(\mathsf{T}, t)$ under the monodromy representation contains the following matrices in $\mathrm{SL}(2, \mathbb{Z})$:

$$A_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ A_2 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{7.7}$$

These matrices generates $\mathrm{SL}(2, \mathbb{Z})$, see Exercise 7.6. This can be used to show that

$$\pi_1(\mathbb{C} \backslash \{\tilde{t}_3, \check{t}_3\}, b) \to \pi_1(\mathsf{T}, b)$$

induced by inclusion is a subjective map and this is a particular case of a general phenomenon, see for instance [Lam81] (7.3.5). Let us explain the above topological picture by the following one parameter family of elliptic curves:

$$E_\psi : y^2 - 4x^3 + 12x - 4\psi = 0.$$

For b a real number between 2 and -2 the elliptic curve E_b intersects the real plane $\mathbb{R}^2$ in two connected pieces which one of them is an oval and we can take it as δ_2 with the anticlockwise orientation. In this example as ψ moves from -2 to 2, δ_2 is born from the point $(-1, 0)$ and ends up in the α-shaped piece which is the intersection of E_2 with $\mathbb{R}^2$. The cycle δ_1 lies in the complex domain and it vanishes on the critical point $(1, 0)$ as ψ moves to 2. It intersects each connected component of $E_b \cap \mathbb{R}^2$ once and it is oriented in such away that $\langle \delta_1, \delta_2 \rangle = 1$.

7.8 Join of topological spaces

We start this section with a definition.

Definition 7.4 The join $X * Y$ of two topological spaces X and Y is the quotient space of the direct product $X \times I \times Y$, where $I = [0,1]$, by the equivalence relation:

$$(x,0,y_1) \sim (x,0,y_2)\ \forall\, y_1,y_2 \in Y,\ x \in X,$$

$$(x_1,1,y) \sim (x_2,1,y)\ \forall\, x_1,x_2 \in X,\ y \in Y.$$

Let X and Y be compact oriented real manifolds and $\pi : X * Y \to I$ be the projection on the second coordinate. The real manifold $X * Y \backslash \pi^{-1}(\{0,1\})$ has a canonical orientation obtained by the wedge product of the orientations of X, I and Y. Let

$$\mathbb{S}^n := \{(x_1,x_2,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

be the n-dimensional sphere with the orientation α such that $x \in \mathbb{S}^n$ as a vector in $\mathbb{R}^{n+1}$ together with α give us the orientation of $\mathbb{R}^n$ given by the coordinate system $(x_1,x_2,\ldots,x_n)$. In terms of differential forms this is given by $\frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}}{d(x_1^2+x_2^2+\cdots+x_{n+1}^2)}$.

Proposition 7.2 *We have*

$$\mathbb{S}^n * \mathbb{S}^m \overset{C^0}{\cong} \mathbb{S}^{n+m+1},\ n,m \in \mathbb{N}_0, \tag{7.8}$$

which is an isomorphism of oriented manifolds outside $\pi^{-1}(\{0,1\})$.

Proof. For the proof of the above diffeomorphism we write $\mathbb{S}^{n+m+1}$ as the set of all $(x,y) \in \mathbb{R}^{n+m+2}$ such that

$$x_1^2 + \cdots + x_{n+1}^2 = 1 - (y_1^2 + \cdots + y_{m+1}^2).$$

Now, let t be the above number and let it varies from 0 to 1. We have the following isomorphism of topological spaces:

$$\mathbb{S}^{n+m+1} \to \mathbb{S}^n * \mathbb{S}^m,\ (x,y) \mapsto \begin{cases} (\frac{x}{\sqrt{t}}, 1-t, \frac{y}{\sqrt{1-t}}) & t \neq 0,1, \\ (0,1,y) & t = 0, \\ (x,0,0) & t = 1. \end{cases}$$

Figure 7.4 shows a geometric construction of $\mathbb{S}^0 * \mathbb{S}^0$. The proof of the statement about orientations is left to the reader. □

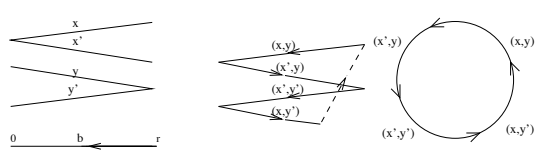

Fig. 7.4 A circle as a join of zero dimensional spheres

7.9 Direct sum of polynomials

Let $f \in \mathbb{C}[x]$ and $g \in \mathbb{C}[y]$ be two polynomials in variables $x := (x_1, x_2, \ldots, x_{n+1})$ and $y := (y_1, y_2 \ldots, y_{m+1})$, respectively. In this section we study the topology of the variety

$$X := \{(x,y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} \mid f(x) = g(y)\}$$

in terms of the topology of the fibrations $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and $g : \mathbb{C}^{m+1} \to \mathbb{C}$. Let C_1 (resp. C_2) denote the set of critical values of f (resp. g). We assume that $C_1 \cap C_2 = \emptyset$, which implies that the variety X is smooth. Fix a regular value $b \in \mathbb{C} \backslash (C_1 \cup C_2)$ of both f and g. Let $\delta_{1b} \in H_n(f^{-1}(b), \mathbb{Z})$ and $\delta_{2b} \in H_m(g^{-1}(b), \mathbb{Z})$ be two vanishing cycles and $t_s,\ s \in [0,1]$ be a path in $\mathbb{C}$ such that it starts from a point in C_1, crosses b and ends in a point of C_2 and never crosses $C_1 \cup C_2$ except at the mentioned cases. We assume that δ_{1b} vanishes along $t_.^{-1}$ when s tends to 0 and δ_{2b} vanishes along $t_.$ when s tends to 1, see Figure 7.5.

Definition 7.5 The cycle

$$\delta_1 * \delta_2 \cong \delta_{1b} *_{t_.} \delta_{2b} := \cup_{s \in [0,1]} \delta_{1t_s} \times \delta_{2t_s} \in H_{n+m+1}(X, \mathbb{Z})$$

is oriented and it is called the join of δ_{1b} and δ_{2b} along $t_.$, or a join cycle for simplicity. Note that its orientation changes when we change the direction of the path $t_.$, see Figure 7.5. We call the triple $(t_., \delta_1, \delta_2) = (t_., \delta_{1t_.}, \delta_{2t_.})$ an admissible triple.

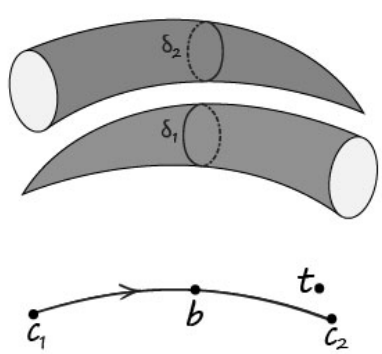

Fig. 7.5 Join of vanishing cycles

Let $b \in \mathbb{C}\backslash(C_1 \cup C_2)$. We take a system of distinguished paths $\lambda_c,\ c \in C_1 \cup C_2$, where λ_c starts from b and ends at c, see Figure 7.6. Let

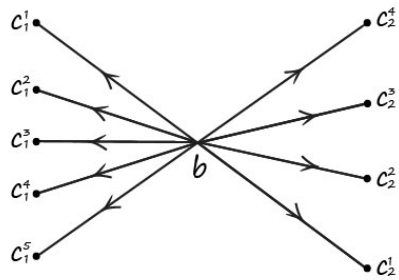

Fig. 7.6 A system of distinguished paths

$$\delta_1^1, \delta_1^2, \cdots, \delta_1^{\mu} \in H_n(f^{-1}(b), \mathbb{Z})$$

and

$$\delta_2^1, \delta_2^2, \cdots, \delta_2^{\mu'} \in H_m(g^{-1}(b), \mathbb{Z})$$

be the corresponding distinguished set of vanishing cycles. Note that many vanishing cycles may vanish along a path in one singularity, see Remark 6.1. In this chapter we care about the ordering of vanishing cycles as this will be important in the intersection number of join cycles.

Theorem 7.4 *Let f and g be two tame polynomials with disjoint set of critical values. The $\mathbb{Z}$-module $H_{n+m+1}(X, \mathbb{Z})$ is freely generated by*

$$\delta_1^i * \delta_2^j,\ i = 1, 2, \ldots, \mu,\ j = 1, 2, \cdots, \mu', \tag{7.9}$$

where we have taken the admissible triples

$$(\lambda_{c_2} \lambda_{c_1}^{-1}, \delta_1^i, \delta_2^j),\ c_1 \in C_1,\ c_2 \in C_2.$$

Proof. The proof which we present for this theorem is similar to the well-known argument in Picard-Lefschetz theory that we have already used it in §6.7, see also [Lam81] or Theorem 2.2.1 of [Mov00]. The homologies below are with $\mathbb{Z}$ coefficients.

The fibration $\pi : X \to \mathbb{C},\ (x, y) \mapsto f(x) = g(y)$ is topologically trivial over $\mathbb{C}\backslash(C_1 \cup C_2)$. Let $Y = f^{-1}(b) \times g^{-1}(b)$. We have

$$0 = H_{n+m+1}(Y) \to H_{n+m+1}(X) \to H_{n+m+1}(X,Y) \tag{7.10}$$

$$\xrightarrow{\partial} H_{n+m}(Y) \to H_{n+m}(X) \to \cdots$$

Since f and g are tame, we have $H_{n+m+1}(Y) = 0$. We take small open disks D_c around each point $c \in C_1 \cup C_2$. Let b_c be a point near c in D_c and $X_c = \pi^{-1}(\lambda_c \cup D_c)$. We have

$$H_{n+m}(Y) \cong H_n(f^{-1}(b)) \otimes_{\mathbb{Z}} H_m(g^{-1}(b))$$

and

$$\begin{aligned} H_{n+m+1}(X,Y) &\cong \oplus_{c\in C_1\cup C_2} H_{n+m+1}(X_c,Y) \\ &\cong \oplus_{c\in C_1\cup C_2} H_{n+m+1}(X_c,Y_{b_c}) \\ &\cong \oplus_{c\in C_1} H_{n+1}(f^{-1}(D_c), f^{-1}(b_c)) \\ &\quad \oplus \oplus_{c\in C_2} H_{m+1}(g^{-1}(D_c), g^{-1}(b_c)). \end{aligned}$$

We look $H_{n+m+1}(X)$ as the kernel of the boundary map ∂ in (7.10). Let us take two cycles Δ_1 and Δ_2 from the pieces of the last direct sum in the above equation and assume that $\partial\Delta = 0$, where $\Delta = \Delta_1 - \Delta_2$. If Δ_1 and Δ_2 belong to different classes, according to $c \in C_1$ or $c \in C_2$, then Δ is the join of two vanishing cycles. Otherwise, $\Delta = 0$ in $H_{n+m+1}(X,\mathbb{Z})$. Note that we have to use the above isomorphisms in backward, so that Δ_i is seen as an $(n+m+1)$-dimensional cycle. Moreover, over the critical value c, Δ_i does not have boundary, despite the way we have depicted the joint cycle in Figure 7.5. $\square$

It is sometimes useful to take $g = b' - g'$, where b' is a fixed complex number and g' is a tame polynomial . The set of critical values of g' is denoted by C_2' and hence the set of critical values of g is $C_2 = b' - C_2'$. We define $t = F(x,y) := f(x) + g'(y)$ and so $X = F^{-1}(b')$. The set of critical values of F is $C_1 + C_2'$ and the assumption that $C_1 \cap (b' - C_2')$ is empty implies that b' is a regular value of F. Let $(t_\cdot, \delta_{1b}, \delta_{2b})$ be an admissible triple and $t_\cdot$ starts from c_1 and ends in $b' - c_2'$. This implies that the path $t_\cdot + c_2'$ starts from $c_1 + c_2'$ and ends in b'.

Proposition 7.3 *The topological cycle $\delta_{1b} * \delta_{2b}$ is a vanishing cycle along the path $t_\cdot + c_2'$ with respect to the fibration $F = t$.*

Proof. Along the path $t_\cdot^{-1}$, the point b' turns into the point $c_1 + c_2'$. $\square$

Let $b \in \mathbb{C}\backslash(C_1 \cup C_2)$. We take a system of distinguished paths λ_c $c \in C_1 \cup C_2$, where λ_c starts from b and ends at c, see Figure 7.5. If the points of the set C_1 (resp. C_2) are enough near (resp. far from) each other then the collection of translations given in Proposition 7.3 gives us a system of paths, which is distinguished after performing a proper homotopy, starting from the points of $C_1 + C_2'$ and ending in b', see Figure 7.7 and Figure 7.8. This together with Theorem 7.2 gives an alternative proof to Theorem 7.4. The corresponding distinguished system of vanishing cycles in this case is given by the join cycles (7.9). These are ordered lexicographically.

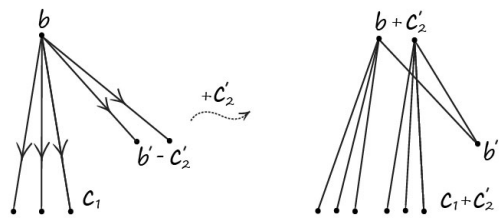

Fig. 7.7 Join of vanishing cycles and distinguished system of paths, I

This ordering will play an essential role in §7.10. We can keep track of this ordering in other situations as we do it in the below discussion.

Let us assume that all the critical values of f and $g' = b' - g$ are real. Moreover, assume that f (resp. g) has nondegenerated critical points with distinct images. For instance, in the case $n = m = 0$ take

$$f := (x-1)(x-2)\cdots(x-m_1),\ g' := (x-m_1-1)(x-m_1-2)\cdots(x-m_1-m_2).$$

Take $b' \in \mathbb{C}$ with $\mathrm{Im}(b') > 0$. We take direct segment of lines which connects the points of C_1 to the points of $b' - C_2'$. The set of joint cycles constructed in this way, is a basis of vanishing cycles associated the direct segment of paths which connect b' to the points of $C_1 + C_2'$, see Figure 7.8.

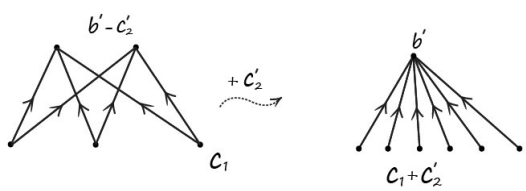

Fig. 7.8 Join of vanishing cycles and distinguished system of paths, II

Remark 7.1 Using the machinery introduced in this section we have formulated Exercise 4.2. We can find a distinguished set of vanishing cycles for $H_n(U,\mathbb{Z})$, where $U : g = 1,\ \ g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}},\ 2 \le m_i \in \mathbb{N}$. For $i = 1,2,\ldots,n+1$ we take the distinguished set of vanishing cycles $\delta_{i,\beta_i},\ \beta_i = 0,1,\ldots,m_i-2$ given in Exercise 6.5 and define the joint cycles

$$\delta_\beta = \delta_{m_1,\beta_1} * \delta_{m_2,\beta_2} * \cdots * \delta_{m_{n+1},\beta_{n+1}} :=$$

$$\cup_{t\in\Delta^n}\delta_{m_1,\beta_1,t_1}\times\delta_{m_2,\beta_2,t_2}\times\cdots\times\delta_{m_{n+1},\beta_{n+1},t_{n+1}}\in H_n(\{g=1\},\mathbb{Z}),\ \beta\in I,$$

where $I:=\{(\beta_1,\dots,\beta_{n+1})\mid 0\le\beta_i\le m_i-2\}$. These are the same cycles as in Exercise 4.2.

7.10 Intersection of topological cycles

Let us consider two tame polynomials $f,g\in\mathbb{C}[x]$. For two oriented paths $t_\cdot,t'_\cdot$ in $\mathbb{C}$ which intersect each other at b transversally the notation $t_\cdot\times_b^+ t'_\cdot$ means that $t_\cdot$ intersects $t'_\cdot$ in the positive direction, i.e. $dt_\cdot\wedge dt'_\cdot$ is the canonical orientation of $\mathbb{C}$. In a similar way we define $t_\cdot\times_b^- t'_\cdot$, see Figure 7.9.

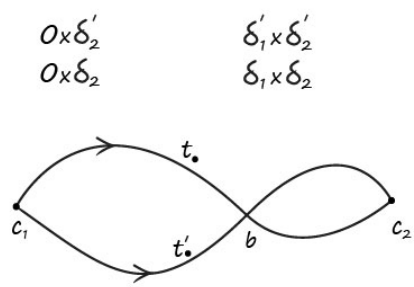

Fig. 7.9 Join over two paths

Theorem 7.5 *Let $(t_\cdot,\delta_1,\delta_2)$ and $(t'_\cdot,\delta'_1,\delta'_2)$ be two admissible triples. Assume that $t_\cdot$ and $t'_\cdot$ intersect each other transversally in their common points. Then*

$$\langle\delta_1*\delta_2,\delta'_1*\delta'_2\rangle=(-1)^{nm+n+m}\sum_b\varepsilon_1(b)\langle\delta_{1b},\delta'_{1b}\rangle\langle\delta_{2b},\delta'_{2b}\rangle,$$

where b runs through all intersection points of $t_\cdot$ and $t'_\cdot$,

$$\varepsilon_1(b)=\begin{cases}1 & t_\cdot\times_b^+ t'_\cdot \text{ and } b \text{ is not a start/end point}\\ -1 & t_\cdot\times_b^- t'_\cdot \text{ and } b \text{ is not a start/end point}\\ (-1)^{\frac{n(n-1)}{2}} & t_\cdot\times_b^+ t'_\cdot \text{ and } b \text{ is a start point}\\ (-1)^{\frac{n(n+1)}{2}+1} & t_\cdot\times_b^- t'_\cdot \text{ and } b \text{ is a start point}\\ (-1)^{\frac{m(m-1)}{2}} & t_\cdot\times_b^+ t'_\cdot \text{ and } b \text{ is an end point}\\ (-1)^{\frac{m(m+1)}{2}+1} & t_\cdot\times_b^- t'_\cdot \text{ and } b \text{ is an end point}\end{cases},$$

and by $\langle 0,0\rangle$ *we mean* 1.

Proof. For simplicity, assume that the start/end points of $t_{\cdot}$ and $t'_{\cdot}$ are critical values with at most nondegenerated singularities. This can be achieved by a versal deformation of critical points of f and g, see also Remark 6.1. Note that if two vanishing cycles vanish along transversal paths in the same singularity then the corresponding thimbles are not necessarily transversal to each other, except when the singularity is nondegenerated.

Let $t_{\cdot}$ intersect $t'_{\cdot}$ transversally at a point b. Let also a_1, a_2, a'_1, a'_2 be the orientation elements of the cycles $\delta_1, \delta_2, \delta'_1, \delta'_2$ and a and a' be the orientation element of $t_{\cdot}$ and $t'_{\cdot}$. We consider two cases:

1. b is not the end/start point of neither $t_{\cdot}$ nor $t'_{\cdot}$. Assume that the cycles δ_1 and δ'_1 (resp. δ_2 and δ'_2) intersect each other at p_1 (resp. p_2) transversally. The cycles $\gamma = \delta_1 * \delta_2$ and $\gamma' = \delta'_1 * \delta_2$ intersect each other transversally at (p_1, p_2). The orientation element of the whole space X obtained by the intersection of γ and γ' is:

$$(a_1 \wedge a \wedge a_2) \wedge (a'_1 \wedge a' \wedge a'_2) = (-1)^{nm+n+m}(a_1 \wedge a'_1) \wedge (a \wedge a') \wedge (a_2 \wedge a'_2)$$

This is $(-1)^{nm+n+m}$ times the canonical orientation of X.

2. $b = c$ is, for instance, the start point of both $t_{\cdot}$ and $t'_{\cdot}$ and δ_1, δ'_1 vanish in the point $p_1 \in \mathbb{C}^{n+1}$ when t tends to c. Assume that the cycles δ_2 and δ'_2 intersect each other transversally at p_2. By assumption, p_1 is a nondegenerated critical point of f and so both cycles γ, γ' are smooth around (p_1, p_2) and intersect each other transversally at (p_1, p_2). The orientation element of the whole space X obtained by the intersection of γ and γ' is:

$$(a_1 \wedge a \wedge a_2) \wedge (a'_1 \wedge a' \wedge a'_2) = (-1)^{(n+1)m}(a_1 \wedge a) \wedge (a'_1 \wedge a') \wedge a_2 \wedge a'_2.$$

Note that $a_1 \wedge a$ has meaning and it is the orientation of the thimble formed by the vanishing of δ_1 at p_1. According to Proposition 7.1, $(a_1 \wedge a) \wedge (a'_1 \wedge a')$ is the canonical orientation of $\mathbb{C}^{n+1}$ multiplied with ε, where $\varepsilon = (-1)^{\frac{n(n+1)}{2}}$ if $t_{\cdot} \times_b^+ t'_{\cdot}$ and $= (-1)^{\frac{n(n+1)}{2}+n+1}$ otherwise. $\square$

One can use Theorem 7.5 to calculate the intersection matrix of $H_n((f+g')^{-1}(b'))$ in the basis given by Theorem 7.4. This calculation in the local case is done by A. M. Gabrielov, see [AGZV88] Theorem 2.11. To state Gabrielov's result in the context of this text take f and g two tame polynomials such that the set C_1 can be separated from C_2 by a real line in $\mathbb{C}$. Then take b a point in that line. The advantage of our calculation is that it works in the global context and the vanishing cycles are constructed explicitly.

Proposition 7.4 *The self intersection of a vanishing cycle of dimension n is given by*

$$(-1)^{\frac{n(n-1)}{2}}(1+(-1)^n). \tag{7.11}$$

Proof. By Proposition 7.3 a joint cycle of two vanishing cycles is also a vanishing cycle. We apply Theorem 7.5 in the case $\delta_1 = \delta'_1$ and $\delta_2 = \delta'_2$ and conclude that the

self-intersection a_n of a vanishing cycle of dimension n satisfies

$$a_{n+m+1} = (-1)^{nm+n+m}((-1)^{\frac{n(n-1)}{2}} a_m + (-1)^{\frac{m(m+1)}{2}+1} a_n),$$

$$a_0 = 2, \;\; n, m \in \mathbb{N}_0.$$

It is easy to see that (7.11) is the only function with the above property. □

Proposition 7.5 (Stabilization) *Let $f(x)$ be an arbitrary tame polynomial, b be a regular value of f and let $\delta_1, \delta_2, \cdots, \delta_\mu$ be a distinguished set of vanishing cycles in $H_n(f^{-1}(b), \mathbb{Z})$. There is a distinguished set of vanishing cycles $\tilde{\delta}_i$ in $H_{n+m+1}(\tilde{f}^{-1}(b), \mathbb{Z})$, where $\tilde{f}(x,y) = f(x) + y_1^2 + y_2^2 + \cdots + y_{m+1}^2$, such that*

$$\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle = (-1)^{nm+n+m+\frac{m(m+1)}{2}+1} \langle \delta_i, \delta_j \rangle, \; i > j,$$

$$\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle = (-1)^{nm+n+m+\frac{m(m-1)}{2}} \langle \delta_i, \delta_j \rangle, \; i < j.$$

This proposition appears in [AGZV88], Theorem 2.14. Note that our system of distinguished paths is oriented anticlockwise whereas this is oriented clockwise in [AGZV88].

Proof. We take $g = y_1^2 + y_2^2 + \cdots + y_{m+1}^2$. Let δ be the unique (up to sign) vanishing cycle in $H_m(g^{-1}(0), \mathbb{Z})$. By Theorem 7.5, the cycles $\tilde{\delta}_i = \delta_i * \delta$ satisfy the desired intersection numbers. □

In proposition 7.5 assume that $n = 0$ and $m = 1$. Choose δ_i's as in §7.6. In this basis (resp. $\tilde{\delta}_i, \; i = 1, 2, \cdots, \mu$) the intersection matrix is:

$$\Psi_0 = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}, \; \tilde{\Psi}_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Proposition 7.6 (Gabrielov [AGZV88] Theorem 2.11) *Let $f = f(x)$ and $g = g(y)$ be two homogeneous tame polynomials and $0 \neq b \in \mathbb{C}$. We choose a distinguished set of vanishing cycles $\delta_i, \; i = 1, 2 \ldots, \mu_1$ (resp. $\gamma_j, \; j = 1, 2 \ldots, \mu_2$) in $H_n(f^{-1}(b), \mathbb{Z})$ (resp. $H_m(g^{-1}(b), \mathbb{Z})$). The intersection matrix in the basis*

$$\delta_i * \gamma_j, \; i = 1, 2, \ldots, \mu_1, \; j = 2, \ldots, \mu_2$$

of $H_{n+m+1}((f+g)^{-1}(b), \mathbb{Z})$ is given by

$$\langle \delta_i * \gamma_j, \delta_{i'} * \gamma_{j'} \rangle =$$

$$\begin{cases} \mathrm{sgn}(j-j')^{n+1}(-1)^{(n+1)(m+1)+\frac{n(n+1)}{2}}\langle\gamma_j,\gamma_{j'}\rangle & \text{if } i'=i \ \&\ j'\neq j, \\ \mathrm{sgn}(i-i')^{m+1}(-1)^{(n+1)(m+1)+\frac{m(m+1)}{2}}\langle\delta_i,\delta_{i'}\rangle & \text{if } j'=j \ \&\ i'\neq i, \\ \mathrm{sgn}(i-i')(-1)^{(n+1)(m+1)}\langle\delta_i,\delta_{i'}\rangle\langle\gamma_j,\gamma_{j'}\rangle & \text{if } (i-i')(j-j')>0, \\ 0 & \text{if } (i-i')(j-j')<0. \end{cases}$$

Proof. This follows from Theorem 7.5. Note that our system of distinguished paths is oriented anticlockwise whereas this is oriented clockwise in [AGZV88]. □

The join $\delta_i * \gamma_j$ is along the unique path (up to homotopy) which connects the critical value 0 of f to the critical value b of $b-g$. For instance, we choose the straight piece of line $t_s = sb,\ 0 \le s \le 1$ as the path for our admissible triples.

Proposition 7.7 *The intersection map in the basis $\delta_\beta,\ \beta \in I$ of Exercise 4.2 is given by:*

$$\langle\delta_\beta,\delta_{\beta'}\rangle = (-1)^{\frac{n(n-1)}{2}}(-1)^{\Sigma_{k=1}^{n+1}\beta'_k-\beta_k}, \tag{7.12}$$

$$\beta = (\beta_1,\beta_2,\ldots,\beta_{n+1}), \beta' = (\beta'_1,\beta'_2,\ldots,\beta'_{n+1})$$

for $\beta_k \le \beta'_k \le \beta_k+1,\ k=1,2,\ldots,n+1, \beta\neq\beta'$, and

$$\langle\delta_\beta,\delta_\beta\rangle = (-1)^{\frac{n(n-1)}{2}}(1+(-1)^n),\ \beta\in I.$$

In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle\delta_\beta,\delta_{\beta'}\rangle = 0$.

Proposition (7.7) is essentially due to F. Pham, see [Pha65] and [AGZV88] p. 66.

Proof. This follows from Proposition 7.6 and induction on n.

Definition 7.6 The Dynkin diagram of a tame polynomial in $n+1$ variables is a graph defined in the following way. Its vertices are in one-to-one correspondence with a distinguished set of vanishing cycles $\delta_i,\ \ i=1,2,\ldots,\mu$. The i-th and j-th vertices of the graph are joined with an edge (of multiplicity $\langle\delta_i,\delta_j\rangle$) if $\langle\delta_i,\delta_j\rangle\neq 0$. For n an odd number, the edge is directed from δ_i to δ_j if $\langle\delta_i,\delta_j\rangle>0$. For n even the graph is not directed and the edge with negative multiplicity is depicted with dashed lines. When $\langle\delta_i,\delta_j\rangle=\pm 1$ then the multiplicities are usually dropped.

The above definition is slightly different from [AGZV88] p. 62 in which one takes a stabilization of a tame polynomial in order to get symmetric intersection pairing. For practical purposes we write down the curve version of Proposition 7.7. Let $f := x^{m_1}$ and $g := b' - y^{m_2}$. The set

$$\delta_i := [\zeta_{m_1}^{i+1} b^{\frac{1}{m_1}}] - [\zeta_{m_1}^{i} b^{\frac{1}{m_1}}],\ i=0,\ldots,m_1-2$$

(resp.

$$\gamma_j := [\zeta_{m_2}^{j+1}(b'-b)^{\frac{1}{m_2}}] - [\zeta_{m_1}^{j}(b'-b)^{\frac{1}{m_2}}],\ j=0,\ldots,m_2-2)$$

is a distinguished set of vanishing cycles for $H_0(\{f=b\},\mathbb{Z})$ (resp. $H_0(\{g=b\},\mathbb{Z})$), where we have fixed a value of $b^{\frac{1}{m_1}}$ and $(b'-b)^{\frac{1}{m_2}}$. See Figure 7.4 for a tentative

picture of the join cycle $\delta_i * \gamma_j$ with $\delta_i = x - y$ and $\gamma_j = x' - y'$. The upper triangle of the intersection matrix in this basis is given by:

$$\langle \delta_i * \gamma_j, \delta_{i'} * \gamma_{j'} \rangle =$$

$$\begin{cases} -1 & \text{if } (i' = i \ \& \ j' = j+1) \vee (i' = i+1 \ \& \ j' = j) \\ +1 & \text{if } (i' = i \ \& \ j' = j-1) \vee (i' = i+1 \ \& \ j' = j+1) \\ 0 & \text{otherwise} \end{cases}.$$

This shows that Figure 7.10 is the associated Dynkin diagram.

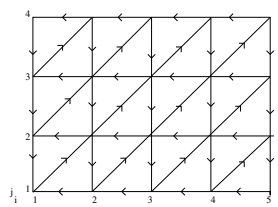

Fig. 7.10 Dynkin diagram of $x^6 + y^5$.

7.11 Exercises

7.1. For a complex manifold of dimension n and a holomorphic nowhere vanishing differential n-form ω on it, the orientation obtained from $\frac{1}{(-2\sqrt{-1})^n}\omega \wedge \overline{\omega}$ differs from the canonical one by $(-1)^{\frac{n(n-1)}{2}}$. Compare also the differential form $\mathrm{Re}(\omega) \wedge \mathrm{Im}(\omega)$ with the canonical orientation. Hint: We can assume that the complex manifold is $(\mathbb{C}^n, 0)$ and $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$.

7.2. For a tame polynomial $f(x_1, \cdots, x_{n+1})$, the Gelfand-Leray form $\frac{dx_1 \wedge dx_2 \wedge \cdots dx_{n+1}}{df}$ restricted to a regular fiber of f is a holomorphic nowhere vanishing differential n-form.

7.3. Recall the notations of Exercise 4.2 in Chapter 5 for $n = 1, m_1 = m_2 = d$, that is, we are dealing with $U : x^d + y^d = 1$. In a similar way, we describe the vanishing cycles $\delta_{\beta,t}$, $t \in \mathbb{C}$ in the curves $U_t : x^d + y^d = t$. We let $t \in [0,1]$ run from 1 to 0 in the real line and we observe that $\delta_{\beta,t}$ vanishes at $0 \in \mathbb{C}^2$. We consider an orientation for the Lefschetz thimble $\Delta_\beta := \cup_{t \in [0,1]} \delta_{\beta,t}$ as it is described in §7.2. This is a two dimensional real submanifold of $\mathbb{C}^2$, provided that we take smooth representations of $\delta_{\beta,t}$'s. From another side we have $x^d + y^d = (x - \zeta_{2d} y)(x - \zeta_{2d}^3 y) \cdots (x - \zeta_{2d}^{2d-1} y)$, and so, we have complex lines $x - \zeta_{2d}^i y = 0$, $i = 1, 3, \ldots 2d-1$ (of real dimension two) which crosses $0 \in \mathbb{C}^2$. Compute the intersection number of these lines with the Lefschetz thimbles.

7.4. Prove that (7.4) for $d = 3$ is an equality. Is this true for $d = 4$? How about arbitrary d?

7.5. In the family of elliptic curves (7.6) fix t_3 and let t_2 varies. Then we get three critical values. Describe the three vanishing cycles, their linear dependence, their relations with the two vanishing cycles in §7.7, and their monodromy around each critical fiber.

7.6. Let $A_1, A_2 \in \mathrm{SL}(2,\mathbb{Z})$ and

$$g_1 := A_2^{-1}A_1^{-1}A_2^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ g_2 := A_1^{-1}A_2^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Prove that $\mathrm{SL}(2,\mathbb{Z})$ is generated with g_1 and g_2. Note that $g_1^2 = g_2^3 = -I$.

7.7. * Let us consider the Dwork family of quintics $X_z, z \in \mathbb{C}$ in $\mathbb{P}^4$. The variety X_z in the homogeneous coordinates is given by

$$z \cdot x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0x_1x_2x_3x_4 = 0.$$

For $z \neq 0, 1, \infty$, X_z is smooth. The group

$$G := \{(\zeta_1, \zeta_2, \cdots, \zeta_5) \mid \zeta_i^5 = 1,\ \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5 = 1\}$$

acts on X_z coordinatewise. Fix $b \in \mathbb{C},\ b \neq 0, 1$. Show that there is a rank 4 subgroup H of $H_3(X_b, \mathbb{Z})$ such that it is invariant under both the monodromy group and G. Moreover, in a basis of H the monodromy around the singularities $z = 0$ and $z = 1$ are respectively given by:

$$M_0 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix},\quad M_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.13}$$

This computation comes originally from [CdlOGP91]. It is one of the main ingredients of the theory of Calabi-Yau modular forms developed in [Mov17a]. Show that the monodromy group, which is generated by M_0 and M_1, has infinite index in $\mathrm{Sp}(4,\mathbb{Z})$. Despite the simplicity of this affirmation, for a long time it was an open problem and it was solved in [BT14].

7.8. Does a join $X * Y$ of two real manifolds X and Y enjoy a structure of a real manifold? The answer to this question in the case of spheres is yes and it follows from (7.8).

7.9. Using Proposition 7.5 construct a symplectic basis of the first homology group of the Riemann surface which is the compactification of the affine curve $x^d + y^2 = 1$.

Chapter 8
Hodge conjecture

One American participant suggested support for some Russian mathematician because "he is working in a good American style." I was puzzled and asked for an explanation. Well, the American answered, "it means that he is traveling a lot to present all his latest results at all our conferences and is personally known to all experts in the field. My opinion is that ISF [International Science Foundation] should better support those who are working in the good Russian style, which is to sit at home working hard to prove fundamental theorems which will remain the cornerstones of mathematics forever! (In an interview with V.I. Arnold, [Lui97]).

8.1 Introduction

In this chapter we present the Hodge conjecture in a form which does not need the Hodge decomposition, that is, using integrals of differential forms over topological cycles. The Hodge conjecture, true or false, is a desire to classify topological cycles supported in algebraic subvarieties. Even if it is false, the desire to classify the cases in which it is true and obtain more criteria to distinguish such cycles in the other cases, is a great challenge in analytical and topological study of algebraic varieties. In this chapter we use a basic terminology of sheaves, complexes etc. The reader who has never seen these before, may consult Bott and Tu's book [BT82] or Voisin's books [Voi02b, Voi03].

8.2 De Rham cohomology

Let X be a C^∞ manifold and $\Omega_X^i,\ i = 0,1,2,\dots$ be the sheaf of C^∞ differentiable i-forms on X. By definition $\mathscr{O}_X = \Omega_X^0$ is the sheaf of C^∞-functions on X. We have the de Rham complex

$$\Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{i-1} \xrightarrow{d} \Omega_X^i \xrightarrow{d} \cdots$$

and the de Rham cohomology of X is defined to be

$$H^m_{\mathrm{dR}}(X) = H^m(\Gamma(X, \Omega_X^\bullet), d) := \frac{\text{global closed } m\text{-forms on } X}{\text{global exact } m\text{-forms on } X}.$$

where $\Gamma(X, \cdot)$ is the set of global sections of the corresponding sheaf. By Poincaré Lemma we know that if X is a unit ball then

$$H^m_{\mathrm{dR}}(X) = \begin{cases} \mathbb{R} & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}.$$

It follows that $\mathbb{R} \to \Omega_X^\bullet$ is the resolution of the constant sheaf $\mathbb{R}$ on a C^∞ manifold X. Since the sheaves Ω_X^m, $m = 0, 1, 2, \ldots$ are fine we conclude that

$$H^m(X, \mathbb{R}) \cong H^m_{\mathrm{dR}}(X),\ i = 0, 1, 2, \ldots,$$

where $H^m(X, \mathbb{R})$ can be interpreted as the Cech cohomology of the constant sheaf $\mathbb{R}$ on X. For the purpose of this chapter we only need to know that an element of $H^m_{\mathrm{dR}}(X)$ is represented by a differential m-form ω in X which is closed, that is, $d\omega = 0$. Further, ω is the zero element of $H^m_{\mathrm{dR}}(X)$ if it is exact, that is, there is a differential $(m-1)$-form η in X such that $\omega = d\eta$.

8.3 Integration

Let $H_m(X, \mathbb{Z})$ be the m-th singular homology of of X. We have the integration map

$$H_m(X, \mathbb{Z}) \times H^m_{\mathrm{dR}}(X) \to \mathbb{R},\ (\delta, \omega) \mapsto \int_\delta \omega,$$

which is defined as follows: let $\delta \in H_m(X, \mathbb{Z})$ be a homology class which is represented by a piecewise smooth m-chain

$$\sum a_i f_i,\ a_i \in \mathbb{Z}$$

and f_i's are C^∞ maps from the standard m-simplex $\Delta^m \subset \mathbb{R}^{m+1}$ to X. Let also ω be a C^∞ global differential form on X. Then

$$\int_\delta \omega := \sum a_i \int_{\Delta^m} f_i^* \omega.$$

By Stokes theorem this definition is well-defined and does not depend on the class of both δ and ω in $H_m(X, \mathbb{Z})$, respectively $H^m_{\mathrm{dR}}(X)$.

8.4 Hodge decomposition

Let X be a complex manifold. All the discussion in the previous chapter is valid replacing $\mathbb{R}$-coefficients with $\mathbb{C}$-coefficients. Let $\Omega_X^{p,q}$ (resp. $Z_X^{p,q}$) be the sheaf of C^∞ differential (p,q)-forms (resp. closed (p,q)-forms) on X. We define

$$H^{p,q} := \frac{\Gamma(X, Z_X^{p,q})}{d\Gamma(\Omega_X^{p+q-1}) \cap \Gamma(X, Z_X^{p,q})}.$$

We have the canonical inclusion:

$$H^{p,q} \hookrightarrow H^m_{\mathrm{dR}}(X)$$

An element $\omega \in H^m_{\mathrm{dR}}(X)$ is in its image if it is represented by a closed (p,q)-form.

Theorem 8.1 (Hodge decomposition) *Let X be a smooth projective variety. We have*

$$H^m_{\mathrm{dR}}(X) = H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{1,m-1} \oplus H^{0,m}, \tag{8.1}$$

which is called the Hodge decomposition.

One can prove the above theorem using harmonic forms, see for instance Voisin's book [Voi02b] page 115 or Green's lectures [GMV94], page 14. The Hodge theory, as we learn it from the literature, starts from the above theorem. For the main purposes of the present book we will not need the above theorem and so we are not going to prove it. We have the conjugation map

$$H^m_{\mathrm{dR}}(X) \to H^m_{\mathrm{dR}}(X),\ \omega \mapsto \bar{\omega} \tag{8.2}$$

which leaves $H^m(X,\mathbb{R})$ invariant and maps $H^{p,q}$ isomorphically to $H^{q,p}$. In order to prove the Hodge decomposition it is enough to prove that:

$$H^m_{\mathrm{dR}}(X) = H^{m,0} + H^{m-1,1} + \cdots + H^{1,m-1} + H^{0,m}. \tag{8.3}$$

The reason for $m = \dim(X)$ is as follows. Let $\alpha_{p,m-p} \in H^{p,m-p},\ p = 0,1,2,\ldots,m$ and $\sum_{p=0}^m \alpha_{p,m-p} = 0$. This equality implies that the wedge product of $\alpha_{p,m-p}$ with any element in $H^m_{\mathrm{dR}}(X)$ is zero, because such a wedge product gives us a (p,q)-form with one of p or q strictly bigger than $\dim(X)$. Since the wedge product is dual to the intersection bilinear map in homology and the latter is nondegenerate, we conclude that $\alpha_{p,m-p} = 0$. For arbitrary m we use the bilinear form (8.4) in the next section and the fact that it is nondegenerate.

8.5 Polarization

Let Y be a hyperplane section of X and let $u \in H^2(X,\mathbb{Z})$ denote the Poincaré dual of the algebraic cycle $[Y] \in H_{2n-2}(X,\mathbb{Z})$, i.e.

$$u \cap [X] = [Y].$$

In the de Rham cohomology $H^2_{\mathrm{dR}}(X)$, u is of type $(1,1)$ and so the Lefschetz decomposition

$$H^m(X,\mathbb{Q}) \cong \oplus_q H^{m-2q}(X,\mathbb{Q})_0$$

for $m \leq n$ is compatible with the Hodge decomposition in the sense that the intersections

$$H_0^{p-q,m-p-q} := H^{p,m-p} \cap H^{m-2q}(X,\mathbb{C})_0, \quad p = 0,1,2,\ldots,$$

gives us a decomposition of $H^{m-2q}(X,\mathbb{C})_0$. We have the bilinear map:

$$\begin{aligned} &H^m_{\mathrm{dR}}(X) \times H^m_{\mathrm{dR}}(X) \to \mathbb{C}, \\ &\langle \omega_1, \omega_2 \rangle := \frac{1}{(2\pi i)^m} \int_X u^{n-m} \wedge \omega_1 \wedge \omega_2, \ m = 0,1,2,\ldots, \end{aligned} \tag{8.4}$$

where $n = \dim_{\mathbb{C}} X$. This is sometimes called the polarization or intersection bilinear form. It is symmetric for m even and alternating otherwise. It is nondegenerate and this follows from the Hard Lefschetz theorem and the fact that the wedge product bilinear map $H^m_{\mathrm{dR}}(X) \times H^{2n-m}_{\mathrm{dR}}(X) \to H^{2n}_{\mathrm{dR}}(X) \cong \mathbb{C}$ is nondegenerate.

Theorem 8.2 *We have*

$$\langle H^{p,m-p}, H^{m-q,q} \rangle = 0 \textit{ unless } p = q, \tag{8.5}$$

$$(-1)^{\frac{m(m-1)}{2}+p} \langle \omega, \bar{\omega} \rangle > 0, \ \forall \omega \in H^{p,m-p} \cap H^m(X,\mathbb{C})_0, \ \omega \neq 0. \tag{8.6}$$

Proof. The first part follows from the fact for ω_k, $k = 1,2$ of type $(p, m-p)$ and $(q, m-q)$, respectively, the $2n$-form $u^{n-m} \wedge \omega_1 \wedge \omega_2$ is identically zero for $p \neq q$.

The proof of the second part uses Harmonic forms and can be found in [Voi02b], Theorem 6.32. We just give a plausiblity argument. Let us assume that ω in a local chart $(z_1, z_2, \ldots, z_n)$ is of the form $f dz^p \wedge d\bar{z}^q$, where $p + q = m$ and f is a local C^∞ function and dz^p (resp. $d\bar{z}^q$) is a wedge product of p (resp. q) dz_i's (resp. $d\bar{z}_i$). This is the case for instance for $p = n = m$. In general, this is not the case and ω is a sum of such terms. Up to positive numbers we have:

$$\begin{aligned} \omega \wedge \bar{\omega} &= |f|^2 dz^p \wedge d\bar{z}^q \wedge d\bar{z}^p \wedge dz^q \\ &= (-1)^{(p+q)q+pq} dz^p \wedge dz^q \wedge d\bar{z}^p \wedge d\bar{z}^q \\ &= (-1)^q (-1)^{\frac{m(m-1)}{2}} dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \\ &= (-1)^q (-1)^{\frac{m(m-1)}{2}} (-2i)^m d\mathrm{Re}(z_{i_1}) \wedge d\mathrm{Im}(z_{i_1}) \wedge \cdots \end{aligned}$$

Note that this argument does not tell us why we have to use the primitive cohomology and what goes wrong if we use the usual cohomology. □

One can mimic the Hodge decomposition and define an (abstract) *Hodge structure*. This together with a polarization will give us the notion of an (abstract) *polarized Hodge structure*. This is left to the reader. The case of curves/Riemann surfaces, that

is $m = n = 1$ is essentially the so-called *Riemann relations*. It gives us the so-called *Siegel domain*, see Exercise 8.5.

Definition 8.1 The filtration

$$0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m_{\mathrm{dR}}(X)$$

with

$$F^p = F^p H^m_{\mathrm{dR}}(X) := H^{m,0} + H^{m-1,1} + \cdots + H^{p,m-p}$$

is called the Hodge filtration.

From an Algebraic Geometry point of view the Hodge filtration is a more natural object than the Hodge decomposition. The pieces of the Hodge decomposition can be recovered from the Hodge filtration in the following way:

$$H^{p,m-p} := F^p \cap \overline{F^{m-p}}$$

Proposition 8.1 *The Hodge filtration with respect to the polarization satisfies the following properties:*

$$\langle F^p, F^q \rangle = 0,\ \forall p,q,\ p+q > m, \tag{8.7}$$

$$(-1)^{\frac{m(m-1)}{2}+p} \langle \omega, \overline{\omega} \rangle > 0,\ \forall \omega \in F^p_0 \cap \overline{F^{m-p}_0},\ \ \omega \neq 0, \tag{8.8}$$

where $F^p_0 := F^p \cap H^m_{\mathrm{dR}}(X)_0$.

Proof. This is just the reformulation of Theorem 8.2. □

8.6 Hodge conjecture I

One of the central conjectures in Hodge theory is the so-called Hodge conjecture. Let m be an even natural number and X a fixed complex compact manifold. Consider a holomorphic map $f : Z \to X$ from a complex compact manifold Z of dimension $\frac{m}{2}$ to X. We have the homology class

$$[Z] \in H_m(X, \mathbb{Z})$$

which is the image of the generator of $H_m(Z, \mathbb{Z})$ (corresponding to the canonical orientation of Z) in $H_m(X, \mathbb{Z})$. The image $\tilde{Z}$ of f in X is a subvariety (probably singular) of X and if $\dim(\tilde{Z}) < \frac{m}{2}$ then using resolution of singularities we can show that $[\tilde{Z}] = 0$. Let us now

$$Z = \sum_{i=1}^{s} r_i Z_i$$

where $Z_i,\ i = 1,2,\dots,s$ is a complex compact manifold of dimension $\frac{m}{2}$, $r_i \in \mathbb{Z}$ and the sum is just a formal way of writing. Let also $f_i : Z_i \to X,\ i = 1,2,\dots,s$ be holomorphic maps. We have then the homology class

$$\sum_{i=1}^{s} r_i[Z_i] \in H_m(X,\mathbb{Z})$$

which is called an algebraic cycle with $\mathbb{Z}$-coefficients, see [BH61].

Definition 8.2 We denote by $\mathrm{Hodge}_m(X,\mathbb{Z})_{\mathrm{alg}}$ the $\mathbb{Z}$-module of algebraic cycles in $H_m(X,\mathbb{Z})$.

Proposition 8.2 *For an algebraic cycle $\delta \in H_m(X,\mathbb{Z})$ we have*

$$\int_\delta \omega = 0, \tag{8.9}$$

For all C^∞ closed (p,q)-form ω on X with $p+q=m,\ p \neq \dfrac{m}{2}$.

Proof. The pull-back of a (p,q)-form with $p+q=m$ and $p \neq \frac{m}{2}$ by f_i is identically zero because at least one of p or q is bigger than $\frac{m}{2}$. □

Any torsion element $\delta \in H_m(X,\mathbb{Z})$ satisfies the property (8.9) and so sometimes it is convenient to consider $H_m(X,\mathbb{Z})$ up to torsions.

Definition 8.3 *A cycle $\delta \in H_m(X,\mathbb{Z})$ with the property (8.9) is called a Hodge cycle. We denote by* $\mathrm{Hodge}_m(X,\mathbb{Z})$ *the $\mathbb{Z}$-module of Hodge cycles in $H_m(X,\mathbb{Z})$. By definition it contains all the torsion elements of $H_m(X,\mathbb{Z})$.*

It is convenient to write down the property of being a Hodge cycle in terms of the Hodge filtration, that is, $\delta \in H_m(X,\mathbb{Z})$ is Hodge if

$$\int_\delta F^{\frac{m}{2}+1} = 0. \tag{8.10}$$

Note that in (8.9) we can assume that $p < \frac{m}{2}$.

Conjecture 8.1 (Hodge conjecture) *For any Hodge cycle $\delta \in H_m(X,\mathbb{Z})$ there is a natural number $a \in \mathbb{N}$ such that $a \cdot \delta$ is an algebraic cycle.*

Note that if δ is a torsion then there is $a \in \mathbb{N}$ such that $a\delta = 0$ and in the way that we have introduced the Hodge conjecture, torsions do not violate the Hodge conjecture.

Definition 8.4 Since algebraic subvarieties of X are canonically oriented, it makes sense to talk about positive elements in $\mathrm{Hodge}_m(X,\mathbb{Z})$. A positive algebraic cycle is namely $\sum_{i=1}^{s} n_i[Z_i]$, $n_i \in \mathbb{N}$, where Z_i's are irreducible algebraic cycles of dimension $\frac{m}{2}$. However, there is no definition of positive elements in the whole homology group $H_m(X,\mathbb{Z})$.

Remark 8.1 If we use l-adic cohomologies and Galois actions instead of de Rham cohomologies and Hodge filtrations then the counterpart of the Hodge conjecture is the so-called *Tate conjecture* which is formulated in [Tat65], see also [Tat94]. In both conjectures one has to find algebraic cycles starting from some cohomological classes. Their equivalence is another challenging problem and it is known for CM abelian varieties, see [Poh68]. For further analogies between these two contexts see Deligne's article [Del71a].

8.7 Hodge conjecture II

In this section all homologies and cohomologies with $\mathbb{Z}$ coefficients are identified with their images in homologies and cohomologies with $\mathbb{C}$ coefficients, and hence, the torsions are killed. We discuss the classical presentation of the Hodge conjecture in the literature, see for instance [Del06]. Let $\delta \in H_m(X,\mathbb{Z})$ be a Hodge cycle and let $\delta^{\mathsf{pd}} \in H^{2n-m}(X,\mathbb{Z})$ be its Poincaré dual. We have

$$\int_\delta \omega = \int_X \omega \wedge \delta^{\mathsf{pd}},\ \omega \in H^m(X,\mathbb{C})$$

and so $\omega \wedge \delta^{\mathsf{pd}} = 0$ for all (p,q)-form ω with $p+q=m,\ p \neq q$. By Hodge decomposition and the fact that the wedge product $H^m(X,\mathbb{C}) \times H^{2n-m}(X,\mathbb{C}) \to H^{2n}(X,\mathbb{C}) \cong \mathbb{C}$ is non-degenerate, we can see that $\delta^{\mathsf{pd}} \in H^{n-\frac{m}{2},n-\frac{m}{2}}$ and so

$$\delta^{\mathsf{pd}} \in H^{n-\frac{m}{2},n-\frac{m}{2}} \cap H^{2n-m}(X,\mathbb{Z}).$$

From now on let $k = n - \frac{m}{2}$.

Definition 8.5 An element of the set $H^{k,k} \cap H^{2k}(X,\mathbb{Z})$ is called a *Hodge class*.

The Poincaré duality gives a bijection between the $\mathbb{Z}$-module of Hodge cycles (up to torsions) and the $\mathbb{Z}$-module of Hodge classes. Note that in terms of the Hodge filtration we can write:

$$H^{k,k} \cap H^{2k}(X,\mathbb{Z}) = F^k \cap H^{2k}(X,\mathbb{Z}).$$

Hodge cycles can be defined over a field in the following way. Let k be a field with $\mathbb{Q} \subset k \subset \mathbb{C}$. We can define as before the set $\mathrm{Hodge}_m(X,k) \subset H_m(X,k)$ of Hodge cycles defined over k. We remark that

$$\dim_{\mathbb{R}} \mathrm{Hodge}_m(X,\mathbb{R}) = h^{\frac{m}{2},\frac{m}{2}}$$

$$\dim_{\mathbb{C}} \mathrm{Hodge}_m(X,\mathbb{C}) = h^{\frac{m}{2},\frac{m}{2}}$$

where $h^{\frac{m}{2},\frac{m}{2}} = \dim H^{\frac{m}{2},\frac{m}{2}}$. We can formulate in a similar way the Hodge conjecture over the field k. In the extreme case $k = \mathbb{R}$ it is trivially false . It is sufficient to give a variety such that

$$\dim_{\mathbb{Q}}\mathrm{Hodge}_m(X,\mathbb{Q}) < h^{\frac{m}{2},\frac{m}{2}}.$$

For instance, the product of two elliptic curves has this property.

8.8 Computational Hodge conjecture

In many practical situations, like Fermat varieties in Chapter 15, we have an explicit basis $\delta_1,\delta_2,\ldots,\delta_\mu$ of $H_m(X,\mathbb{Q})$. Each δ_i is homeomorphic to the sphere $\mathbb{S}^m$ and may not be embedded smoothly in X. In these situations, we are able to write a Hodge cycle in terms of δ_i's.

Conjecture 8.2 (Computational Hodge conjecture) *Given a Hodge cycle*

$$\delta = \sum_{i=1}^{\mu} n_i\delta_i,\ n_i \in \mathbb{Z},$$

construct an algebraic cycle Z such that Z is homologous to δ.

Even in cases where the Hodge conjecture is well-known this conjecture might get non-trivial. For instance, in Chapter 9 we will discuss the Lefschetz $(1,1)$ theorem which is the most well-known case of the Hodge conjecture. Its proof does not shed light on how one has to construct algebraic cycles, see Exercise 9.2. There is another reason why the computational Hodge conjecture is harder. Most of the well-known verifications of the Hodge conjecture are based on the dimension computation as follows. We first compute the dimension of the space of Hodge cycles $s := \dim\mathrm{Hodge}_m(X,\mathbb{Q})$. Then we find a bunch of algebraic cycles $Z_i,\ \ i = 1,2,\ldots,\dim(Z_i) = \frac{m}{2}$ such that their topological class $[Z_i]$ are linearly independent over $\mathbb{Q}$. This proves the Hodge conjecture in this case, however, it does not tell us how one for a given Hodge cycle δ constructs the corresponding $\mathbb{Q}$-linear combination of Z_i's. For further detail of this discussion see the case of Fermat varieties discussed in Chapter 15.

8.9 The $\mathbb{Z}$-module of algebraic cycles

Let X be a smooth projective variety of dimension n and let m be an even number between 0 and $2n$. We consider irreducible algebraic cycle $Z_i,\ i = 1,2,\ldots,s$ of dimension $\frac{m}{2}$ and we ask the following question: Is it possible to check the $\mathbb{Z}$-linear dependence $\sum_{i=1}^{s} r_i[Z_i] = 0$ of $[Z_i] \in H_m(X,\mathbb{Z})$, in case it occurs, by means of algebraic methods? In other words, if $\sum n_iZ_i$ is homologous to zero then can we express this property in the framework of algebraic geometry over an arbitrary algebraically closed field of characteristic zero? The answer to these question lies in the intersection theory of Z_i's. For simplicity we assume $m = n = \dim(X)$. If two algebraic cycles Z_i and Z_j intersect each other transversely, and in particular if the intersection

points of Z_i and Z_j are smooth points of both Z_i and Z_j, then the intersection number $Z_i \cdot Z_j$ is simply the number of such intersection points. In the other extreme case if $Z_i = Z_j$ then the intersection number $Z_i \cdot Z_i$ is an integer which can be computed through adjunction formula, see §17.6. This number coincides with the topological intersection $\langle [Z_i], [Z_i] \rangle$. In general, the intersection of Z_i with Z_j may have many irreducible components of different dimensions. However, we can still define a number $Z_i \cdot Z_j \in \mathbb{Z}$ in the framework of algebraic geometry which coincides with $\langle [Z_i], [Z_j] \rangle$ in the topological context. For all these see A. Weil's book *Foundations of Algebraic Geometry* (1946) and W. Fulton's book *Intersection Theory* (1984).

Theorem 8.3 *Let $Z_1, Z_2, \ldots, Z_s$ be algebraic cycles of dimension $\frac{n}{2}$ in a smooth projective variety X of dimension n. If the $s \times s$ intersection matrix $[Z_i \cdot Z_j]$ has non-zero determinant then the homology classes $[Z_i]$'s are linearly independent in $H_n(X, \mathbb{Z})$.*

Proof. First, let us recall that $Z_i \cdot Z_j$ is the intersection number between the topological cycles $[Z_i]$ and $[Z_j]$. Any $\mathbb{Z}$-linear relation between $[Z_i]$ gives us a $s \times 1$ matrix $v = [v_i]$ such that $[Z_i \cdot Z_j]v = 0$. This implies that $\det([Z_i \cdot Z_j]) = 0$. $\square$

It is sometimes convenient to formulate the if and only if version of Theorem 8.3. This is as follows. Let $X \subset \mathbb{P}^N$ be a smooth projective variety of even dimension n and let Z_∞ be a transversal intersection of a linear subspace $\mathbb{P}^{\frac{n}{2}+1}$ with X. We call $[Z_\infty]$ the algebraic cycle induced by the embedding $X \subset \mathbb{P}^N$. Its Poincaré dual is given by $u^{\frac{n}{2}}$, where u is defined in §8.5.

Let Y be a smooth hyperplane section of X. Recall that an algebraic cycle Z of dimension $\frac{n}{2}$ is called primitive if the intersection of Z with Y is zero in $H_{n-2}(X, \mathbb{Z})$. If $H_{n-2}(X, \mathbb{Z}) \cong \mathbb{Z}$, for instance of X is a smooth hypersurface in $\mathbb{P}^{n+1}$, $n \geq 2$ then the definition of a primitive cycle can be reformulated in the following way.

Definition 8.6 An algebraic cycle Z of dimension $\frac{n}{2}$ is called primitive if

$$Z \cdot Z_\infty = 0.$$

Definition 8.7 For an algebraic cycle Z_i of of dimension $\frac{n}{2}$ let us define

$$\check{Z}_i := \frac{(Z_\infty \cdot Z_\infty)Z_i - (Z_i \cdot Z_\infty)Z_\infty}{\gcd(Z_\infty \cdot Z_\infty, Z_i \cdot Z_\infty)}. \tag{8.11}$$

This new algebraic cycle is characterized by the equality $\langle \check{Z}_i, Z_\infty \rangle = 0$ and the fact that it is not a multiple of a linear combination of Z_i and Z_∞ with $\mathbb{Z}$-coefficients.

The image of $\check{Z}_i$ under $H_n(X, \mathbb{Z}) \to H_{n-2}(Y, \mathbb{Z})$ is zero and and so $\check{Z}_i$ induces an element $[\check{Z}_i] \in H_n(X, \mathbb{Z})_0$.

Theorem 8.4 *Let $Z_1, Z_2, \ldots, Z_s$ be primitive algebraic cycles of dimension $\frac{n}{2}$ in a smooth projective variety X of dimension n. The $s \times s$ intersection matrix $[Z_i \cdot Z_j]$ has non-zero determinant if and only if the homology classes $[Z_i]$'s are linearly independent in $H_n(X, \mathbb{Z})$.*

Proof. The 'only if' part is as in Theorem 8.3. Let us prove the 'if' part. If there is a vector $v \in \mathbb{Z}^s$ such that $[Z_i \cdot Z_j]v = 0$ then for $Z := \sum_{i=1}^s v_i Z_i$ we have

$$Z \cdot Z = v^{\mathrm{tr}}[Z_i \cdot Z_j]v = 0.$$

This together with (8.6) in Theorem 8.2 implies that $[Z]$ is a torsion class in $H_n(X,\mathbb{Z})$. Multiplying Z with a natural number we get a $\mathbb{Z}$-linear relation between $[Z_i]$'s which contradicts the hypothesis. Note that the Poincaré dual of Z is a primitive Hodge class. □

Theorem 8.4 motivates us to make the following definition:

Definition 8.8 Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic zero. We define $\mathrm{Hodge}_n(X,\mathbb{Z})_{0,\mathrm{alg}}$ to be the $\mathbb{Z}$-module of primitive algebraic cycles of dimension $\frac{n}{2}$ modulo $\sum_{i=1}^s v_i Z_i$, where $v = [v_i]$ is a $s \times 1$ matrix with integer entries such that $[Z_i \cdot Z_j]v = 0$.

Note that $\mathrm{Hodge}_n(X,\mathbb{Z})_{0,\mathrm{alg}}$ is defined in the framework of algebraic geometry and it does not have torsions. By Theorem 8.4 if the base field is $\mathbb{C}$ then $\mathrm{Hodge}_n(X,\mathbb{Z})_{0,\mathrm{alg}}$ is the $\mathbb{Z}$-module of primitive algebraic cycles in $H_n(X,\mathbb{Z})_{\mathrm{free}}$ and

$$\mathrm{Hodge}_n(X,\mathbb{Z})_{0,\mathrm{alg}} \subset \mathrm{Hodge}_n(X,\mathbb{Z})_{0,\mathrm{free}}. \tag{8.12}$$

Hodge conjecture claims that the quotient in the above inclusion is a torsion group.

Theorem 8.5 *Let $s \in \mathbb{N}$ be the maximum number of primitive algebraic cycles $Z_1, Z_2, \ldots, Z_s$ of dimension $\frac{n}{2}$ in X such that the determinant $N := \det[Z_i \cdot Z_j]$ of the $s \times s$ intersection matrix is non-zero. Then Z_i's generate the $\mathbb{Z}$-module $\frac{1}{N}\mathrm{Hodge}_n(X,\mathbb{Z})_{0,\mathrm{alg}}$. More precisely, for any other primitive algebraic cycle Z of dimension $\frac{n}{2}$ in X we have the following identity in homology:*

$$Z = [Z_1, Z_2, \ldots, Z_s] \begin{bmatrix} Z_1 \cdot Z_1, & Z_1 \cdot Z_2, & \cdots & Z_1 \cdot Z_s \\ Z_2 \cdot Z_1, & Z_2 \cdot Z_2, & \cdots & Z_2 \cdot Z_s \\ \vdots & \vdots & \ddots & \vdots \\ Z_s \cdot Z_1, & Z_s \cdot Z_2, & \cdots & Z_s \cdot Z_s \end{bmatrix}^{-1} \begin{bmatrix} Z_1 \cdot Z \\ Z_2 \cdot Z \\ \vdots \\ Z_s \cdot Z \end{bmatrix}. \tag{8.13}$$

Proof. By our hypothesis the intersection matrix of the cycles $Z_1, Z_2, \ldots, Z_s, Z$ has zero determinant, and hence, by Theorem 8.4 such cycles are $\mathbb{Z}$-linear dependent in homology. Using the same theorem for the cycles $Z_1, Z_2, \ldots, Z_s$, we know that the coefficient of Z in this linear combination is not zero, and hence, we can write $Z = [Z_1, Z_2, \ldots, Z_s]C$ for some $s \times 1$ matrix with rational coefficients. The intersection of this equality with all Z_i will give the desired equality. □

It is natural to look for the maximum number s for which $|N|$ is the minimum possible. The ideal situation would be when $N = \pm 1$. This does not seem to be realistic. The Hodge conjecture is false when it is formulated over integers (integral Hodge conjecture). In our formulation, see Conjecture 8.1, this means that we cannot take

$a = 1$. In other words, a Hodge cycle $\delta \in H_m(X,\mathbb{Z})$ may not be an algebraic cycle, however, we expect that a multiple of it is algebraic. It is natural to look for a counterexample δ which is a torsion. The first example of torsion non-algebraic homology elements was obtained by Atiyah and Hirzebruch in [AH62] In Exercise 5.4, item 2 we have formulated how one has to look for counterexamples which are not torsions. This has been the main object of study in [Kol92] [Tot97]. In [Kol92] the author shows that if $X \subset \mathbb{P}^4$ is a very general threefold of degree d and $p \geq 5$ is a prime number such that p^3 divides d, the degree of every curve C contained in X is divisible by p. This provides a counterexample to the Hodge conjecture over $\mathbb{Z}$ not involving torsion classes, since it implies that the generator α of $H^4(X,\mathbb{Z})$ is not algebraic whereas $d \cdot \alpha$ is algebraic. In [SV05] Voisin and Soulé remarks that the methods of Atiyah-Hirzebruch and Totaro cannot produce non-algebraic p-torsion classes for prime numbers $p > \dim_{\mathbb{C}}(X)$. They show that for every prime number $p \geq 3$ there exist a fivefold Y and a non-algebraic p-torsion class in $H^6(Y,\mathbb{Z})$. The Hodge decomposition is also valid for Kähler manifolds, however, the Hodge conjecture is not valid in this case, see [Zuc77, Voi02a]

Our discussion in this section is closely related to Grothendieck's standard conjectures, see [Gro69, Kle94]. Theorem 8.4 can be formulated as follows: the numerical and homological equivalence formulated using 'primitive' algebraic cycles are equivalent notions. The general statement on algebraic cycles is a consequence of standard conjectures, see [Kle94] Proposition 5.1, and the Hodge conjecture, see [Kle94] Corollary 5.3. That is why Theorem 8.3 could not be stated in the 'if and only if' format.

8.10 Hodge index theorem

One of the beautiful applications of the positivity of the intersection form in the Hodge decomposition of cohomologies (Theorem 8.2) is the following:

Theorem 8.6 (Hodge index theorem) *Let X be a smooth projective variety of even dimension n and let Y be a transversal hyperplane section of X. Assume that*

1. *Z is a primitive algebraic cycle with $\mathbb{R}$-coefficients of dimension $\frac{n}{2}$ in X, that is, its intersection with Y is zero in $H_{n-2}(X,\mathbb{R})$,*
2. *Z is not numerically equivalent to zero, that is, its intersection with at least one primitive algebraic cycle of dimension $\frac{n}{2}$ and with $\mathbb{R}$-coefficients is not zero.*

Then

$$(-1)^{\frac{n}{2}} Z \cdot Z > 0.$$

For the surface version of this theorem see Theorem 1.9 of Hartshorne's book [Har77]. For another equivalent version of this theorem see Voisin's book [Voi02b], 6.3.2.

Proof. We apply Theorem 8.2 to the Poincaré dual of Z which is in the intersection of the middle piece $H_0^{\frac{n}{2},\frac{n}{2}}$ of $H^n_{\mathrm{dR}}(X)_0$ and $H^n(X,\mathbb{R})_0$. $\square$

An immediate consequence of Theorem 8.6 is the following:

Proposition 8.3 *Let $Z_1, Z_2, \ldots, Z_s$ be primitive algebraic cycles of dimension $\frac{n}{2}$ in a smooth projective variety X of dimension n. Assume that the $s \times s$ intersection matrix $[Z_i \cdot Z_j]$ has non-zero determinant. Then $(-1)^{\frac{n}{2}}[Z_i \cdot Z_j]$ is a positive definite matrix.*

Proof. Let $v \in \mathbb{R}^s$ and $Z := \sum_{i=1}^{s} v_i Z_i$. Using Theorem 8.4 we know that the homology class of Z in $H_n(X, \mathbb{R})$ is not zero. We have $Z \cdot Z = v^{\mathsf{tr}}[Z_i \cdot Z_j]v$ and the proposition follows from Hodge index theorem. □

8.11 Exercises

8.1. Show that the wedge product in the de Rham cohomology corresponds to the cup product in the singular cohomology.

8.2. ** Let X be a real analytic manifold. In the de Rham cohomology of this variety we do not have any kind of (p,q)-form and so it is natural to ask: Is any element of $H_m(X, \mathbb{Z})$ a $\mathbb{Z}$-linear combination of the topological classes of analytic subvarieties of X of dimension m? Discuss this question in few examples.

8.3. Show that the top de Rham cohomology of an oriented compact manifold is one dimensional.

8.4. Show that (8.3) is equivalent to the Hodge decomposition.

8.5 (Siegel domain). A polarized Hodge structure of weight one is the following data.

1. A lattice $(V_{\mathbb{Z}}, \langle \cdot, \cdot \rangle)$ of rank $2g$. This is a free $\mathbb{Z}$-module $V_{\mathbb{Z}}$ of rank $2g$ together with a unimodular skew-symmetric bilinear map $\langle \cdot, \cdot \rangle : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$.
2. A decomposition $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = H^{10} \oplus H^{01}$ such that $\overline{H^{10}} = H^{01}$ and

$$\sqrt{-1}\langle \omega, \bar{\omega} \rangle > 0, \ \forall \omega \in H^{10}, \ \omega \neq 0. \tag{8.14}$$

The first cohomology of a compact Riemann surface enjoys a Hodge structure of weight 1. This is mainly known as *Riemann's bilinear relations*. Define canonical isomorphisms between Hodge structures of weight one and show that the moduli of such objects is $\mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g$, where $\mathbb{H}_g$ is the *Siegel domain*

$$\mathbb{H}_g := \left\{ z \in \mathrm{Mat}(g \times g, \mathbb{C}) \middle| z^{\mathsf{tr}} = z, \ \mathrm{Im}(z) \text{ is a positive matrix} \right\}$$

and $\mathrm{Sp}(2g, \mathbb{Z})$ is the symplectic group acting on $\mathbb{H}_g$ in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az+b)(cz+d)^{-1}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}), \ z \in \mathbb{H}_g.$$

A similar discussion for arbitrary polarized Hodge structures is formulated under the name *period domain*, see [Gri70].

Chapter 9
Lefschetz $(1,1)$ theorem

Of course, the most famous construction issue in algebraic geometry is that of algebraic cycles. Here the famous Hodge conjecture stated first by Hodge some seventy years ago remains the central unsolved problem. The problem is not only unsolved, but other than the Lefschetz $(1,1)$ theorem of the 1920s and its implications, I do not know of a single instance of the construction of an algebraic cycle in a given Hodge class, (P. A. Griffiths in [Gri03] page 3).

9.1 Introduction

Lefschetz $(1,1)$ theorem is the strongest evidence for the Hodge conjecture. It was first stated by S. Lefschetz for curves lying inside projective surfaces and for its original announcement one only needs to know what is a differential 2-form of the first kind on X. In a modern terminology this is a global everywhere holomorphic differential 2-form on X. Lefschetz stated his theorem in the following way: "Pour que Γ_2 soit algébrique, il faut et il suffit que les périodes correspondantes des intégrales de première espèce soient nulles", (S. Lefschetz [Lef24b] page 370). Here Γ_2 is an element in $H_2(X,\mathbb{Z})$. Interestingly enough, another statement is his fundamental theorem and not the mentioned one. "Deux courbes équivalentes sont homologues en tant que cycles, et réciproquement", (S. Lefschetz [Lef24b] page 369). "Perhaps the most illuminating chapter in the book [Lefschetz's book [Lef24b]] is the one dealing with systems of curves on a surface. Here we find a fundamental theorem that two curves are algebraically equivalent if, and only if, they are homologous in the ordinary sense of analysis situs, (Review of [Lef68] by J. W. Alexander in Bull. Amer. Math. Soc. 31 (1925), 558-559). Both these theorems are immediate consequences of a short exact sequence that we discuss it in §9.2. Using his theorems, Lefschetz was able to define the Picard number $\rho = \rho(X)$ in a topological context. "Le nombre de courbes algébriques, indépendantes, a un maximum $\rho \leq R_2$ [the second Betti number]. L'entier ρ n'est autre que celui introduit par M. Picard dans la théorie des intégrales de différentielles totales de troisième espèce (Picard et

Simart, vol. II, Chap. X). Son rapport avec la théorie de l'équivalence des courbes est une des plus belles découvertes de M. Severi (Math. Ann., 1906)", (S. Lefschetz in [Lef24b] page 372). In [Gri79] P. Griffiths gives a moderm presentation of the original proof of Lefschetz $(1,1)$ theorem. This is namely using the theory of Poincaré's *normal functions*, see [Lef68] page 869 for some historical remarks, and [Lew99] Lectures 6 and 14 for a nice presentation of this. Griffiths wanted to use this theory and prove the Hodge conjecture for a wider class of varieties, however, the power of this method is limited to a small class of varieties such as surfaces and cubic fourfolds, for the latter see Zucker's article [Zuc77]. This proof and the modern one which we present it in §9.2 seem to be useless from a computational point of view. In general it is hard to write down explicitly a complete list of curves generating the Picard group of a surface. For the discussion of this for the Fermat surface of degree 12 see Exercise 9.2. In this way the meaning of "constructing algebraic cycles" in the present text is different from what it meant decades ago, for instance from what Griffiths meant in the quotation at the beginning of this chapter.

The present chapter is for the sake of completeness and if the reader is not familiar with its prerequisites, such as sheaf theory, Cech cohomology etc., and does not want to learn these notions from elsewhere, then he or she may skip it. In this way the isomorphism (9.6) must be considered as a definition, and one must assume the Hodge conjecture for surfaces, which is the Lefschetz $(1,1)$ theorem, and its original announcement under Theorem 9.1 and Theorem 9.2.

9.2 Lefschetz $(1,1)$ theorem

In this section we give a sketch of the modern proof of Lefschetz $(1,1)$ theorem. According to P. Deligne in [Del94], this proof is due to K. Kodaira and D. C. Spencer in [KS53]. Here is the announcement of Lefschetz's theorems in his own style.

Theorem 9.1 *On an algebraic surface X a 2-dimensional homology cycle δ is the homology class of an algebraic curve if and only if*

$$\int_\delta \omega = 0, \tag{9.1}$$

for all holomorphic differential 2*-forms in X.*

The differential forms in (9.1) are also called differential forms of the first kind. If we denote by Ω_X^2 the sheaf of regular differnetial forms in X then they correspond to global sections of Ω_X^2. The dimension of the space $H^0(X,\Omega_X^2)$ of such 2-forms is the Hodge number h^{20} of X, which is also called the geometric genus p_g of X.

Definition 9.1 Let X be a projective variety. For two divisors $Y_i,\ \ i=1,2$ in X we say that Y_1 and Y_2 are algebraically dependent if Y_1-Y_2 is the divisor of a rational function on X.

Lefschetz's "Théorème fondamental" reads as follows.

Theorem 9.2 *For divisors in a hypersurface X of dimension ≥ 2, algebraic dependence and homology in X are equivalent relations.*

Both theorems are consequences of the long exact sequence

$$\cdots \to H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X^*) \xrightarrow{\mathrm{cl}} H^2(X,\mathbb{Z}) \xrightarrow{p} H^2(X,\mathscr{O}_X) \to \cdots \tag{9.2}$$

derived from the short exact sequence:

$$0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0. \tag{9.3}$$

The map cl is called the Chern class map. Here $\mathbb{Z}$ is the sheaf of constant functions with values in $\mathbb{Z}$, $\mathscr{O}_X$ is the sheaf of regular functions in X, $\mathscr{O}_X^*$ is the sheaf of regular invertible functions in X and $\mathscr{O}_X \to \mathscr{O}_X^*$ is the exponential map $f \mapsto e^{2\pi i f}$. We have considered the usuall topology of of X and not the Zariski topology. Therefore, the cohomologies in (9.2) are computed using the usuall topology. By Serre's GAGA theorem, see [Ser56], in $H^1(X,\mathscr{O}_X)$, and by a slight modification of the same theorem, in $H^1(X,\mathscr{O}_X^*)$ too, both usual and Zariski topology lead us to the same cohomology groups. However, for $H^i(X,\mathbb{Z}),\ \ i \geq 1$ we have to use the usual topology, see Exercise 9.3. At the time Lefschetz proved his theorem, there was no Cech cohomology theory, and so the modern argument using the long exact sequence of (9.3) didn't exist too.

Let Y be a divisor in X. We associate to Y a line bundle $L_Y \in H^1(X,\mathscr{O}_X^*)$ in X in a natual way. Consider the short exact sequence (9.3) and the corresponding long exact sequence (9.2).

Proposition 9.1 *Let Y be a divisor in X. We have*

$$\mathrm{cl}(L_Y) = \textit{ Poincaré dual of } [Y].$$

Proof. See [GH94], page 141, Proposition 1.

Proof (Sketch of the proof of Theorem 9.2). Since X is a smooth hypersurface of dimension $n \geq 2$, we have $H^1(X,\mathscr{O}_X) = 0$ and so the long exact sequence (9.2) tells us that $H^1(X,\mathscr{O}_X^*) \subset H^2(X,\mathbb{Z})$. Now the theorem follows from Proposition 9.1.

Proof (Sketch of the proof of Theorem 9.1). In $H^2(X,\mathbb{C})$ we have the Hodge filtration

$$\{0\} = F^3 \subset F^2 \subset F^1 \subset F^0 = H^2(X,\mathbb{C})$$

and a canonical isomorphism $F^0/F^1 \cong H^2(X,\mathscr{O}_X)$. Therefore, we have a canonical projection

$$\check{p} : H^2(X,\mathbb{C}) \to H^2(X,\mathscr{O}_X). \tag{9.4}$$

The theorem follows from the fact that the map p and $\check{p}$ are the same, see [GH94] page 163.

Theorem 9.3 (Lefschetz $(1,1)$ theorem) *Let X be a projective variety. Every class in*

$$H^2(X,\mathbb{Z}) \cap F^1(H^2(X,\mathbb{C}))$$

is Poincaré dual of a divisor in X. Here, we write $H^2(X,\mathbb{Z})$ to denote its image in $H^2(X,\mathbb{C})$.

Proof. This is a direct consequence of Proposition 9.1 and the fact that p and $\check{p}$ in (9.4) are the same map.

Note that Lefschetz $(1,1)$ theorem is valid over integers. It follows from the long exact sequence (9.2) that any torsion element in $H^2(X,\mathbb{Z})$ is Poincaré dual of a divisor in X.

Proposition 9.2 *Let X be a projective variety. Every class in*

$$H^{2n-2}(X,\mathbb{Q}) \cap F^{n-1}(H^{2n-2}(X,\mathbb{C}))$$

is Poincaré dual of an algebraic cycle of dimension one (sum of curves) in X.

Proof. This follows from Lefschetz $(1,1)$-theorem and the hard Lefschetz theorem which says that

$$H^2(X,\mathbb{Q}) \to H^{2n-2}(X,\mathbb{Q}),\ \alpha \mapsto \alpha \cup u^{n-2}$$

is an isomorphism, where u is the Poincaré dual of $[Y]$. Note that in general the hard Lefschetz theorem is not valid over integers.

9.3 Picard and Neron-Severi groups

Let X be a smooth algebraic variety over complex numbers.

Definition 9.2 The Picard group is the group of line bundles in X. Taking local sections of line bundles it can be also defined as the following Cech cohomology group:

$$\mathrm{Pic}(X) := H^1(X, \mathscr{O}_X^*).$$

The Picard group can be defined as the group of divisors modulo divisors of rational functions in X. Therefore, we sometime denote an element of $\mathrm{Pic}(X)$ as a sum $D = \sum_i n_i D_i$, where n_i's are integers and D_i's are divisors. For Picard's orginal approach to $\mathrm{Pic}(X)$ see Exercise 9.6.

Let us consider the long exact sequence (9.2). We define

$$\mathrm{Pic}^0(X) := \ker\left(H^1(X, \mathscr{O}_X^*) \xrightarrow{\mathrm{cl}} H^2(X,\mathbb{Z})\right)$$

and the Neron-Severi group

$$\mathrm{NS}(X) := \frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)}.$$

Using the long exact sequence (9.2) it follows that the Neron-Severi group can be considered as a subgroup of $H^2(X,\mathbb{Z})$, and so, it is finitely generated. Its rank

$$\rho = \rho(X) := \mathrm{rank}(\mathrm{NS}(X))$$

is called the Picard number of X. Another useful notation in [PS06] is

$$\rho_0 = \rho_0(X) := \mathrm{rank} H^2(X,\mathbb{Z}) - \rho \tag{9.5}$$

This is sometimes called the dimension of the transcendental lattice in $H^2(X,\mathbb{Z})$.

For a smooth hypersurface X in $\mathbb{P}^{n+1}$ with $n \geq 2$, we have $H^1(X,\mathscr{O}_X) = 0$ and so $\mathrm{Pic}^0(X) = 0$. This implies that the Picard and Neron-Severi groups are the same. From the perspective of the present book this is exactly the group of two dimensional Hodge classes in X. Further, if $n \geq 3$ then $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ and so $\mathrm{NS}(X) \cong \mathbb{Z}$. Therefore, in the following we only consider the case $n = 2$.

Let us return to our topological context. Let X be an arbitrary smooth projective surface and let U be an affine subset of X and Y_i, $i = 1,2,\ldots,s$ be the irreducible components of $Y := X \backslash U$. We only assume that Y_i's are smooth. For instance, we can take a smooth hyperplane section of X, which has just one irreducible component, and call it Y. What we can capture using the topological methods developed in this book is the free part of the following group:

$$\mathrm{NS}(X)_0 := \{D \in \mathrm{NS}(X)_{\mathrm{free}} \mid D \cdot Y_i = 0, \ \ \forall i\}.$$

The intersection $D \cdot Y$ is the usuall intersection of curves in a surface.

Proposition 9.3 *We have*

$$\begin{aligned} \mathrm{NS}(X)_0 &\cong \mathrm{Hodge}_2(X,\mathbb{Z})_0 \\ &:= \frac{\{\delta \in H_2(U,\mathbb{Z}) \mid \int_\delta \omega = 0, \ \textit{for all holomorphic } 2\textit{-form in } X\ \}}{\{\delta \in H_2(U,\mathbb{Z}) \mid \int_\delta \omega = 0, \ \textit{for all holomorphic } 2\textit{-form in } U\ \}}, \end{aligned} \tag{9.6}$$

$$D \mapsto [D].$$

Proof. We know that $\mathrm{NS}(X) \subset H_2(X,\mathbb{Z})$, $D \mapsto [D]$. It is enough to show that a cycle $\delta \in H_2(X,\mathbb{Z})$ is in the image of $H_2(U,\mathbb{Z}) \to H_2(X,\mathbb{Z})$ if and only if $\langle \delta, [Y_i] \rangle = 0$ for all i. The 'only if' part is trivial and so it is enough to prove the 'if' part. If Y is irreducible then this affirmation follows from the Gysin sequence (5.18). In general, we first use the Gysin sequence for the pair $(X, X\backslash Y_1)$ and then conclude that the support of δ is in $X\backslash Y_1$. Then we use the Gysin sequence for $(X\backslash Y_1, X\backslash(Y_1 \cup Y_2))$, and so on. $\square$

Note that the isomorphism (9.6) does not give any information about the torsion elements of the Picard group. Actualy, we have

$$\mathrm{NS}(X)_{\mathrm{tor}} \cong H^2(X,\mathbb{Z})_{\mathrm{tor}}.$$

Finally, let U be an smooth affine variety which is the zero set of a tame polynomial. In this case the irreducible components Y_i of Y are linearly independent in $\mathrm{NS}(X)$, see Exercise 9.4. Therefore

$$\rho(X) = \mathrm{rank}(\mathrm{NS}(X)_0) + \text{ number of components of } Y. \tag{9.7}$$

For a curve C in X, the following sum

$$C - \sum_{i=1}^{s} c_i Y_i,$$

where

$$[c_1,\, c_2, \cdots,\, c_s] := [C\cdot Y_1,\, C\cdot Y_2,\, \cdots,\, C\cdot Y_s] \begin{bmatrix} Y_1\cdot Y_1, & Y_1\cdot Y_2, & \cdots & Y_1\cdot Y_s \\ Y_2\cdot Y_1, & Y_2\cdot Y_2, & \cdots & Y_2\cdot Y_s \\ \vdots & \vdots & \ddots & \vdots \\ Y_s\cdot Y_1, & Y_s\cdot Y_2, & \cdots & Y_s\cdot Y_s \end{bmatrix}^{-1},$$

belongs to $\mathrm{NS}(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}$.

9.4 Exercises

9.1. Describe explicitly the Chern class map cl in (9.2).

9.2. * Lefschetz $(1,1)$-theorem from a computational point of view might get complicated. Consider the Fermat surface X given in the homogenous coordinates by the equation

$$x^d + y^d + z^d + w^d = 0.$$

In this surface we have the line $\mathbb{P}^1 : x - \zeta_{2d} y = z - \zeta_{2d} w = 0$ and its orbit under the permutations of the coordinates and multiplication of the coordinates with powers of ζ_d. This gives us in total $3d^2$ lines in X. In [SSvL10] the authors have proved that if $d \leq 100$ and $(d,6) = 1$, then $\mathrm{NS}(X)$ is generated by lines in X. In [Deg15] it has been proved that for this affirmation we only need the hypothesis $d \leq 4$ or $(d,6) = 1$. For $d = 6,8,9,10$ we have Aoki-Shioda curves introduced in [Aok87] which together with lines generate $\mathrm{NS}(X)$, for more details see Chapter 17. For $d = 12$ the lines and the Aoki-Shioda curves are not enough to generate $\mathrm{NS}(X)$ and so one has to look for other curves in X. Such an algebraic curve D is characterized by the fact that the period

$$\int_D \omega_\beta, \;\; \beta = (0,5,7), \;\; \text{or } (0,3,8)$$

is not zero, for further details on these integrals see Chapter 15. These differential forms appear in Problem 21.10 part 2 in Chapter 21. N. Aoki in his talk [Aok15] has

announced an explicit set of generators in this case, see [AS10] for the description of some generators. As a simple start-up to think on this kind of problems, find a basis of $\mathrm{NS}(X)$ for $d = 1,2,3,4$.

9.3. For examples of a smooth projective variety X and $m \in \mathbb{N}$, show that the Cech cohomologies $H^m(X,\mathbb{Z})$ with Zariski topology and usual topology, are different.

9.4. Let $f(x,y,z)$ be a tame polynomial and assume that 0 is a regular value of f, and hence, $U : f = 0$ is smooth. Let also X be a compactification of U and $X \backslash U$ is a union of smooth curves. Show that Y_i's are linearly independent in the Neron-Severi group of X. Hint: $H^1(U,\mathbb{Z}) = 0$ and then $H^1(X,\mathbb{Z}) = 0$ and $H^1(X,\mathscr{O}_X) = 0$.

9.5 (Differential forms of the second kind). Let X be a smooth projective hypersurface in $\mathbb{P}^3$ and let ω be a meromorphic differential form on X. We say that ω is of the second kind if one of the following is satisfied:

1. ω has no residues along all the components Y_i of the polar set Y of ω, that is,
$$\int_{\sigma(\delta)} \omega = 0$$
for all $\delta \in H_1(U_i,\mathbb{Z})$. Here U_i is Y_i minus its intersection with other components of Y and σ is the map in the Gysin sequence (4.7) for the pair $(X \backslash Y) \subset (X \backslash Y) \cup U_i$.
2. For all $p \in X$ there is a meromorphic differential 1-form η on X such that $\omega - d\eta$ has no poles at p.

Prove that the above statements are equivalent. For further discussion see [PS06] Vol. II, pages 160, 204. For a singular surface X the second statement can be taken as the definition of differential forms of the second kind.

9.6 (Picard's ρ_0 formula). * Let X be a smooth projective hypersurface in $\mathbb{P}^3$. Recall the definition of differential forms of the second kind in Exercise 9.5. Following [PS06], Vol. II, pages 162, 186, we define ρ_0 to be the dimension of

$$\frac{\text{differential 2-forms of the second kind on } X}{\text{exact differential 2-forms on } X}. \tag{9.8}$$

Prove that
$$\rho_0 + \rho := \mathrm{rank} H^2(X,\mathbb{Z}),$$
where ρ is the Picard number of X. This combined with a decomposition of $\mathrm{rank} H^2(X,\mathbb{Z})$, see page 408 of the same book, is Picard's 'formule fondamentale' or Picard's ρ_0-formula.

9.7. This problem is taken from [PS06] Vol. II, page 414, 415. Using Noether's theorem in Chapter 14 show that for a general surface $X \subset \mathbb{P}^3$ of degree d we have
$$\rho_0 = (d-1)^3 - (d-1)^2 + (d-1).$$

9.8. This problem is taken from [PS06] Vol. II, page 415. For a surface in $\mathbb{P}^3$ given in the affine coordinates (x,y,z) by

$$z^d = x^d + P(y),$$

where P is a general polynomial of degree $d \geq 4$ in one variable y, we have

$$\rho = (d-1)^2 + 1, \quad \rho_0 = (d-1)(d-2)^2.$$

In general, computing Picard number of a given surface is a hard task. For more examples see [Sch15].

9.9. In [PS06], Vol. II pages 240, 241, Picard and Simart give a definition of Picard number of a surface in $\mathbb{P}^3$ using logarithmic differential forms. Prove that this definition is equivalemt to the definition in §9.3.

Chapter 10
De Rham cohomology and Brieskorn module

In 1992 there was a special colloquium in honour of Felix Hausdorff in Bonn. Brieskorn started his talk by citing Hausdorff. Hausdorff had spoken the following words at the grave of the mathematician Eduard Study and had cited Friedrich Nietsche from "Zaratustra" with the words.

"Trachte ich denn nach dem Glück?
Ich trachte nach meinem Werke."

(Do I aim for happiness? I do aim for my work.), (G.-M. Greuel in [Gre98] page xxii).

10.1 Introduction

The objective of the present chapter is to develop one of the basic ingredients in the study of multiple integrals. This is the algebraic de Rham cohomology of affine hypersurfaces which is essentially the study of the integrand of multiple integrals. We work with a tame polynomial f and its Brieskorn module, both defined over a general ring R instead of the field of complex numbers. We have tried to keep as much as possible the algebraic language. The origin of many propositions and their topological interpretations will be explained in Chapter 13. When one works with affine varieties in an algebraic context, one does not need the whole algebraic geometry of schemes and one needs only a basic theory of commutative algebra such as rings, ideals, polynomials etc. This is also the case in this chapter and so from algebraic geometry of schemes we only use some standard notations. Exercise 2.8 explains in simple words the final objective of the present chapter in a purely algebraic framework and for the most well-studied example of a tame polynomial.

In this chapter, we would like to capture the de Rham cohomology of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ without pushing our discussion into category theory and hypercohomology. The main strategy is to use a transversal hyperplane section Y and study the de Rham cohomology of the affine variety $U := X \backslash Y \subset \mathbb{C}^{n+1}$ given

by a polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$. Knowing that we have the exact sequence

$$0 \to H^n_{\mathrm{dR}}(X)_0 \to H^n_{\mathrm{dR}}(U) \to H^{n-1}_{\mathrm{dR}}(Y)_0 \to 0, \tag{10.1}$$

we capture both de Rham cohomologies $H^n_{\mathrm{dR}}(X)_0, H^{n-1}_{\mathrm{dR}}(Y)_0$ in the single object $H^n_{\mathrm{dR}}(U)$. Looking in this way, we generalize f to the wider class of tame polynomials with coefficients in a ring R. The advantage is that we first capture a framework for many historical multiple integrals, such as elliptic and hyperelliptic integrals, and second we develop a purely algebraic framework for de Rham cohomologies. Our approach in this chapter is a combination of the works of Atiyah and Hodge in [HA55], Griffiths in [Gri69], Brieskorn in [Bri70] and the author's works [Mov06, Mov07b], and it has new features proper for the main purposes of the present book. For the implementation of the algorithms introduced in this chapter see Chapter 20.

10.2 The base ring

We consider a commutative ring R with multiplicative identity element 1. We assume that R is without zero divisors, i.e. if for some $a, b \in \mathsf{R},\ ab = 0$ then $a = 0$ or $b = 0$. We also assume that R is Noetherian, i.e. it does not contain an infinite ascending chain of ideals. Equivalently, every ideal of R is finitely generated/every set of ideals contains a maximal element. This is named after Emmy Noether the daughter of the mathematician Max Noether.

A multiplicative system in a ring R is a subset S of R containing 1 and closed under multiplication. The localization M_S of an R-module M is defined to be the R-module formed by the quotients $\frac{a}{s},\ a \in M,\ s \in S$. If $S = \{1, a, a^2, \ldots\}$ for some $a \in \mathsf{R},\ a \neq 0$ then the corresponding localization is denoted by M_a. Note that by this notation $\mathbb{Z}_a,\ a \in \mathbb{Z}, a \neq 0$ is no more the set of integers modulo $a \in \mathbb{N}$. By $\check{M}$ we mean the dual of the R-module

$$\check{M} := \{a : M \to \mathsf{R},\ a \text{ is } \mathsf{R} \text{ linear }\}.$$

Usually, we denote a basis or set of generators of M as a column matrix with entries in M. We denote by k the field obtained by the localization of R over $\mathsf{R} \setminus \{0\}$ and by $\bar{\mathsf{k}}$ the algebraic closure of k. In many arguments we need that the characteristic of k to be zero. If this is the case then we mention it explicitly.

For the main purposes of the present text we will need the following examples. 1. R is the field of complex numbers. With this we will explain the original and topological interpretation of many objects such as de Rham cohomology. 2. We take R a localization of a polynomial ring $\mathbb{Q}[t_1, t_2, \ldots, t_n]$. This case is particularly useful to explain the Gauss-Manin connection and the fact that it is defined over rational numbers. The reader may follow the content of this and the next chapter only for these examples of rings.

The properties of R mentioned above are assumed throughout the chapter. There are other properties of R that we will mention when we need them. These are:

1. A natural number d, which will be the degree of a tame polynomial, is invertible in R, see Definition 10.12 and examples of homogeneous tame polynomials in §10.7.
2. The ring R is Cohen-Macaulay, see Proposition 10.3.
3. All the nonzero integers are invertible in R, that is $\mathbb{Q} \subset \mathsf{R}$, see Theorem 10.1.

10.3 Differential forms

In this section we collect many concepts of Differential Geometry in the framework of rings and ideals. Any book on Differential Geometry and Differential Topology will most probably contain all the material of this section in the C^∞-category. The author's favorite source is S. Shahshahani's lecture notes [Sha16] which was originally written in Persian.

Let $\check{\mathsf{R}}$ be another ring as in §10.2 and $\mathsf{R} \hookrightarrow \check{\mathsf{R}}$ be an injective morphism of rings. For most of the discussion in the the present text we will assume that both R and $\check{\mathsf{R}}$ are k-algebras and are finitely generated. The example $\check{\mathsf{R}} = \mathsf{R}[x] = \mathsf{R}[x_1, \cdots, x_{n+1}]$ will be our protagonist. Another example $\check{\mathsf{R}} = \mathsf{R}[x]/\langle f \rangle$ will be used in §10.6.

Definition 10.1 We denote by $\Omega_{\check{\mathsf{R}}/\mathsf{R}}$ the $\check{\mathsf{R}}$-module of relative (Kähler) differentials, that is, $\Omega_{\check{\mathsf{R}}/\mathsf{R}}$ is the quotient of the $\check{\mathsf{R}}$-module freely generated by symbols $dr,\ r \in \check{\mathsf{R}}$, modulo its submodule generated by

$$\begin{aligned}
&dr,\ r \in \mathsf{R},\\
&d(ab) - adb - bda,\\
&d(a+b) - da - db,\ a,b \in \check{\mathsf{R}}.
\end{aligned}$$

If $\check{\mathsf{R}}$ is a finitely generated R-module then the $\check{\mathsf{R}}$-module $\Omega_{\check{\mathsf{R}}/\mathsf{R}}$ is finitely generated too. It is equipped with the derivation

$$d : \check{\mathsf{R}} \to \Omega_{\check{\mathsf{R}}/\mathsf{R}}, \qquad r \mapsto dr.$$

The $\check{\mathsf{R}}$-module $\Omega_{\check{\mathsf{R}}/\mathsf{R}}$is characterized uniquely by its universal property: for any R-linear derivation $D : \check{\mathsf{R}} \to M$ with the $\check{\mathsf{R}}$-module M (that is, D is R-linear and $D(r_1 r_2) = r_1 D r_2 + D(r_1) r_2$ for all $r_1, r_2 \in \check{\mathsf{R}}$), there is a unique $\check{\mathsf{R}}$-linear map $\psi : \Omega_{\check{\mathsf{R}}/\mathsf{R}} \to M$ such that $D = \psi \circ d$, see Exercise 10.1.

Let $L = \mathrm{Spec}(\check{\mathsf{R}})$ and $\mathsf{T} = \mathrm{Spec}(\mathsf{R})$ be the corresponding affine varieties and $L \to \mathsf{T}$ be the map obtained by $\mathsf{R} \hookrightarrow \check{\mathsf{R}}$. Here, for a ring R, $\mathrm{Spec}(\mathsf{R})$ is the spectrum of R. This is the set of all prime ideals of R. We will mainly use the Algebraic Geometry notation

$$\Omega^1_{L/\mathsf{T}} := \Omega_{\check{\mathsf{R}}/\mathsf{R}}.$$

The adjective relative for elements of $\Omega^1_{L/\mathsf{T}}$ is used because of the morphism π.

Definition 10.2 We define

$$\Omega^i_{L/\mathsf{T}} := \bigwedge_{k=1}^{i} \Omega_{\check{\mathsf{R}}/\mathsf{R}}.$$

It is the i-th wedge product of $\Omega^1_{L/\mathsf{T}}$ over $\check{\mathsf{R}}$, that is, $\Omega^i_{L/\mathsf{T}}$ is the quotient of the $\check{\mathsf{R}}$-module freely generated by the symbols $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_i$ modulo its submodule generated by elements which make $\wedge$ $\check{\mathsf{R}}$-linear in each ω_i and

$$\omega_1 \wedge \cdots \wedge \omega_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_i = 0, \text{ for } \omega_j = \omega_{j+1}.$$

It is convenient to define

$$\Omega^0_{L/\mathsf{T}} := \check{\mathsf{R}}.$$

The differential operator

$$d : \Omega^i_{L/\mathsf{T}} \to \Omega^{i+1}_{L/\mathsf{T}}$$

is defined by assuming that it is R-linear and

$$d(ada_1 \wedge \cdots \wedge da_i) = da \wedge da_1 \wedge \cdots \wedge da_i, \quad a, a_1, \ldots, a_i \in \check{\mathsf{R}},$$

see [Har77], page 172. Sometimes it is convenient to remember that d's are defined relative to R. In this case we write

$$d = d_{\mathsf{R}}.$$

One can easily verify that d is in fact well-defined and satisfies all the properties of the classical differential operator on differential forms on manifolds, see for instance Exercises 10.2, 10.3 and 10.4 in this chapter.

Definition 10.3 In the case $\check{\mathsf{R}} := \mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$ we use the notation A instead of L. In this case the set of (relative) differential i-forms is:

$$\Omega^i_{\mathsf{A}/\mathsf{T}} := \left\{ \sum f_{k_1,k_2,\ldots,k_i} dx_{k_1} \wedge dx_{k_2} \wedge \cdots \wedge dx_{k_i} \,\middle|\, f_{k_1,k_2,\ldots,k_i} \in \mathsf{R}[x] \right\}.$$

We are not going to use the language of sheaves and cohomologies. We will mainly use the notation of a sheaf for the set of global sections of the same sheaf. For instance, $\mathscr{O}_{\mathsf{T}} := \mathsf{R}$ and $\Omega_{L/\mathsf{T}} := \Omega_{\check{\mathsf{R}}/\mathsf{R}}$. Whenever we need sheaf theoretic language we mention it explicitly in order to avoid confusion.

Whenever R is the subring of $\check{\mathsf{R}}$ generated by 1 (either $\mathbb{Z}$ or $\mathbb{Z}/N\mathbb{Z}$ for some $N \in \mathbb{N}$) then we write L instead of L/T. For instance, in Chapter 12, we will need the differential forms on R. In this case we simply write $\Omega^1_{\mathsf{T}} := \Omega^1_{\mathsf{R}}$.

Definition 10.4 If we have a surjective morphism of rings

$$i : \check{\mathsf{R}}_1 \to \check{\mathsf{R}}_2.$$

then we say that $L_2 := \mathrm{Spec}(\check{\mathsf{R}}_2)$ is an affine subvariety of $L_1 := \mathrm{Spec}(\check{\mathsf{R}}_1)$. We also say that L_2 is given by the vanishing of the ideal $\ker(i) \subset \check{\mathsf{R}}_1$.

Definition 10.5 Let t be a point in T, that is, a prime ideal in R. The fiber of L/T over t is defined to be $L_t := \mathrm{Spec}(\check{\mathsf{R}}/t)$. It is a subvariety of L. Since we have an injection $\mathsf{R}/t \hookrightarrow \check{\mathsf{R}}/t$, it has canonically defined over$\mathsf{T}_t := \mathrm{Spec}(\mathsf{R}/t)$, and so, we can write L_t/T_t.

10.4 Vector fields

Let L/T be as in the previous section.

Definition 10.6 The $\check{\mathsf{R}}$-module of (relative) vector fields in L is by definition the dual of the $\check{\mathsf{R}}$-module $\Omega^1_{L/\mathsf{T}}$:

$$\Theta_{L/\mathsf{T}} := (\Omega^1_{L/\mathsf{T}})^\vee := \left\{ f : \Omega^1_{L/\mathsf{T}} \to \check{\mathsf{R}} \mid f \text{ is } \check{\mathsf{R}} \text{ linear} \right\}. \tag{10.2}$$

Note that by definition we have a canonical surjective map $\Omega^1_L \to \Omega^1_{L/\mathsf{T}}$ and hence a canonical inclusion $\Theta_{L/\mathsf{T}} \subset \Theta_L$. We use the notations

$$i_v(\omega) = \omega(v) := v(\omega), \;\; v \in \Theta_{L/\mathsf{T}}, \; \omega \in \Omega^1_{L/\mathsf{T}}, \tag{10.3}$$

where we have used the definition of v as an $\check{\mathsf{R}}$-linear map.

Definition 10.7 For a vector field $v \in \Theta_{L/\mathsf{T}}$ the contraction of differential forms with v

$$i_v : \Omega^m_{L/\mathsf{T}} \to \Omega^{m-1}_{L/\mathsf{T}} \tag{10.4}$$

is defined by the following properties:

1. i_v is $\check{\mathsf{R}}$-linear.
2. For $m = 1$ it is just the pairing of differential 1-forms and vector fields in (10.3).
3. It satisfies the following with respect to the wedge product:

$$i_v(\omega_1 \wedge \omega_2) = i_v(\omega_1) \wedge \omega_2 + (-1)^i \omega_1 \wedge i_v(\omega_2), \;\; \omega_1 \in \Omega^i_{L/\mathsf{T}}, \; \omega_2 \in \Omega^j_{L/\mathsf{T}}.$$

One can prove that such an operator exists and is unique, see Exercise (10.5) in this Chapter.

Definition 10.8 Let $L_1 = \mathrm{Spec}(\check{\mathsf{R}}_1)$ be an affine variety and let $L_2 = \mathrm{Spec}(\check{\mathsf{R}}_2)$ be an affine subvariety of L_1, that is, we have a surjective morphism of rings $i : \check{\mathsf{R}}_1 \to \check{\mathsf{R}}_2$. Let $v \in \Theta_{L_1}$ be a vector field. We say that v is tangent to L_2 if

$$df(v) \in \ker(i), \quad \forall f \in \ker(i).$$

Proposition 10.1 *We have*

$$\Theta_{L/\mathsf{T}} = \{v \in \Theta_L \mid v \textit{ is tangent to the fibers of } L \to \mathsf{T}\}. \tag{10.5}$$

Proof. By definition of $\Theta_{L/\mathsf{T}}$ we have a canonical inclusion $\Theta_{L/\mathsf{T}} \subset \Theta_L$, and so, the equality in (10.5) makes sense. The proof is left to the reader, see Exercise 10.6. □

10.5 Cohen-Macaulay rings

To make this section self-sufficient we recall some facts from commutative algebra. For this purpose we have used [Eis95]. Let R be a commutative Noetherin ring with the multiplicative identity 1.

Definition 10.9 The dimension of R is the maximum s of the lengths of chains

$$0 \neq I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_s$$

of prime ideals in R. For a prime ideal $I \subset \mathsf{R}$ we define $\dim(I) = \dim(\frac{\mathsf{R}}{I})$ and $\operatorname{codim}(I) = \dim(\mathsf{R}_I)$ ([Eis95] page 225), where R_I is the localization of R over the complement of I in R.

Definition 10.10 A sequence of elements $a_1, a_2, \ldots, a_{n+1} \in \mathsf{R}$ is called a regular sequence if

$$\langle a_1, a_2, \ldots, a_{n+1} \rangle \neq \mathsf{R}$$

and for $i = 1, 2, \ldots, n+1$, a_i is a non-zero divisor in $\frac{\mathsf{R}}{\langle a_1, a_2, \ldots, a_{i-1} \rangle}$ ([Eis95] page 17). For $I \neq \mathsf{R}$, the depth of the ideal I is the length of a (indeed any) maximal regular sequence in I.

Definition 10.11 The ring R is called Cohen-Macaulay if the codimension and the depth of any proper ideal of R coincide ([Eis95] page 452).

If the Cohen-Macaulay ring R is a domain, i.e. it is finitely generated over a field, then we have

$$\dim(I) + \operatorname{codim}(I) = \dim(\mathsf{R}) \tag{10.6}$$

(this follows from [Eis95], Theorem A, page 221) but in general the equality does not hold. If R is a Cohen-Macaulay ring then $\mathsf{R}[x] = \mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$ is also Cohen-Macaulay ([Eis95] page 452 Proposition. 18.9). In particular, any polynomial ring with coefficients in a field and its localizations are Cohen-Macaulay.

10.6 Tame polynomials

Let $n \in \mathbb{N}_0$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_{n+1}) \in \mathbb{N}^{n+1}$. For a ring R we denote by $\mathsf{R}[x]$ the polynomial ring with coefficients in R and the variable $x := (x_1, x_2, \ldots, x_{n+1})$. We consider

$$\mathsf{R}[x] := \mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$$

as a graded algebra with $\deg(x_i) = \nu_i$. For $n = 0$ (resp. $n = 1$ and $n = 2$) we use the notations x (resp. x, y and x, y, z).

A polynomial $f \in \mathsf{R}[x]$ is called a homogeneous polynomial of degree d with respect to the grading ν if f is a linear combination of monomials of the type

$$x^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}},\ \deg(x^\beta) = \nu \cdot \beta := \sum_{i=1}^{n+1} \nu_i \beta_i = d$$

with coefficients in R. For an arbitrary polynomial $f \in \mathsf{R}[x]$ one can write in a unique way $f = \sum_{i=0}^{d} f_i,\ f_d \neq 0$, where f_i is a homogeneous polynomial of degree i. The number d is called the degree of f. The Jacobian ideal of f is defined to be:

$$\mathrm{jacob}(f) := \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_{n+1}} \rangle \subset \mathsf{R}[x].$$

The Tjurina ideal is

$$\mathrm{tjurina}(f) := \mathrm{jacob}(f) + \langle f \rangle \subset \mathsf{R}[x].$$

We also define the R-modules

$$\mathsf{V}_f := \frac{\mathsf{R}[x]}{\mathrm{jacob}(f)},\ \mathsf{W}_f := \frac{\mathsf{R}[x]}{\mathrm{tjurina}(f)}.$$

These modules may be called the Milnor module and Tjurina module of f, analogous to the objects with the same name in singularity theory, see [Bri70]. In practice one considers V_f as an $\mathsf{R}[f]$-module. If we introduce the new parameter s and define

$$\tilde{f} := f - s \in \tilde{\mathsf{R}}[x],\ \tilde{\mathsf{R}} := \mathsf{R}[s]$$

then $\mathsf{W}_{\tilde{f}}$ as $\tilde{\mathsf{R}}$-module is isomorphic to V_f as $\mathsf{R}[f]$-module. We have introduced V_f because the main machineries of the present text are first developed for $f - s,\ f \in \mathbb{C}[x]$ in the literature of singularities, see [Mov07b].

Definition 10.12 A homogeneous polynomial $g \in \mathsf{R}[x]$ in the weighted ring

$$\mathsf{R}[x],\ \deg(x_i) = \nu_i,\ i = 1, 2, \ldots, n+1$$

has an isolated singularity at the origin if the R-module V_g is freely generated of finite rank. We also say that g is a (homogeneous) tame polynomial in $\mathsf{R}[x]$.

In the case $\mathsf{R} = \mathbb{C}$, a homogeneous polynomial g has an isolated singularity at the origin if the zero set of the ideal $\langle \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n+1}} \rangle$ is the single point $0 \in \mathbb{C}^{n+1}$. This justifies the definition geometrically. The homogeneous polynomial $g \in \mathbb{C}[x]$ has an isolated singularity at the origin if and only if the projective variety induced by $\{g = 0\}$ in $\mathbb{P}^{\nu}$ is a V-manifold/quasi-smooth variety, see Steenbrink [Ste77b]. For the case $\nu_1 = \nu_2 = \cdots = \nu_{n+1} = 1$ the notions of V- and smooth manifolds are equivalent.

Definition 10.13 A polynomial $f \in \mathsf{R}[x]$ is called a tame polynomial if there exist natural numbers $\nu_1, \nu_2, \dots, \nu_{n+1} \in \mathbb{N}$ such that V_g is a freely generated R-module of finite rank (g has an isolated singularity at the origin), where $g = f_d$ is the last homogeneous piece of f in the graded algebra $\mathsf{R}[x]$, $\deg(x_i) = \nu_i$.

In practice, we fix a weighted ring $\mathsf{R}[x]$, $\deg(x_i) = \nu_i \in \mathbb{N}$ and a homogeneous tame polynomial $g \in \mathsf{R}[x]$. The perturbations $g + g_1$, $\deg(g_1) < \deg(g)$ of g are tame polynomials.

In the context of the article [Bro88] the polynomial mapping $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is tame if there is a compact neighborhood U of the critical points of f such that the norm of the Jacobian vector of f is bounded away from zero on $\mathbb{C}^n \backslash U$. It has been proved in the same article (Proposition 3.1) that f is tame if and only if the Milnor number of f is finite and the Milnor numbers of $f^w := f - (w_1 x_1 + \cdots + w_{n+1} x_{n+1})$ and f coincide for all sufficiently small $(w_1, \cdots, w_{n+1}) \in \mathbb{C}^{n+1}$. This and Proposition 10.7 imply that every tame polynomial in the sense of this article is also tame in the sense of [Bro88]. However, the inverse may not be true, for instance take $f = x^2 + y^2 + x^2 y^2$, see [Sch05] for other examples.

Throughout the present text we will work with a fixed homogeneous tame polynomial g of degree d and its deformation $f = g + \cdots$ by monomials of degree $< d$. Later in Proposition 10.7 we will see that the R-module V_f is freely generated by x^β, $\beta \in I$. We use the following notations. We fix a basis

$$x^I := \left\{ x^\beta \middle| \beta \in I \right\}$$

of monomials for the R-module V_g, see Exercise 10.7. We also define

$$A_\beta := \sum_{i=1}^{n+1} (\beta_i + 1) \frac{\nu_i}{d}, \tag{10.7}$$

$$\mu := \#I = \operatorname{rank} \mathsf{V}_g, \tag{10.8}$$

$$\widehat{dx_i} := dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}, \tag{10.9}$$

$$\eta := \sum_{i=1}^{n+1} (-1)^{i-1} \frac{\nu_i}{d} x_i \widehat{dx_i}, \tag{10.10}$$

$$dx := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}, \tag{10.11}$$

$$\eta_\beta := x^\beta \eta, \tag{10.12}$$

$$\omega_\beta := x^\beta dx, \ \ \beta \in I, \tag{10.13}$$

One may call μ the Milnor number of g. J. Milnor in [Mil68] proves that in the case $\mathsf{R} = \mathbb{C}$ there are small neighborhoods $U \subset \mathbb{C}^{n+1}$ and $S \subset \mathbb{C}$ of the origins such that $g : U \to S$ is a C^∞ fiber bundle over $S \backslash \{0\}$ whose fiber is of homotopy type of a bouquet of μ n-spheres. We have seen a similar statement for tame polynomials in Chapter 7.

Let

$$\mathsf{T} := \mathrm{Spec}(\mathsf{R}),\ \mathsf{A} := \mathrm{Spec}(\mathsf{R}[x])$$

and denote by $\pi : \mathsf{A} \to \mathsf{T}$ the canonical morphism obtained by the inclusion $\mathsf{R} \subset \mathsf{R}[x]$. Any $\omega \in \Omega^i_{\mathsf{A}/\mathsf{T}}$ can be written as $\omega = \sum P_k dx_{k_1} \wedge dx_{k_2} \wedge \cdots \wedge dx_{k_i},\ k_1 < k_2 < \cdots < k_i$ with $P_k \in \mathsf{R}[x]$. We define $\deg(P_k dx_{k_1} \wedge dx_{k_2} \wedge \cdots \wedge dx_{k_i}) = \deg(P_k) + \nu_{k_1} + \cdots + \nu_{k_i}$, for which have have assumed $\deg(dx_j) = \nu_j$. With the above rules, $\Omega^i_{\mathsf{A}/\mathsf{T}}$ turns into a graded $\mathsf{R}[x]$-module and we can talk about homogeneous differential forms and decomposition of a differential form into homogeneous pieces. A geometric way to look at this is the following: the multiplicative group $\mathsf{R}^* = \mathsf{R} \setminus \{0\}$ acts on A by:

$$(x_1, x_2, \ldots, x_{n+1}) \to (\lambda^{\nu_1} x_1, \lambda^{\nu_2} x_2, \ldots, \lambda^{\nu_{n+1}} x_{n+1}),\ \lambda \in \mathsf{R}^*.$$

We also denote the above map by $\lambda : \mathsf{A} \to \mathsf{A}$. The polynomial form $\omega \in \Omega^i_{\mathsf{A}/\mathsf{T}}$ is weighted homogeneous of degree m if

$$\lambda^*(\omega) = \lambda^m \omega,\ \lambda \in \mathsf{R}^*.$$

For the homogeneous polynomial g of degree d this means that

$$g(\lambda^{\nu_1} x_1, \lambda^{\nu_2} x_2, \ldots, \lambda^{\nu_{n+1}} x_{n+1}) = \lambda^d g(x_1, x_2, \ldots, x_{n+1}),\ \forall \lambda \in \mathsf{R}^*.$$

The reader who wants to follow the present text in a geometric context may assume that $\mathsf{R} = \mathbb{C}[t_1, t_2, \ldots, t_s]$ and hence identify T and A with their geometric points, that is,

$$\mathsf{T} = \mathbb{C}^s,\ \mathsf{A} = \mathbb{C}^{n+1} \times \mathbb{C}^s.$$

The map $\pi : \mathsf{A} \to \mathsf{T}$ is now the projection on the last s coordinates.

10.7 Examples of tame polynomials

In this section we list few examples of tame polynomials. In all the examples we discuss we need that the degree d of the homogeneous tame polynomial to be invertible in R, see Exercise 10.11.

Example 10.1 Consider the case $n = 0,\ \deg(x) = 1$. For $g = x^d$ we have

$$\mathsf{V}_g = \oplus_{i=0}^{d-2} \mathsf{R} \cdot x^i \oplus \oplus_{i=d-1}^{\infty} \mathsf{R}/(d \cdot \mathsf{R}) \cdot x^i$$

and so g is tame if and only if d is invertible in R. A basis of the R-module V_g is given by $x^I = \{1, x, x^2, \dots, x^{d-2}\}$. In this case

$$f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0,\ t_i \in \mathsf{R}$$

is a tame polynomial in $\mathsf{R}[x]$ (for simplicity we have used $x = x_1$).

Example 10.2 One of the most important class of tame polynomials are the so called hyperelliptic polynomials

$$f := y^2 + t_d x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \in \mathsf{R}[x, y],$$

$$\deg(x) := 2,\ \deg(y) := d,$$

with $g := y^2 + t_d x^d$. If d is even then we can also take $\deg(x) = 1,\ \deg(y) := \frac{d}{2}$. We assume that t_d and $2d$ are invertible in R. A basis of the R-module V_g (and hence of V_f) is given by $x^I := \{1, x, x^2, \dots, x^{d-2}\}$. In this example we have:

$$A_i = \frac{1}{2} + \frac{i+1}{d},\ \eta := \frac{1}{d}xdy - \frac{1}{2}ydx,$$

and

$$\frac{x^i dx}{y} = -2\frac{x^i dx \wedge dy}{df} = \frac{-2}{A_i}\nabla_{\frac{\partial}{\partial t_0}}(x^i \eta). \tag{10.14}$$

The first equality will be explained in §10.11 (Definition 10.16) and last equalities will be explained in §12.2. Note that $\frac{x^i dx}{y}$ are classical integrands of hyperelliptic integrals discussed in Chapter 2 and Chapter 3. For further details see [BM00].

Example 10.3 In the weighted ring $\mathsf{R}[x]$, $\deg(x_i) = \nu_i \in \mathbb{N}$ for a given degree $d \in \mathbb{N}$, we would like to have at least one homogeneous tame polynomial of degree d. For instance, if

$$m_i := \frac{d}{\nu_i} \in \mathbb{N}_{\geq 2},\ \ i = 1, 2, \dots, n+1$$

and all m_i's are invertible in R then the homogeneous polynomial

$$g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$$

is tame. A basis of the R-module V_g is given by

$$I = \{x^\beta \mid 0 \leq \beta_i \leq m_i - 2,\ i = 1, 2, \dots, n+1\}.$$

One may try to find homogeneous tame polynomials for other d's, see Exercise 10.10.

Example 10.4 Let us consider all monomials $x^\alpha,\ \alpha \in J$ of degree $\leq d$ and for each monomial a parameter t_α. The polynomial

$$f = \sum_{\alpha \in J} t_\alpha x^\alpha \in \mathsf{R}[x],$$

over the ring $\mathsf{R} = \mathbb{Q}[t_\alpha,\ \alpha \in J]$ is called a complete polynomial. We want to see when f is a tame polynomial. Let $\tilde{\mathsf{R}} \subset \mathsf{R}$ be the polynomial ring generated by the coefficients of the last homogeneous piece g of f. Let also $\tilde{\mathsf{k}}$ be the field obtained by the localization of $\tilde{\mathsf{R}}$ over $\tilde{\mathsf{R}}\backslash\{0\}$. Assume that the polynomial $g \in \tilde{\mathsf{k}}[x]$ has an isolated singularity at the origin and so it has an isolated singularity at the origin as a polynomial in a localization $\tilde{\mathsf{R}}_a$ of $\tilde{\mathsf{R}}$ for some $a \in \mathsf{R}$. The variety $\{a = 0\}$ contains the locus of parameters for which g has not an isolated zero at the origin. It may contain more points. To find such an a we choose a monomial basis $x^\beta,\ \beta \in I$ of $\tilde{\mathsf{k}}[x]/\mathrm{jacob}(g)$ and write all $x_i x^\beta,\ \beta \in I,\ i = 1,2,\ldots,n+1$ as a $\tilde{\mathsf{k}}$-linear combination of x^β's and a residue in $\mathrm{jacob}(g)$. The product of the denominators of all the coefficients (in $\tilde{\mathsf{k}}$) used in the mentioned equalities is a candidate for a. The obtained a depends on the choice of the monomial basis. Now, a complete polynomial is tame over $\mathsf{R}_a[x]$. An arbitrary tame polynomial f over any ring is a specialization of a unique complete tame polynomial, called the completion of f.

Example 10.5 The parameter space of few tame polynomials can be interpreted as moduli spaces. Here are two examples:

$$\begin{aligned} &f := y^2 - 4x^3 + t_2 x + t_3, \\ &f := y^3 - x^4 + yP(x) + Q(x),\ \deg(P),\ \deg(Q) \leq 2. \end{aligned}$$

For the first tame polynomial, $f = 0$ is the universal family of (E, ω), where E is an elliptic curve and ω is a regular differential 1-form on E, see for instance [Mov12]. For the second tame polynomial, $f = 0$ is the universal family of (C, p, v), where C is a non-hyperelliptic curve of genus 3, p is a point of C such that $4P$ is a canonical divisor and v is a vector in the Zariski tangent space of C at p, see for instance [Tho15].

Example 10.6 Let us consider the homogeneous polynomial $g := -4x^3 + y^2 w - \frac{1}{2} w^4$ of degree 24 in the weighted ring

$$\mathsf{R}[x, y, w],\ \deg(x) = 8,\ \deg(y) = 9,\ \deg(w) = 6,\ \ \mathsf{R} := \mathbb{Q}[a, b, c, d]. \tag{10.15}$$

This polynomial has an isolated singularity at the origin and so

$$f := y^2 w - 4x^3 + 3axw^2 + bw^3 + cxw - \frac{1}{2}(dw^2 + w^4) \tag{10.16}$$

is tame. It has been extensively studied in [CD12, CD11, DMWH16].

10.8 De Rham lemma

In this section we state the de Rham lemma for a homogeneous tame polynomial. Originally, a similar Lemma was stated for a germ of holomorphic function f :

$(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ in [Bri70], page 110. For the definition of a Cohen-Macaulay ring and depth of an ideal see §10.5.

Proposition 10.2 *Let* R *be a Cohen-Macaulay ring and* g *be a homogeneous tame polynomial in* $\mathsf{R}[x]$. *The depth of the Jacobian ideal* $\mathrm{jacob}(g) \subset \mathsf{R}[x]$ *of* g *is* $n+1$.

Proof. Let $I := \mathrm{jacob}(g) \subset \mathsf{R}[x]$ we have:

$$\mathrm{codim}(I) := \mathrm{dim}\mathsf{R}[x]_I = \mathrm{dim}\mathsf{k}[x]_{\bar{I}} = \mathrm{dim}\mathsf{k}[x] - \mathrm{dim}\bar{I} = n+1.$$

Here $\bar{I}$ is the Jacobian ideal of g in $\mathsf{k}[x]$, where k is the quotient field of R. In the second and last equalities we have used the fact that g is tame and hence I does no contain any non-zero element of R and $\mathrm{dim}\bar{I} := \mathrm{dim}(\frac{\mathsf{k}[x]}{\bar{I}}) = 0$. We have also used $\mathrm{dim}(\mathsf{k}[x]) = n+1$ ([Eis95] Theorem A, page 221). We conclude that the depth of $\mathrm{jacob}(g) \subset \mathsf{R}[x]$ is $n+1$. □

Proposition 10.3 (de Rham Lemma) *Let* R *be a Cohen-Macaulay ring and* g *be a homogeneous tame polynomial in* $\mathsf{R}[x]$. *An element* $\omega \in \Omega^i_{\mathsf{A}/\mathsf{T}}, i \le n$ *is of the form* $dg \wedge \eta,\ \eta \in \Omega^{i-1}_{\mathsf{A}/\mathsf{T}}$ *if and only if* $dg \wedge \omega = 0$. *This means that the following sequence is exact*

$$0 \to \Omega^0_{\mathsf{A}/\mathsf{T}} \overset{dg\wedge\cdot}{\to} \Omega^1_{\mathsf{A}/\mathsf{T}} \overset{dg\wedge\cdot}{\to} \cdots \overset{dg\wedge\cdot}{\to} \Omega^n_{\mathsf{A}/\mathsf{T}} \overset{dg\wedge\cdot}{\to} \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}. \tag{10.17}$$

In other words

$$H^i(\Omega^\bullet_{\mathsf{A}/\mathsf{T}}, dg \wedge \cdot) = 0,\ i = 0,1,\ldots,n.$$

Proof. We have proved the depth of $\mathrm{jacob}(g) \subset \mathsf{R}[x]$ is $n+1$. Knowing this the above proposition follows from the main theorem of [Sai76]. See also [Eis95] Corollary 17.5 page 424, Corollary 17.7 page 426 for similar topics. □

The sequence in (10.17) is also called the Koszul complex.

Proposition 10.4 *The following sequence is exact*

$$0 \overset{d}{\to} \Omega^0_{\mathsf{A}/\mathsf{T}} \overset{d}{\to} \Omega^1_{\mathsf{A}/\mathsf{T}} \overset{d}{\to} \cdots \overset{d}{\to} \Omega^n_{\mathsf{A}/\mathsf{T}} \overset{d}{\to} \Omega^{n+1}_{\mathsf{A}/\mathsf{T}} \overset{d}{\to} 0.$$

In other words

$$H^i_{\mathrm{dR}}(\mathsf{A}/\mathsf{T}) := H^i(\Omega^\bullet_{\mathsf{A}/\mathsf{T}}, d) = 0,\ i = 1,2,\ldots,n+1.$$

Proof. This is [Eis95], Exercise 16.15 c, page 414. □

Note that in the above proposition we do not need R to be Cohen-Macaulay. Later, we will need the following proposition.

Proposition 10.5 *Let* R *be a Cohen-Macaulay ring. If for* $\omega \in \Omega^i_{\mathsf{A}/\mathsf{T}},\ 1 \le i \le n-1$ *we have*

$$d\omega = dg \wedge \omega_1,\ \textit{for some } \omega_1 \in \Omega^i_{\mathsf{A}/\mathsf{T}} \tag{10.18}$$

then there is an $\omega' \in \Omega^{i-1}_{\mathsf{A}/\mathsf{T}}$ *such that*

$$d\omega = dg \wedge d\omega'.$$

Proof. Since g is homogeneous, in (10.18) we can assume that

$$\deg_x(\omega_1) = \deg_x(d\omega) - d \text{ and so } \deg_x(\omega_1) < \deg_x(d\omega) \leq \deg_x(\omega).$$

We take differential of (10.18) and use Proposition 10.3. Then we have $d\omega_1 = dg \wedge \omega_2$, and again we can assume that $\deg_x(\omega_2) < \deg_x(\omega_1)$. We obtain a sequence of differential forms $\omega_k,\ k = 0,1,2,3,\ldots,\ \omega_0 = \omega$ with decreasing degrees and $d\omega_{k-1} = dg \wedge \omega_k$. Therefore, for some $k \in \mathbb{N}$ we have $\omega_k = 0$. We claim that for all $0 \leq j \leq k$ we have $d\omega_j = dg \wedge d\omega'_j$ for some $\omega'_j \in \Omega^{i-1}_{\mathsf{A}/\mathsf{T}}$. We prove our claim by decreasing induction on j. For $j = k$ it is already proved. Assume that it is true for j. Then by Proposition 10.4 we have

$$\omega_j = dg \wedge \omega'_j + d\omega'_{j-1},\ \omega'_{j-1} \in \Omega^{i-1}_{\mathsf{A}/\mathsf{T}}.$$

Putting this in $d\omega_{j-1} = dg \wedge \omega_j$, our claim is proved for $j-1$. □

Now, we are ready to state the de Rham lemma for tame polynomials.

Proposition 10.6 (de Rham lemma for tame polynomials) *Proposition 10.3 is valid replacing g with an arbitrary tame polynomial f.*

Proof. If there is $\omega \in \Omega^i_{\mathsf{A}/\mathsf{T}},\ i \leq n$ such that $df \wedge \omega = 0$ then $dg \wedge \omega' = 0$, where ω' is the last homogeneous piece of ω. We apply Proposition 10.3 and conclude that $\omega = df \wedge \omega_1 + \omega_2$ for some $\omega_1 \in \Omega^{i-1}_{\mathsf{A}/\mathsf{T}}$ and $\omega_2 \in \Omega^i_{\mathsf{A}/\mathsf{T}}$ with $\deg(\omega_2) < \deg(\omega)$ and $df \wedge \omega_2 = 0$. We repeat this argument for ω_2. Since the degree of ω_2 is decreasing, at some point we will get $\omega_2 = 0$ and then the desired form of ω. □

Recall that in §10.6 we fixed a monomial basis x^I for the R-module V_g.

Proposition 10.7 *For a tame polynomial f, the R-module V_f is freely generated by x^I.*

Proof. Let $f = f_0 + f_1 + f_2 + \cdots + f_{d-1} + f_d$ be the homogeneous decomposition of f in the graded ring $\mathsf{R}[x]$, $\deg(x_i) = \nu_i$ and $g := f_d$ be the last homogeneous piece of f. Let also $F = f_0 x_0^d + f_1 x_0^{d-1} + \cdots + f_{d-1} x_0 + g$ be the homogenization of f. We claim that the set x^I generates freely the $\mathsf{R}[x_0]$-module $V := \mathsf{R}[x_0, x]/\langle \frac{\partial F}{\partial x_i} \mid i = 1,2,\ldots,n+1\rangle$. More precisely, we prove that every element $P \in \mathsf{R}[x_0, x]$ can be written in the form

$$P = \sum_{\beta \in I} C_\beta x^\beta + R,\ R := \sum_{i=1}^{n+1} Q_i \frac{\partial F}{\partial x_i}, \tag{10.19}$$

$$\deg_x(R) \leq \deg_x(P),\ C_\beta \in \mathsf{R}[x_0],\ Q_i \in \mathsf{R}[x_0, x]. \tag{10.20}$$

Since x^I is a basis of V_g, we can write

$$P = \sum_{\beta \in I} c_\beta x^\beta + R', \tag{10.21}$$

$$R' = \sum_{i=1}^{n+1} q_i \frac{\partial g}{\partial x_i},\ c_\beta \in \mathsf{R}[x_0],\ q_i \in \mathsf{R}[x_0, x].$$

We can choose q_i's so that

$$\deg_x(R') \leq \deg_x(P). \tag{10.22}$$

If this is not the case then we write the non-trivial homogeneous equation of highest degree obtained from (10.21). Note that $\frac{\partial g}{\partial x_i}$ is homogeneous. If some terms of P occur in this new equation then we have already (10.22). If not we subtract this new equation from (10.21). We repeat this until getting the first case and so the desired inequality. Now we have

$$\frac{\partial g}{\partial x_i} = \frac{\partial F}{\partial x_i} - x_0 \sum_{j=0}^{d-1} \frac{\partial f_j}{\partial x_i} x_0^{d-j-1},$$

and so

$$P = \sum_{\beta \in I} c_\beta x^\beta + R_1 - P_1, \tag{10.23}$$

where

$$R_1 := \sum_{i=1}^{n+1} q_i \frac{\partial F}{\partial x_i},\ P_1 := x_0 (\sum_{i=1}^{n+1} \sum_{j=0}^{d-1} q_i \frac{\partial f_j}{\partial x_i} x_0^{d-j-1}).$$

From (10.22) we have

$$\deg_x(P_1) \leq \deg_x(P) - 1,\ \deg_x(R_1) \leq \deg_x(P).$$

We write again $q_i \frac{\partial f_j}{\partial x_i}$ in the form (10.21) and substitute it in (10.23). By degree conditions this process stops and at the end we get the equation (10.19) with the conditions (10.20).

Now let us prove that x^I generates the $\mathsf{R}[x_0]$-module V freely. If the elements of x^I are not $\mathsf{R}[x_0]$-independent then we have

$$\sum_{\beta \in I} C_\beta x^\beta = 0$$

in V for some $C_\beta \in \mathsf{R}[x_0]$ or equivalently

$$\sum_{\beta \in I} C_\beta x^\beta = dF \wedge \omega \tag{10.24}$$

for some $\omega = \sum_{i=1}^{n+1} Q_i[x, x_0] \widehat{dx_i},\ Q_i \in \mathsf{R}[x, x_0]$, where d is the differential with respect to $x_i,\ i = 1, 2, \ldots, n+1$ and hence $dx_0 = 0$. Since F is homogeneous in (x, x_0), we can assume that in the equality (10.24) the $\deg_{(x,x_0)}$ of the left hand side is $d + \deg_{(x,x_0)}(\omega)$. Let $\omega = \omega_0 + x_0 \omega_1$ and ω_0 does not contain the variable x_0. In the equation obtained from (10.24) by putting $x_0 = 0$, the right hand side side must

be zero otherwise we have a non-trivial relation between the elements of x^I in V_g. Therefore, we have $dg \wedge \omega_0 = 0$ and so by de Rham lemma (Proposition 10.3)

$$\omega_0 = dg \wedge \omega' = dF \wedge \omega' + x_0(\frac{g-F}{x_0}) \wedge \omega',$$

with $\deg_x(\omega_0) = d + \deg(\omega')$. Substituting this in ω and then ω in (10.24) we obtain a new ω with the property (10.24) and strictly less $\deg_x$. $\square$

Proposition 10.7 implies that f and its last homogeneous piece have the same Milnor number.

10.9 The discriminant of a polynomial

Definition 10.14 Let A be the R-linear map in V_f induced by multiplication by f. According to (10.7), V_f is freely generated by x^I and so we can talk about the matrix of A in the basis x^I. For a new parameter s define

$$S(s) := \det(A - s \cdot I_{\mu \times \mu}),$$

where $I_{\mu \times \mu}$ is the identity μ times μ matrix and $\mu = \#I$. It has the property $S(f)\mathsf{V}_f = 0$. We define the discriminant of f to be

$$\Delta = \Delta_f := S(0) \in \mathsf{R}.$$

The discriminant has the following property

$$\Delta_f \cdot \mathsf{W}_f = 0. \tag{10.25}$$

In general Δ_f is not the the simplest element in R with the property (10.25). For the comparison of Δ_f with the notion of discriminant in the zero dimensional case see Exercise 10.12.

Proposition 10.8 *Let* R *be an algebraically closed field. We have* $\Delta_f = 0$ *if and only if the affine variety* $\{f = 0\} \subset \mathsf{R}^{n+1}$ *is singular.*

Proof. $\Leftarrow$: If $\Delta_f \neq 0$ then A is surjective and $1 \in \mathsf{R}[x]$ can be written in the form $1 = \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i} q_i + qf$. This implies that the variety $Z := \{\frac{\partial f}{\partial x_i} = 0,\ i = 1,2,\ldots,n+1,\ f = 0\}$ is empty.

$\Rightarrow$: If $\{f = 0\}$ is smooth then the variety Z is empty and so by Hilbert's Nullstellensatz there exists $\tilde{f} \in \mathsf{R}[x]$ such that $f\tilde{f} = 1$ in V_f. This means that A is invertible and so $\Delta_f \neq 0$. $\square$

In the case of R equal to $\mathbb{C}[t_1,t_2,\dots,t_s]$, the above Proposition applied to specializations f_t of f implies that the affine variety $\{\Delta_f(t)=0\}\subset\mathbb{C}^s$ is the locus of parameters t such that the affine variety $\{f_t=0\}\subset\mathbb{C}^{n+1}$ is singular.

Definition 10.15 For a tame polynomial f we say that the affine variety $\{f=0\}$ is smooth if the discriminant Δ_f of f is not zero.

Proposition 10.9 *Assume that f is a tame polynomial over a Cohen-Macaulay ring R and $\Delta_f\neq 0$. If*

$$df\wedge\omega_2=f\omega_1,$$

$$\textit{for some } \omega_2\in\Omega^n_{\mathsf{A}/\mathsf{T}},\ \omega_1\in\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$$

then

$$\omega_2=f\omega_3+df\wedge\omega_4,\ \omega_1=df\wedge\omega_3,$$

$$\textit{for some } \omega_3\in\Omega^n_{\mathsf{A}/\mathsf{T}},\ \omega_4\in\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}.$$

Proof. If ω_1 is not zero in W_f then the multiplication by f R-linear map in V_f has a non trivial kernel and so $\Delta_f=0$ which contradicts the hypothesis. Now let $\omega_1=df\wedge\omega_3$ and so $df\wedge(f\omega_3-\omega_2)=0$. The de Rham lemma for f (Proposition 10.6) finishes the proof. □

The example below shows that the above proposition is not true for singular affine varieties. For a homogeneous polynomial g in the graded ring $\mathsf{R}[x]$, $\deg(x_i)=\nu_i$ we have

$$g=\sum_{i=1}^{n+1}\frac{\nu_i}{d}x_i\frac{\partial g}{\partial x_i}\ \text{ equivalently } gdx=dg\wedge\eta \tag{10.26}$$

and so the matrix A in the definition of the discriminant of g is the zero matrix. In particular, the discriminant of $g-s\in\mathsf{R}[s][x]$ is $(-s)^\mu$.

Assume that $2d$ is invertible in R. For the hypergeometric polynomial $f:=y^2-p(x)\in\mathsf{R}[x,y]$, $\deg(p)=d$ we have $\mathsf{V}_f\cong\mathsf{V}_p$ and under this isomorphism the multiplication by f linear map in V_f coincide with the multiplication by p map in V_p. Therefore,

$$\Delta_f=\Delta_p.$$

Below we have collected some discriminants in the zero dimensional case.

d	$d^d\cdot\Delta_f$
2	$4t_0-t_1^2$
3	$27t_0^2-18t_0t_1t_2+4t_0t_2^3+4t_1^3-t_1^2t_2^2$
4	$256t_0^3-192t_0^2t_1t_3-128t_0^2t_2^2+144t_0^2t_2t_3^2-27t_0^2t_3^4+144t_0t_1^2t_2-6t_0t_1^2t_3^2-80t_0t_1t_2^2t_3+$ $18t_0t_1t_2t_3^3+16t_0t_2^4-4t_0t_2^3t_3^2-27t_1^4+18t_1^3t_2t_3-4t_1^3t_3^3-4t_1^2t_2^3+t_1^2t_2^2t_3^2$

Table 10.1 Discriminant of $p=x^d+t_{d-1}x^{d-1}+t_{d-2}x^{d-2}+\cdots+t_0$

10.10 The double discriminant of a tame polynomial

Let $f \in \mathsf{R}[x]$ be a tame polynomial. We consider a new parameter s and the tame polynomial $f - s \in \mathsf{R}[s][x]$. The discriminant Δ_{f-s} of $f - s$ as a polynomial in s has degree μ and its coefficients are in R. Its leading coefficient is $(-1)^{\mu}$ and so if μ is invertible in R then it is tame (as a polynomial in s) in $\mathsf{R}[s]$. Now, we take again the discriminant of Δ_{f-s} with respect to the parameter s and obtain

$$\check{\Delta} = \check{\Delta}_f := \Delta_{\Delta_{f-s}} \in \mathsf{R}$$

which is called the double discriminant of f. We consider a tame polynomial f as a function from $\bar{\mathsf{k}}^{n+1}$ to $\bar{\mathsf{k}}$. The set of critical point of f is defined to be $P = P_f := Z(\mathrm{jacob}(f))$ and the set of critical values of f is $C = C_f := f(P_f)$. It is easy to see that:

Proposition 10.10 *The tame polynomial f has μ distinct critical values (and hence distinct critical points) if and only of its double discriminant is not zero.*

Note that μ is the maximum possible number for $\#C_f$.

10.11 De Rham cohomology and Brieskorn modules

Let $f \in \mathsf{R}[x]$ be a tame polynomial as in §10.6. The following quotients

$$\mathsf{H}' = \mathsf{H}'_f := \frac{\Omega^n_{\mathsf{A}}}{f\Omega^n_{\mathsf{A}} + df \wedge \Omega^{n-1}_{\mathsf{A}} + \pi^{-1}\Omega^1_{\mathsf{T}} \wedge \Omega^{n-1}_{\mathsf{A}} + d\Omega^{n-1}_{\mathsf{A}}} \tag{10.27}$$

$$\cong \frac{\Omega^n_{\mathsf{A}/\mathsf{T}}}{f\Omega^n_{\mathsf{A}/\mathsf{T}} + df \wedge \Omega^{n-1}_{\mathsf{A}/\mathsf{T}} + d\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}}, \tag{10.28}$$

$$\mathsf{H}'' = \mathsf{H}''_f := \frac{\Omega^{n+1}_{\mathsf{A}}}{f\Omega^{n+1}_{\mathsf{A}} + df \wedge d\Omega^{n-1}_{\mathsf{A}} + \pi^{-1}\Omega^1_{\mathsf{T}} \wedge \Omega^n_{\mathsf{A}}} \tag{10.29}$$

$$\cong \frac{\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}}{f\Omega^{n+1}_{\mathsf{A}/\mathsf{T}} + df \wedge d\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}} \tag{10.30}$$

are R-modules and play the role of de Rham cohomology of the affine variety

$$\{f = 0\} := \mathrm{Spec}\left(\frac{\mathsf{R}[x]}{f \cdot \mathsf{R}[x]}\right).$$

Here $\pi : \mathsf{A} \to \mathsf{T}$ is the projection corresponding to $\mathsf{R} \subset \mathsf{R}[x]$. We have assumed that $n > 0$. In the case $n = 0$ we define:

$$\mathsf{H}' = \mathsf{H}'_f := \frac{\mathsf{R}[x]}{f \cdot \mathsf{R}[x] + \mathsf{R}}$$
$$\mathsf{H}'' = \mathsf{H}''_f := \frac{\Omega^1_{\mathsf{A}/\mathsf{T}}}{f \cdot \Omega^1_{\mathsf{A}/\mathsf{T}} + \mathsf{R} \cdot df}.$$

We will use H or $H^n_{\mathrm{dR}}(\{f=0\})$ to denote one of the modules H' or H''. We note that for an arbitrary polynomial f such modules may not coincide with the de Rham cohomology of the affine variety $\{f=0\}$ defined by Grothendieck, Atiyah and Hodge (see [Gro66]). For instance, for $f = x(1+xy) - t \in \mathsf{R}[x,y],\ \mathsf{R} = \mathbb{C}[t]$ the differential forms $y^{k+1}dx + xy^k dy,\ k > 0$ are not zero in the corresponding H' but they are relatively exact and so zero in $H^1_{\mathrm{dR}}(\{f=0\})$ (see [Bon03]).

One may call H' and H'' the Brieskorn modules associated to f in analogy to the local modules introduced by Brieskorn in 1970. In fact, the classical Brieskorn modules for $n > 0$ are

$$H' = H'_f := \frac{\Omega^n_{\mathsf{A}/\mathsf{T}}}{df \wedge \Omega^{n-1}_{\mathsf{A}/\mathsf{T}} + d\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}},$$
$$H'' = H''_f := \frac{\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}}{df \wedge d\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}}$$

and for $n = 0$

$$H' := \frac{\mathsf{R}[x]}{\mathsf{R}[f]},$$
$$H'' := \frac{\Omega^1_{\mathsf{A}/\mathsf{T}}}{\mathsf{R}[f] \cdot df}.$$

We consider them as $\mathsf{R}[f]$-modules. The $\mathsf{R}[f]$-module H'_f is isomorphic to the $\mathsf{R}[s]$-module $\mathsf{H}'_{\tilde{f}}$, where $\tilde{f} = f - s \in \mathsf{R}[s][x]$ and s is a new parameter. A similar statement is true for the other Brieskorn module.

We have the following well-defined R-linear map

$$\mathsf{H}' \to \mathsf{H}'',\ \omega \mapsto df \wedge \omega \tag{10.31}$$

which is an inclusion by Proposition 10.9. When we write $\mathsf{H}' \subset \mathsf{H}''$ then we mean the inclusion obtained by the above map. We have

$$\frac{\mathsf{H}''}{\mathsf{H}'} = \mathsf{W}_f.$$

Definition 10.16 For $\omega \in \mathsf{H}''$ we define the Gelfand-Leray form

$$\frac{\omega}{df} := \frac{\omega'}{\Delta} \in \mathsf{H}'_\Delta, \text{ where } \Delta \cdot \omega = df \wedge \omega', \tag{10.32}$$

where H'_Δ is the localization of H' over $\{1, \Delta, \Delta^2, \cdots\}$. Our definition of Gelfand-Leray form is done in the ring $\mathsf{R}[x]$, that is, we have not used divisions over elements of $\mathsf{R}[x]$. If we allow such divisions then we may also define it in the following way:

$$\begin{aligned}\frac{g dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}}{df} &:= \frac{g dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge df}{f_{x_{n+1}} df} \\ &= (-1)^n \frac{g dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{f_{x_{n+1}}},\end{aligned}$$

where $f_{x_{n+1}}$ is the partial derivative of f with respect to x_{n+1}. This is actually the way Picard and Simart in [PS06] represented the integrands of multiple integrals for $n = 2$. This kind of writing differential forms can be also found in many places in the Physics literature, see for instance [CdlOGP91]. Note that for the hypergeometric polynomial we get the classical differential forms $\frac{x^i dx}{y}$ as it is written in (10.14).

10.12 Main theorem

Recall the definition of ω_β and η_β from §10.6. Let us first state the main results of this section.

Theorem 10.1 *Let R be a Cohen-Macaulay ring of characteristic zero and $\mathbb{Q} \subset \mathsf{R}$. If f is a tame polynomial in $\mathsf{R}[x]$ then the $\mathsf{R}[f]$-modules H'' and H' are free and $\omega_\beta, \beta \in I$ (resp. $\eta_\beta,\ \beta \in I$) form a basis of H'' (resp. H'). More precisely, in the case $n > 0$ every $\omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$ (resp. $\omega \in \Omega^{n}_{\mathsf{A}/\mathsf{T}}$) can be written*

$$\omega = \sum_{\beta \in I} p_\beta(f) \omega_\beta + df \wedge d\xi, \tag{10.33}$$

$$p_\beta \in \mathsf{R}[f],\ \xi \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}},\ \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta$$

(resp.

$$\omega = \sum_{\beta \in I} p_\beta(f) \eta_\beta + df \wedge \xi + d\xi_1, \tag{10.34}$$

$$p_\beta \in \mathsf{R}[t],\ \xi, \xi_1 \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}},\ \deg(p_\beta) \leq \frac{\deg(\omega)}{d} - A_\beta$$

).

A similar statement holds for the case $n = 0$. We leave its formulation and proof to the reader. We will prove the above theorem in §10.13 and §10.14.

Corollary 10.1 *Let* R *be of characteristic zero and* $\mathbb{Q} \subset \mathsf{R}$. *If* f *is a tame polynomial in* $\mathsf{R}[x]$ *then the* R*-modules* H' *and* H'' *are free and* $\eta_\beta, \beta \in I$ *(resp.* $\omega_\beta,\ \beta \in I$*) form a basis of* H' *(resp.* H''*).*

Note that in the above corollary $\{f = 0\}$ may be singular. We call $\eta_\beta, \beta \in I$ (resp. $\omega_\beta,\ \beta \in I$) the canonical basis of H' (resp. H'').

Proof. We prove the corollary for H'. The proof for H'' is similar. We consider the following canonical exact sequence

$$0 \to fH' \to H' \to \mathsf{H}' \to 0.$$

Using this, Theorem 10.1 implies that H' is generated by $\eta_\beta,\ \beta \in I$. It remains to prove that η_β's are R-linear independent. If $a := \Sigma_{\beta \in I} r_\beta \eta_\beta = 0,\ r_\beta \in \mathsf{R}$ in H' then $a = fb,\ b \in H'$. We write b as an $\mathsf{R}[f]$-linear combination of η_β's and we obtain $r_\beta = fc_\beta(f)$ for some $c_\beta(f) \in \mathsf{R}[f]$. This implies that for all $\beta \in I,\ r_\beta = 0$. $\square$

Theorem 10.1 is proved first for the case $\mathsf{R} = \mathbb{C}$ in [Mov07b]. In this article we have used a topological argument to prove that the forms $\omega_\beta, \beta \in I$ (resp. $\eta_\beta,\ \beta \in I$) are $\mathsf{R}[f]$-linear independent. It is based on the following facts: 1. η_β's generates the $\mathbb{C}[f]$-module H', 2. $\#I = \mu$ is the dimension of $H^n_{\mathrm{dR}}(\{f = c\})$ for a regular value $c \in \mathbb{C} - C$, 3. H' restricted to $\{f = 0\}$ is isomorphic to $H^n_{\mathrm{dR}}(\{f = c\})$. In the forthcoming sections we present an algebraic proof.

10.13 Proof of the main theorem for a homogeneous tame polynomial

Let $f = g$ be a homogeneous tame polynomial with an isolated singularity at the origin. We explain the algorithm which writes every element of H'' of g as an $\mathsf{R}[g]$-linear combination of ω_β's. Recall that

$$dg \wedge d(P\widehat{dx_i, dx_j}) = (-1)^{i+j+\varepsilon_{i,j}}\left(\frac{\partial g}{\partial x_j}\frac{\partial P}{\partial x_i} - \frac{\partial g}{\partial x_i}\frac{\partial P}{\partial x_j}\right)dx,$$

where $\varepsilon_{i,j} = 0$ if $i < j$ and $= 1$ if $i > j$ and $\widehat{dx_i, dx_j}$ is dx without dx_i and dx_j (we have not changed the order of $dx_1, dx_2, \ldots$ in dx).

Proposition 10.11 *In the case* $n > 0$*, for a monomial* $P = x^\beta$ *we have*

$$\frac{\partial g}{\partial x_i} \cdot P dx = \tag{10.35}$$

$$\frac{d}{d \cdot A_\beta - \nu_i}\frac{\partial P}{\partial x_i} g dx + dg \wedge d\Big(\sum_{j \neq i} \frac{(-1)^{i+j+1+\varepsilon_{i,j}}\nu_j}{d \cdot A_\beta - \nu_i} x_j P\widehat{dx_i, dx_j}\Big).$$

Proof. The proof is a straightforward calculation:

$$\sum_{j\neq i}\frac{(-1)^{i+j+1+\varepsilon_{i,j}}\nu_j}{d\cdot A_\beta-\nu_i}dg\wedge d(x_jP\widehat{dx_i,dx_j})=$$

$$\frac{-1}{d\cdot A_\beta-\nu_i}\sum_{j\neq i}(\nu_j\frac{\partial g}{\partial x_j}\frac{\partial(x_jP)}{\partial x_i}-\nu_j\frac{\partial g}{\partial x_i}\frac{\partial(x_jP)}{\partial x_j})dx=$$

$$\frac{-1}{d\cdot A_\beta-\nu_i}((d\cdot g-\nu_ix_i\frac{\partial g}{\partial x_i})\frac{\partial P}{\partial x_i}-P\frac{\partial g}{\partial x_i}\sum_{j\neq i}\nu_j(\beta_j+1))dx=$$

$$\frac{-1}{d\cdot A_\beta-\nu_i}(d\cdot g\frac{\partial P}{\partial x_i}-\nu_i\beta_iP\frac{\partial g}{\partial x_i}-P\frac{\partial g}{\partial x_i}\sum_{j\neq i}\nu_j(\beta_j+1))dx$$

□

The only case in which $dA_\beta-\nu_i=0$ is when $n=0$ and $P=1$. In the case $n=0$ for $P\neq 1$ we have

$$\frac{\partial g}{\partial x}\cdot Pdx=\frac{d}{d\cdot A_\beta-\nu}\frac{\partial P}{\partial x}gdx=\frac{d}{\nu}x^{\beta-1}gdx$$

and if $P=1$ then $\frac{\partial g}{\partial x_i}\cdot Pdx$ is zero in H''. Based on this observation the following works for $n=0$.

We use the above Proposition to write every $Pdx\in\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$ in the form

$$Pdx=\sum_{\beta\in I}p_\beta(g)\omega_\beta+dg\wedge d\xi, \tag{10.36}$$

$$p_\beta\in\mathsf{R}[g],\ \xi\in\Omega^{n-1}_{\mathsf{A}/\mathsf{T}},\ \deg(p_\beta(g)\omega_\beta),\ \deg(dg\wedge d\xi)\leq\deg(Pdx).$$

- Input: The homogeneous tame polynomial g and $P\in\mathsf{R}[x]$ representing $[Pdx]\in H''$.
- Output: $p_\beta,\beta\in I$ and ξ satisfying (10.36)

We write

$$Pdx=\sum_{\beta\in I}c_\beta x^\beta\cdot dx+dg\wedge\eta,\ \deg(dg\wedge\eta)\leq\deg(Pdx). \tag{10.37}$$

Then we apply (10.35) to each monomial component $\tilde{P}\frac{\partial g}{\partial x_i}$ of $dg\wedge\eta$ and then we write each $\frac{\partial\tilde{P}}{\partial x_i}dx$ in the form (10.37). The degree of the components which make Pdx not to be of the form (10.36) always decreases and finally we get the desired form.

To find a similar algorithm for H' we note that if $\eta\in\Omega^n_{\mathsf{A}/\mathsf{T}}$ is written in the form

$$\eta=\sum_{\beta\in I}p_\beta(g)\eta_\beta+dg\wedge\xi+d\xi_1, \tag{10.38}$$

$$p_\beta\in\mathsf{R}[g],\ \xi,\xi_1\in\Omega^{n-1}_{\mathsf{A}/\mathsf{T}},$$

where each piece in the right hand side of the above equality has degree less than $\deg(\eta)$, then

$$d\eta = \sum_{\beta\in I}(p_\beta(g)A_\beta + p'_\beta(g)g)\omega_\beta - dg\wedge d\xi \tag{10.39}$$

and the inverse of the map

$$\mathsf{R}[g] \to \mathsf{R}[g],\ p(g) \mapsto A_\beta.p(g) + p'(g)\cdot g$$

is given by

$$\sum_{i=0}^{k} a_i g^i \to \sum_{i=1}^{k} \frac{a_i}{A_\beta + i} g^i. \tag{10.40}$$

Now let us prove that there is no $\mathsf{R}[g]$-relation between ω_β's in H''_g. This also implies that there is no $\mathsf{R}[g]$-linear relation between η_β's in H'_g. If such a relation exists then we take its differential and since $dg\wedge\eta_\beta = g\omega_\beta$ and $d\eta_\beta = A_\beta\omega_\beta$ we obtain a non-trivial relation in H''_g.

Since $g = dg\wedge\eta$ and x^β are R-linear independent in V_g, the existence of a non trivial $\mathsf{R}[g]$-relation between ω_β's in H''_g implies that there is a $0\neq\omega\in H''_g$ such that $g\omega = 0$ in H''_g. Therefore, we have to prove that H''_g has no torsion. Let $a\in\mathsf{R}[x]$ and

$$g\cdot a\cdot dx = dg\wedge d\omega_1,\ \text{ for some } \omega_1\in\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}. \tag{10.41}$$

Since g is homogeneous, we can assume that a is also homogeneous. Now, the above equality, Proposition 10.3 and

$$dg\wedge(a\eta - d\omega_1) = 0$$

imply that

$$a\eta = d\omega_1 + dg\wedge\omega_2,\ \text{ for some } \omega_2\in\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}.$$

We take differential of the above equality and we conclude that

$$(\sum_{i=1}^{n+1}\frac{\nu_i}{d} + \frac{\deg(a)}{d})a\cdot dx = 0 \ \text{ in } H''_g.$$

Since $\mathbb{Q}\subset\mathsf{R}$, we get $adx = 0$ in H''_g.

Remark 10.1 The reader may have already noticed that Theorem 10.1 is not at all true if R has characteristic different from zero. In the formulas (10.40) and (10.35) we need to divide over $d\cdot A_\beta - \nu_i$ and $A_\beta + i$. Moreover, to prove that H''_g has no torsion we must be able to divide on $\sum_{i=1}^{n+1}\frac{\nu_i}{d} + \frac{\deg(a)}{d}$.

10.14 Proof of the main theorem for an arbitrary tame polynomial

For simplicity we assume that $n > 0$. We explain an algorithm which writes every element of H'' of f as an $\mathsf{R}[f]$-linear combination of ω_β's. We write an element $\omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}, \deg(\omega) = m$ in the form

$$\omega = \sum_{\beta \in I} p_\beta(g)\omega_\beta + dg \wedge d\psi,$$

$$p_\beta \in \mathsf{R}[g],\ \psi \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}},\ \deg(p_\beta(g)\omega_\beta) \leq m,\ \deg(d\psi) \leq m - d.$$

This is possible because g is homogeneous. Now, we write the above equality in the form

$$\omega = \sum_{\beta \in I} p_\beta(f)\omega_\beta + df \wedge d\psi + \omega',$$

with

$$\omega' = \sum_{\beta \in I} (p_\beta(g) - p_\beta(f))\omega_\beta + d(g-f) \wedge d\psi.$$

The degree of ω' is strictly less than m and so we repeat what we have done at the beginning and finally we write ω as an $\mathsf{R}[f]$-linear combination of ω_β's.

The algorithm for H' is similar and uses the fact that for $\eta \in \Omega^n_{\mathsf{A}/\mathsf{T}}$ one can write

$$\eta = \sum_{\beta \in I} p_\beta(g)\eta_\beta + dg \wedge \psi_1 + d\psi_2 \tag{10.42}$$

and each piece in the right hand side of the above equality has degree less than $\deg(\eta)$.

Let us now prove that the forms $\omega_\beta, \beta \in I$ (resp. $\eta_\beta,\ \beta \in I$) are $\mathsf{R}[f]$-linear independent. If there is an $\mathsf{R}[f]$-relation between ω_β's in H''_f, namely

$$\sum_{\beta \in I} p_\beta(f)\omega_\beta = df \wedge d\omega,\ \omega \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}}, \tag{10.43}$$

then by taking the last homogeneous piece of the relation, we obtain a non-trivial $\mathsf{R}[g]$-relations between ω_β's in H''_g or

$$dg \wedge d\omega_1 = 0,\ \omega_1 \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}},$$

where $\omega = \omega_1 + \omega'_1$ with $\deg(\omega'_1) < \deg(\omega_1) = \deg(\omega)$. The first case does not happen by the proof of our theorem in the $f = g$ case (see §10.13). In the second case we use Proposition 10.3 and Proposition 10.5 and obtain

$$d\omega_1 = dg \wedge d\omega_2,$$

$$\omega_2 \in \Omega^{n-2}_{\mathsf{A}/\mathsf{T}},\ \deg(d\omega_1) = d + \deg(d\omega_2).$$

Now

$$df \wedge d\omega = df \wedge d(\omega_1 + \omega_1') = df \wedge (d(g-f) \wedge d\omega_1 + d\omega_1').$$

This means that we can substitute ω with another one and with less $\deg_x$. Taking ω the one with the smallest degree and with the property (10.43), we get a contradiction. In the case of H_f' the proof is similar and it is left to the reader.

10.15 Exercises

10.1. Prove the universal property of the differential map $d : \check{\mathsf{R}} \to \Omega_{\check{\mathsf{R}}/\mathsf{R}}$.

10.2. Prove the following properties of the wedge product: For $\alpha \in \Omega^i_{L/\mathsf{T}},\ \beta \in \Omega^j_{L/\mathsf{T}},\ \gamma \in \Omega^r_{L/\mathsf{T}}$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma),$$
$$\alpha \wedge \beta \wedge \gamma = (-1)^{ij+jr+ir} \gamma \wedge \beta \wedge \alpha,$$

10.3. Prove that $d \circ d = 0$, where $d : \Omega^i_{L/\mathsf{T}} \to \Omega^{i+1}_{L/\mathsf{T}}$ is the differential map.

10.4. For $\alpha \in \Omega^i_{L/\mathsf{T}},\ \beta \in \Omega^j_{L/\mathsf{T}}$ we have:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^i \alpha \wedge (d\beta).$$

10.5. Prove that there is a unique map i_v in (10.4) satisfying the properties 1,2,3 listed in §10.4.

10.6. Complete the proof of (10.5).

10.7. Show that if V_g is a free R-module of finite rank then we can choose a basis of monomials for V_g.

10.8. Let us consider the case $\mathsf{R} = \mathbb{C}$. Show that the canonical map from the Brieskorn module H' to the classical de Rham cohomology of $\{f = 0\}$ defined by C^∞ forms is an isomorphism.

10.9. A homogeneous tame polynomial in $\mathbb{C}[x,y],\ \deg(x) = \deg(y) = 1$ has an isolated singularity at the origin if and only if in its decomposition into linear factors, there is no factor of multiplicity greater than one.

10.10. In the weighted polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$ with $\text{weight}(x_i) = \nu_i \in \mathbb{N}$ and for a given degree $d \in \mathbb{N}$, we may look for at least one homogeneous polynomial $g \in \mathbb{C}[x]$ with an isolated singularity at the origin and of degree d. In the case where $\nu_1, \nu_2, \ldots, \nu_{n+1}$ divide d we have the polynomial g in (15.9) with $m_i := \frac{d}{\nu_i}$. How about other d's? Produce few examples of such polynomials for $n = 1, 2, \ldots$ and particular values of $m_1, m_2, \ldots$.

10.11. Prove or disprove: if g is a homogeneous tame polynomial in the ring $\mathsf{R}[x_1, x_2, \cdots, x_{n+1}]$, $\text{weight}(x_i) = \nu_i \in \mathbb{N}$ then the degree of g is invertible in R.

10.12. In the zero dimensional case $n = 0$ the discriminant of a monic polynomial $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \in \mathsf{R}[x]$ is defined as follows:

$$\tilde{\Delta}_f := \prod_{1 \le i \ne j \le d} (x_i - x_j) = \prod_{i=1}^{d} f'(x_i) \in \mathsf{R},$$

where $f' = \frac{\partial f}{\partial x}$ is the derivative of f and x_i's are roots of f. Show that the multiplication of $\tilde{\Delta}_f$ with the number d^d is equal to the discriminant Δ_f of f in Definition 10.14.

10.13. Rewrite Exercise 2.8 and solve it using the techniques of this chapter and without any integral sign.

10.14. Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface given by $f = 0$ in an affine chart $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$, where $f := x^d + x_2^d + \cdots + x_{n+1}^d + \cdots$ and $\cdots$ means monomials of degree $< d$. Show that the differential forms

$$\frac{x^\beta dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}}{df}, \quad \sum_{i=1}^{n+1} \beta_i < d - (n+1) \tag{10.44}$$

restricted to X are holomorphic everywhere in X, that is, they have no poles.

10.15. There are many statements in [PS06], Vol. II, for singular surfaces $U := \{f(x,y,z) = 0\}$. For instance, in page 181 of this book we find a statement which suggests that the second de Rham cohomology of U must be defined using differential forms $\frac{Pdx \wedge dy \wedge dz}{df}$ with $P \in \mathbb{C}[x,y,z]$ vanishing on non-isolated singular locus of U. Why is this condition on P natural? In page 320 of the same book we find the statement that ρ_0 (Picard number) of

$$U := \left\{(x,y,z) \in \mathbb{C}^3 \mid z^2 = f(x)g(y)\right\} \tag{10.45}$$

is $\le 4pq$, where f, g are polynomials of degree $2p+1$ and $2q+1$ in x, y, respectively. For $f = g$ we have the curve $z = f(x)$, $x = y$ in U and ρ_0 is reduced by one, see page 322 of the same book. The tame polynomial f in Example 10.6 has zero discriminant and despite the fact that its fibers are singular, we can still talk about the second de Rham cohomologies, see [DMWH16] page 133. Write a report on the topic of de Rham cohomologies, Picard numbers, etc. for singular varieties.

Chapter 11
Hodge filtrations and Mixed Hodge structures

...what is the analogue [of l-adic cohomology and the action of Galois for singular or non-compact varieties] in the complex case? One clue is given by the existence, in l-adic cohomology, of an increasing filtration, the weight filtration W, for which the i-th quotient W_i/W_{i-1} is a subquotient of the cohomology of a projective nonsingular variety. We hence expect in the complex case a filtration W such that the i-th quotient has a Hodge decomposition of weight i. Another clue, coming from works of Griffiths and Grothendieck, is that the Hodge filtration is more important than the Hodge decomposition. Both clues force the definition of mixed Hodge structures, suggest that they form an abelian category, and suggest also how to construct them, (P. Deligne in [RS14] page 182).

11.1 Introduction

The theory of mixed Hodge structures, introduced by P. Deligne in a series of articles [Del71a, Del71b, Del74], is without doubt one of the breakthrough achievements in Hodge theory. Despite classical Hodge structures which have origin in the study of multiple integrals, mixed Hodge structures have origin in motives and analogies between de Rham cohomologies with Hodge structures, and l-adic cohomologies with Galois actions, see [Del71a]. It is beyond the limited objective of the present text to go into the detail of all these developments. Instead, we focus on one of the most simple examples of mixed Hodge structures, that is, the one living in the middle algebraic de Rham cohomology of the affine variety $\{f = 0\}$ for a tame polynomial f. Its elements are constructed by polynomials and it is natural to make some distinctions between such elements. The mixed Hodge structure in this case is going to do this job. In the case of curves this is essentially the old story of differential forms of the first kind (holomorphic everywhere), of the second kind (with poles but no residues) and the third kind (with poles which might have residues).

We define the Gauss-Manin system M associated to f which plays the same role as H and it has the advantage that the Hodge and weight filtrations in M are defined

explicitly. Our approach is by looking at differential forms with poles along $\{f=0\}$. The main role of the Hodge and weight filtrations in the present text is to distinguish between differential forms.

11.2 Gauss-Manin system M

The Gauss-Manin system for a tame polynomial f is defined to be:

$$\begin{aligned}\mathsf{M}_f = \mathsf{M} &:= \frac{\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}[\frac{1}{f}]}{\Omega^{n+1}_{\mathsf{A}/\mathsf{T}} + d(\Omega^{n}_{\mathsf{A}/\mathsf{T}}[\frac{1}{f}])} \\ &\cong \frac{\Omega^{n+1}_{\mathsf{A}}[\frac{1}{f}]}{\Omega^{n+1}_{\mathsf{A}} + d(\Omega^{n}_{\mathsf{A}}[\frac{1}{f}]) + \pi^{-1}\Omega^{1}_{\mathsf{T}} \wedge \Omega^{n}_{\mathsf{A}}[\frac{1}{f}]},\end{aligned}$$

where $\Omega^i_{\mathsf{A}}[\frac{1}{f}]$ is the set of polynomials in $\frac{1}{f}$ with coefficients in Ω^i_{A}, etc.. It has a natural filtration given by the pole order along $\{f=0\}$, namely

$$\mathsf{M}_i := \{[\frac{\omega}{f^i}] \in \mathsf{M} \mid \omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}\},$$

$$\mathsf{M}_1 \subset \mathsf{M}_2 \subset \cdots \subset \mathsf{M}_i \subset \cdots \subset \mathsf{M}_\infty := \mathsf{M}.$$

It is useful to identify H' by its image under $df \wedge \cdot$ in H'' and define $\mathsf{M}_0 := \mathsf{H}'$. Moreover, we have $\mathsf{H}'' \cong \mathsf{M}_1$, $[\omega] \mapsto [\frac{\omega}{f}]$. From now on we will use H for one of M_i's. Note that in M we have

$$[\frac{d\omega}{f^{i-1}}] = [\frac{(i-1)df \wedge \omega}{f^i}],\ \omega \in \Omega^n_{\mathsf{A}/\mathsf{T}}, i = 2,3,\ldots \tag{11.1}$$

$$[\frac{df \wedge d\omega}{f^i}] = [d(\frac{df \wedge \omega}{f^i})] = 0,\ \omega \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}}, i = 1,2,\ldots \tag{11.2}$$

Proposition 11.1 *If the discriminant of the tame polynomial f is not zero then the differential form $\frac{\omega}{f^i}$, $i \in \mathbb{N}$, $\omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$ is zero in M if and only if ω is of the form*

$$fd\omega_1 - (i-1)df \wedge \omega_1 + df \wedge d\omega_2 + f^i\omega_3,$$

$$\omega_1 \in \Omega^n_{\mathsf{A}/\mathsf{T}},\ \omega_2 \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}},\ \omega_3 \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}.$$

Proof. Let

$$\frac{\omega}{f^i} = d(\frac{\omega_1}{f^s}) \mod \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}. \tag{11.3}$$

If $s = i-1$ then ω has the desired form. If $s \geq i$ then $df \wedge \omega_1 \in f\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$ and so by Proposition 10.9 we have $\omega_1 = f\omega_3 + df \wedge \omega_2$ and so

$$\frac{\omega}{f^i} = d(\frac{f\omega_3 + df \wedge \omega_2}{f^s}), \quad \mod \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}. \tag{11.4}$$

If $s = i$ then we obtain the desired form for ω. If $s > i$ we get $df \wedge d\omega_2 + (s-1)df \wedge \omega_3 \in f\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$ and so again by Proposition 10.9 we have $d\omega_2 + (s-1)\omega_3 = f\omega_4 + df \wedge \omega_5$. We calculate ω_3 from this equality and substitute it in (11.4) and obtain

$$\frac{\omega}{f^i} = \frac{1}{s-1} d(\frac{f\omega_4 + df \wedge \omega_5}{f^{s-1}}) \quad \mod \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}.$$

We repeat this until getting the situation $s = i$. □

The structure of M and its relation with H is described in the following proposition.

Proposition 11.2 *We have the well-defined canonical maps*

$$\mathsf{H}'' \to \mathsf{M}_1, \quad \omega \mapsto [\frac{\omega}{f}], \tag{11.5}$$

$$\mathsf{W}_f \to \mathsf{M}_i/\mathsf{M}_{i-1}, \quad \omega \mapsto [\frac{\omega}{f^i}],\ i = 1, 2, \ldots. \tag{11.6}$$

If the discriminant of the tame polynomial f is not zero then they are isomorphism of R*-modules.*

Proof. The fact that they are well-defined follows from the equalities (11.1) and (11.2). The non-trivial part of the second part is that they are injective. This follows from Proposition 11.1. □

11.3 Two filtrations of M

Recall that $\{x^\beta \mid \beta \in I\}$ is a monomial basis of the R-module V_g and $\omega_\beta,\ \beta \in I$ is a basis of the R-module H''.

Definition 11.1 We define the degree of $\frac{\omega}{f^k},\ k \in \mathbb{N}, \omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$ to be $\deg_x(\omega) - \deg_x(f^k)$. By definition we have $\deg(\frac{\omega_\beta}{f^k}) = d(A_\beta - k)$. The degree of $\alpha \in \mathsf{M}$ is defined to be the minimum of the degrees of $\frac{\omega}{f^k} \in \alpha$.

In order to define the mixed Hodge structure of M we need the following proposition.

Proposition 11.3 *Every element of degree s of* M *can be written as an* R*-linear sum of the elements*

$$\frac{\omega_\beta}{f^k},\ \beta \in I,\ 1 \le k,\ A_\beta \le k, \tag{11.7}$$

$$\deg(\frac{\omega_\beta}{f^k}) \le s.$$

Proof. Let us be given an element $\frac{\omega}{f^k}$ of degree s in M. According to Corollary 10.1, we write $\omega = \sum_{\beta\in I} a_\beta \omega_\beta + df\wedge d\omega_2 + f\omega_1$ and so

$$\frac{\omega}{f^k} = \sum_{\beta\in I} a_\beta \frac{\omega_\beta}{f^k} + \frac{\omega_1}{f^{k-1}} \text{ in } \mathsf{M}.$$

We repeat this argument for ω_1. At the end we get $\frac{\omega}{f^k}$ as a R-linear combination of $\frac{\omega_\beta}{f^i}$, $\beta\in I$, $k\in\mathbb{N}$. An alternative way is to say that ω can be written as an $\mathsf{R}[f]$-linear combinations of ω_β, $\beta\in I$ modulo $df\wedge d\Omega^{n-1}_{\mathsf{A}/\mathsf{T}}$ (see Theorem 10.1). The degree conditions (10.33) implies that the we have used only $\frac{\omega_\beta}{f^i}$ with $\deg(\frac{\omega_\beta}{f^i}) \le \deg(\frac{\omega}{f^k})$.

Now, we have to get rid of elements of type $\frac{\omega_\beta}{f^k}$, $A_\beta > k$. Given such an element, in M we have:

$$\begin{aligned}\frac{\omega_\beta}{f^k} &= \frac{1}{A_\beta}\frac{d\eta_\beta}{f^k} = \frac{k}{A_\beta}\frac{df\wedge\eta_\beta}{f^{k+1}}\\ &= \frac{k}{A_\beta}\frac{f\omega_\beta + (g-f)\omega_\beta + d(f-g)\wedge\eta_\beta}{f^{k+1}}\end{aligned}$$

and so

$$\frac{\omega_\beta}{f^k} = \frac{A_\beta}{A_\beta - k}\frac{(g-f)\omega_\beta + d(f-g)\wedge\eta_\beta}{f^{k+1}}. \tag{11.8}$$

The degree of the right hand side of (11.8) is less than $d(A_\beta - k)$, which is the degree of the left hand side. We write the right hand side in terms of $\frac{\omega_{\beta'}}{f^s}$, $\beta'\in I, s\in\mathbb{N}$ and repeat (11.8) for these new terms. Since each time the degree of the new elements $\frac{\omega'_\beta}{f^s}$ decreases, at some point we get the desired form for $\frac{\omega_\beta}{f^k}$. $\square$

Now, we can define two natural filtration on M_Δ (the localization of M over the multiplicative group $\{1,\Delta,\Delta^2,\cdots\}$).

Definition 11.2 We define $W_n = W_n\mathsf{M}_\Delta$ to be the R_Δ-submodule of M_Δ generated by

$$\frac{\omega_\beta}{f^k},\ \beta\in I,\ A_\beta < k$$

and call

$$0 =: W_{n-1} \subset W_n \subset W_{n+1} := \mathsf{M}_\Delta$$

the weight filtration of M_Δ. We also define $F^i = F^i\mathsf{M}_\Delta$ to be the R_Δ-submodule of of M_Δ generated by

$$\frac{\omega_\beta}{f^k},\ \beta\in I,\ A_\beta \le k \le n+1-i \tag{11.9}$$

and call

$$0 = F^{n+1} \subset F^n \subset F^{n-1} \subset \cdots \subset F^0$$

the Hodge filtration of M_Δ. The pair $(F^\bullet, W_\bullet)$ is called the mixed Hodge structure of M_Δ.

Since for $j = 0,1,2,\ldots,\infty$ we have the the inclusion $\mathsf{H} := \mathsf{M}_j \subset \mathsf{M}_\Delta$, we define the mixed Hodge structure of H to be the intersection of the (pieces) of the mixed Hodge structure of M_Δ with H:

$$W_i\mathsf{H} := W_i\mathsf{M}_\Delta \cap \mathsf{H},\ F^j\mathsf{H} := F^j\mathsf{M}_\Delta \cap \mathsf{H},$$

$$i = n-1, n, n+1,\ j = 0,1,2,\ldots,n+1.$$

The Hodge filtration induces a filtration on $\mathrm{Gr}_i^W := W_i/W_{i-1},\ i = n, n+1$ and we set

$$\begin{aligned}\mathrm{Gr}_F^j\mathrm{Gr}_i^W &:= F^j\mathrm{Gr}_i^W / F^{j+1}\mathrm{Gr}_i^W \\ &= \frac{(F^j \cap W_i) + W_{i-1}}{(F^{j+1} \cap W_i) + W_{i-1}}.\end{aligned} \tag{11.10}$$

for $j = 0,1,2,\ldots,n+1$.

For the original definition of the mixed Hodge structure in the complex context $\mathsf{R} = \mathbb{C}$ see Deligne's original articles [Del71a, Del71b, Del74] or Voisin's books [Voi02b, Voi03]. In fact we have used Griffiths-Steenbrink theorem, see [Ste77b], in order to formulate the above definition. We are going to discuss this theorem in §11.6

Since R is a principal ideal domain and $\mathsf{H} := \mathsf{H}',\ \mathsf{H}''$ is a free R-module (Corollary 10.1), any R-sub-module of H is also free and in particular the pieces of mixed Hodge structure of H are free R-modules.

Definition 11.3 *A set $B = \cup_{k=0}^n B_n^k \cup \cup_{k=1}^n B_{n+1}^k \subset \mathsf{H}$ is a basis of H compatible with the mixed Hodge structure if it is a basis of the R-module H and moreover each B_m^k form a basis of $\mathrm{Gr}_F^k\mathrm{Gr}_m^W\mathsf{H}$.*

11.4 Homogeneous tame polynomials

Below for simplicity we use d to denote the differential operator with respect to the variables $x_1, x_2, \ldots, x_{n+1}$. Let us consider a homogeneous polynomial g in the graded ring $\mathsf{R}[x]$, $\deg(x_i) = \nu_i$. We have the equality

$$g = \sum_{i=1}^{n+1} w_i x_i \frac{\partial g}{\partial x_i}$$

which is equivalent to

$$gdx = dg \wedge \eta.$$

We have also

$$g\omega_\beta = dg \wedge \eta_\beta, \tag{11.11}$$
$$d\eta_\beta = A_\beta \omega_\beta. \tag{11.12}$$

The discriminant of the polynomial g is zero. We define $f := g - s \in \mathsf{R}[s][x]$ which is tame and its discriminant is $(-s)^\mu$. We have

$$\frac{s\omega_\beta}{f^k} = \frac{-f\omega_\beta + dg \wedge \eta_\beta}{f^k} = (-1 + \frac{A_\beta}{k-1})\frac{\omega_\beta}{f^{k-1}}$$

in M. Therefore

$$\frac{\omega_\beta}{f^k} = \frac{1}{s^{k-1}}(-1+\frac{A_\beta}{k-1})(-1+\frac{A_\beta}{k-2})\cdots(-1+\frac{A_\beta}{1})\frac{\omega_\beta}{f} \tag{11.13}$$
$$= s^{-k}\frac{(A_\beta - k+1)_{k-1}}{(k-1)!}\eta_\beta = s^{-k}\frac{\langle A_\beta \rangle_k}{(k-1)!}\eta_\beta \tag{11.14}$$

in M, where $(x)_y = \langle x+y \rangle_{y+1} := x(x+1)\cdots(x+y-1)$ is the Pochhammer symbol. Note that under the canonical inclusion $\mathsf{H}' \subset \mathsf{H}''$ of the Brieskorn modules of f we have

$$s\omega_\beta = \eta_\beta.$$

We have also

$$\nabla_{\frac{\partial}{\partial s}}\eta_\beta = \frac{A_\beta}{s}\eta_\beta,$$
$$\nabla_{\frac{\partial}{\partial s}}(\omega_\beta) = \frac{(A_\beta - 1)}{s}\omega_\beta.$$

The meaning of these equalities will be clarified in Chapter (12).

Theorem 11.1 *For a weighted homogeneous polynomial $g \in \mathsf{R}[x]$ with an isolated singularity at the origin, the set*

$$B = \cup_{k=1}^n B_{n+1}^k \cup \cup_{k=0}^n B_n^k$$

with

$$B_{n+1}^k = \{\eta_\beta \mid A_\beta = n-k+1\},$$
$$B_n^k = \{\eta_\beta \mid n-k < A_\beta < n-k+1\},$$

is a basis of the R-module H' associated to $g - t \in \mathsf{R}[t][x]$ compatible with the mixed Hodge structure. The same is true for H'' replacing η_β with ω_β.

Proof. This theorem with the classical definition of the mixed Hodge structures is proved by Steenbrink in [Ste77b]. In our context it is a direct consequence of Definition 11.2 and the equality (11.13). □

Theorem 11.2 *In the Brieskorn module H'' of the tame polynomial $g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} - 1$ we have the following identity:*

$$\omega_\beta = \frac{\prod_{i=1}^{n+1} \left\langle \frac{\beta_i+1}{m_i} \right\rangle_{\left[\frac{\beta_i+1}{m_i}\right]+1}}{\left\langle A_\beta \right\rangle_{\sum_{i=1}^{n+1}\left[\frac{\beta_i+1}{m_i}\right]+1}} \omega_{\tilde{\beta}}, \tag{11.15}$$

where

$$\tilde{\beta}_i = \beta_i - \left[\frac{\beta_i+1}{m_i}\right] m_i, \quad i = 1,2,\ldots,n+1$$

and $\langle x \rangle_y := (x-1)(x-2)\cdots(x-y+1)$, $\langle x \rangle_1 := 1$.

Proof. For an arbitrary monomial x^β with $\beta_i > m_i - 2$ for some i, we use Proposition 10.11 and in the Brieskorn module H'' of g we have:

$$\begin{aligned}\omega_\beta = x^\beta dx &= x_i^{m_i-1} \cdot x^{\check{\beta}} dx \\ &= \frac{\beta_i + 1 - m_i}{m_i \cdot A_{\check{\beta}} - 1} x^{\tilde{\beta}} dx = \frac{\frac{\beta_i+1}{m_i} - 1}{A_\beta - 1} \omega_{\tilde{\beta}},\end{aligned}$$

where for $j = 1,2,\ldots,n+1$, $j \neq i$ we have $\tilde{\beta}_j = \beta_j$ and $\tilde{\beta}_i = \beta_i - m_i$. Note that the right hand side in the above equality might be zero and we have $A_{\tilde{\beta}} = A_\beta - 1, A_{\check{\beta}} = A_\beta - 1 + \frac{1}{m_i}$. Repeating this argument we get the desired equality. □

11.5 Weighted projective spaces

In this and the next section we are going to explain the origin of the definition of mixed Hodge structures in §11.3. For this reason we proceed in the complex context $\mathsf{R} = \mathbb{C}$. We first recall some terminology on weighted projective spaces which are the natural projective spaces for the varieties defined by tame polynomials. For a first reading of this and the next section, the reader may consider the classical projective space of dimension $n+1$. We have used [Dol82, Ste77b] as our main source on weighted projective spaces. The material of the present section is necessary in order to say that the Hodge filtration used in Definition 11.2 is the same one as in Chapter 8.

Let n be a natural number and $\nu := (\nu_1, \nu_2, \ldots, \nu_{n+1})$ be a vector of natural numbers whose greatest common divisor is one. The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ in the following way:

$$(x_1, x_2, \ldots, x_{n+1}) \to (\lambda^{\nu_1} x_1, \lambda^{\nu_2} x_2, \ldots, \lambda^{\nu_{n+1}} x_{n+1}),\ \lambda \in \mathbb{C}^*.$$

We also denote the above map by λ. The quotient space

$$\mathbb{P}^\nu := \mathbb{C}^{n+1}/\mathbb{C}^*$$

is called the projective space of weight ν. If $\nu_1 = \nu_2 = \cdots = \nu_{n+1} = 1$ then $\mathbb{P}^\nu$ is the usual projective space $\mathbb{P}^n$, (since n is a natural number, $\mathbb{P}^n$ will not mean a zero dimensional weighted projective space). One can give another interpretation of $\mathbb{P}^\nu$ as follow: Let $G_{\nu_i} := \{e^{\frac{2\pi\sqrt{-1}m}{\nu_i}} \mid m \in \mathbb{Z}\}$. The group $\Pi_{i=1}^{n+1} G_{\nu_i}$ acts discretely on the usual projective space $\mathbb{P}^n$ as follows:

$$(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1}), [x_1 : x_2 : \cdots : x_{n+1}] \to [\varepsilon_1 x_1 : \varepsilon_2 x_2 : \cdots : \varepsilon_{n+1} x_{n+1}]. \tag{11.16}$$

The quotient space $\mathbb{P}^n / \Pi_{i=1}^{n+1} G_{\nu_i}$ is canonically isomorphic to $\mathbb{P}^\nu$. This canonical isomorphism is given by

$$[x_1 : x_2 : \cdots : x_{n+1}] \in \mathbb{P}^n / \Pi_{i=1}^{n+1} G_{\nu_i} \to [x_1^{\nu_1} : x_2^{\nu_2} : \cdots : x_{n+1}^{\nu_{n+1}}] \in \mathbb{P}^\nu.$$

Let g be a homogeneous polynomial of degree d in the ring $\mathbb{C}[x]$ with weight$(x_i) := \nu_i$. The set $g = 0$ induces a hypersurface D in $\mathbb{P}^\nu$, $\nu = (\nu_1, \nu_2, \ldots, \nu_{n+1})$. If g has an isolated singularity at $0 \in \mathbb{C}^{n+1}$ then Steenbrink has proved that D is a V-manifold/quasi-smooth variety. A V-manifold may be singular but it has many common features with smooth varieties (see [Ste77b, Dol82]). We skip the context of V-manifolds and instead we use the following strategy to deal with V-manifolds. Let $\tilde{g} = g(x_1^{\nu_1}, x_2^{\nu_2}, \cdots, x_{n+1}^{\nu_{n+1}})$. This is a homogeneous polynomial of degree d in the usual sense (all weights equals to one). If g has an isolated singularity at the origin then $\tilde{g}$ has also this property and so $\tilde{g} = 0$ induces a smooth variety $\tilde{D}$ in $\mathbb{P}^n$. We can see easily that $\tilde{D}$ is invariant under (11.16), and D is the quotient of $\tilde{D}$ under this action. The following definition seems to be the simplest way to deal with the cohomology of V-manifolds in our context:

Definition 11.4 The de Rham cohomologies of $D \subset \mathbb{P}^\nu$ is the invariant part of the de Rham cohomologies of $\tilde{D}$ under the action of the discrete group $\Pi_{i=1}^{n+1} G_{\nu_i}$ on $\tilde{D}$ given in (11.16), that is,

$$H^m_{\mathrm{dR}}(D) := \left\{ \omega \in H^m_{\mathrm{dR}}(\tilde{D}) \middle| \varepsilon^* \omega = \omega, \ \forall \varepsilon \in \Pi_{i=1}^{n+1} G_{\nu_i} \right\}. \tag{11.17}$$

A similar definition is made for the de Rham cohomology $H^m_{\mathrm{dR}}(\mathbb{P}^\nu - D)$ of the complement of D in $\mathbb{P}^\nu$.

Definition 11.5 Let d be a natural number. The polynomial (resp. the polynomial form) ω in $\mathbb{C}^{n+1}$ is weighted homogeneous of degree d if

$$\lambda^*(\omega) = \lambda^d \omega, \ \lambda \in \mathbb{C}^*.$$

For a polynomial g this means that

$$g(\lambda^{\nu_1} x_1, \lambda^{\nu_2} x_2, \ldots, \lambda^{\nu_{n+1}} x_{n+1}) = \lambda^d g(x_1, x_2, \ldots, x_{n+1}), \ \forall \lambda \in \mathbb{C}^*$$

which is the same as to say that g is homogeneous of degree d in the weighted ring $\mathbb{C}[x]$. For a polynomial form ω of degree dk, $k \in \mathbb{N}$ in $\mathbb{C}^{n+1}$ we have $\lambda^* \frac{\omega}{g^k} =$

$\frac{\omega}{g^k}$ for all $\lambda \in \mathbb{C}^*$. Therefore, $\frac{\omega}{g^k}$ induce a meromorphic form on $\mathbb{P}^\nu$ with poles of order k along D. If there is no confusion we denote it again by $\frac{\omega}{g^k}$. We denote by $H^0(\mathbb{P}^\nu, \Omega^n(*D))$ the set of such differential forms. For fixed k, we also denote by $H^0(\mathbb{P}^\nu, \Omega^n(kD))$ the set of differential forms $\frac{\omega}{g^i}$, $i \leq k$. The polynomial form

$$\eta = \sum_{i=1}^{n+1} (-1)^{i-1} \nu_i x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}, \tag{11.18}$$

is of degree $\sum_{i=1}^{n+1} \nu_i$.

Let $\mathbb{P}^{(1,\nu)} = \{[x_0 : x_1 : \cdots : x_{n+1}] \mid (x_0, x_1, \cdots, x_{n+1}) \in \mathbb{C}^{n+2}\}$ be the projective space of weight $(1, \nu)$, $\nu = (\nu_1, \ldots, \nu_{n+1})$. One can consider $\mathbb{P}^{(1,\nu)}$ as a compactification of $\mathbb{C}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1})\}$ by replacing x_i with

$$\frac{x_i}{x_0^{\nu_i}},\ i = 1, 2, \cdots, n+1. \tag{11.19}$$

The projective space at infinity $\mathbb{P}^\nu_\infty = \mathbb{P}^{(1,\nu)} \backslash \mathbb{C}^{n+1}$ is of weight $\nu := (\nu_1, \nu_2, \ldots, \nu_{n+1})$. Let f be a tame polynomial of degree d in §10 over $\mathbb{C}$ and let g be its last homogeneous part. We take the homogenization

$$F = x_0^d f\left(\frac{x_1}{x_0^{\nu_1}}, \frac{x_2}{x_0^{\nu_2}}, \ldots, \frac{x_{n+1}}{x_0^{\nu_{n+1}}}\right)$$

of f and so we can regard $\{f = 0\}$ as an affine subvariety in $\{F = 0\} \subset \mathbb{P}^{(1,\nu)}$.

11.6 De Rham cohomology of hypersurfaces

Now, we are ready to state a classical theorem of Griffiths in [Gri69]. Its generalization for quasi-homogeneous spaces is due to Steenbrink in [Ste77b]. Since in the present text we only deal with the de Rham cohomology of smooth manifolds, the reader may consider all the weights ν_i equal to one and so $\mathbb{P}^\nu$ is the usual projective space $\mathbb{P}^{n+1}$. The reader may also consult Voisin's book [Voi03] Chapter 6 on this topic.

Theorem 11.3 ([Gri69, Ste77b]) *Let $g(x_1, x_2, \cdots, x_{n+1})$ be a weighted homogeneous polynomial of degree d, weight $\nu = (\nu_1, \nu_2, \ldots, \nu_{n+1})$ and with an isolated singularity at $0 \in \mathbb{C}^{n+1}$. We have*

$$H^n_{\mathrm{dR}}(\mathbb{P}^\nu - D) \cong \frac{H^0(\mathbb{P}^\nu, \Omega^n(*D))}{dH^0(\mathbb{P}^\nu, \Omega^{n-1}(*D))} \tag{11.20}$$

and under the above isomorphism

$$Gr_F^{n+1-k} Gr_{n+1}^W H^{n+1}(\mathbb{P}^\nu - D, \mathbb{C}) := F^{n-k+1}/F^{n-k+2} \cong \tag{11.21}$$

$$\frac{H^0(\mathbb{P}^\nu, \Omega^n(kD))}{dH^0(\mathbb{P}^\nu, \Omega^{n-1}((k-1)D)) + H^0(\mathbb{P}^\nu, \Omega^n((k-1)D))},$$

where $0 = F^{n+1} \subset F^n \subset \cdots \subset F^1 \subset F^0 = H^n_{\mathrm{dR}}(\mathbb{P}^\nu - D)$ *is the Hodge filtration of* $H^n(\mathbb{P}^\nu - D, \mathbb{C})$. *Let* $\{x^\beta \mid \beta \in I\}$ *be a basis of monomials for the Milnor vector space*

$$\mathbb{C}[x_1, x_2, \cdots, x_{n+1}]/ < \frac{\partial g}{\partial x_i} \mid i = 1, 2, \ldots, n+1 > .$$

A basis of (11.21) is given by

$$\frac{x^\beta \eta}{g^k},\ \beta \in I,\ A_\beta = k, \tag{11.22}$$

where

$$\eta = \sum_{i=1}^{n+1} (-1)^{i-1} \nu_i x_i \widehat{dx_i}.$$

In the above theorem we are talking about the Hodge filtration of $H^n_{\mathrm{dR}}(\mathbb{P}^\nu - D)$ and so far we have not defined it. We break its definition into two steps. First, let us consider the classical context, where all ν_i's equal one. In this case D is a smooth hypersurface in $\mathbb{P}^n$. In general, if D is normal crossing divisor in a projective variety M then $H^m_{\mathrm{dR}}(M - D)$ is isomorphic to the hypercohomology $\mathbb{H}^m(M, \Omega^\bullet(\log D)), m \geq 1$, where $\Omega^\bullet(\log D)$ is the sheaf of meromorphic differential forms in M with logarithmic poles along D. The i-th piece of the Hodge filtration of $H^m_{\mathrm{dR}}(M - D)$ under this isomorphism is given by $\mathbb{H}^m(M, \Omega^{\bullet \geq i}(\log D))$, see [Del71b]. By definition we have $F^0/F^1 \cong H^n(M, \mathscr{O}_M)$ and so in the situation of the above theorem $F^0 = F^1$. For arbitrary ν we use Definition (11.4) and define the Hodge filtration of $H^m_{\mathrm{dR}}(\mathbb{P}^\nu - D)$ the invariant part of the Hodge filtration of $H^m_{\mathrm{dR}}(\mathbb{P}^n - \tilde{D})$.

The essential ingredient in the proof of Theorem 11.3 is Bott's vanishing theorem for weighted projective spaces:

Theorem 11.4 Let X be a smooth hypersurface in $\mathbb{P}^\nu$. For $k > 0, p > 0$ and $q \geq 0$ we have

$$H^p(\mathbb{P}^\nu, \Omega^q_{\mathbb{P}^\nu}(kX)) = 0$$

where $\Omega^q_{\mathbb{P}^\nu}(kX)$ is the sheaf of meromorphic differential forms in $\mathbb{P}^\nu$ with pole order at most k along X.

11.7 Residue map

"La notion de résidu des intégrales doubles de fonctions rationnelles est due à M. Pioncaré (Sur les résidus des intégrales doubles, Acta Mathematica, t. II). Le point de vue auquel nous nous plaçons ici est celui qui a été adopté par M. Picard, dans son Mémoire sur les fonctions algébriques de deux variables (Journal de Mathematiques, 1889), dans le tome II de son *Traité d'Analysis* (p. 256), et dans le tome

CXXIV des *Comptes rendus*", (Picard and Simart in [PS06], Vol. I, page 52). The above quotation from Picard-Simart book leads us to the first appearances of higher dimensional residues in mathematics. In this section we give a modern account of this. Note that at the time of Poincaré and Picard, neither singular homology nor de Rham cohomology were developed.

Recall the Leray-Thom-Gysin isomorphism in 4.6 and the corresponding Gysin sequence 4.7. Let us write this in de Rham cohomology:

$$\cdots \leftarrow H^{m+1}_{\mathrm{dR}}(X) \leftarrow H^{m+1-c}_{\mathrm{dR}}(Y) \overset{\sigma^*}{\leftarrow} H^m_{\mathrm{dR}}(X-Y) \leftarrow H^m_{\mathrm{dR}}(X) \leftarrow \cdots \tag{11.23}$$

We have used $m+1$ instead of m.

Definition 11.6 The map $\mathrm{Resi} = \mathrm{Resi}_Y := \sigma^*$ is called the residue map. In other words, let $\omega \in H^m_{\mathrm{dR}}(X-Y) := \check{H}_m(X-Y,\mathbb{Z}) \otimes \mathbb{C}$, where $\check{H}_m(X-Y,\mathbb{Z})$ is the dual of $H_m(X-Y,\mathbb{Z})$. The composition $\omega \circ \sigma : H_{m+1-c}(Y,\mathbb{Z}) \to \mathbb{C}$ defines a linear map and its complexification is an element in $H^{m+1-c}_{\mathrm{dR}}(Y)$. It is denoted by $\mathrm{Resi}_Y(\omega)$ and called the residue of ω in Y. It is uniquely characterized by the equality

$$\int_\delta \mathrm{Resi}_Y(\omega) = \int_{\sigma(\delta)} \omega, \quad \omega \in H^m_{\mathrm{dR}}(X-Y), \ \ \delta \in H_{m+1-c}(Y,\mathbb{Z}). \tag{11.24}$$

Now let us consider the case in which X is a complex manifold of dimension n and Y is a complex submanifold of codimension one of X. We consider the case in which ω in the n-th de Rham cohomology of $X-Y$ is represented by a meromorphic C^∞ differential form ω' in X with poles of order at most one along Y. Let $f=0$ be the defining equation of Y and, let U_α be a neighborhood of a point $p \in Y$ in X in which we write $\omega' = \frac{df}{f} \wedge \omega_\alpha$. For two such neighborhoods U_α and U_β with non empty intersection we have $\omega_\alpha = \omega_\beta$ restricted to Y. Therefore, we get a $(n-1)$-form on Y which in the de Rham cohomology of Y represents $2\pi\sqrt{-1} \cdot \mathrm{Resi}_Y\,\omega$, see [Gri69] for details. This is called the Poincaré residue. For the rest of discussion in this section we will redefine the residue map to be the old one multiplied with $2\pi\sqrt{-1}$.

Now, let us return back to the context of §11.6. First we consider the case in which all ν_i's are equal to one. We have the residue map

$$\mathrm{Resi} : H^n(\mathbb{P}^\nu - D, \mathbb{C}) \to H^{n-1}(D,\mathbb{C})_0$$

which is an isomorphism of Hodge structures of weight -2, that is, it maps the k-th piece of the Hodge filtration of $H^n(\mathbb{P}^\nu - D,\mathbb{C})$ to the $(k-1)$-th piece of the Hodge filtration of $H^{n-1}(D,\mathbb{C})_0$. Here the sub index 0 means the primitive cohomology.

Proposition 11.4 *For a monomial x^β with $A_\beta = k \in \mathbb{N}$, the meromorphic form $\frac{x^\beta dx}{f^k}$ has a pole of order one at infinity and its Poincaré residue at infinity is $\frac{x^\beta \eta}{g^k}$. If $A_\beta < k$ then it has no poles at infinity.*

Proof. Let us write the above form in the homogeneous coordinates (11.19). We use $d(\frac{x_i}{x_0^{\nu_i}}) = x_0^{-\nu_i} dx_i - \nu_i x_i x_0^{-\nu_i - 1} dx_0$ and

$$\frac{x^\beta dx}{f^k} =$$

$$= \frac{(\frac{x_1}{x_0^{v_1}})^{\beta_1}\cdots(\frac{x_{n+1}}{x_0^{v_{n+1}}})^{\beta_{n+1}} d(\frac{x_1}{x_0^{v_1}})\wedge\cdots\wedge d(\frac{x_{n+1}}{x_0^{v_{n+1}}})}{f(\frac{x_1}{x_0^{v_1}},\cdots,\frac{x_{n+1}}{x_0^{v_{n+1}}})^k}$$

$$= \frac{x^\beta \eta_{(1,v)}}{x_0^{(\sum_{i=1}^{n+1}\beta_i v_i)+(\sum_{i=1}^{n+1} v_i)+1-kd}(x_0\tilde{F}+g(x_1,x_2,\cdots,x_{n+1}))^k}$$

$$= \frac{x^\beta \eta_{(1,v)}}{x_0(x_0\tilde{F}+g(x_1,x_2,\cdots,x_{n+1}))^k}$$

$$= \frac{dx_0}{x_0}\wedge\frac{x^\beta\eta}{(x_0\tilde{F}+g)^k}$$

where $x_0\tilde{F}+g$ is the homogenization of f. The last equality is up to forms without pole at $x_0=0$. The restriction of $\frac{x^\beta\eta}{(x_0\tilde{F}+g)^k}$ to $x_0=0$ gives us the desired form. If $A_\beta<k$ then the second equality above tells us that $\frac{\omega_\beta}{f^k}$ has no poles at infinity $x_0=0$. □

Recall from §11.2 the Gauss-Manin system of a tame polynomial f with non-zero discriminant Δ over the ring R. We have a well-defined map

$$\mathrm{Resi}:\mathsf{M}\to\mathsf{H}'_\Delta, \tag{11.25}$$

where H'_Δ is the localization of H' over $\{1,\Delta,\Delta^2,\cdots\}$. It is defined in the following way. We use (10.25) and (11.6) to reduce the pole order of an element of M, till we get the pole order one. For this we have to divide over Δ many times. Then we use (11.5) in order to get an element ω of H''_Δ. Finally, we use the Gelfand-Leray form of ω, see (10.32), for which we have to divide over Δ once again. Over complex numbers, that is $\mathsf{R}=\mathbb{C}$, we have $\mathsf{M}\cong H^{n+1}_{\mathrm{dR}}(\mathbb{C}^{n+1}\backslash L)$ and $\mathsf{H}'_\Delta\cong H^n_{\mathrm{dR}}(L)$, where $L:=\{f=0\}$, and this is the usual residue map defined earlier. If $f=g-s$ then the computation of the residue map is done in (11.13). The following is a part of Theorem 11.3 and it follows from our computations with Brieskorn modules.

Proposition 11.5 *A basis of* (11.20) *is given by* (11.22) *with* $k=1,2,\ldots,n$.

Proof. We consider $\mathbb{P}^v$ in Theorem 11.3 as the projective space at infinity of $\mathbb{C}^{n+1}$ and so $\mathbb{P}^{n+1}=\mathbb{C}^{n+1}\cup\mathbb{P}^v$, see Figure 11.1. We also consider the affine variety $L:=\{g-s=0\}$, $s\in\mathbb{C},\ s\neq 0$. It intersects $\mathbb{P}^v$ at D transversely. The differential form $\frac{x^\beta\eta}{(g-s)^k},\ \beta\in I,\ A_\beta=k$ has a pole order k along L and a pole order one along $\mathbb{P}^v$. Its residue along L (resp. $\mathbb{P}^v$) is η_β (resp. the projectivization of $\frac{x^\beta\eta}{g^k}$). Now the proposition follows from the fact that $\eta_\beta,\ \beta\in I$ form a basis of $H^n_{\mathrm{dR}}(L)$, see Theorem 10.1. The details are left to the reader. □

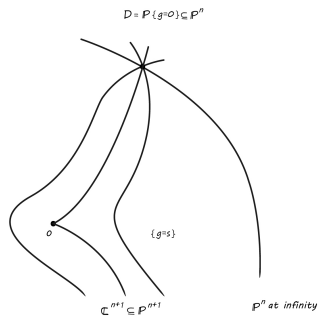

Fig. 11.1 Fibrations and infinity

11.8 A basis of Brieskorn module

In this section we aim to generalize Theorem 11.1 for arbitrary tame polynomial with non-zero discriminant. For simplicity, we state the result over a field k. In this case the canonical inclusions become identities $\mathsf{H} := \mathsf{M}_0 = \mathsf{M}_1 = \cdots = \mathsf{M}_i = \cdots$.

Theorem 11.5 *Let $f \in \mathsf{k}[x]$ be a tame polynomial with non-zero discriminant. There exists a map $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ such that*

$$B_n^k := \left\{ \frac{\omega_\beta}{f^{n+1-k}} \;\middle|\; n+1-k-\frac{d_\beta+1}{d} < A_\beta < n+1-k \right\}, \tag{11.26}$$

$$B_{n+1}^k := \left\{ \frac{\omega_\beta}{f^{n+1-k}} \;\middle|\; A_\beta = n+1-k \right\} \tag{11.27}$$

is a basis of $\mathrm{Gr}_F^k \mathrm{Gr}_n^W \mathsf{H}$ *and* $\mathrm{Gr}_F^k \mathrm{Gr}_{n+1}^W \mathsf{H}$*, respectively.*

Definition 11.7 We call the basis in Theorem 11.5 the Griffiths-Steenbrink basis of H. The reason for this naming is that the main ingredient in its proof is the Definition 11.2 which is derived from Theorem 11.3.

Remark 11.1 There are two different ways to derive results from Theorem 11.5 for arbitrary ring R. Either we take k to be the quotient field of R or we take a closed point $t \in \mathsf{T}$, $\Delta(t) \neq 0$ and define k to be the residue field of t. In both cases there exists $\tilde{\Delta}$ divisible by Δ such that Theorem 11.5 is valid over the ring $\mathsf{R}_{\tilde{\Delta}}$ (in the second case $\tilde{\Delta}(t) \neq 0$). Our proof of Theorem 11.5 in the first case, also computes $\tilde{\Delta}$. This is not true in the second case.

Let $f = f_0 + f_1 + f_2 + \cdots + f_{d-1} + f_d$ be the homogeneous decomposition of f in the graded ring $\mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$, $\deg(x_i) = \alpha_i$ and let $F = f_0 x_0^d + f_1 x_0^{d-1} +$

$\cdots + f_{d-1}x_0 + g$ be the homogenization of f. Recall that we have fixed a basis $x^I := \{x^\beta | \beta \in I\}$ for the free R-module V_g.

Proposition 11.6 *The* $\mathsf{R}[x_0]$*-module*

$$V := \frac{\mathsf{R}[x_0, x]}{\langle \frac{\partial F}{\partial x_i} \mid i = 1, 2, \ldots, n+1 \rangle}$$

is freely generated by x^I.

Proof. The proof is the same as in Proposition 10.7. □

Let

$$A_F : V \to V,\ A_F(G) = \frac{\partial F}{\partial x_0} G,\ G \in V.$$

Proposition 11.7 *The matrix of* A_F *in the basis* x^I *is of the form* $d \cdot [x_0^{K_{\beta,\beta'}} c_{\beta,\beta'}]$, *where*

$$0 \le K_{\beta,\beta'} := d - 1 + \deg(x^\beta) - \deg(x^{\beta'})$$

and $A := [c_{\beta,\beta'}]$ *is the multiplication by* f *in* V_f *(see §10.9). In particular, if* $A_{\beta'} - A_\beta \ge 1$ *then* $c_{\beta,\beta'} = 0$ *and*

$$\det(A_F) = \Delta \cdot x_0^{(d-1)\mu}.$$

Proof. Since the polynomial F is weighted homogeneous, we have $\sum_{i=0}^{n+1} \alpha_i x_i \frac{\partial F}{\partial x_i} = d \cdot F$ and so $x_0 \frac{\partial F}{\partial x_0} = d \cdot F$ in V (note that $\alpha_0 = 1$ by definition). We write $f \cdot x^\beta$ in terms of x^I and partial derivatives of f with respect to x_i, $i = 1, 2, \ldots, n+1$ and homogenize it. We get

$$F \cdot x^\beta = \sum_{\beta' \in I} x^{\beta'} c_{\beta,\beta'}(x_0) + \sum_{i=1}^{n+1} \frac{\partial F}{\partial x_i} q_i, \tag{11.28}$$

$$c_{\beta,\beta'}(x_0) \in \mathsf{R}[x_0],\ q_i \in \mathsf{R}[x_0, x].$$

Since the left hand side is homogeneous of degree $d + \deg(x^\beta)$, the pieces of the right hand side are also homogeneous of the same degree. The proposition follows immediately. □

For the next proposition we work over a field.

Proposition 11.8 *Let* $f \in \mathsf{k}[x]$ *be a tame polynomial with non-zero discriminant. There exists a map* $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ *such that the* k*-vector space*

$$\tilde{V} := \frac{\mathsf{k}[x_0, x]}{\langle \frac{\partial F}{\partial x_i} \mid i = 0, 1, \ldots, n+1 \rangle}$$

is freely generated by

$$\{x_0^{\beta_0}x^{\beta}, 0 \le \beta_0 \le d_\beta - 1, \beta \in I\} \tag{11.29}$$

A theoretical, but short, proof for this proposition is given in [Mov07b], Lemma 6.3. We prefer the following algorithmic proof which is an adaptation from [Mov07a],§7.

Proof. We introduce a kind of Gaussian elimination in A_F and simplify it. For this reason we introduce the operation $GE(\beta_1,\beta_2,\beta_3)$. Meantime, we also compute $\tilde{\Delta} \in \mathsf{k}$ which encodes all the denominators which we need in the algorithm. For $\beta \in I$ let $(A_F)_\beta$ be the β-th row of A_F.

- Input: A_F, $\beta_1,\beta_2,\beta_3 \in I$ with $A_{\beta_1} \le A_{\beta_2}$ with $\tilde{\Delta} \in \mathsf{R}$. Output: a new matrix A_F and a new $\tilde{\Delta} \in \mathsf{R}$. We replace $(A_F)_{\beta_2}$ with

$$-\frac{(A_F)_{\beta_2,\beta_3}}{(A_F)_{\beta_1,\beta_3}}(A_F)_{\beta_1} + (A_F)_{\beta_2}$$

  and we set $\tilde{\Delta}$ to be the input $\tilde{\Delta}$ times $c \in \mathsf{k}$, where, $(A_F)_{\beta_1,\beta_3} = c \cdot x_0^{K_{\beta_1,\beta_3}}$. Since for all $\beta_4 \in I$ we have

$$K_{\beta_2,\beta_3} + K_{\beta_1,\beta_4} = K_{\beta_1,\beta_3} + K_{\beta_2,\beta_4}$$

  the obtained matrix A_F is of the form $[x_0^{K_{\beta,\beta'}} c'_{\beta,\beta'}]$ and $c'_{\beta_2,\beta_3} = 0$.

Now, we give an algorithm which calculates d_β's.

- Input: A_F. Output: $d_\beta, \beta \in I$ and $\tilde{\Delta} \in \mathsf{k}$. We identify I with $\{1,2,\ldots,\mu\}$ and assume that

$$\beta_1 \le \beta_2 \Rightarrow A_{\beta_1} \ge A_{\beta_2}.$$

  The algorithm has μ steps indexed by $\beta = \mu, \mu-1, \ldots, 1$. In $\beta = \mu$ we have the input A_F and $\tilde{\Delta}$ is the discriminant Δ. In the step β we find the first β_1 such that $(A_F)_{\beta,\beta_1} \ne 0$ and put $d_{\beta_1} = d - 1 + \deg(x^\beta) - \deg(x^{\beta_1})$. For $\beta_2 = \beta - 1, \ldots, 1$ we perform $GE(\beta,\beta_2,\beta_1)$, and get a new A_F with $\tilde{\Delta} \in \mathsf{k}$. □

Proof (of Theorem 11.5). The main idea behind the proof is that Definition 11.2 is derived from Theorem 11.3. We have canonical identifications:

$$\mathrm{Gr}_F^j \mathrm{Gr}_{n+1}^W \mathsf{H} \cong (\mathsf{W}_f)_{A_\beta = n+1-j} \cong (\mathsf{k}[x]/\mathrm{jacob}(g))_{(n+1-j)d-n-1} \tag{11.30}$$

$$\mathrm{Gr}_F^j \mathrm{Gr}_n^W \mathsf{H} \cong (\mathsf{W}_f)_{A_\beta < n+1-j,\ A_\beta \notin \mathbb{N}} \cong (\mathsf{k}[x_0,x]/\mathrm{jacob}(F))_{(n+1-j)d-n-2} \tag{11.31}$$

where V_* means the subspace of V generated by monomials with condition $*$ (for the second vector spaces) or monomials of degree $*$ (for the third vector spaces). These together Proposition 11.8 finish the proof.

11.9 Exercises

11.1. This is the reformulation of the notion of mixed Hodge structure in the one dimensional case and using the classical terminology of differential 1-forms on curves. Let L be the curve $x^{m_1}+y^{m_2}=1$ in $\mathbb{C}^2$, where $m_1,m_2\geq 2$ are positive integers. Let also $\omega_\beta := x^{\beta_1}y^{\beta_2}(xdy-ydx)$ for $\beta=(\beta_1,\beta_2)\in I:=\{0,1,\dots,m_1-2\}\times\{0,1,\dots,m_2-2\}$ and $A_\beta := \frac{\beta_1+1}{m_1}+\frac{\beta_2+1}{m_2}$.

1. Show that there is a compactification X of L by adding
$$\#\left\{\beta\in I\middle| A_\beta=1\right\}+1$$
points to L. We call these points at infinity.
2. Show that the differential forms
$$\omega_\beta,\ \ A_\beta<1, \tag{11.32}$$
are holomorphic at points at infinity. These are called differential forms of the first kind. Conclude that the genus of X is the number of differential forms in (11.32).
3. Show that the differential forms
$$\omega_\beta,\ \ 1<A_\beta,$$
are meromorphic at the points at infinity and they do not have residues at such points. These are called differential forms of the second kind.
4. Compute the residues of differential forms
$$\omega_\beta,\ A_\beta=1$$
at the points at infinity, except one, and show that the corresponding matrix of residues has non-zero determinant. These are called differential forms of the third kind.

11.2. Show that in Theorem 11.5 and Proposition 11.8 we have
$$\sum_{\beta\in I} d_\beta=(d-1)\cdot\mu, \tag{11.33}$$
where d and μ are respectively the degree and the Milnor number of f.

Chapter 12
Gauss-Manin connection for tame polynomials

Grâce à l'article de Manin [Man64], les équations de Picard-Fuchs redeviennent à la mode, sous le nouveau nom de "connexion de Gauss-Manin" (dû à Grothendieck, je crois)-pourquoi "connexion"?-sans doute parce que "équations différentielles" sonne moins bien, aux oreilles d'un géomètre, que "connexion sur un fibré vectoriel", (F. Pham in [Pha79] page 18).

12.1 Introduction

The objective of the present chapter is to introduce the Gauss-Manin connection of the fibration induced by a tame polynomial f. In 1958 Yu. Manin in [Man64] solved the Mordell conjecture over function fields and A. Grothendieck after reading his article invented the name Gauss-Manin connection. The name of Gauss in this concept mainly represent the works of many people, starting from Euler, Cauchy, Jacobi, Riemann, Picard and many others, who were interested in integrals which depend on parameters, and so their differentiation is a natural job to do. As we read in the above quotation from F. Pham, Gauss-Manin connections are the resuscitation of the old Picard-Fuchs equations. I did not find any simple exposition of Gauss-Manin connection, the one which could be understandable by Gauss's mathematics. The reader who is good at integral manipulation is invited to do Exercises 12.2 and 12.1 in order to get the core idea behind the concepts of Gauss-Manin connection and Picard-Fuchs equation. The idea is very simple. Many times an integral depends on some parameters and so the resulting integration is a function in that parameters. For instance, take the elliptic integral $\int_\delta \frac{dx}{P(x)}$ which depends on a coefficient of P, let us call it t. In any course in calculus we learn that the integration and derivation with respect to t commute:

$$\frac{\partial}{\partial t}\int_\delta \frac{dx}{\sqrt{P(x)}} = \int_\delta \frac{\partial}{\partial t}(\frac{1}{\sqrt{P(x)}})dx. \tag{12.1}$$

In Exercise 2.8 and Chapter 10 we have learned that the right hand side of the above equality can be written as a linear combination of two integrals

$$\int_\delta \frac{dx}{\sqrt{P(x)}}, \int_\delta \frac{xdx}{\sqrt{P(x)}}.$$

This is the historical origin of the notion of Gauss-Manin connection, that is, derivation of integrals with respect to parameters and simplifying the result in terms of integrals which cannot be simplified more.

An algebraic description of the Gauss-Manin connection was first done in Katz-Oda article [KO68]. The first computational approach in the case of hypersurfaces is due to Griffiths and Dwork, see [Gri70] and the references therein. A complete computation of the Gauss-Manin connection in the case of elliptic curves (see Exercise 12.2) appears in the works of Griffiths and Sasai, see [Sas74, Gri66], see also [Pha79] page 8. For the two variable polynomial $f(x,y)-s$ with the parameter s such a calculation or parts of it is done by many people in the context of planar differential equations, see for instance [Gav01] and the references therein. The multi variable case of such a calculation can be interesting from the Hodge theory point of view and it is discussed in the author's articles [Mov07b, Mov07a] for a tame polynomial in the sense of Chapter 10. Our arguments in the present chapter work for a tame polynomial f defined on a general ring as it is described in Chapter 10.

12.2 Gauss-Manin connection

In this section we define the so called Gauss-Manin connection of the R-module H. The Tjurina module of f can be rewritten in the form

$$\begin{aligned}\mathsf{W}_f &:= \frac{\Omega^{n+1}_{\mathsf{A}}}{df\wedge\Omega^n_{\mathsf{A}} + f\Omega^{n+1}_{\mathsf{A}} + \pi^{-1}\Omega^1_{\mathsf{T}}\wedge\Omega^n_{\mathsf{A}}}\\ &\cong \frac{\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}}{df\wedge\Omega^n_{\mathsf{A}/\mathsf{T}} + f\Omega^{n+1}_{\mathsf{A}/\mathsf{T}}}.\end{aligned}$$

Looking in this way, we have the well defined differential map

$$d : \mathsf{H}' \to \mathsf{W}_f.$$

Let Δ be the discriminant of f. We define the Gauss-Manin connection on H' as follows:

$$\nabla : \mathsf{H}' \to \Omega^1_{\mathsf{T}_\Delta} \otimes_{\mathsf{R}} \mathsf{H}'$$

$$\nabla\omega = \frac{1}{\Delta}\sum_i \alpha_i \otimes \beta_i,$$

where

$$\Delta d\omega - \sum_i \alpha_i \wedge \beta_i \in f\Omega_{\mathsf{A}}^{n+1} + df \wedge \Omega_{\mathsf{A}}^n,\ \alpha_i \in \Omega_{\mathsf{T}}^1,\ \beta_i \in \Omega_{\mathsf{A}}^n,$$

and $\Omega_{\mathsf{T}_\Delta}^1$ is the localization of Ω_{T}^1 on the multiplicative set $\{1, \Delta, \Delta^2, \ldots\}$. From an Algebraic Geometry point of view this is the set of differential forms defined in

$$\mathsf{T}_\Delta := \mathrm{Spec}(\mathsf{R}_\Delta) = \mathsf{T} \setminus \{\Delta = 0\}.$$

To define the Gauss-Manin connection on H'' we use the fact that $\frac{\mathsf{H}''}{\mathsf{H}'} = \mathsf{W}_f$ and define

$$\nabla : \mathsf{H}'' \to \Omega_{\mathsf{T}_\Delta}^1 \otimes_{\mathsf{R}} \mathsf{H}'',$$

$$\nabla(\omega) = \nabla(\frac{\Delta \cdot \omega}{\Delta}) = \frac{\nabla(\Delta \cdot \omega) - d\Delta \otimes \omega}{\Delta}, \tag{12.2}$$

where $\Delta \cdot \omega = df \wedge \eta,\ \eta \in \mathsf{H}'$.

The operator ∇ satisfies the Leibniz rule, i.e.

$$\nabla(p \cdot \omega) = p \cdot \nabla(\omega) + dp \otimes \omega,\ p \in \mathsf{R},\ \omega \in \mathsf{H}$$

and so it is a connection on the module H. It defines the operators

$$\nabla_i = \nabla : \Omega_{\mathsf{T}_\Delta}^i \otimes_{\mathsf{R}} \mathsf{H} \to \Omega_{\mathsf{T}_\Delta}^{i+1} \otimes_{\mathsf{R}} \mathsf{H}.$$

by the rules

$$\nabla_i(\alpha \otimes \omega) = d\alpha \otimes \omega + (-1)^i \alpha \wedge \nabla\omega,\ \alpha \in \Omega_{\mathsf{T}_\Delta}^i,\ \omega \in \mathsf{H}.$$

If there is no danger of confusion we will use the symbol ∇ for these operators too.

Proposition 12.1 *The connection ∇ is an integrable connection, i.e. $\nabla \circ \nabla = 0$.*

Proof. We have

$$d\omega = \sum_i \alpha_i \wedge \beta_i,\ \alpha_i \in \Omega_{\mathsf{T}_\Delta}^1,\ \beta_i \in \Omega_{\mathsf{A}}^n$$

modulo $f\Omega_{\mathsf{A}}^{n+1} + df \wedge \Omega_{\mathsf{A}}^n$. We take the differential of this equality and so we have

$$\sum_i d\alpha_i \wedge \beta_i - \alpha_i \wedge d\beta_i = 0$$

again modulo $f\Omega_{\mathsf{A}}^{n+1} + df \wedge \Omega_{\mathsf{A}}^n$. This implies that

$$\begin{aligned} \nabla \circ \nabla(\omega) &= \nabla(\sum_i \alpha_i \otimes \beta_i) \\ &= \sum_i d\alpha_i \otimes \beta_i - \alpha_i \wedge \nabla\beta_i \\ &= 0. \end{aligned}$$

□

12.3 Picard-Fuchs equations

It is is useful to look at the Gauss-Manin connection in the following way: We have the operator

$$\Theta_{\mathsf{T}} \to \mathrm{Lei}(\mathsf{H}_\Delta),\ v \mapsto \nabla_v,$$

where Θ_{T} is the set of vector fields in T, ∇_v is the composition

$$\mathsf{H}_\Delta \overset{\nabla}{\to} \Omega^1_{\mathsf{T}_\Delta} \otimes_{\mathsf{R}_\Delta} \mathsf{H}_\Delta \overset{v\otimes 1}{\to} \mathsf{R}_\Delta \otimes_{\mathsf{R}_\Delta} \mathsf{H}_\Delta \cong \mathsf{H}_\Delta,$$

and $\mathrm{Lei}(\mathsf{H}_\Delta)$ is the set of additive maps ∇_v from H_Δ to itself which satisfy

$$\nabla_v(r\omega) = r\nabla_v(\omega) + dr(v)\cdot\omega,\ v \in \Theta_{\mathsf{T}},\ \omega \in \mathsf{H}_\Delta,\ r \in \mathsf{R}_\Delta.$$

In this way H_Δ is a (left) Θ_{T}-module (differential module):

$$v\cdot\omega := \nabla_v(\omega),\ v \in \Theta_{\mathsf{T}},\ \omega \in \mathsf{H}_\Delta.$$

Note that we can now iterate ∇_v, i.e. $\nabla_v^s = \nabla_v \circ \nabla_v \circ \cdots \circ \nabla_v$ s-times, and this is different from $\nabla \circ \nabla$ introduced before.

Definition 12.1 For a given vector field $v \in \Theta_{\mathsf{T}}$ and $\omega \in \mathsf{H}$ consider

$$\omega, \nabla_v(\omega), \nabla_v^2(\omega), \cdots \in \mathsf{H}_\Delta.$$

Since the R_Δ-module H_Δ is free of of rank μ, there exists a positive integer $m \le \mu$ and $p_i \in \mathsf{R},\ i = 0,1,2,\ldots,m$ such that

$$p_0\omega + p_1\nabla_v(\omega) + p_2\nabla_v^2(\omega) + \cdots + p_m\nabla_v^m(\omega) = 0 \tag{12.3}$$

This is called the Picard-Fuchs equation of ω along the vector field v. Since R is a unique factorization domain, we assume that there is no common factor between p_i.

Apart from our computer implementation of Picard-Fuchs equations in this book see also [Lai16] for other algorithms.

12.4 Gauss-Manin connection matrix

Let $\omega_1, \omega_2, \ldots, \omega_\mu$ be a basis of H and define $\omega = [\omega_1, \omega_2, \ldots, \omega_\mu]^{\mathsf{tr}}$. The Gauss-Manin connection in this basis can be written in the following way:

$$\nabla\omega = A \otimes \omega,\ A \in \frac{1}{\Delta}\mathrm{Mat}(\mu, \Omega^1_{\mathsf{T}}) \tag{12.4}$$

The integrability condition translates into $dA = A \wedge A$.

12.5 Calculating Gauss-Manin connection

Let

$$\tilde{d} : \Omega_{\mathsf{A}}^{\bullet} \to \Omega_{\mathsf{A}}^{\bullet+1}$$

be the differential map with respect to variable x, i.e. $\tilde{d}r = 0$ for all $r \in \mathsf{R}$, and

$$\check{d} : \Omega_{\mathsf{A}}^{\bullet} \to \Omega_{\mathsf{A}}^{\bullet+1}$$

be the differential map with respect to the elements of R. It is the pull-back of the differential in T. We have

$$d = \tilde{d} + \check{d},$$

where d is the total differential mapping. Let s be a new parameter and $S(s)$ be the discriminant of $f - s$. We have

$$S(f) = \sum_{i=1}^{n+1} p_i \frac{\partial f}{\partial x_i},\ p_i \in \mathsf{k}[x]$$

or equivalently

$$S(f)dx = df \wedge \eta_f,\ \eta_f = \sum_{i=1}^{n+1} (-1)^{i-1} p_i \widehat{dx_i}. \tag{12.5}$$

Note that $S(f)$ is the substitution of the polynomial f in the variable s of the polynomial $S(s)$ and so it is not the discriminant of $f - f = 0$ which is meaningless. To calculate ∇ of

$$\omega = \sum_{i=1}^{n+1} P_i \widehat{dx_i} \in \mathsf{H}'$$

we assume that ω has no $dr,\ r \in \mathsf{R}$, but the ingredient polynomials of ω may have coefficients in R. Let $\Delta = S(0)$ and

$$\tilde{d}\omega = P \cdot dx.$$

We have

$$\begin{aligned} S(f)d\omega &= S(f)\tilde{d}\omega + S(f)\sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i} \\ &= \tilde{d}f \wedge (P \cdot \eta_f) + S(f)\sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}. \end{aligned}$$

This implies that

$$\begin{aligned}\Delta d\omega &= (\Delta - S(f))(d\omega - \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) + \\ &\quad df \wedge (P\cdot\eta_f) + (\Delta \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) - \check{d}f \wedge (P\cdot\eta_f) \\ &= (\Delta \sum_{i=1}^{n+1} \check{d}P_i \wedge \widehat{dx_i}) - \check{d}f \wedge (P\cdot\eta_f) \\ &= \sum_j dt_j \wedge \left(\Delta(\sum_{i=1}^{n+1} \frac{\partial P_i}{\partial t_j}\widehat{dx_i}) - \frac{\partial f}{\partial t_j}\cdot P\cdot\eta_f\right).\end{aligned}$$

all the equalities are in $\Omega^1_{\mathsf{T}} \otimes \mathsf{H}'$. We conclude that

$$\nabla(\omega) = \tag{12.6}$$

$$\frac{1}{\Delta}\left(\sum_j dt_j \otimes \left(\sum_{i=1}^{n+1}\left(\Delta\frac{\partial P_i}{\partial t_j} - (-1)^{i-1}\frac{\partial f}{\partial t_j}\cdot P\cdot p_i\right)\widehat{dx_i}\right)\right),$$

where

$$P = \sum_{i=1}^{n+1}(-1)^{i-1}\frac{\partial P_i}{\partial x_i}.$$

It is useful to define

$$\frac{\partial\omega}{\partial t_j} = \sum_{i=1}^{n+1}\frac{\partial P_i}{\partial t_j}\widehat{dx_i}.$$

Then

$$\nabla(\omega) = \frac{1}{\Delta}\left(\sum_j dt_j \otimes \left(\Delta\frac{\partial\omega}{\partial t_j} - \frac{\partial f}{\partial t_j}\cdot P\cdot\eta_f\right)\right). \tag{12.7}$$

The calculation of ∇ in H'' can be done using

$$\nabla(P\cdot dx) = \frac{df\wedge\nabla(P\eta_f) - d\Delta\otimes P dx}{\Delta},\ P dx \in \mathsf{H}''$$

which is derived from (12.2). Note that we calculate $\nabla(P\cdot\eta_f)$ from (12.6). We lead to the following explicit formula

$$\nabla(P\cdot dx) = \tag{12.8}$$

$$\frac{1}{\Delta}\left(\sum_j dt_j \otimes \left(\tilde{d}f \wedge \frac{\partial(P\eta_f)}{\partial t_j} - \frac{\partial f}{\partial t_j}Q_P - \frac{\partial\Delta}{\partial t_j}P\right)\right),$$

where

$$Q_P = \sum_{i=1}^{n+1}(\frac{\partial P}{\partial x_i}p_i + P\frac{\partial p_i}{\partial x_i}).$$

To be able to calculate the iterations of the Gauss-Manin connection along a vector field v in T, it is useful to introduce the operators:

$$\nabla_{v,k} : \mathsf{H} \to \mathsf{H},\ k = 0, 1, 2, \ldots$$

$$\nabla_{v,k}(\omega) = \nabla_v(\frac{\omega}{\Delta^k})\Delta^{k+1} = \Delta \cdot \nabla_v(\omega) - k \cdot d\Delta(v) \cdot \omega.$$

It is easy to show by induction on k that

$$\nabla_v^k = \frac{\nabla_{v,k-1} \circ \nabla_{v,k-2} \circ \cdots \circ \nabla_{v,0}}{\Delta^k}. \tag{12.9}$$

Remark 12.1 The formulas (12.8) and (12.7) for the Gauss-Manin connection usually produce polynomials of huge size, even for simple examples. Specially when we want to iterate the Gauss-Manin connection along a vector field, the size of polynomials is so huge that even with a computer (of the time of writing this text) we get the lack of memory problem. However, if we write the result of the Gauss-Manin connection, in the canonical basis of the R-module H, and hence reduce it modulo to those differential forms which are zero in H, we get polynomials of reasonable size.

12.6 $\mathsf{R}[\theta]$ structure of H''

In this section we consider the $\mathsf{R}[s]$-modules H'' and H', where $s\omega := f\omega$. We have the following well-defined map:

$$\theta : H'' \to H',\ \theta\omega = \eta,\ \text{where}\ \ \omega = d\eta.$$

We have used the fact that $H^n_{\mathrm{dR}}(\mathsf{A}/\mathsf{T}) = 0$ (see Proposition 10.4). It is well-defined because:

$$df \wedge d\eta_1 = d\eta_2 \Rightarrow \eta_2 = df \wedge \eta_1 + d\eta_3,\ \text{for some}\ \eta_3 \in \Omega^{n-1}_{\mathsf{A}/\mathsf{T}}.$$

Using the inclusion $H' \to H''$, $\omega \mapsto df \wedge \omega$, both H' and H'' are now $\mathsf{R}[s,\theta]$-modules. The relation between $\mathsf{R}[s]$ and $\mathsf{R}[\theta]$ structures is given by:

Proposition 12.2 *We have:*

$$\theta \cdot s = s \cdot \theta - \theta \cdot \theta$$

and for $n \in \mathbb{N}$

$$\theta^n s = s\theta^n - n\theta^{n+1}.$$

Proof. The map $d : H' \to H''$ satisfies

$$d \cdot s = s \cdot d + df,$$

where s stands for the mapping $\omega \mapsto s\omega$ and df stands for the mapping $\omega \mapsto df \wedge \omega,\ \omega \in H'$. Composing the both sides of the above equality by θ we get the first statement. The second statement is proved by induction. $\square$

For a homogeneous polynomial g with an isolated singularity at the origin we have $d\eta_\beta = A_\beta \omega_\beta$ and $dg \wedge \eta_\beta = g\omega_\beta$ and so

$$\theta \omega_\beta = \frac{s}{A_\beta} \omega_\beta.$$

Remark 12.2 The action of θ on H'' is inverse to to the action

$$\frac{d}{df} : H' \to H'_{\Delta(s)}, \quad \omega \mapsto \frac{d\omega}{df},$$

where $\Delta(s)$ is the discriminant of $f - s = 0$. This is the Gauss-Manin connection with respect to the parameter s in $f - s = 0$: (we have composed the Gauss-Manin connection with $\frac{\partial}{\partial s}$). This arises the following question: is it possible to construct similar structures for H' and H''?

12.7 Gauss-Manin system

The Gauss-Manin connection on M is the map

$$\nabla : \mathsf{M} \to \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} \mathsf{M}$$

which is obtained by derivation with respect to the elements of R (the derivation of x_i is zero). By definition it maps M_i to $\Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} \mathsf{M}_{i+1}$. For any vector field in T, ∇_v is given by

$$\nabla_v : \mathsf{M} \to \mathsf{M},\ \nabla_v([\frac{Pdx}{f^i}]) := [\frac{v(P) \cdot f - iP \cdot v(f)}{f^{i+1}} dx],\ P \in \mathsf{R}[x], \tag{12.10}$$

where $v(P)$ is the differential of P with respect to elements in R and along the vector field v ($v(\cdot) : \mathsf{R} \to \mathsf{R},\ p \mapsto dp(v)$). In the case $i = 0$ it is given by

$$\begin{aligned} \nabla_v([\frac{df \wedge \omega}{f}]) &= [\frac{f \cdot v(df \wedge \omega) - v(f) \cdot df \wedge \omega}{f^2}] \\ &= [\frac{v(df \wedge \omega) + d(v(f) \cdot \omega)}{f}] \end{aligned}$$

and so ∇_v maps M_0 to M_1. The operator ∇_v is also called the Gauss-Manin connection along the vector field v. By definition of ∇_v in (12.10) and Proposition 11.7, we

have

$$\deg(\nabla_v(\alpha)) \leq \deg(\alpha),\ \alpha \in \mathsf{M}. \tag{12.11}$$

To see the relation of the Gauss-Manin connection of this section with the Gauss-Manin connection of §12.2 we need the following proposition:

Proposition 12.3 *Suppose that the discriminant Δ of the tame polynomial f is not zero. Then the multiplication by Δ in M maps M_i to M_{i-1} for all $i \in \mathbb{N}$.*

Proof. The multiplication by Δ in W is zero and so for a given $\frac{\omega}{f^i}$ we can write

$$\Delta \frac{\omega}{f^i} = \frac{f\omega_1 + df \wedge \omega_2}{f^i} = \frac{\omega_1}{f^{i-1}} + \frac{1}{i-1}(\frac{d\omega_2}{f^{i-1}} - d(\frac{\omega_2}{f^{i-1}}))$$

which is equal to $\frac{\omega_1 + \frac{1}{i-1} d\omega_2}{f^{i-1}}$ in M. $\square$

Now, it is easy to see that $\Delta \cdot \nabla_v : \mathsf{H} \to \mathsf{H},\ \mathsf{H} = \mathsf{H}', \mathsf{H}''$ of this section and §12.2 coincide. Recall that for a R-module M and $a \in M$, M_a denotes the localization of M over the multiplicative set $\{1, a, a^2, \cdots\}$. As a corollary of Proposition 12.3 we have:

Proposition 12.4 *The inclusion $\mathsf{H} \to \mathsf{M}$ induces an isomorphism of R-modules $\mathsf{M}_\Delta \cong \mathsf{H}_\Delta$. Further, for any $v \in \Theta_\mathsf{T}$ this isomorphism commutes with ∇_v.*

Now, we discuss the process of pole order reduction for elements in M and the residue map. Let $[\frac{\omega}{f^i}] \in \mathsf{M}_i$ we consider it in the quotient $\mathsf{M}_i/\mathsf{M}_{i-1}$ and we use the isomorphism (11.6). The fundamental property of the discriminant Δ is that $\Delta \mathsf{W} = 0$, §10.9. We get a sequence of equalities in M_Δ:

$$\frac{\omega}{f^i} = \frac{1}{\Delta}\frac{\omega_1}{f^{i-1}} = \cdots = \frac{1}{\Delta^{i-1}}\frac{\omega_{i-1}}{f} = \frac{1}{\Delta^i}\omega_i,\ \omega_i \in \mathsf{H}' \tag{12.12}$$

Definition 12.2 The residue map is defined in the following way:

$$\mathrm{Resi} : \mathsf{M} \to \mathsf{H}'_\Delta, \quad \mathrm{Resi}(\frac{\omega}{f^i}) := \omega_i$$

where ω_i is defined in (12.12).

A consequence of Proposition 12.4 is the following:

Proposition 12.5 *The residue map* Resi *and the Gauss-Manin connection ∇_v along the vector field v commute, that is,*

$$\mathrm{Resi}(\nabla_v([\frac{\omega}{f^k}]) = \nabla_v(\mathrm{Resi}([\frac{\omega}{f^k}]), \tag{12.13}$$

$$v \in \Theta_\mathsf{T},\ \ \omega \in \omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}.$$

Let us now consider a tame polynomial $f \in \mathsf{R}[x]$. We introduce a new parameter s and consider the tame polynomial $f - s \in \mathsf{R}[s][x]$.

Proposition 12.6 *In* H'' *we have the equality*

$$\operatorname{Resi}\left(\frac{\omega}{(f-s)^k}\right) = \frac{1}{(k-1)!}\nabla^{k-1}_{\frac{\partial}{\partial s}}\omega, \quad \omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}, \tag{12.14}$$

where we have used the Gauss-Manin connection $\nabla_v : \mathsf{H}'' \to \mathsf{H}''_\Delta$ *and in the right hand side of the above equality* ω *is an element of* H''.

Proof. This follows from the definition of the Gauss-Manin connection in M given in (12.10). □

Note that the parameter s does not appear in both f and ω. For hypersurfaces $X \subset \mathbb{P}^{n+1}$ given by homogeneous polynomials, the process of differentiation of differential forms with poles along X and then the pole order reduction, is mainly known as Griffiths-Dwork method.

12.8 Griffiths transversality

In the free module H we have introduced the mixed Hodge structure and the Gauss-Manin connection. It is natural to ask whether there is any relation between these two concepts or not. The answer is given by the next theorem. First, we give a definition.

Definition 12.3 A vector field v in T is called a basic vector field if for any $p \in \mathsf{R}$ there is $k \in \mathbb{N}$ such that $v^k(p) = 0$, where v^k is the k-th iteration of $v(\cdot) : \mathsf{R} \to \mathsf{R},\ p \mapsto dp(v)$.

For $\mathsf{R} = \mathbb{C}[t_1, t_2, \ldots, t_s]$, the vector fields $\frac{\partial}{\partial t_i},\ i = 1, 2, \ldots, s$ are basic.

Theorem 12.1 *Let* $(W_\bullet, F^\bullet)$ *be the mixed Hodge structure of* H. *The Gauss-Manin connection on* H *satisfies:*

1. *Griffiths transversality:*

$$\nabla(F^i) \subset \Omega^1_{\mathsf{T}_\Delta} \otimes_{\mathsf{R}} F^{i-1},\ i = 1, 2, \ldots, n.$$

2. *No residue at infinity: We have*

$$\nabla(W_n) \subset \Omega^1_{\mathsf{T}_\Delta} \otimes_{\mathsf{R}} W_n.$$

3. *Residue killer: For a tame polynomial* f *of degree* d, $\omega \in \mathsf{H}$ *and a basic vector field* $v \in \Theta_{\mathsf{T}}$ *such that* $\deg(v(f)) < d$ *there exists* $k \in \mathbb{N}$ *such that* $\nabla^k_v \omega \in W_n \mathsf{M}_\Delta$.

Griffiths transversality has been proved in [Gri68b, Gri68a] for Hodge structures. For a recent text see also [Voi02b]. The proof for mixed Hodge structures is similar and can be found in [Zuc84, Zuc87].

Proof. It is enough to prove the theorem for the Gauss-Manin connection ∇_v along a vector field $v \in \Theta_{\mathsf{T}}$ and the mixed Hodge of M_Δ.

For the Griffiths transversality, we have to prove that ∇_v maps $F^i\mathsf{M}_\Delta$ to $F^{i-1}\mathsf{M}_\Delta$. By Leibniz rule, it is enough to take an element $\omega = \frac{\omega_\beta}{f^k}$, $A_\beta \leq k$ in the set (11.9) and prove that $\nabla_v\omega$ is in $F^{i-1}\mathsf{M}$. This follows from (12.10) and:

$$\nabla_v \frac{\omega_\beta}{f^k} = \frac{v(f)\omega_\beta}{f^{k+1}},$$

$$\deg(\frac{v(f)\omega_\beta}{f^{k+1}}) \leq \deg(\frac{\omega_\beta}{f^k}) = d(A_\beta - k) \leq 0$$

For the second part of the theorem we have to prove that ∇_v maps $W_n\mathsf{M}_\Delta$ to $W_n\mathsf{M}_\Delta$. This follows follows form (12.11) and the fact that $\frac{\omega_\beta}{f^k}$, $A_\beta < k$ generate $W_n\mathsf{M}_\Delta$

For the third part of the theorem we proceed as follows: For $\omega \in \mathsf{M}$ we use Proposition 11.7 and write ω as a R-linear combination of $\frac{\omega_\beta}{f^k}$, $\beta \in I$, $A_\beta \leq k$. By the second part of the theorem, it is enough to prove that for $\frac{\omega_\beta}{f^k}$, $A_\beta = k$ and $p \in \mathsf{R}$, there exists some $s \in \mathbb{N}$ such that

$$\nabla_v^s(p[\frac{\omega_\beta}{f^k}]) \in W_n\mathsf{M}_\Delta.$$

Since $\deg(v(f)) < d$ we have

$$\deg \nabla_v[\frac{\omega_\beta}{f^k}] < \deg(\frac{\omega_\beta}{f^k}) = 0$$

and so $\nabla_v[\frac{\omega_\beta}{f^k}] \in W_n\mathsf{M}_\Delta$. Now modulo $W_n\mathsf{M}_\Delta$ we have

$$\nabla_v^k(p[\frac{\omega_\beta}{f^k}]) = v^k(p) \cdot [\frac{\omega_\beta}{f^k}]$$

and the affirmation follows from the fact that v is a basic vector field. □

Definition 12.4 We say that a polynomial $g \in \mathsf{R}[x]$ does not depend on R (or parameters in R) if all the coefficients of g lies in the kernel of the map $d : \mathsf{R} \to \Omega^1_{\mathsf{T}}$. In other words, $v(g) = 0$ for all vector field $v \in \Theta_{\mathsf{T}}$.

In the case $\mathsf{R} := \mathbb{Q}[t_1, t_2, \ldots, t_s]$ the above definition simply means that in g the parameters t_i, $i = 1, 2, \ldots, s$ do not appear. In this case, for a tame polynomial f over R such that its last homogeneous piece g does not depend on R, all $v = \frac{\partial}{\partial t_i}$'s are basic and $\deg(v(f)) \leq \deg(f)$. In practice, we use this as an example for the third part of Theorem 12.1.

An immediate consequence of Theorem 11.5 is the following.

Theorem 12.2 ([Mov07b], Theorem 0.1) *Let $f \in \mathsf{k}[x]$ be a tame polynomial with non-zero discriminant. There exists a map $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ such that*

$$\nabla^k \eta_\beta,\ \beta \in I,\ A_\beta = k \tag{12.15}$$

form a basis of $\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}_{n+1}^W \mathsf{H}$ *and the forms*

$$\nabla^k \eta_\beta, \quad k - \frac{d_\beta + 1}{d} < A_\beta < k \tag{12.16}$$

form a basis of $\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}_n^W \mathsf{H}$*, where* $\nabla^k = \nabla \circ \nabla \circ \cdots \circ \nabla$ *k times and* $\nabla = \nabla_{\frac{\partial}{\partial s}}$ *is the Gauss-Manin connection of the tame polynomial* $f - s \in \mathsf{k}[s][x]$.

Proof. This follows from Theorem 11.5 and the following identities in M of $f - s$:

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} \frac{\omega_\beta}{(f-s)^k} &= \frac{-k\omega_\beta}{(f-s)^{k+1}} \\ \nabla_{\frac{\partial}{\partial s}} \eta_\beta &= A_\beta \omega_\beta. \end{aligned}$$

□

12.9 Tame polynomials with zero discriminant

For a tame polynomial f with zero discriminant, $\Delta = 0$, we can still use the methods of this book and compute the Gauss-Manin connection. This is already of historical interest, as in [PS06] we find many computations for singular surfaces, see Exercise 10.15 for some comments.

Let A be the linear map in Definition 10.14 (this is different from the Gauss-Manin connection matrix). For $P \in \ker(A)$ we have $fPdx = df \wedge \omega_P$ for some $\omega_P \in \Omega^n_{\mathsf{A}/\mathsf{T}}$ and so

$$\begin{aligned} \frac{Pdx}{f^i} &= \frac{fPdx}{f^{i+1}} = \frac{df \wedge \omega_P}{f^{i+1}} \\ &= \frac{1}{i}\left(\frac{d\omega_P}{f^i} - d\left(\frac{\omega_P}{f^i}\right)\right) \\ &= \frac{1}{i}\frac{d\omega_P}{f^i} \quad \text{in } \mathsf{M}. \end{aligned}$$

We conclude that

$$\frac{Pdx - \frac{1}{i}d\omega_P}{f^i} = 0, \quad \text{in } \mathsf{M}. \tag{12.17}$$

For $i = 1$ we conclude that there are many R-linear relations for the generators $\omega_\beta,\ \beta \in I$ of the Brieskorn module H. The description of relations of H with the de Rham cohomology of the desingularized $f = 0$ is an interesting problem that we do not deal with it in this book. For simplicity, we proceed our discussion for the tame polynomial in Example 10.6 with $c = 0$:

$$f = y^2w - 4x^3 + 3axw^2 + bw^3 - \frac{1}{2}(dw^2 + w^4) \tag{12.18}$$

and explain how to compute its Gauss-Manin connection. A similar discussion is valid for arbitrary c, see [DMWH16]. The discriminant of the tame polynomial (10.16) with $a,b,d \in \mathsf{R} := \mathbb{Q}[a,b,d]$ is zero. In order to obtain a tame polynomial with non-zero discriminant, we introduce a new parameter s and work with the ring $\tilde{\mathsf{R}} := \mathsf{R}[s]$ and the tame polynomial $\tilde{f} = f - s$. The R-module $\mathsf{V}_g := \mathsf{R}[x,y,w]/\mathrm{Jacob}(g)$ is free of rank 10. In fact, the set of monomials

$$I := \{xw^3, xw^2, w^3, xy, xw, w^2, y, x, w, 1\} \tag{12.19}$$

form a basis for both V_g and V_f. Therefore, a basis of the Brieskorn module H is given by:

$$\alpha dx \wedge dy \wedge dw, \alpha \in I. \tag{12.20}$$

We use the algorithms in this chapter to calculate the Gauss-Manin connection of $\tilde{f}$. This means that we take the 10×1 matrix ω formed by (12.20) and calculate the 10×10 matrices $A(s), B(s), D(s)$ and $S(s)$ with entries in $\mathsf{R}[s]$ satisfying the equality

$$\nabla\omega = \frac{1}{\Delta(s)}\tilde{A}(s) \otimes \omega,$$

$$\tilde{A}(s) = A(s)da + B(s)db + D(s)dd + S(s)ds.$$

The data of $\Delta(s), A(s), \cdots$ as a text file is about 50 kilobytes. Here d stands for both differential and a parameter. Therefore, dd means the differential of $d \in \mathsf{R}$. For our example (12.18) we have $\Delta(s) = s(\Delta + \Delta_1 s + \cdots),\ \Delta_i \in \mathsf{R}$ with

$$\Delta := a(a^6d^6 - 2a^3b^2d^6 - 2a^3d^7 + b^4d^6 - 2b^2d^7 + d^8).$$

It turns out that $A(0) = B(0) = D(0) = 0$. Therefore, we get the calculation of the Gauss-Manin connection for f:

$$\nabla\Omega = \frac{1}{\Delta}\tilde{A} \otimes \Omega,$$

$$\tilde{A} = A \cdot da + B \cdot db + D \cdot dd,$$

where $A = \lim_{s\to 0} \frac{A(s)}{s}$ and so on. The matrices A, B, C and D are 10×10 matrices with entries in R. Using these calculations we can check that:

Proposition 12.7 *We have*

1. *The* R*-module generated by*

$$\alpha dx \wedge dy \wedge dw, \alpha = xw^3, xw^2, w^3, xw, w^2, \tag{12.21}$$

   is invariant under $\Delta \cdot \nabla$.
2. $\omega = \alpha dx \wedge dy \wedge dw, \alpha = xy, y$ *is a flat section, that is,* $\nabla(\omega) = 0$.

Using (12.17) we can also check that:

Proposition 12.8 *We have the following equalities in* M*:*

$$\alpha dx \wedge dy \wedge dw = 0 \ in \quad \mathsf{M}, \tag{12.22}$$

where α *is one of the six polynomials:*

$$\begin{aligned}
&25xw^3 - 36bxw^2 - 18a^2w^3 + 11dxw,\\
&xy,\\
&y,\\
&-75bxw^3 + (108b^2 - 19d)xw^2 + 54a^2bw^3 - 9bdxw + 12a^2dw^2 - 5d^2x,\\
&-150axw^3 + 216abxw^2 + (108a^3 - 17d)w^3 - 18adxw + 24bdw^2 - 7d^2w,\\
&-900abxw^3 + (1296ab^2 - 114ad)xw^2 + (648a^3b - 51bd)w^3 - 108abdxw +\\
&\qquad (72a^3d + 72b^2d - 11d^2)w^2 - 6ad^2x - 9bd^2w - d^3.
\end{aligned}$$

12.10 Exercises

12.1. Prove that for the Legendre, resp. Weierstrass, family of elliptic curves E_t : $y^2 - x(x-1)(x-t)$, resp. $E_t : y^2 - x^3 + 3tx - 2t$, the periods $I(t) := \int_{\delta_t} \frac{dx}{y}$, $\delta_t \in H_1(E_t, \mathbb{Z})$ satisfy the Picard-Fuchs equation $L(I) = 0$, where

$$L := 1 + (8t-4)\frac{\partial}{\partial t} + 4t(t-1)\frac{\partial^2}{\partial^2 t}, \tag{12.23}$$

resp.

$$L := (27t+4) + 144t(2t-1)\frac{\partial}{\partial t} + 144t^2(t-1)\frac{\partial^2}{\partial^2 t}, \tag{12.24}$$

see [KZ01] for some discussion on these Picard-Fuchs equations.

12.2. Let $P(x) := 4(x-t_1)^3 + t_2(x-t_1) + t_3$. We have

$$\begin{pmatrix} d\left(\int \frac{dx}{\sqrt{P(x)}}\right) \\ d\left(\int \frac{xdx}{\sqrt{P(x)}}\right) \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}t_1\frac{\alpha}{\Delta} - \frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ dt_1 - \frac{1}{6}t_1\frac{d\Delta}{\Delta} - (\frac{3}{2}t_1^2 + \frac{1}{8}t_2)\frac{\alpha}{\Delta}, & \frac{3}{2}t_1\frac{\alpha}{\Delta} + \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \int \frac{dx}{\sqrt{P(x)}} \\ \int \frac{xdx}{\sqrt{P(x)}} \end{pmatrix}$$

where

$$\Delta := 27t_3^2 - t_2^3, \ \alpha := 3t_3dt_2 - 2t_2dt_3.$$

The above data is the Gauss-Manin connection of the family of elliptic curves $y^2 = P(x)$ before the invention of cohomology theories (before 1900). The parameter t_1 for the purpose of the Exercise is superficial, however, it plays an important role in the derivation of the Eisenstein series in [Mov12]. Hint: Instead of the differential d use the differentiations $\frac{\partial}{\partial t_i}$, $i = 1,2,3$.

12.3. In Exercise 12.2 let us put $P(x) = 4(x-t_1)(x-t_2)(x-t_3)$. Then the Gauss-Manin connection matrix is given by

$$\frac{dt_1}{2(t_1-t_2)(t_1-t_3)}\begin{pmatrix} -t_1, & 1 \\ t_2t_3-t_1(t_2+t_3), & t_1 \end{pmatrix} + \frac{dt_2}{2(t_2-t_1)(t_2-t_3)}\begin{pmatrix} -t_2, & 1 \\ t_1t_3-t_2(t_1+t_3), & t_2 \end{pmatrix} +$$

$$+\frac{dt_3}{2(t_3-t_1)(t_3-t_2)}\begin{pmatrix} -t_3, & 1 \\ t_1t_2-t_3(t_1+t_2), & t_3 \end{pmatrix}. \tag{12.25}$$

12.4. Let $f := y^2-(x-t_1)(x-t_2)\cdots(x-t_s)$ be a tame polynomial in two variables x,y and with parameters $t_1,t_2,\ldots,t_s$. Show that

$$\nabla_{\frac{\partial}{\partial t_1}} \circ \nabla_{\frac{\partial}{\partial t_2}}\,\omega = \frac{1}{2}\frac{1}{t_2-t_1}\left(\nabla_{\frac{\partial}{\partial t_2}}\,\omega - \nabla_{\frac{\partial}{\partial t_1}}\,\omega\right)$$

for $\omega = x^i dx\wedge dy \in \mathsf{H}''$. Hint: First use the relation of the Gauss-Manin connection with integrals, see Proposition 13.1 and the equality (12.1), and then try to prove it using the definitions in the present chapter (this exercise is proposed by J. Shaffaf and Kh. M. Shokri).

12.5. Recall the discussion in §12.6. Show that $\theta s^n = (s-\theta)^n\theta = (-1)^n n!\theta^{n+1} + s(\cdots)$, where $\cdots$ is a polynomial in s and θ with $\mathbb{Z}$ coefficients.

Chapter 13
Multiple integrals and periods

You were able to make [mechanics] almost interesting; I have always wondered how you went about this, because I was never able to do it when it was my turn. But you also escaped, you introduced us not only to hydrodynamics and turbulence, but to many other theories of mathematical physics and even of infinitesimal geometry; all this in lectures, the most masterly I have heard in my opinion, where there was not one word too many nor one word too little, and where the essence of the problem and the means used to overcome it appeared crystal clear, with all secondary details treated thoroughly and at the same time consigned to their right place, (J. Hadamard in the obituary of Émile Picard, [OR16]).

13.1 Introduction

Picard is without doubt the founder of the study of double integrals and his expertise in many branches of Mathematics and Physics might have led him to invest a good portion of his life on this topic. He gave the name period to these integrals. "M. Picard a donné à ces integrales le nom de périodes; je ne saurais l'en blâmer puisque cette dénomination lui a permis d'exprimer dans un langage plus concis les intéressants résultats auxquels il est parvenu. Mais je crois qu'il serait fâcheux qu'elle s'introduisit définitivement dans la science et qu'elle serait propre à engendrer de nombreuses confusions", (H. Poincaré's remarks on the name period used for integrals, see [Poi87] page 323). The most relevant critic to the name "period" is that multiple integrals in general are not periods of any reasonable periodic function. The name seems to merge from the case of abelian integrals which are periods of Riemann's theta functions introduced in 3. The first systematic attempt to make multiple integrals into periods of periodic functions was formulated by P. Griffiths in the 1970's, see [Gri70, Mov08] and the references therein. This produced the concept of period domain which is not in general a Hermitian symmetric domain, and hence, the construction of functions with prescribed multiple integrals as periods

fails. Nowadays, the name period is so common and well-accepted that it does not make sense to talk about whether it is a good name for integrals or not.

In this chapter we unify the material of Chapters 10, 7 and 12 in order to study integrals of algebraic differential forms over topological cycles. This will be a modern approach to some of problems Picard dealt with.

13.2 Integrals

We fix an element $\hat{\Delta}$ of the polynomial ring $\mathsf{k}[t]$, $t = (t_1, t_2, \ldots, t_s)$ a multi parameter, and we assume that the ring R used in Chapter 10 is the localization of $\mathsf{k}[t]$ over the multiplicative group generated by $\hat{\Delta} \in \mathsf{k}[t]$, that is, division over $\hat{\Delta}$ is allowed. Here, k is any subfield of $\mathbb{C}$, for instance take $\mathsf{k} = \mathbb{Q}$ or $\mathsf{k} = \mathbb{C}$. We call R the parameter ring. Our tame polynomials

$$f \in \mathsf{R}[x] := \mathsf{k}[t_1, t_2, \cdots, t_s, x_1, x_2, \ldots, x_{n+1}, \frac{1}{\hat{\Delta}}]$$

are usually polynomial in t too, and in order to make them tame in the sense of Definition 10.12, we have to allow divisions over $\hat{\Delta}$, see for instance Example 10.2 and Example 10.4. We will freely use the notations related to f introduced in §10.6. Let $\mathsf{T}, \mathsf{T}_\Delta,\ L, L_t$ be as in (7.2). For a fixed value $t \in \mathsf{T}$, we denote by f_t the polynomial obtained by replacing t in f. We will frequently use a continuous family of cycles

$$\delta := \{\delta_t\}_{t \in U},\ \delta_t \in H_n(L_t, \mathbb{Z}),$$

where U is a small neighborhood in T_Δ. Let us recall the Brieskorn modules H' and H'' in (10.28) and (10.30). The integral

$$\int_\delta \omega := \int_{\delta_t} \omega\Big|_{L_t}, \qquad \omega \in \mathsf{H}'$$

is well-defined, that is, it does not depend on the choice of the differential form (resp. cycle) in the class of ω (resp. in the homology class of δ). We have also the integral

$$\int_\delta \omega := \int_\delta \frac{\omega}{df} \qquad \omega \in \mathsf{H}'',$$

where the Gelfand-Leray form $\frac{\omega}{df}$ is defined in §10.8. The above integrals are holomorphic function in U and can be extended to a multi-valued holomorphic function in T_Δ. If $\omega = P(x)dx$ and we use the naive definition of Gelfand-Leray form after Definition 10.16 then we get:

$$\int_\delta \omega = \int_\delta \frac{P(x_1, x_2, \ldots, x_{n+1}) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{f_{x_{n+1}}} \tag{13.1}$$

where $f_{x_{n+1}}$ is the derivation of f with respect to x_{n+1}. In the three variable case $(x,y,z) := (x_1,x_2,x_3)$ we get Picard's famous double integrals in the book [PS06]:

$$\iint \frac{P(x,y,z)dxdy}{f'_z}. \tag{13.2}$$

13.3 Integrals and Gauss-Manin connections

The origin of the Gauss-Manin connection in Chapter 12, and its main property in relation with integrals, is the following proposition.

Proposition 13.1 *Let U be a small open set in T_Δ and $\delta := \{\delta_t\}_{t\in U}, \delta_t \in H_n(L_t,\mathbb{Z})$ be a continuous family of topological n-dimensional cycles. Then for $\omega \in \mathsf{H}$ we have*

$$d(\int_\delta \omega) = \sum_{i=1}^{\mu} \alpha_i \int_\delta \omega_i, \tag{13.3}$$

where

$$\nabla\omega = \sum_{i=1}^{\mu} \alpha_i \otimes \omega_i, \tag{13.4}$$

$$\alpha_i \in \Omega^1_{\mathsf{T}_\Delta},\ \omega_i \in \mathsf{H}$$

and ω_i's form a R-basis of H.

Note that the differential operator d in (13.3) is with respect to parameters in t. For examples of the equality (13.3) see Exercise 12.2. The equality (13.4) in its explicit format is the following:

$$d\omega - \sum_{i=1}^{\mu} \alpha_i \wedge \omega_i \in f\Omega^{n+1}_{\mathsf{A}} + df \wedge \Omega^n_{\mathsf{A}}, \tag{13.5}$$

$$\alpha_i \in \Omega^1_{\mathsf{T}_\Delta},\ \omega_i \in \Omega^n_{\mathsf{A}}.$$

Proof. We can assume that δ_t is represented by a smooth oriented submanifold of L_t. In our case this follows from Theorem 7.2 which says that a distinguished set of vanishing cycles generate the n-th homology of L_t. We may further assume that δ_t is a vanishing cycle in a smooth point c of the variety $\{\Delta = 0\}$. We will not need this for the proof. We conclude that there exists an $(n+1)$-dimensional real thimble

$$D_t = \cup_{s\in[0,1]}\{\gamma_t(s)\} \times \delta_{\gamma_t(s)} \subset \mathbb{C}^s \times \mathbb{C}^{n+1}$$

such that γ_t is a path in T connecting c to t and $\delta_{\gamma_t(s)}$ is the trace of δ_t when it vanishes along γ_t. Note that the orientation induced by D_t in δ_t is the original orientaton of δ_t. In order to define the action of the Gauss-Manin connection ∇ on $\omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$ we first write (13.5). Since $f\,|_{D_t} = 0$, the integral of the elements of $f\Omega^{n+1}_{\mathsf{A}/\mathsf{T}} + df \wedge \Omega^n_{\mathsf{A}/\mathsf{T}}$

on D_t is zero and we have

$$\begin{aligned}\int_{\delta_t} \omega &= \int_{D_t} d\omega \\ &= \sum_{i=1}^{\mu} \int_{D_t} \alpha_i \wedge \omega_i \\ &= \int_{\gamma_t(s)} \left(\sum_{i=1}^{\mu} \alpha_i \int_{\delta_{\gamma_t(s)}} \omega_i \right).\end{aligned}$$

In the first equality we have used Stokes Lemma and in the last equality we have used Fubini's theorem. Taking the differential of the above equality we get the desired result. □

Let $v \in \Theta_{\mathsf{T}}$ be a vector field in T. For instance, take $v = \frac{\partial}{\partial t_i}$, $i = 1, 2, \ldots, s$. In this case for a holomorphic function y in t,

$$\frac{\partial}{\partial t_i}(y) := \frac{\partial y}{\partial t_i}$$

is the usual derivation with respect to t_i. From (13.3) it follows that

$$v\left(\int_{\delta} \omega\right) = \int_{\delta} \nabla_v \omega, \tag{13.6}$$

$$\omega \in \mathsf{H},\ v \in \Theta_{\mathsf{T}}$$

for any continuous family of cycles $\delta = \{\delta_t\}_{t \in U}$ in a small neighborhood in T_Δ. For a fixed v, the operator $\nabla_v : \mathsf{H} \to \mathsf{H}_\Delta$ with the above property is unique. This follows from the fact that if $\omega \in \mathsf{H}$ restricted to all regular fibers of f is exact then ω is zero in H, this in turn, is a consequence of Corollary 10.1. If we want to prove an equality for the Gauss-Manin connection of a tame polynomial f over the function field introduced at the beginning of this chapter then we may use (13.6). This will be considered a proof using transcendental methods. A proof of the same equality for an arbitrary R of Chapter 10 demands only algebraic methods.

13.4 Period matrix

In this section let us take $\mathsf{R} = \mathbb{C}$, that is, the tame polynomial $f \in \mathbb{C}[x]$ does not depend on any parameter. Further, assume that its discriminant is non-zero, and hence, $L := \{f = 0\}$ is a smooth affine variety.

Definition 13.1 Let $\delta = [\delta_1, \delta_2, \ldots, \delta_\mu]^{\mathsf{tr}}$ is a basis of the $\mathbb{Z}$-module $H_n(L, \mathbb{Z})$ and $\omega := [\omega_1, \omega_2, \ldots, \omega_\mu]^{\mathsf{tr}}$ be a basis of the $\mathbb{C}$-vector space H. By Theorem 7.2 we know that δ can be chosen as a distinguished set of vanishing cycles. The period matrix (in the basis δ and ω) is defined in the following way.

$$\mathbf{P} = \left[\int_\delta \omega^{\mathsf{tr}}\right] = \begin{pmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_\mu \\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_\mu \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_\mu}\omega_1 & \int_{\delta_\mu}\omega_2 & \cdots & \int_{\delta_\mu}\omega_\mu \end{pmatrix}. \tag{13.7}$$

If R is the parameter ring as in §13.2 then $\mathbf{P} = \mathbf{P}(t)$ is a matrix of holomorphic functions in $t \in U$. In this case we also call it the period map. The classical de Rham theorem states that for a smooth manifold X the map

$$H^m_{\mathrm{dR}}(X) \to H_m(X,\mathbb{R})^\vee, \quad \omega \mapsto \left(\delta \mapsto \int_\delta \omega\right)$$

is an isomorphism. If we choose a basis ω and δ for $H^m_{\mathrm{dR}}(X)$ and $H_m(X,\mathbb{R})$, respectively, then this is equivalent to say that the corresponding period matrix $\mathbf{P}$ has non-zero determinant.

Theorem 13.1 (De Rham theorem for tame polynomials) *Let f be tame polynomial with non-zero discriminant over complex numbers. Then the canonical map induced by inclusion $\mathsf{H}' \to H^n_{\mathrm{dR}}(L)$ is an isomorphism, and hence, the period matrix* (13.7) *of f has non-zero determinant.*

Note that for a tame polynomial as above all the Brieskorn modules and M_i turn out to be the same object that we denote it by H. This theorem for arbitrary smooth affine varieties is called the Atiyah-Hodge theorem, see [HA55] for the original article and [Nar68] for a simplified version of its proof. In the case of hypersurfaces in the weighted projective spaces, this also follows from Griffiths work [Gri69] and its generalization by Steenbrink in [Ste77b]. A more self-content proof can be produced following the arguments in Exercise 13.2.

13.5 Period matrix and Picard-Fuchs equations

Let R be the parameter ring as in §13.2. Let also $\omega = [\omega_1, \omega_2, \ldots, \omega_\mu]^{\mathsf{tr}}$ be a basis of of the free R-module H. In this basis we can write the Gauss-Manin connection matrix of ∇:

$$\nabla\omega = A \otimes \omega, \tag{13.8}$$

$$A \in \frac{1}{\Delta}\mathrm{Mat}(\mu, \Omega^1_{\mathsf{T}}).$$

After integrating the above equality we get solutions to the linear differential equation:

$$dY = A \cdot Y \tag{13.9}$$

Here, Y is either a $\mu \times 1$ or $\mu \times \mu$ unknown matrix, whose entries are holomorphic functions defined in a small open neighborhood U in T_Δ.

Definition 13.2 A $\mu \times \mu$ matrix Y as above is called a fundamental system of solutions or fundamental matrix if $\det(Y) \neq 0$.

It is easy to show that if Y is a fundamental system then any other $\mu \times 1$ solution of (13.9) is a $\mathbb{C}$-linear combination of the columns of Y. This justifies the name in the above definition.

Proposition 13.2 *A fundamental system of solutions for the linear differential equation* (13.9) *is given by the transpose* $\mathbf{P}^{\mathsf{tr}}$ *of the period matrix.*

Proof. From Proposition 13.1 it follows that $\mathbf{P}^{\mathsf{tr}}$ satisfies (13.9). Its determinant is not identically zero. This follows from Theorem 13.1. □

We saw in §12.3 that for $\omega \in \mathsf{H}$ and $v \in \Theta_{\mathsf{T}}$ a vector field in T, there exists $m \leq \mu$ and $p_i \in \mathsf{R}$, $i = 0, 1, 2, \ldots, m$ such that we have the Picard-Fuchs equation of ω along v:

$$p_0\omega + p_1\nabla_v(\omega) + p_2\nabla_v^2(\omega) + \cdots + p_m\nabla_v^m(\omega) = 0 \tag{13.10}$$

For a continuous family of vanishing cycles $\delta_t \in H_n(L_t, \mathbb{Z})$, $t \in U$ we take the integral of the above equality over δ_t, use the equality (13.6) and conclude that the analytic functions

$$\int_{\delta_t} \omega, \quad \delta_t \in H_n(L_t, \mathbb{Z}) \tag{13.11}$$

satisfy the linear differential equation:

$$p_0(t)y + p_1(t)y' + p_2(t)y'' + \cdots + p_m(t)y^{(m)} = 0, \tag{13.12}$$

$$y' := dy(v) = v(y).$$

The number m is called the order of the differential equation (13.12).

Proposition 13.3 *The holomorphic functions* (13.11) *span the* $\mathbb{C}$*-vector space of the solutions of (13.12). If* $m = \mu$ *then these integrals for a basis of* $H_n(L_t, \mathbb{Z})$ *form a basis of the solution space of (13.12).*

Proof. This follows from the fact that the period matrix (13.7) is a fundamental system for the linear differential equation (13.9). □

For examples of Picard-Fuchs equations see Exercise 12.1.

Remark 13.1 We have seen that the period matrix $Y = \mathbf{P}^{\mathsf{tr}}$ satisfies the differential equation (13.9). Fix a point $t_0 \in \mathsf{T}_\Delta$, let γ be a path in T which connects t_0 to $t \in \mathsf{T}_\Delta$. We have the equality

$$\mathbf{P}(t)^{\mathsf{tr}} = \left(I + \int_\gamma A + \int_\gamma AA + \int_\gamma AAA + \cdots \right) \mathbf{P}(t_0)^{\mathsf{tr}}, \tag{13.13}$$

where we have used iterated integrals, for further details see [Hai87] Lemma 2.5, and I is the $\mu \times \mu$ identity matrix. Note that the above series is convergent and the sum is homotopy invariant but its pieces are not homotopy invariant.

13.6 Homogeneous polynomials

For a homogeneous polynomial $g(x) = g(x_1, x_2, \ldots, x_{n+1}) \in \mathsf{k}[x]$ with an isolated singularity at the origin let $f = g - s$ which is a tame polynomial in $\mathsf{R}[x]$, $\mathsf{R} = \mathsf{k}[s]$. Its discriminant is $(-s)^\mu$. We have

$$\nabla_{\frac{\partial}{\partial s}}(\omega_\beta) = \frac{(A_\beta - 1)}{s}\omega_\beta$$

and so

$$\frac{\partial}{\partial s}\int_{\delta_s}\omega_\beta = \frac{A_\beta - 1}{s}\int_{\delta_s}\omega_\beta.$$

Therefore

$$\int_{\delta_s}\omega_\beta = \left(\int_{\delta_1}\omega_\beta\right)\cdot s^{A_\beta - 1}. \tag{13.14}$$

Here, we have chosen a branch of s^{A_β} whose evaluation on $s = 1$ is 1. Using $\eta_\beta = s\omega_\beta$ in H'' of $g - s$ we obtain:

$$\int_{\delta_s}\eta_\beta = \left(\int_{\delta_1}\eta_\beta\right)\cdot s^{A_\beta}, \tag{13.15}$$

$$\int_{\delta_s}\mathrm{Resi}(\frac{\omega_\beta}{f^k}) = \left(\int_{\delta_1}\eta_\beta\right)\frac{(A_\beta - k + 1)_{k-1}}{(k-1)!}s^{A_\beta - k} \tag{13.16}$$

13.7 Residues and integrals

Let us be given a tame polynomial f, $t \in \mathsf{T}_\Delta := \mathsf{T}\backslash\{\Delta = 0\}$ and $\omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}$. We can associate to $\frac{\omega}{f^k}$, $k \in \mathbb{N}$ its residue in L_t which is going to be an element of $H^n_{\mathrm{dR}}(L_t)$. For this, see Definition 12.2. This gives us a global section of the n-th cohomology bundle of the fibration L_t, $t \in \mathsf{T}_\Delta$. The main property of the residue map is the following:

Proposition 13.4 *We have*

$$2\pi\sqrt{-1}\int_{\delta_t}\mathrm{Resi}(\frac{\omega}{f^k}) = \int_{\sigma(\delta_t)}\frac{\omega}{f^k}, \tag{13.17}$$

$$\omega \in \Omega^{n+1}_{\mathsf{A}/\mathsf{T}}, \quad \delta_t \in H_n(L_t, \mathbb{Z}),$$

where $\sigma : H_n(L_t, \mathbb{Z}) \to H_{n+1}(\mathbb{C}^{n+1}\backslash L_t, \mathbb{Z})$ *is the map which is explained in the Gysin sequence* (4.7).

Proof. The pole reduction in (12.12) is modulo differential forms whose integrals on $\sigma(\delta_t)$ are zero. In the last step, by definition we have $\Delta\omega_{i-1} = df \wedge \omega_i$ and we use the equality

$$\int_{\sigma(\delta_t)} \frac{df}{f} \wedge \omega_i = 2\pi\sqrt{-1} \int_{\delta_t} \omega_i.$$

This follows from integration by parts. $\square$

Let v be a vector field in T and $\nabla_v : \mathsf{M} \to \mathsf{M}_\Delta$ be the composition of the Gauss-Manin connection with the vector field v as in §12.3. The following identities give us the origin of Proposition 12.5.

$$\begin{aligned} v \int_\delta \mathrm{Resi}(\frac{\omega}{f^k}) &= v \int_{\sigma(\delta)} \frac{\omega}{f^k} \\ &= \int_{\sigma(\delta)} \nabla_v(\frac{\omega}{f^k}) \\ &= \int_\delta \mathrm{Resi}(\nabla_v(\frac{\omega}{f^k})). \end{aligned}$$

Let us now consider a tame polynomial $f \in \mathsf{R}[x]$. We introduce a new parameter s and consider the tame polynomial $f - s \in \mathsf{R}[s][x]$. The origin of Proposition 12.6 is the following:

$$\begin{aligned} \int_\delta \nabla^{k-1}_{\frac{\partial}{\partial s}} \omega &= \frac{\partial^{k-1}}{\partial s^{k-1}} \int_\delta \omega \\ &= \frac{\partial^{k-1}}{\partial s^{k-1}} \int_\delta \mathrm{Resi}(\frac{\omega}{f-s}) = \frac{\partial^{k-1}}{\partial s^{k-1}} \int_{\sigma(\delta)} \frac{\omega}{f-s} \\ &= \int_{\sigma(\delta)} (k-1)! \frac{\omega}{f^k} = (k-1)! \int_\delta \mathrm{Resi}(\frac{\omega}{(f-s)^k}). \end{aligned}$$

Let $X_t = L_t \cup Y_t$ be the compactification of L_t, where Y_t is the projective variety in $\mathbb{P}^\nu$ given by $g = 0$.

Proposition 13.5 *For a cycle $\delta_t \in H_n(L_t, \mathbb{Z})$ at infinity we have*

$$\begin{aligned} \int_{\sigma(\delta)} \frac{x^\beta dx}{f^k} &= 0, \qquad \text{if } A_\beta < k, \\ \int_{\sigma(\delta)} \frac{x^\beta dx}{f^k} &= 2\pi\sqrt{-1} \int_{\delta'} \frac{x^\beta \eta}{g^k}, \qquad \text{if } A_\beta = k \end{aligned}$$

for some cycle $\delta' \in H_n(\mathbb{P}^\nu \backslash Y_t, \mathbb{Z})$.

Proof. This follows from Proposition 11.4. In order to describe the cycle δ' we need the map σ in the Gysin sequence (4.7) for the pairs $(\mathbb{C}^{n+1}, \mathbb{C}^{n+1} \backslash L_t)$, (X_t, L_t) and $(\mathbb{P}^\nu, \mathbb{P}^\nu \backslash Y_t)$. $\square$

If the last homogeneous part g of f does not depend on any parameter of R then for $A_\beta = k$ the integral $\int_{\delta_t} \frac{x^\beta dx}{f^k}$ is constant. Now it is evident that W_n is the set of differential forms which do not have any residue at infinity. This gives another proof of Theorem 12.1, part 2. The topological interpretation of part 3 is as follows. For

simplicity we take $\mathsf{R} = \mathbb{Q}[t_1, t_2, \ldots, t_s]$ and assume that g does not depend on the parameters in R. We use Proposition 11.3 and write an $\omega \in \mathsf{M}$ in the form

$$\omega = \sum_{A_\beta = k,\ k\in\mathbb{N},\ \beta\in I} a_{\beta,k} \frac{\omega_\beta}{f^k} + \sum_{A_\beta < k,\ k\in\mathbb{N},\ \beta\in I} b_{\beta,k} \frac{\omega_\beta}{f^k} \tag{13.18}$$

with $a_{\beta,k}, b_{\beta,k} \in \mathsf{R}$. For a cycle at infinity $\delta_t \in H_n(L_t, \mathbb{Z})$ we see that

$$\int_\delta \omega = \sum_{A_\beta = k,\ k\in\mathbb{N},\ \beta\in I} a_{\beta,k} \int_\delta \frac{\omega_\beta}{f^k},$$

which is a polynomial in $t_1, t_2, \ldots, t_s$. According to Proposition 13.5 the integrals $\int_\delta \frac{\omega_\beta}{f^k}$ are constant numbers and so the above polynomial has complex coefficients. Therefore, the m-th derivation of $\int_\delta \omega$ with respect to t_i must be zero for m bigger than the degree in t_i of $\int_\delta \omega$.

13.8 Integration over joint cycles

The objective of the present section is to introduce techniques to simplify integrals and in the best case to calculate them. For simplicity, we take tame polynomials over $\mathbb{C}$ but the whole discussion is valid for tame polynomials depending on parameters as it is explained at the beginning of the present chapter.

Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$ and $g \in \mathbb{C}[y_1, y_2, \ldots, y_{m+1}]$ be two tame polynomials in $n+1$, respectively $m+1$, variables. Recall the definition of an admissible triple from §7.9. By abuse of notation we will use β_1 and β_2 for the exponents of monomials in x and y, respectively, therefore they are multi indices and no more the entries of a single β.

Proposition 13.6 *Let ω_1 (resp. ω_2) be an $(n+1)$-form (resp. $(m+1)$-form) in $\mathbb{C}^{n+1}$ (resp. $\mathbb{C}^{m+1}$). Let also $(t_s,\ s \in [0,1], \delta_{1b}, \delta_{2b})$ be an admissible triple and*

$$I_1(t_s) = \int_{\delta_{1,t_s}} \frac{\omega_1}{df},$$
$$I_2(t_s) = \int_{\delta_{2,t_s}} \frac{\omega_2}{dg}.$$

Then

$$\int_{\delta_{1b} *_{t_s} \delta_{2b}} \frac{\omega_1 \wedge \omega_2}{d(f-g)} = \int_{t_s} I_1(t_s) I_2(t_s) dt_s.$$

Proof. We have

$$\begin{aligned}\omega_1 \wedge \omega_2 &= df \wedge \frac{\omega_1}{df} \wedge dg \wedge \frac{\omega_2}{dg} \\ &= d(f-g) \wedge \frac{\omega_1}{df} \wedge dg \wedge \frac{\omega_2}{dg}\end{aligned}$$

and so restricted to the variety

$$X := \{(x,y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} \mid f(x) - g(y) = 0\}$$

we have

$$\frac{\omega_1 \wedge \omega_2}{d(f-g)} = \frac{\omega_1}{df} \wedge dt \wedge \frac{\omega_2}{dg},$$

where t is the holomorphic function on X defined by $t(x,y) := f(x) = g(y)$. Now, the proposition follows by Fubini's theorem. □

Recall the B-function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 s^{a-1}(1-s)^{b-1} ds,\ a,b, \in \mathbb{C}$$

and its multi-parameter form:

$$B(a_1, a_2, \cdots, a_r) = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_r)}{\Gamma(a_1+a_2+\cdots+a_r)}.$$

For a homogeneous tame polynomial f let

$$\mathsf{p}(\beta,\delta) = \mathsf{p}(\{f=1\},\beta,\delta) := \int_\delta \frac{x^\beta dx}{df}, \tag{13.19}$$

$$\delta \in H_n(\{f=1\},\mathbb{Z}).$$

This is the constant term of the integral formula (13.15).

Proposition 13.7 *Let $f(x)$ and $g(y)$ be two tame homogeneous polynomials. Let also $(t_s,\ s \in [0,1], \delta_{1b}, \delta_{2b})$ be an admissible triple, x^{β_1} be a monomial in x and y^{β_2} be a monomial in y. We have*

$$\mathsf{p}(\{f+g=1\},(\beta_1,\beta_2),\delta_1 *_{t.} \delta_2) =$$

$$\mathsf{p}(\{f=1\},\beta_1,\delta_1)\mathsf{p}(\{g=1\},\beta_2,\delta_2)B(A_{\beta_1},A_{\beta_2}).$$

Proof. In Proposition 13.6 let us replace g with $-g+1$. We use (13.14) and we have

$$\begin{aligned}\int_{\delta_1 *_{t.} \delta_2} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{d(f+g)} &= \mathsf{p}(\{f=1\},\ \beta_1,\delta_1)\mathsf{p}(\{g=1\},\beta_2,\delta_2) \cdot \\ &\quad \int_0^1 s^{A_{\beta_1}-1}(1-s)^{A_{\beta_2}-1} ds \\ &= \mathsf{p}(\{f=1\},\ \beta_1,\delta_1)\mathsf{p}(\{g=1\},\beta_2,\delta_2) \cdot B(A_{\beta_1},A_{\beta_2}).\end{aligned}$$

□

As a corollary of the above proposition we have:

Proposition 13.8 *For zero dimensional cycles*

$$\delta_i = [a_i] - [b_i] \in H_0(\{x_i^{m_i} - 1\}, \mathbb{Z})$$

we have

$$\mathsf{p}(\{x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1\}, (\beta_1, \beta_2, \ldots, \beta_{n+1}), \delta_1 * \delta_2 * \cdots * \delta_{n+1}) =$$

$$(\int_{\delta_1} \frac{x_1^{\beta_1} dx_1}{dx_1^{m_1}})(\int_{\delta_2} \frac{x_2^{\beta_2} dx_2}{dx_2^{m_2}}) \cdots (\int_{\delta_{n+1}} \frac{x_{n+1}^{\beta_{n+1}} dx_{n+1}}{dx_{n+1}^{m_{n+1}}}) \cdot$$

$$B(\frac{\beta_1 + 1}{m_1}, \frac{\beta_2 + 1}{m_2}, \cdots, \frac{\beta_{n+1} + 1}{m_{n+1}}) =$$

$$\frac{1}{m_1 m_2 \cdots m_{n+1}} (a_1^{\beta_1+1} - b_1^{\beta_1+1})(a_2^{\beta_2+1} - b_2^{\beta_2+1}) \cdots (a_{n+1}^{\beta_{n+1}+1} - b_{n+1}^{\beta_{n+1}+1}) \cdot$$

$$B(\frac{\beta_1 + 1}{m_1}, \frac{\beta_2 + 1}{m_2}, \cdots, \frac{\beta_{n+1} + 1}{m_{n+1}})$$

Proof. Successive use of Proposition 13.7 will give us the desired equality of the proposition. □

Now, we describe some simple rules for reducing a higher dimensional integral to a lower dimensional one.

Proposition 13.9 *Let $f(x) = f(x_1, x_2, \ldots, x_{n+1})$ be a tame polynomial and $g(y) = g(y_1, y_2, \ldots, y_{m+1})$ be a homogeneous tame polynomial. Let $\delta_1 \in H_n(\{f = 1\}, \mathbb{Z})$, $\delta_2 \in H_m(\{g = 1\}, \mathbb{Z})$, x^{β_1} be a monomial in x and y^{β_2} be a monomial in y. Let also $t_s,\ s \in [0,1]$ be a path in the $\mathbb{C}$-plane which connects a critical value of f to 0 (the unique critical value of g). We assume that δ_1 vanishes along $t_\cdot^{-1}$ and δ_2 vanishes along $t_\cdot$. Then we have*

$$\int_{\delta_1 *_{t_\cdot} \delta_2} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{d(f-g)} = \begin{cases} \frac{\mathsf{p}(\beta_2, \delta_2)}{\mathsf{p}(\beta_3, \delta_3)} \int_{\delta_1 *_{t_\cdot} \delta_3} \frac{x^{\beta_1} z^{\beta_3} dx \wedge dz}{d(f - z^q)} & A_{\beta_2} \notin \mathbb{N}, \\ \\ \mathsf{p}(\beta_2, \delta_2) \int_{\tilde{\delta}_1} \theta(\frac{f^{A_{\beta_2}-1} x^{\beta_1} dx}{df}) & A_{\beta_2} \in \mathbb{N} \end{cases}.$$

In the first case q and β_3 are given by the equality $A_{\beta_2} = \frac{\beta_3+1}{q}$ and δ_3 is any cycle in $H_0(\{z^q = 1\}, \mathbb{Z})$ with $\mathsf{p}(\beta_3, \delta_3) \neq 0$. In the second case, $\tilde{\delta}_1 \in H_n(\{f = 0\}, \mathbb{Z})$ is the monodromy of δ_1 along the path $t_s,\ s \in [0,1]$ and θ is the operator in §12.6.

Proof. Using Proposition 13.6 we have:

$$\int_{\delta_1 * \delta_2} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{d(f-g)} = \mathsf{p}(\beta_2, \delta_2) \int_{t_s} t_s^{A_{\beta_2}-1} I_1(t_s) dt_s$$
$$= \mathsf{p}(\beta_2, \delta_2) \int_{t_s} t_s^{\frac{\beta_3+1}{q}-1} I_1(t_s) dt_s,$$

where $I_1(t_s) := \int_{\delta_{1,t_s}} \frac{x^{\beta_1} dx}{df}$. We consider two cases. If $A_{\beta_2} \notin \mathbb{N}$ then we can choose a cycle $\delta_3 \in H_0(\{z^q = 1\}, \mathbb{Z})$ such that $\mathsf{p}(\beta_3, \delta_3) \neq 0$ and so

$$t^{\frac{\beta_3+1}{q}} = \frac{1}{\mathsf{p}(\beta_3, \delta_3)} I_3(t),\ I_3(t) := \int_{\delta_3} \frac{z^{\beta_3} dz}{dz^q}.$$

We again use Proposition 13.6 and get the desired equality.

If $A_{\beta_2} \in \mathbb{N}$ then z^{β_3} is zero in H'' of the tame one variable polynomial $z^q - t$ and we cannot repeat the argument of the first part. In this case we have

$$= \mathsf{p}(\beta_2, \delta_2) \int_{t_s} (\int_{\delta_{1,t_s}} \frac{f^{A_{\beta_2}-1} x^{\beta_1} dx}{df}) dt_s$$
$$= \mathsf{p}(\beta_2, \delta_2) \int_{\Delta} f^{A_{\beta_2}-1} x^{\beta_1} dx$$
$$= \mathsf{p}(\beta_2, \delta_2) \int_{\tilde{\delta}_1} \theta(\frac{f^{A_{\beta_2}-1} x^{\beta_1} dx}{df}),$$

where

$$\Delta := \cup_{s \in [0,1]} \delta_{1,t_s} \in H_{n+1}(\mathbb{C}^{n+1}, f^{-1}(0), \mathbb{Z})$$

is the Lefschetz thimble with the boundary $\tilde{\delta}_1$. $\square$

Proposition 13.10 *With the notations of Proposition 13.9 we have*

$$\int_{\sigma(\delta_1 *_{t_.} \delta_2)} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{(f-g)^k} = \begin{cases} \frac{\mathsf{p}(\beta_2, \delta_2)}{\mathsf{p}(\beta_3, \delta_3)} \int_{\sigma(\delta_1 *_{t_.} \delta_3)} \frac{x^{\beta_1} z^{\beta_3} dx \wedge dz}{(f-z^q)^k}, & A_{\beta_2} \notin \mathbb{N} \\ \mathsf{p}(\beta_2, \delta_2) \frac{(-1)^{A_{\beta_2}-1} (A_{\beta_2}-1)!(k-A_{\beta_2}-1)!}{(k-1)!} \int_{\sigma(\tilde{\delta}_1)} \frac{x^{\beta_1} dx}{f^{k-A_{\beta_2}}}, & k > A_{\beta_2} \in \mathbb{N} \\ \mathsf{p}(\beta_2, \delta_2) \frac{2\pi i (-1)^{A_{\beta_2}-1} (A_{\beta_2}-1)!}{(k-1)!} \int_{\tilde{\delta}_1} \theta^{A_{\beta_2}-k+1} (\frac{x^{\beta_1} dx}{df}), & k \leq A_{\beta_2} \in \mathbb{N} \end{cases}.$$

Proof. We only prove the case $k > A_{\beta_2} \in \mathbb{N}$. The other cases are similar. We introduce a new parameter s and assume that f is of the form $\tilde{f} - s$. We have

$$\begin{aligned}
\int_{\sigma(\delta_1 *_{t_\cdot} \delta_2)} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{(f-g)^k} &= \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_{\sigma(\delta_1 *_{t_\cdot} \delta_2)} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{(f-g)} \\
&= \mathsf{p}(\beta_2, \delta_2) \frac{2\pi i}{(k-1)!} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_{\tilde{\delta}_1} \theta \left(\frac{f^{A_{\beta_2}-1} x^{\beta_1} dx}{df} \right) \\
&= \mathsf{p}(\beta_2, \delta_2) \frac{(2\pi i)(-1)^{A_{\beta_2}-1}(A_{\beta_2}-1)!}{(k-1)!} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_{\tilde{\delta}_1} \theta^{A_{\beta_2}} \left(\frac{x^{\beta_1} dx}{df} \right) \\
&= \mathsf{p}(\beta_2, \delta_2) \frac{(-1)^{A_{\beta_2}-1}(A_{\beta_2}-1)!(k-A_{\beta_2}-1)!}{(k-1)!} \int_{\sigma(\tilde{\delta}_1)} \frac{x^{\beta_1} dx}{f^{k-A_{\beta_2}}}.
\end{aligned}$$

In the second equality we have used Proposition 13.9 and in the third equality we have used Exercise 12.5. The fourth equality uses Remark 12.2. Note that the integrand is independent of s because $df = d\tilde{f}$, and so, the derivation $\frac{\partial}{\partial s}$ goes inside the integral as $\frac{d}{df}$. $\square$

Proposition 13.11 *Let $f(x) = f(x_1, x_2, \ldots, x_{n+1})$ be a tame polynomial and $g(y) = g(y_1, y_2, \ldots, y_{m+1})$ be a homogeneous tame polynomial. Let $\delta_1 \in H_n(\{f=1\}, \mathbb{Z})$, $\delta_2 \in H_m(\{g=1\}, \mathbb{Z})$, x^{β_1} be a monomial in x and y^{β_2} be a monomial in y. Let also $t_s,\ s \in [0,1]$ be a path in the $\mathbb{C}$-plane which connects a critical value of f to 0 (the unique critical value of g). We assume that δ_1 vanishes along $t_\cdot^{-1}$ and δ_2 vanishes along $t_\cdot$. Then we have*

$$\int_{\sigma(\delta_1 *_{t_\cdot} \delta_2)} \frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{(f-g)^k} =$$

$$\begin{cases}
\left(\int_{\delta_3} \frac{z^{\beta_3} dz}{dz^q}) \right)^{-1} \left(\int_{\sigma(\delta_1 *_{t_\cdot} \delta_3)} \frac{x^{\beta_1} z^{\beta_3} dx \wedge dz}{(f-z^q)^k} \right) \left(\int_{\delta_2} \frac{y^{\beta_2} dy}{dg} \right), & A_{\beta_2} \notin \mathbb{N} \\
\frac{(-1)^{A_{\beta_2}-1}(A_{\beta_2}-1)!(k-A_{\beta_2}-1)!}{(k-1)!} \left(\int_{\sigma(\tilde{\delta}_1)} \frac{x^{\beta_1} dx}{f^{k-A_{\beta_2}}} \right) \left(\int_{\delta_2} \frac{y^{\beta_2} dy}{dg} \right), & k > A_{\beta_2} \in \mathbb{N} \\
\frac{2\pi i (-1)^{A_{\beta_2}-1}(A_{\beta_2}-1)!}{(k-1)!} \left(\int_{\tilde{\delta}_1} \theta^{A_{\beta_2}-k+1} (\frac{x^{\beta_1} dx}{df}) \right) \left(\int_{\delta_2} \frac{y^{\beta_2} dy}{dg} \right), & k \leq A_{\beta_2} \in \mathbb{N}
\end{cases}.$$

In the first case q and β_3 are given by the equality $A_{\beta_2} = \frac{\beta_3+1}{q}$ and δ_3 is any cycle in $H_0(\{z^q=1\}, \mathbb{Z})$ with $\int_{\delta_3} \frac{z^{\beta_3} dz}{dz^q} \neq 0$. In the second case, $\tilde{\delta}_1 \in H_n(\{f=0\}, \mathbb{Z})$ is the monodromy of δ_1 along the path $t_s,\ s \in [0,1]$ and θ is the operator in defined in §12.6.

This is just Proposition 13.10 written in a different way. The content of this section has been used by J. Duque in his Ph.D thesis [Duq20] in order to obtain algebraic relations between Gauss hypergeometric functions.

13.9 Taylor series of periods

Periods in general are transcendental numbers and rigorous proofs to this statement in concrete examples are hard and, in most of the cases, difficult open conjectures. Therefore, it does not make sense to talk about computing periods similar to what we did in the case of Fermat varieties, see Proposition 13.8. However, when they depend on parameters then they are holomorphic functions and so we can write their Taylor series at a point with computable periods and with reasonable "nice-looking" coefficients. Such series range from the historical example of Gauss hypergeometric function to periods of Calabi-Yau varieties used in String Theory and Gelfand-Kapranov-Zelevinsky (GKZ) hypergeometric functions, for some examples see [GAZ10]. This has been a good candidate for the concept of computing periods.

In this section we want to compute the periods of the full family of hypersurfaces near the Fermat variety. We do this in the general framework of tame polynomials. The constant terms of these periods encode the Hodge structure of the Fermat variety and the linear terms encode the so-called infinitesimal variation of Hodge structures. All these are written in terms of cohomologies and it would be interesting to know whether our computation of higher order terms has some cohomological interpretation.

Let $g_0 \in \mathbb{C}[x]$ be a (weighted) homogeneous tame polynomial of degree d. Our main example for this is

$$g_0 = x_1^{m_1} + \cdots + x_{n+1}^{m_{n+1}}.$$

with $\deg(x_i) := \frac{d}{m_i}$, where $d := [m_1, m_2, \ldots, m_{n+1}]$. Let also

$$f_t := g_0 - s - \sum_{\alpha \in I} t_\alpha x^\alpha. \tag{13.20}$$

Here, $\{x^\alpha \mid i \in I\}$ is any set of monomials $x^\alpha = x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}$ of degree $\leq d$, $t = (t_\alpha, \alpha \in I)$ is a set of parameters and s is a fixed non-zero complex number. We define $\mathsf{R} := \mathbb{C}[t]_{\hat{\Delta}}$, where $\hat{\Delta} \in \mathbb{C}[t]$ is defined in such a way that f is tame over R, see Example 10.4. Note that the last homogeneous piece g of f might depend on parameters and f evaluated at $t = 0$ is $g_0 - s$. Since g_0 is tame, we can choose $\hat{\Delta}$ in such way that $\hat{\Delta}$ evaluated at $t = 0$ is not zero. We fix a continuous family of cycles $\delta = \{\delta_t\}_{t \in U}$, $\delta_t \in H_n(L_t, \mathbb{Z})$, where L_t is the affine variety in $\mathbb{C}^{n+1}$ given by $f_t = 0$ and U is a small neighborhood of $0 \in \mathsf{T} := \mathbb{C}^{\#I} - \{\hat{\Delta} = 0\}$. The following theorem writes the Taylor series of periods over δ_t with coefficients which are periods over $\delta_0 \in H_n(\{g_0 = 1\}, \mathbb{Z})$. Note that for δ_0 we have put $s = 1$. Recall the definition of the residue map in Definition 12.2.

Theorem 13.2 *For $k \in \mathbb{N}$ and a monomial $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}}$ we have*

$$(k-1)!\cdot s^k \int_{\delta_t} \mathrm{Resi}\left(\frac{x^\beta dx}{f^k}\right) = \sum_{a:I\to\mathbb{N}_0} \left(\frac{1}{s^{|a|}a!}\langle A_{\beta+a^*}\rangle_{k+|a|}\cdot s^{A_{\beta+a^*}}\cdot \int_{\delta_0} x^{\beta+a^*}\eta\right) t^a. \tag{13.21}$$

Here, the sum runs through all #I-tuples $a=(a_\alpha,\ \ \alpha\in I)$ of non-negative integers, $\delta_0\in H_n(L_t,\mathbb{Z})$ with $t=0,\ s=1$, and

$$\begin{aligned}
t^a :&= \prod_{\alpha\in I} t_\alpha^{a_\alpha},\ \ |a| := \sum_{\alpha\in I} a_\alpha,\\
a! &:= \prod_{\alpha\in I} a_\alpha!,\ \ a^* := \sum_\alpha a_\alpha\cdot\alpha,\\
A_\beta &:= \sum_{i=1}^{n+1} \frac{\nu_i(\beta_i+1)}{d},\quad \beta\in\mathbb{N}_0^{n+1},\\
\langle x\rangle_y &:= (x-y+1)_{y-1} = (x-1)(x-2)\cdots(x-y+1),\ y\geq 1,\ \ \langle x\rangle_1 := 1.
\end{aligned}$$

Our formula (13.21) using the classical Pochhammer symbol $(x)_y := x(x+1)\cdots(x+y-1)$ is not so nice-looking, that is why we have used $\langle x\rangle_y$ notation.

Proof. Let $\sigma : H_n(L_t,\mathbb{Z})\to H_{n+1}(\mathbb{C}^{n+1}\backslash L_t,\mathbb{Z})$ be the Thom-Leray-Gysin map. Let also

$$\int_{\sigma(\delta_t)} \frac{x^\beta dx}{f^k} = \sum_a c_a t^a.$$

We have

$$\begin{aligned}
c_a &= \frac{1}{a!}\frac{\partial^a}{\partial t^a}\int_{\sigma(\delta_t)} \frac{x^\beta dx}{f^k}\bigg|_{t=0}\\
&= \frac{1}{a!}\int_{\sigma(\delta_t)} \frac{\partial^a}{\partial t^a}\frac{x^\beta dx}{f^k}\bigg|_{t=0}\\
&= \frac{(k)_{|a|}}{a!}\int_{\sigma(\delta_0)} \frac{x^{\beta+\sum_\alpha a_\alpha\alpha}dx}{(g_0-s)^{k+|a|}}\\
&= \frac{(k)_{|a|}}{a!}\frac{\langle A_{\beta+a^*}\rangle_{k+|a|}}{(k+|a|-1)!}s^{A_{\beta+a^*}-k-|a|}\int_{\delta_0}\eta_{\beta+a^*}\\
&= \frac{\langle A_{\beta+a^*}\rangle_{k+|a|}}{a!\cdot(k-1)!}s^{A_{\beta+a^*}-k-|a|}\int_{\delta_0}\eta_{\beta+a^*}.
\end{aligned}$$

In the fourth equality we have used (11.13) or its integral version (13.16). In the last two equalities we have not written the $2\pi\sqrt{-1}$ factors. □

For the homogeneous tame polynomial $x_1^{m_1}+x_2^{m_2}+\cdots+x_{n+1}^{m_{n+1}}-1$ we have computed the periods $\int_{\delta_0}$ appearing in (13.21) in Proposition 13.8. This computation is also done in Proposition 15.1 (note that in this proposition $\omega=\eta$). In this case note that the coefficients of the Taylor series (13.21) are explicit numbers in the field extension of $\mathbb{Q}$ by ζ_{2d} and all the values of the Γ function on rational numbers.

For applications of the formula (13.21), it is convenient to write down the integrals over δ_0 of differential forms $x^\beta\eta$ with $0 < A_\beta < n+1$, or differential forms which appear in Griffiths-Steenbrink theorem (Theorem 11.3). For this we use Theorem 11.2 and the equality (11.13) (see also its integral version (13.16)). Let $\delta_0 \in H_n(L_0,\mathbb{Z})$, where L_0 is the affine variety $g_0 = 1$, where $g_0 := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$. For an arbitrary monomial x^β we have:

$$\begin{aligned} s^{A_\beta}\int_{\delta_0} x^\beta\eta &= \frac{\prod_{i=1}^{n+1}\left\langle \frac{\beta_i+1}{m_i}\right\rangle_{\left[\frac{\beta_i+1}{m_i}\right]+1}}{\left\langle A_\beta\right\rangle_{\sum_{i=1}^{n+1}\left[\frac{\beta_i+1}{m_i}\right]+1}} s^{A_\beta}\int_{\delta_0} x^{\bar\beta}\eta \\ &= \frac{\prod_{i=1}^{n+1}\left\langle \frac{\beta_i+1}{m_i}\right\rangle_{\left[\frac{\beta_i+1}{m_i}\right]+1}}{\left\langle A_\beta\right\rangle_{\sum_{i=1}^{n+1}\left[\frac{\beta_i+1}{m_i}\right]+1}} \frac{[A_{\bar\beta}]!\cdot s^{[A_\beta]+1}}{\langle A_{\bar\beta}\rangle_{[A_{\bar\beta}]+1}} \int_{\sigma(\tilde\delta_0)} \frac{x^{\bar\beta}dx}{(g_0-s)^{[A_{\bar\beta}]+1}}, \end{aligned}$$

where $\bar\beta_i = \beta_i - \left[\frac{\beta_i+1}{m_i}\right] m_i, \quad i = 1,2,\ldots,n+1$. Here, $\tilde\delta_0$ is the monodromy of the original δ_0 for $t = 0,\ s = 1$ along a circle centered at $s = 0$ and in the s-plane, which connects 1 to $|s|$ in the real line, and then $|s|$ to s in the anticlockwise direction. Inserting the above formula inside (13.21) makes it even longer. The last equality is valid only for $A_\beta \notin \mathbb{N}$ because otherwise we will have division over 0 due to $\langle A_{\bar\beta}\rangle_{[A_{\bar\beta}]+1}$.

If $A_{\beta+a^*} = 1,2,\cdots,k+|a|-1$ or for some $i = 1,2,\ldots,n+1$ with $\frac{(\beta+a^*)_i+1}{m_i} \in \mathbb{N}$ then the coefficient of the monomial t^a in (13.2) is zero.

13.10 Periods of projective hypersurfaces

In this section we reformulate Theorem 13.2 for weighted projective hypersurfaces so that its independence from the affine chart $x_0 = 1$ becomes clear. We will need this reformulation in Chapter 17 in which we will discuss deformations of Hodge cycles.

We consider the family of hypersurfaces $X_t \subset \mathbb{P}^{1,\nu}$ given by the homogeneous polynomial of degree d:

$$f_t := x_0^{m_0} + x_1^{m_1} + \cdots + x_{n+1}^{m_{n+1}} - \sum_{\alpha\in I} t_\alpha x^\alpha,\ \ t = (t_\alpha)_{\alpha\in I} \in \mathsf{T} \tag{13.22}$$

in the weighted ring $\mathbb{C}[x],\ \deg(x_i) = \nu_i = \frac{d}{m_i} \in \mathbb{N},\ \ i = 0,1,2,\ldots,n+1,\ \ \nu_0 := 1$. Let

$$\Omega := \sum_{j=0}^{n+1} \nu_i(-1)^j x_j\, dx_0\wedge\cdots\wedge\widehat{dx}_j\wedge\cdots\wedge dx_{n+1}.$$

For $\beta \in \mathbb{N}_0^{n+2}$, let us define $\bar{\beta} \in \mathbb{N}_0^{n+2}$ by the rules:

$$0 \le \bar{\beta}_i \le m_i - 1, \;\; \beta_i \equiv_{m_i} \bar{\beta}_i.$$

For a rational number r let $[r]$ be the integer part of r, that is $[r] \le r < [r]+1$, and $\{r\} := r - [r]$. Let also $(x)_y := x(x+1)(x+2)\cdots(x+y-1)$, $(x)_0 := 1$ be the Pochhammer symbol.

Theorem 13.3 *For a monomial $x^\beta = x_0^{\beta_0} x_1^{\beta_1} \cdots x_{n+1}^{\beta_{n+1}}$ with $k := \sum_{i=0}^{n+1} \frac{\beta_i+1}{m_i} \in \mathbb{N}$ and a continuous family of cycles $\delta_t \in H_n(X_t, \mathbb{Z})_0$, $t \in (\mathsf{T}, 0)$, we have*

$$(k-1)! \int_{\delta_t} \operatorname{Resi}\left(\frac{x^\beta \Omega}{f_t^k}\right) = \sum_{a: I \to \mathbb{N}_0} \left(\frac{1}{a!} D_{\beta+a^*} \mathsf{p}_{\overline{\beta+a^*}}\right) t^a, \tag{13.23}$$

where

$$\begin{aligned}
t^a :&= \prod_{\alpha \in I} t_\alpha^{a_\alpha}, \;\; |a| := \sum_{\alpha \in I} a_\alpha, \\
a! &:= \prod_{\alpha \in I} a_\alpha!, \;\; a^* := \sum_\alpha a_\alpha \cdot \alpha, \\
D_{\check{\beta}} &:= \left(k + |a| - 1 - \sum_{i=0}^{n+1} \left[\frac{\check{\beta}_i + 1}{m_i}\right]\right)! \cdot \prod_{i=0}^{n+1} \left(\left\{\frac{\check{\beta}_i+1}{m_i}\right\}\right)_{\left[\frac{\check{\beta}_i+1}{m_i}\right]} \\
\mathsf{p}_{\check{\beta}} &:= \int_{\delta_0} \operatorname{Resi}\left(\frac{x^{\check{\beta}} \Omega}{(x_0^{m_0} + x_1^{m_1} + \cdots + x_{n+1}^{m_{n+1}})^{\check{k}}}\right), \;\; \check{k} := \sum_{i=0}^{n+1} \frac{\check{\beta}_i + 1}{m_i} \in \mathbb{N}.
\end{aligned}$$

Here, the sum runs through all #I-tuples $a = (a_\alpha, \;\; \alpha \in I)$ of non-negative integers.

Proof. The first step in the proof is our general formula for the Taylor series of periods computed in Theorem 13.2. From this theorem it comes most of our notations. For this we restrict the integrand (13.23) to the affine chart $x_0 = 1$ and get the integrand in (13.21). We further need to set $s := -1$. Note that $\nu_0 = 1$. The second step is to use Theorem 11.2 in order to handle $\int_{\delta_0} \eta_{\beta+a^*}$, that is, to get integrands $\eta_{\overline{\beta+a^*}}$ instead of $\eta_{\beta+a^*}$. This is where the second factor in the rational number $D_{\beta+a^*}$ pops up. Note that in this integral δ_0 is in the n-th homology group of $-sx_0^{m_0} + x_1^{m_1} + \cdots + x_{n+1}^{m_{n+1}} = 0$ with $s = 1$ and not $s = -1$. Let $\check{\beta} := \beta + a^*$. If one of $\check{\beta}_i + 1$ is divisible by m_i then $\int_{\delta_0} \eta_{\check{\beta}} = 0$, or equivalently $D_{\check{\beta}} = 0$, that is why we can assume that none of $\check{\beta}_i + 1$'s is divisible by m_i. In particular, for $r_0 := \frac{\check{\beta}_0+1}{m_0}$ we have

$$\begin{aligned}
A_{\check{\beta}} + r_0 = k + |a| \Rightarrow \left[A_{\check{\beta}}\right] + [r_0] &= k + |a| - 1, \\
\{A_{\check{\beta}}\} + \{r_0\} &= 1, \\
\langle A_{\check{\beta}} \rangle_{k+|a|} &= (-r_0 + 1)_{k+|a|-1}.
\end{aligned}$$

The third step is to write the integrand $\eta_{\overline{\beta+a^*}}$ back in a form which appears in the Griffiths theorem (Theorem 11.3). For this we use (11.13) or its integral version (13.16), the equalities

$$\begin{aligned} A_{\overline{\check{\beta}}} &= A_{\check{\beta}} - \sum_{i=1}^{n+1}[\frac{\check{\beta}_i+1}{m_i}] = k+|a|-\{r_0\}-N_a, \quad N_a := \sum_{i=0}^{n+1}[\frac{\check{\beta}_i+1}{m_i}], \\ \left[A_{\overline{\check{\beta}}}\right] &= k+|a|-1-N_a, \\ \{A_{\overline{\check{\beta}}}\} &= \{A_{\check{\beta}}\} = 1-\{r_0\}, \end{aligned}$$

and we get

$$\begin{aligned} (-1)^{A_\beta}\int_{\delta_0}\eta_{\overline{\check{\beta}}} &= (-1)^{A_\beta}\frac{(-1)^{[A_{\overline{\check{\beta}}}]+1-A_{\overline{\check{\beta}}}}[A_{\overline{\check{\beta}}}]!}{(A_{\overline{\check{\beta}}}-[A_{\overline{\check{\beta}}}])_{[A_{\overline{\check{\beta}}}]}}\mathsf{p}_{\overline{\check{\beta}}} \\ &= \frac{(-1)^{k+|a|-[r_0]}\,(k+|a|-1-N_a)\,!}{(1-\{r_0\})_{k+|a|-1-N_a}}\mathsf{p}_{\overline{\check{\beta}}}. \end{aligned}$$

The theorem now follows from the fact that $(-1)^{k+|a|-1}(-r_0+1)_{k+|a|-1}$ is equal to

$$(\{r_0\})_{[r_0]}\cdot(-1)^{k+|a|-1-N_a}(1-\{r_0\})_{k+|a|-1-N_a}\cdot(-1)^{N_a-[r_0]}(k+|a|-\{r_0\}-N_a)_{N_a-[r_0]}$$

□

It is worth to rewrite the linear part of the Taylor series (13.23) in Theorem (13.3) for hypersurfaces X_t, where α runs through $\mathbb{N}_0^{n+1}$ with $\sum_{i=0}^{n+1}\alpha_i = d$ and $0 \le \alpha_i \le m_i - 2$. This is

$$\int_{\delta_t}\mathrm{Resi}\left(\frac{x^\beta dx}{f_t^k}\right) = \mathsf{p}_\beta + \frac{1}{(k-1)!}\sum_{\alpha\in I}\left(D_{\beta+\alpha}\mathsf{p}_{\overline{\beta+\alpha}}\right)t_\alpha + \cdots. \tag{13.24}$$

This linear term can be interpreted using Infinitesimal Variation of Hodge structures (IVHS) developed by P. Griffiths and his school in [CGGH83]. This mainly uses cohomologies of sheaves of differential forms and vector fields, Kodaira-Spencer theory, etc, see [Mov17b]. For further applications of Hodge theory in Algebraic Geometry it seems to be interesting to interpret the second order (and in general higher order) approximations in (13.21) in terms of some cohomological data. The formula in Theorem 13.3 becomes a little bit simpler if we assume that δ_0 is a Hodge cycle.

Theorem 13.4 *In Theorem 13.3 if $\delta_0 \in H_n(X_0,\mathbb{Z})_0$ is a Hodge cycle then we have*

$$\frac{(k-1)!}{\frac{n}{2}!}\int_{\delta_t}\mathrm{Resi}\left(\frac{x^\beta\Omega}{f_t^k}\right) = \sum\left(\frac{1}{a!}D_{\beta+a^*}\mathsf{p}_{\overline{\beta+a^*}}\right)t^a, \tag{13.25}$$

where the sum runs over $a : I \to \mathbb{N}_0$ *with*

$$\sum_{i=0}^{n+1} \left[\frac{\check{\beta}_i + 1}{m_i} \right] = k + |a| - \frac{n}{2} - 1, \;\; \check{\beta} := \beta + a^* \tag{13.26}$$

and every term is as in Theorem 13.3 except

$$D_{\check{\beta}} := \prod_{i=0}^{n+1} \left(\left\{ \frac{\check{\beta}_i + 1}{m_i} \right\} \right)_{\left[\frac{\check{\beta}_i + 1}{m_i} \right]}.$$

Proof. We have just used the fact that for a Hodge cycle we have $\mathsf{p}_{\overline{\beta + a^*}} = 0$ if the condition (13.26) is not satisfied. For $\check{k}$ as in the end of Theorem 13.3 only those equal to $\frac{n}{2} + 1$ will remain in the sum. For $\check{k} < \frac{n}{2} + 1$, this follows from the definition of a Hodge cycle, and for $\check{k} > \frac{n}{2} + 1$ by complex conjugation. Here, we are using one of the peculiar properties of the Fermat variety. Its Hodge decomposition is defined over $\mathbb{Q}$. $\square$

If $\delta_0 \in H_n(X_0, \mathbb{Z})_0$ is a Hodge cycle, $\beta \in \mathbb{N}_0^{n+2}$ with $\sum_{i=0}^{n+1} \frac{\beta_i + 1}{m_i} = \frac{n}{2}$ (in our notation of the affine chart $x_0 = 1$, $\frac{n}{2} - 1 < A_\beta < \frac{n}{2}$) and for some $i = 0, 1, \ldots, n+1$ we have $\beta_i + \alpha_i > m_i - 2$ then $\mathsf{p}_{\overline{\beta + \alpha}} = 0$. In case $\mathsf{p}_{\beta + \alpha}$ is non-zero, we have $D_{\beta + \alpha} = 1$. We conclude that the integral (13.24) is of the form

$$\frac{n}{2} \sum_{\alpha \in I} \mathsf{p}_{\beta + \alpha} t_\alpha + \cdots,$$

where we redefine $\mathsf{p}_{\beta + \alpha}$ to be zero if for some $i = 0, 1, \ldots, n+1$ we have $\beta_i + \alpha_i > m_i - 2$. Note that $\mathsf{p}_\beta = 0$ by definition of a Hodge cycle. We will use this in Chapter 16 in order to define an invariant of Hodge cycles.

13.11 Exercises

13.1. Let $f \in \mathsf{R}[x]$ be a monic polynomial in one variable. This is automatically a tame polynomial. Recall that $\delta_t \in H_0(L_t, \mathbb{Z})$ is the set of all finite sums $\sum_i r_i [x_i]$, where $r_i \in \mathbb{Z}, \sum_i r_i = 0$ and x_i's are the roots of f_i. For $\omega \in \mathsf{H}'$ we have the zero dimensional integrals

$$\int_\delta \omega := \sum_i r_i \omega(x_i),$$

Write down the discussion of Picard-Fuchs equation and Gauss-Manin connection of f using elementary algebra, for some formulas in this case see [GM07].

13.2. Prove Theorem 13.1 using the following hint. Let us consider a tame polynomial in $\mathsf{R}[x]$, where R is a parameter ring as in the beginning of §13.2, and assume

that its discriminant is not zero and the polynomial in Theorem 13.1 is a specialization of f for $0 \in \mathsf{T}$. Further, assume that the last homogeneous piece g of f does not depend on any parameter. Concerning these assumptions see the discussion of the completion of a tame polynomial in Example 10.4. Prove the following:

1. The trace $\mathrm{Tr}(A)$ of the Gauss-Manin connection matrix A in (13.8) is of the form $\frac{a}{2}\frac{d\Delta}{\Delta}$, where $a \in \mathbb{Z}$ and Δ is the discriminant of f.
2. Prove that the determinant of the period matrix satisfies the differential equation

$$d\,(\det \mathbf{P}) = \mathrm{Tr}(A) \cdot \det \mathbf{P}$$

   and so

$$(\det \mathbf{P})^2 = c \cdot \Delta^a$$

   for some constant $c \in \mathbb{C}$.
3. The period matrix of the tame polynomial $g-1$ has non-zero determinant. In the case $g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$ this follows from Proposition 15.1 and the problem in §21.4.

There is a topological argument which shows that $\det(\mathbf{P})^2$ is a one-valued function in T_Δ. This is as follows. If δ' is another basis of $H_n(L_t, \mathbb{Z})$ obtained by the monodromy of δ then $\delta' = A\delta$, $A\Psi_0 A^{\mathsf{tr}} = \Psi_0$, where Ψ_0 is the intersection matrix of $H_n(L_t, \mathbb{Z})$ in the basis δ. This implies that $\det(A)^2 = 1$ and so $\det(\mathbf{P})^2$ is a one-valued function in T_Δ. Since our integrals have a finite growth at infinity and $\{\Delta = 0\}$, see for instance [Gri70], we conclude that $\det(\mathbf{P})^2$ is rational function in T with poles along $\{\Delta = 0\}$.

13.3. In the Picard-Simart book [PS06], Vol. II, page 168 we find the following: let $f \in \mathbb{C}[x,y,z]$ and assume that $\{f = 0\}$ is smooth. Show that an integral of the form

$$\int \frac{P(x,y,z)dx \wedge dy}{(x-a)^k f_z}, \;\; k \in \mathbb{N},\; a \in \mathbb{C},\; P \in \mathbb{C}[x,y,z]$$

is reduced to an integral of the same type with $k = 1$. Here, the integration takes place over a topological 2-cycle of $\{f = 0\} \backslash \{z = a\}$. Prove this statement. Hint: For the reduction of pole order see page 167 of the same book.

13.4. Check Theorem 13.3 in the following way. Use the methods in Chapter 12, and show that the Picard-Fuchs equation of the differential form $\frac{dx \wedge dy}{f}$ with respect to the families of elliptic curves

$$x^3 + y^2 + 1 - tx = 0, \;\; x^3 + y^3 + 1 - tx = 0,$$

is given respectively by

$$7t \cdot I + 48t^2 \cdot I' + (16t^3 - 108) \cdot I'' = 0, \;\; 2t^2 I + (4t^3 - 27) I' = 0. \tag{13.27}$$

From another side, write few coefficients of the Taylor series in Theorem 13.3 for these two examples, and verify that it satisfies the Picard-Fuchs equations in (13.27).

Chapter 14
Noether-Lefschetz theorem

A la vérité, la démonstration de M. Nöther fondée sur une énumeration de constantes ne peut être regardée que comme rendant le théorèm très vraisemblable,... (Picard and Simart in [PS06] page 414.)

14.1 Introduction

In this chapter we are going to prove a celebrated theorem of Max Noether which states that a generic surface X of degree $d \geq 4$ in a projective space of dimension three has no curves apart from a hypersurface section of X. "Noether, it would seem, stated this theorem but never completely proved it. Instead, he gave a plausibility argument, based on ... [a Hilbert scheme argument]", (Griffiths and Harris in [GH85] page 31). This seems to be the case for many theorems of Algebraic Geometry of that time. A rigorous proof came after the breakthrough work of Lefschetz in 1924 [Lef24a] in which Picard-Lefschetz theory was founded. For this reason it is nowadays called Noether-Lefschetz theorem. It basically uses the fact that the monodromy group of the full family of all surfaces of degree d acts irreducibly on the primitive part of the second cohomology of the surface. We are going to reproduce this proof in §14.3. After 1950's, and specially after Grothendieck's scheme theoretic foundation of Algebraic Geometry, there was a tremendous effort to give algebraic proofs for theorems of algebraic geometry, previously proved by topological and transcendental methods. This was also the case for Noether-Lefschetz theorem. Therefore, algebraic geometers were not satisfied by Lefschetz's proof. The next rigorous proof were given after the invention of infinitesimal variation of Hodge structures (IVHS) by Griffiths and his coauthors in [CGGH83], and even a purely algebraic proof by Harris and Griffiths in [GH85].

In this chapter we give two proofs of Noether-Lefschetz theorem, the first one is topological and is due to Lefschetz, and the second one is based on the period computation that we performed in Theorem 13.2, and it must be considered a simplification of IVHS, in the sense that we avoid using unnecessary language of sheaves and

cohomologies, and so it is more elementary than IVHS. It is worth to remark that Noether-Lefschetz theorem appears in the book [PS06] page 414 and an immediate equivalent statement is highlighted there: the Picard number of a generic surface is one.

Theorem 14.1 (Noether-Lefschetz theorem) *For $d \geq 4$ a generic surface $X \subset \mathbb{P}^3$ of degree d has Picard group* $\mathrm{Pic}(X) \cong \mathbb{Z}$*, that is, every curve $C \subset X$ is a complete intersection of X with another surface.*

Using the language of line bundles, the isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$ is equivalent to say that $\mathrm{Pic}(X)$ is generated by restriction of the standard line bundle $\mathscr{O}(1)$ of $\mathbb{P}^3$. Let $\mathbb{P}^N$ be the projectivization of the parameter space of surfaces in $\mathbb{P}^3$. Here by a generic surface we mean that there is a countable union V of proper subvarieties of $\mathbb{P}^N$ such that X is in $\mathbb{P}^N - V$. A slight modification of the proof of Theorem 14.1 gives us a similar result for higher dimensional hypersurfaces: for a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d > 2 + \frac{2}{n}$, we have $\mathrm{Hodge}_n(X, \mathbb{Z}) \cong \mathbb{Z}$. A generator of this $\mathbb{Z}$-module is given by the homology class of the intersection of a linear $\mathbb{P}^{\frac{n}{2}+1}$ with X. It is not clear to the author whether the following stronger affirmation is true or not: for a generic hypersurface as above any algebraic cycle of dimension $\frac{n}{2}$ in X is the intersection of X with $\frac{n}{2}$ other hypersurfaces, see the last step of the proof of Noether-Lefschetz theorem in §14.3. Similar questions for curves inside general threefolds of degree $d \geq 6$ have been asked by Harris and Griffiths in [GH85] page 32. The reader can also consult Voisin's book [Voi03] 3.3.2 on Noether-Lefschetz theorem.

14.2 Hilbert schemes

For the proof of Noether-Lefschetz theorem, we need varieties which parameterize curves in $\mathbb{P}^3$. This is done using Hilbert schemes. Let $\mathsf{H}_{g,n}$ be the Hilbert scheme of curves of genus g and degree n in $\mathbb{P}^3$. We know that $\mathsf{H}_{g,n}$ is a projective variety, see for instance [ACG11]. It may not be irreducible. Let also T be the open subset of $\mathbb{P}^N$ parameterizing smooth degree d surfaces in $\mathbb{P}^3$. We have the following incidence variety

$$\Sigma_{g,n,d} := \{(C, X) \in \mathsf{H}_{g,n} \times \mathsf{T} \mid C \subset X\}$$

which is again a projective variety. For particular classes of (g,n), we have an irreducible component $\tilde{\Sigma}_{g,n,d}$ which parametrizes the pairs (C,X), where C is a complete intersection of X with another surface. The closure $\check{\Sigma}_{g,n,d}$ of $\Sigma_{g,n,d} - \tilde{\Sigma}_{g,n,d}$ is still a projective variety and the projection

$$\pi : \check{\Sigma}_{g,n,d} \to \mathsf{T} \tag{14.1}$$

is a proper map.

Definition 14.1 The image $\mathrm{NL}_{g,n,d}$ of (14.1) is a finite union of irreducible subvarieties of T. The union

$$\mathrm{NL}_d := \cup_{g=0}^{\infty} \cup_{n=1}^{\infty} \mathrm{NL}_{g,n,d}$$

is called the Noether-Lefschetz locus in T.

In other words, a component V of the Noether-Lefschetz locus parameterizes pairs (X,C), where C is a curve inside the surface $X = X_t \subset \mathbb{P}^3$ which is not a complete intersection of X with another surface and it cannot be deformed for $t \notin V$. Note that for a fixed t there might be a family of such curves and not a single curve, see Figure 14.1.

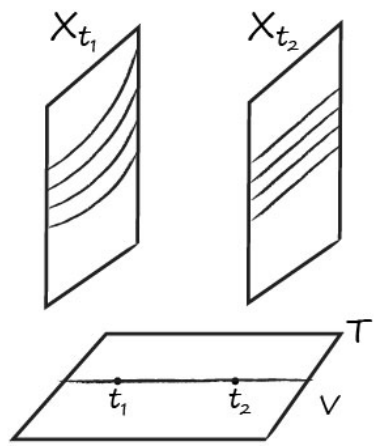

Fig. 14.1 Family of curves inside family of surfaces.

14.3 Topological proof of Noether-Lefschetz theorem

The main ingredient of the topological proof of Noether-Lefschetz theorem is the consideration of the topological class $[C] \in H_2(X,\mathbb{Z})$ of a curve $C \subset X$ and this is what led Lefschetz to prove the celebrated Lefschetz $(1,1)$ theorem, see Chapter 9. The proof is of global nature in the sense that it uses the full family of hypersurfaces $\mathsf{X} \to \mathsf{T}$.

Let us consider a component Σ of $\Sigma_{g,n,d}$ which maps onto T, that is, the canonical projection $\pi : \Sigma \to \mathsf{T}$ is surjective. Since we have a full family of hypersurfaces over T, and a full family of curves of genus g and degree n over $\mathsf{H}_{g,n}$, we have also a

morphism of algebraic varieties $f : V \to \Sigma$ such that the fiber of f over a point $s \in \Sigma$ is $C_s \times X_t$, where C_s is a curve of genus g and degree n, and X_t is a surface in $\mathbb{P}^3$ with $\pi(s) = t$ and $C_s \subset X_t$. We have also a subvariety $\check{V} \subset V$ such that the fiber of f over $s \in \Sigma$ is $C_s \times C_s \subset C_s \times X_t$. We use Ehresmann's fibration theorem (Theorem 6.1) for the map $f : (V, \check{V}) \to \Sigma$ and conclude that there is a Zariski open subset $\check{U} \subset \Sigma$ such that f over $\check{U}$ is a C^∞ fiber bundle. In particular, we get a continuous family of cycles $\delta_s := [C_s] \in H_2(X_t, \mathbb{Z}),\ s \in \check{U}$. We would like to have a similar family of cycles over $U := \pi(\check{U}) \subset \mathsf{T}$ and not over $\check{U}$. For this we make the following modifications:

1. We replace $\check{U}$ with a $\dim(\mathsf{T})$-dimensional subvariety of $\check{U}$ such that $\pi : \check{U} \to \mathsf{T}$ is still dominant and its generic fiber is zero dimensional, and hence a finite number of points.
2. We replace again $\check{U}$ with a Zariski open set such that $\pi : \check{U} \to \mathsf{T}$ over its image is étale, that is in addition to the previous property, it is locally an isomorphism.

Now let U be the image of $\check{U}$ under π. Since π is proper, this is a Zariski open subset of T. For $t \in U$ we get the proper subspace W of $H_2(X_t, \mathbb{Z})$ generated by $\delta_s, s \in \pi^{-1}(t)$ which is invariant under the monodromy group, because its rank is less that the Hodge number h^{11} of X_t, see Figure 14.2. This property will be unchanged if we replace U with T. Remember that T parametrizes only smooth surfaces. Moreover, W contains the homology class of the intersection of a linear $\mathbb{P}^2$ with X_t. The reason is as follows. We consider the continuous family of cycles

$$\delta_t := \sum_{s \in \pi^{-1}(t)} \delta_s \in H_2(X_t, \mathbb{Z}),\ t \in \mathsf{T}. \tag{14.2}$$

The monodromy of the topological cycle $\delta_b,\ b \in \mathsf{T}$ along any path in T which connects b to $t \in \mathsf{T}$, is δ_t. Using the Veronese embedding we can regard X_t as a hyperplane section of $\mathbb{P}^3$. By Proposition 6.5, δ_b is in the image of the intersection map $H_4(\mathbb{P}^3, \mathbb{Z}) \to H_2(X, \mathbb{Z})$. The homology group $H_4(\mathbb{P}^3, \mathbb{Z})$ is generated by the homology class of a linear $\mathbb{P}^2$ and so, for some $a \in \mathbb{Z}$ the homology class of $\sum_{s \in \pi^{-1}(t)} C_s - a \cdot \mathbb{P}^2 \cap X_t$ in $H_2(X_t, \mathbb{Z})$ is zero. Since $H^1(X_t, \mathscr{O}_{X_t}) = 0$, we can use Theorem 9.2 and conclude that $\sum_{s \in \pi^{-1}(t)} C_s$ is an intersection of X_t with another surface. In particular, $a \neq 0$ and so $\delta_t \neq 0$.

Let $V \subset H_2(X_t, \mathbb{Z})$ be generated by vanishing cycles. For any $\delta_s,\ \ s \in \ \pi^{-1}(t)$ as above by Theorem 6.5 and Picard-Lefschetz formula we know that either δ_s is invariant under the monodromy group or the action of the monodromy group on δ_s generates V. In the first case as above we conclude that C_s is a hypersurface intersection of X_t. In the second case the action of the monodromy on δ_s generates the whole $H_2(X_t, \mathbb{Z})$ and hence $W = H_2(X_t, \mathbb{Z})$. This is a contradiction. $\square$

14.4 Proof of Noether-Lefschetz theorem using integrals

Our second proof of Noether-Lefschetz theorem is of local nature as it just uses a small neighborhood of a point $t \in \mathsf{T}$ such that X_t is smooth, and the Taylor expansion

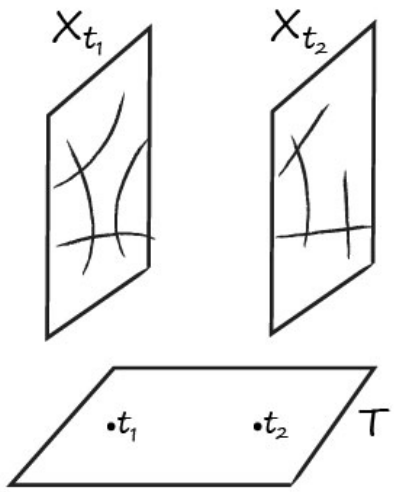

Fig. 14.2 Monodromy of curves.

of periods in its neighborhood, see Theorem 13.2. Actually, we will only use the first order approximation of periods which is formulated under the name "infinitesimal variation of Hodge structures" (IVHS) by P. Griffiths and his school, see [CGGH83]. For simplicity, we will only work with the Fermat point $0 \in \mathsf{T}$, however, the whole argument can be reproduced in general, see Exercise 14.1.

Let us consider the modifications 1,2 of $\pi : \Sigma \to \mathsf{T}$ in §14.3. Over T we have a $\deg(\pi)$ families of cycles and we look at only one of these families $\delta_t \in H_2(X_t, \mathbb{Z})$, $t \in (\mathsf{T}, 0)$. This time we do not use the global property of δ_t, instead, we use the fact that the following local holomorphic functions are identically zero:

$$\int_{\delta_t} F_t^2 = 0, \;\; F_t^2 := H^0(X, \Omega^2_{X_t}), \;\; t \in (\mathsf{T}, 0), \tag{14.3}$$

where $H^0(X, \Omega^2_{X_t})$ is the set of all global 2-forms on X_t. This is a piece of the Hodge filtration $F_t^2 \subset F_t^1 \subset F_t^0 := H^2_{\mathrm{dR}}(X_t)_0$. We make derivations of this equality with respect to parameters in T and we conclude that

$$\int_{\delta_t} F_t^2 + \Theta_{\mathsf{T}} F_t^2 = 0, \tag{14.4}$$

where Θ_{T} is the set of vector fields in T. By Griffiths transversality

$$F_t^2 + \Theta_{\mathsf{T}} F_t^2 \subset F_t^1 \tag{14.5}$$

and we claim that over $t = 0$ the equality happens. Since F_t^1 and its complex conjugate generate the whole primitive cohomology $H^2_{\mathrm{dR}}(X_t)_0$, we conclude that δ_t is in the one dimensional subspace of $H_2(X_t, \mathbb{Z})$ generated by $\mathbb{P}^2 \cap X_t$. This is the same as to say that an integer multiple of C_t is homologous to an integer multiple of $\mathbb{P}^2 \cap X_t$. The argument is now similar to the end of the topological proof in §14.3.

Now we prove that in (14.5) the equality happens for $t = 0$. For this we work in the affine chart $w = 1$ and use the notations I, A_β etc., of the tame polynomial $f := x^d + y^d + z^d - 1$. We have explicit expression for generators of F_t^i as in Definition 11.2. In the following it is assumed that $t = 0$. The equalities (14.3) and (14.4) turn out to be:

$$\int_{\delta_t} \frac{x^\beta dx}{f} = 0, \;\; \beta \in I, \;\; A_\beta < 1$$

$$\int_{\delta_t} \frac{x^{\beta+\alpha} dx}{f^2} = 0, \; \beta \in I, \;\; A_\beta < 1, \; \alpha \in J$$

respectively, where J is the set of triples of integers $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $0 \le \alpha_i \le d-2$. Any $\check{\beta} \in I$ with $1 < A_{\check{\beta}} < 2$ can be written as a sum $\beta + \alpha$ with α and β as above and so the equality in (14.5) follows. $\square$

14.5 Exercises

14.1. Rewrite the proof in §14.4 for an arbitrary point t of T. In this case X_t is a smooth surface of degree d.

14.2 (Surfaces with Picard number equal to one). Both proofs of the Noether-Lefschetz theorem do not give any explicit example of a surface with Picard number equal to one. Actually, if for a given surface X one proves that its Picard number is one (which is a difficult problem and there is no general method to prove such kind of statements) then this is already a proof of Noether-Lefschetz theorem. In [Shi81a] T. Shioda proves that for $d \ge 5$ a prime number the following surface in $\mathbb{P}^3$

$$w^d + xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$$

has the Picard number equal to one. Give an sketch of the proof.

Chapter 15
Fermat varieties

Cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet, (Fermat's last theorem in latin, see [Nag51], p. 252 or [Oes88] page 165).

15.1 A brief history

We are going to discuss Hodge cycles for the Fermat variety $X_n^d \subset \mathbb{P}^{n+1}$ of dimension n and degree d:

$$X = X_n^d : \quad x_0^d + x_1^d + \cdots + x_{n+1}^d = 0. \tag{15.1}$$

It is a generalization of the classical Fermat curve $x^d + y^d - z^d = 0$ in $\mathbb{P}^2$. Taking this as a Diophantine equation we have the celebrated *Fermat last theorem* and this is the origin of this naming. Since the main emphasis of the present text is the study of multiple integrals, and these are usually written using affine varieties, we will take the affine chart $x_0 = 1$ of the Fermat variety, and we will consider a more general variety given by

$$L = L_n^m : \quad x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1. \tag{15.2}$$

The original Fermat variety (15.1) can be recovered form L with $d := m_1 = m_2 = \cdots = m_{n+1}$, the projectivization of L using the variable x_0 and then performing the linear transformation $x_0 \to \zeta_{2d} x_0$. The variety (15.2) as a Diophantine equation over finite field has a long history for which we refer the reader to the Weil's article [Wei49]. In this case we may insert coefficients near each monomial $x_i^{m_i}$ which in the complex context do not produce new varieties. The analysis of the zeta function of (15.2) in [Wei49] produced the so-called *Weil conjectures* which ultimately solved by P. Deligne. A detailed topological study of (15.2) is due to F. Pham in [Pha65].

In the present text, we study the Fermat variety for two main reasons. First, we are interested in studying deformations of Hodge cycles of the Fermat variety. Second, the Hodge conjecture is still unsolved for Fermat varieties of arbitrary degree, for instance for Fermat fourfolds of degree 12.

The present chapter is supposed to be independent of the previous ones, as far as the reader take many statements within the chapter as definitions. For instance, our algorithm to compute the $\mathbb{Z}$-module of Hodge cycles can be taken as its definition. In the literature we mainly find Shioda's seminal works [Shi79b, Shi79a, Shi81b] on Hodge cycles of the Fermat variety (15.1). This is basically based on the action of μ_d^{n+2} on the Fermat variety and the decomposition of its cohomology into one dimensional pieces. Our approach is different, because it is based on the integral computation in Proposition 15.1. It writes Hodge cycles in terms of Lefschetz's vanishing cycles, and so, it is purely topological. We aim to generalize this approach to arbitrary hypersurfaces in the forthcoming works, using the Gauss-Manin connection of families of hypersurfaces. That is why we avoid to use the automorphisms of the Fermat variety.

15.2 Multiple integrals for Fermat varieties

Let $\Gamma(t)$ be the classical Γ-function

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x}\, dx,$$

$$\mathrm{Re}(t) > 0.$$

This is not a multiple integral in the sense that its integrand is not an algebraic function in x (it involves the exponential function). However, the beta function

$$B(a,b) := \int_0^1 s^{a-1}(1-s)^{b-1} ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{15.3}$$

$$\mathrm{Re}(a), \mathrm{Re}(b) > 0,$$

with a and b rational numbers, will appear in a natural way in our study of multiple integrals. We will also need its multi parameter version:

$$\begin{aligned} B(a_1, a_2, \cdots, a_{n+1}) &:= \int_{\Delta_n} t_1^{a_1-1} t_2^{a_2-1} \cdots t_{n+1}^{a_{n+1}-1} dt_1 \wedge dt_2 \cdots \wedge dt_n \\ &= \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{n+1})}{\Gamma(a_1+a_2+\cdots+a_{n+1})}, \end{aligned} \tag{15.4}$$

$$\mathrm{Re}(a_1), \mathrm{Re}(a_2), \cdots, \mathrm{Re}(a_{n+1}) > 0,$$

where

$$\Delta_n := \left\{ (t_1, t_2, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; t_i \geq 0, \quad \sum_{i=1}^{n+1} t_i = 1 \right\}$$

is the standard n-dimensional simplex (n-simplex). Note that the integration domain Δ_n is used in the definition of singular homology and cohomology.

Now, we are going to explain how the values of B appear in the computation of periods of the Fermat variety. Let

$$I_{m_i} := \{0, 1, 2, \ldots, m_i - 2\}$$

and let $\zeta_{m_i} = e^{\frac{2\pi\sqrt{-1}}{m_i}}$ be an m_i-th primitive root of unity. For $\beta \in I := I_1 \times I_2 \times \cdots \times I_{n+1}$ and $a \in \{0,1\}^{n+1}$ let

$$\Delta_{\beta+a} : \Delta_n \to L$$

$$\Delta_{\beta+a}(t) = \left(t_1^{\frac{1}{m_1}} \zeta_{m_1}^{\beta_1+a_1},\ t_2^{\frac{1}{m_2}} \zeta_{m_2}^{\beta_2+a_2},\ \ldots, t_{n+1}^{\frac{1}{m_{n+1}}} \zeta_{m_{n+1}}^{\beta_{n+1}+a_{n+1}} \right),$$

where for a positive number r and a natural number s, $r^{\frac{1}{s}}$ is the unique positive s-th root of r. The formal sum

$$\delta_\beta := \sum_a (-1)^{\sum_{i=1}^{n+1}(1-a_i)} \Delta_{\beta+a} \tag{15.5}$$

induces a non-zero element in $H_n(L, \mathbb{Z})$ (see Exercise 4.2 in Chapter 4). In Chapter 10 we have used the differential n-form $\eta_{\beta'}$ and $(n+1)$-forms $\omega_{\beta'}$ in $\mathbb{C}^{n+1}$. Since in this chapter we will never use $\omega_{\beta'}$, and this is our favorite notation for an integrand, we rename $\eta_{\beta'}$ to $\omega_{\beta'}$:

$$\omega_{\beta'} := x^{\beta'} \left(\sum_{i=1}^{n+1} \frac{(-1)^{i-1}}{m_i} x_i dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \right), \tag{15.6}$$

where $x^{\beta'} := x_1^{\beta'_1} x_2^{\beta'_2} \cdots x_{n+1}^{\beta'_{n+1}}$ is a monomial. Historically, differential forms are invented to be integrated over topological cycles. We are also going to integrate $\omega_{\beta'}$ over δ_β. The whole discussion of the present chapter is motivated from the following simple integral computation. It is first done in Deligne's lectures notes in [DMOS82] for the classical Fermat variety with $m_1 = m_2 = \cdots = m_{n+1}$.

Proposition 15.1 *We have*

$$\int_{\delta_\beta} \omega_{\beta'} = \frac{(-1)^n}{\prod_{i=1}^{n+1} m_i} \cdot B\left(\frac{\beta'_1+1}{m_1}, \frac{\beta'_2+1}{m_2}, \cdots, \frac{\beta'_{n+1}+1}{m_{n+1}} \right) \cdot \tag{15.7}$$

$$\prod_{i=1}^{n+1} \left(\zeta_{m_i}^{(\beta_i+1)(\beta'_i+1)} - \zeta_{m_i}^{\beta_i(\beta'_i+1)} \right). \tag{15.8}$$

We have split the result of our integral computation into the apparently transcendental part (15.7) and algebraic part (15.8).

Proof. The proof essentially uses the integration formula (15.4):

$$\begin{aligned}
\int_{\Delta_{\beta+a}} \omega_{\beta'} &= \int_{\Delta_{\beta+a}} x^{\beta'} \left(\sum_{i=1}^{n+1} \frac{(-1)^{i-1}}{m_i} x_i dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \right) \\
&= \left(\prod_{j=1}^{n+1} \zeta_{m_j}^{(\beta'_j+1)(\beta_j+a_j)} \right) \\
&\quad \cdot \sum_{i=1}^{n+1} \frac{(-1)^{i-1}}{m_i} \int_{\Delta_n} (t_1^{\frac{\beta'_1}{m_1}} \cdots t_{n+1}^{\frac{\beta'_{n+1}}{m_{n+1}}}) t_i^{\frac{1}{m_i}} dt_1^{\frac{1}{m_1}} \wedge \cdots \wedge \widehat{dt_i^{\frac{1}{m_i}}} \wedge \cdots \wedge dt_{n+1}^{\frac{1}{m_{n+1}}} \\
&= \left(\prod_{j=1}^{n+1} \zeta_{m_j}^{(\beta'_j+1)(\beta_j+a_j)} \right) \left(\prod_{j=1}^{n+1} \frac{1}{m_j} \right) \\
&\quad \cdot (-1)^n B(\frac{\beta'_1+1}{m_1}, \frac{\beta'_2+1}{m_2}, \cdots, \frac{\beta'_{n+1}+1}{m_{n+1}}) \sum_{i=1}^{n+1} \frac{\frac{\beta_i+1}{m_i}}{\sum_{j=1}^{n+1} \frac{\beta_j+1}{m_j}}.
\end{aligned}$$

In the last equality we have used $\Gamma(a+1) = a\Gamma(a)$ and the consequent equality for the beta function. $\square$

The periods of the one parameter family $g = s$, where

$$g := x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} \tag{15.9}$$

turns out to be given by $s^{\sum_{j=1}^{n+1} \frac{\beta'_j+1}{m_j}} \int_{\delta_\beta} \omega_{\beta'}$. For $s = -1$ we recover the periods of the classical Fermat variety (16.37). An immediate consequence of Proposition 15.1 is

$$\overline{\omega_\beta} = \omega_{m-2-\beta} \quad \text{in } H^n_{\mathrm{dR}}(L), \tag{15.10}$$

where $m-2-\beta := (m_1-2-\beta_1, m_2-2-\beta_2, \cdots)$. Note that $A_{m-2-\beta} = n+1-A_\beta$.

15.3 De Rham cohomology of affine Fermat varieties

Let us recall the polynomial g in (15.9). The Milnor vector space

$$\mathsf{V}_g := \mathbb{C}[x]/\mathrm{jacob}(g)$$

has the following basis of monomials

$$x^\beta, \ \beta \in I, \tag{15.11}$$

where

$$I := \{\beta \in \mathbb{Z}^{n+1} \mid 0 \le \beta_i \le m_i - 2\}.$$

We define

$$\mu := \dim_{\mathbb{C}} \mathsf{V}_g = \Pi_{i=1}^{n+1}(m_i - 1)$$

and

$$A_\beta := \sum_{i=1}^{n+1} \frac{(\beta_i + 1)}{m_i}, \quad \beta \in I,$$

for which we have

$$0 < A_\beta < n+1, \quad \forall \beta \in I.$$

We also define d to be the lowest common multiple of m_i's:

$$d := [m_1, m_2, \ldots, m_{n+1}]$$

and

$$\nu_i := \frac{[m_1, m_2, \ldots, m_{n+1}]}{m_i}.$$

The number μ is the Milnor number of the singularity $g = 0$, see for instance [AGZV88]. In the weighted ring $\mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$, $\deg(x_i) := \nu_i$, g is a homogeneous polynomial of degree d. A particular case of Theorem 10.1 is the following proposition:

Proposition 15.2 *Let us consider the affine variety*

$$L := \{g = 1\} \subset \mathbb{C}^{n+1}$$

The set of differential forms

$$\omega_\beta, \ \beta \in I \tag{15.12}$$

restricted to L form a basis of the n-th de Rham cohomolgy $H^n_{\mathrm{dR}}(L)$ of L.

A reader who does not know what is de Rham cohomology may take $H^n_{\mathrm{dR}}(L)$ the $\mathbb{C}$-vector space generated by ω_β, $\beta \in I$ and proceed reading the text without loss. A reader who does know the de Rham cohomology using C^∞ forms and the duality theorem between the de Rham cohomology and singular homology might try to give an elementary solution to the problem in §21.4. This together with the fact that δ_β, $\beta \in I$ form a basis of the $\mathbb{Z}$-module $H_n(L, \mathbb{Z})$ (see Exercise 15.3) gives an alternative proof to Proposition 15.2.

The de Rham cohomology $H^n_{\mathrm{dR}}(L)$ carries two natural filtrations:

$$0 =: W_{n-1} \subset W_n \subset W_{n+1} := H^n_{\mathrm{dR}}(L),$$

$$0 = F^{n+1} \subset F^n \subset F^{n-1} \subset \cdots \subset F^0 := H^n_{\mathrm{dR}}(L)$$

called respectively the weight and Hodge filtration. The following proposition is a particular case of Theorem 11.1. It can be served as the definition of the Hodge and

weight filtration of $H^n_{\mathrm{dR}}(L)$, if one is not familiar with the original definitions and one does not want to get bothered with the content of Chapter 10 and Chapter 11.

Proposition 15.3 *A basis of W_n is given by*

$$\omega_\beta, \quad A_\beta \notin \mathbb{N}, \ \ \beta \in I \tag{15.13}$$

and a basis of $\frac{W_{n+1}}{W_n}$ is given by

$$\omega_\beta, \quad A_\beta \in \mathbb{N}, \ \ \beta \in I. \tag{15.14}$$

A a basis of the $(n+1-k)$-th piece F^{n+1-k} of the Hodge filtration of $H^n_{\mathrm{dR}}(L)$ is given by

$$\omega_\beta, \ \ A_\beta \leq k, \ \ \beta \in I. \tag{15.15}$$

For a reformulation of Proposition 15.3 for $n = 1$ and using classical terminology of differential forms of the first, second and third type see Exercise 11.1. We say that the basis $\omega_\beta, \ \beta \in I$ of $H^n_{\mathrm{dR}}(L)$ is compatible with both the weight and Hodge filtrations. It follows that

$$\omega_\beta, \quad A_\beta = k, \ \ \beta \in I, \tag{15.16}$$

form a basis of

$$\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}^W_{n+1}H^n_{\mathrm{dR}}(L) := \frac{F^{n+1-k}+W_n}{F^{n+2-k}+W_n},$$

and

$$\omega_\beta \in I, \quad k-1 < A_\beta < k. \tag{15.17}$$

form a basis of

$$\mathrm{Gr}_F^{n+1-k}\mathrm{Gr}^W_n H^n_{\mathrm{dR}}(L) := \frac{F^{n+1-k}\cap W_n}{F^{n+2-k}\cap W_n}.$$

The notation Gr was already in use in Chapter 11.

15.4 Hodge numbers

We can use Proposition 15.3 and we can compute the Hodge numbers of the underlying varieties. First, note that the set I is invariant under the transformation

$$\beta \mapsto m-\beta-2 := (m_1-\beta_1-2, m_2-\beta_2-2, \cdots),$$

which gives us the following identity:

$$A_{m-\beta-2} = n+1-A_\beta \ \ \forall \beta \in I.$$

This, in turn, gives us a symmetric sequence of numbers

$$h_0^{0,n}, \ h_0^{1,n-1}, \ \cdots, h_0^{n-1,1}, \ h_0^{n,0}$$

$$h_0^{0,n-1},\ h_0^{1,n-2},\ \cdots, h_0^{n-2,1},\ h_0^{n-1,0}$$

where

$$h_0^{k-1,n-k+1} := \#\{\beta \in I \mid k-1 < A_\beta < k\}, \tag{15.18}$$

$$h_0^{k-1,n-k} := \#\{\beta \in I \mid A_\beta = k\}. \tag{15.19}$$

These numbers are related to the classical Hodge numbers of the weighted projective varieties X and Y in (16.37) and (16.36) of Exercise 16.1, respectively. Note that these varieties are in general singular. We have

$$h^{i,j} = h_0^{i,j}, \quad \text{for } i \neq j. \tag{15.20}$$

For n even (resp. odd) $h^{\frac{n}{2},\frac{n}{2}} - h_0^{\frac{n}{2},\frac{n}{2}}$ (resp. $h^{\frac{n}{2}-1,\frac{n}{2}-1} - h_0^{\frac{n}{2}-1,\frac{n}{2}-1}$) depends on the desingularization of X (resp. Y). The case $\nu_0 = \nu_1 = \cdots = \nu_{n+1} = 1$ is easier to describe. In this case both Y and X are smooth projective varieties and (15.19) and (15.18) are the Hodge numbers of the primitive cohomologies $H^{n-1}_{\mathrm{dR}}(Y)_0$ and $H^n_{\mathrm{dR}}(X)_0$. In this case, for n even (resp. odd) $h^{\frac{n}{2},\frac{n}{2}} = h_0^{\frac{n}{2},\frac{n}{2}} + 1$ (resp. $h^{\frac{n}{2}-1,\frac{n}{2}-1} = h_0^{\frac{n}{2}-1,\frac{n}{2}-1} + 1$). Here are some tables of Hodge numbers of weighted hypersurfaces obtained by Proposition 15.3. A smooth surface of degree 4 is a K3 surface and its

d	1	2	3	4	5	6	7	8	9	10
$h^{0,2}$	0	0	0	1	4	10	20	35	56	84
$h^{1,1}$	1	2	7	20	45	86	147	232	345	490
$h^{2,0}$	0	0	0	1	4	10	20	35	56	84

Table 15.1 Hodge numbers of smooth surfaces of degree d in $\mathbb{P}^3$.

Hodge numbers 1,20,1 appear in the above table. The Hodge numbers $1, 101, 101, 1$

d	1	2	3	4	5	6	7	8	9	10
$h^{0,3}$	0	0	0	0	1	5	15	35	70	126
$h^{1,2}$	0	0	5	30	101	255	540	1015	1750	2826
$h^{2,1}$	0	0	5	30	101	255	540	1015	1750	2826
$h^{3,0}$	0	0	0	0	1	5	15	35	70	126

Table 15.2 Hodge numbers of smooth hypersurfaces of degree d in $\mathbb{P}^4$.

of a smooth quintic $X \subset \mathbb{P}^4$ are the most famous Hodge numbers in the above table. Such an X is a Calabi-Yau threefold, and in the mirror symmetry of string theory it is used as the A-model variety, for further details see [Mov17a] and the references therein. We will focus on sextic fourfold hypersurfaces. The corresponding Hodge numbers are $1, 426, 1752, 426, 1$. It is interesting to fix the degree and let the dimension grow. Below, we have computed Hodge numbers of cubic and quartic hypersurfaces in $\mathbb{P}^{n+1}$.

d	1	2	3	4	5	6	7	8	9	10
$h^{0,4}$	0	0	0	0	0	1	6	21	56	126
$h^{1,3}$	0	0	1	21	120	426	1161	2667	5432	10116
$h^{2,2}$	1	2	21	142	581	1752	4333	9332	18153	32662
$h^{3,1}$	0	0	1	21	120	426	1161	2667	5432	10116
$h^{4,0}$	0	0	0	0	0	1	6	21	56	126

Table 15.3 Hodge numbers of smooth hypersurfaces of degree d in $\mathbb{P}^5$.

d	3	6	9	12	15	18	21	24	27	30
$h^{0,2}$	0	1	11	39	94	185	321	511	764	1089
$h^{1,1}_0$	0	19	92	255	544	995	1644	2527	3680	5139
$h^{2,0}$	0	1	11	39	94	185	321	511	764	1089

Table 15.4 $n = 2, v_0 = v_1 = v_2 = 1,\ v_3 = 3.$

n	
1	1, 1
2	0, 7, 0
3	0, 5, 5, 0
4	0, 1, 21, 1, 0
5	0, 0, 21, 21, 0, 0
6	0, 0, 8, 71, 8, 0, 0
7	0, 0, 1, 84, 84, 1, 0, 0
8	0, 0, 0, 45, 253, 45, 0, 0, 0
9	0, 0, 0, 11, 330, 330, 11, 0, 0, 0
10	0, 0, 0, 1, 220, 925, 220, 1, 0, 0, 0

Table 15.5 Hodge numbers of smooth cubic hypersurfaces in $\mathbb{P}^{n+1}$.

n	
1	3, 3
2	1, 20, 1
3	0, 30, 30, 0
4	0, 21, 142, 21, 0
5	0, 7, 266, 266, 7, 0
6	0, 1, 266, 1108, 266, 1, 0
7	0, 0, 156, 2304, 2304, 156, 0, 0
8	0, 0, 55, 2850, 8954, 2850, 55, 0, 0
9	0, 0, 11, 2277, 19855, 19855, 2277, 11, 0, 0
10	0, 0, 1, 1221, 28314, 73790, 28314, 1221, 1, 0, 0

Table 15.6 Hodge numbers of smooth quartic hypersurfaces in $\mathbb{P}^{n+1}$.

Let us introduce another number which will be interpreted later as the dimension of the moduli space of hypersurfaces. This is namely

$$r := \#\left\{\beta \in I \,\middle|\, \sum_{i=0}^{n+1} \frac{1}{m_i} < A_\beta < 1 + \sum_{i=0}^{n+1} \frac{1}{m_i}\right\} \tag{15.21}$$

where m_0 is the lowest common multiple of $m_1, m_2, \ldots, m_{n+1}$. There are some interesting cases where this number is a Hodge number. This is when $k := \sum_{i=0}^{n+1} \frac{1}{m_i}$ is a

natural number. In this case

$$r = h_0^{n-k,k}$$

and

$$h_0^{n,0} = \cdots = h_0^{n-k-2,k+2} = 0, \;\; h_0^{n-k-1,k+1} = 1.$$

These varieties must be considered as generalizations of the classical smooth Calabi-Yau hypersurfaces in $\mathbb{P}^{n+1}$:

$$d := m_0 = m_1 = \cdots = m_{n+1} = n+2, \;\; r = h_0^{n-1,1}, \;\; h_0^{n,0} = 1.$$

The Hodge numbers $h_0^{p,q}$, $p+q=n$ of a smooth hypersurface given by $\{F(x) = 0\} \subset \mathbb{P}^{n+1}$, F a homogeneous polynomial of degree d in $x = (x_0, x_1, ..., x_{n+1})$, is the dimension of the homogeneous piece of degree $(q+1)d-(n+2)$ of the Jacobian ring $\mathbb{C}[x]/\text{jacob}(F)$. If we take F to be Fermat, it is an elementary problem to show that $h^{n,0} < h^{n-1,1} < \cdots < h^{\frac{n}{2}+1,\frac{n}{2}-1} < h_0^{\frac{n}{2},\frac{n}{2}}$. If $h^{p,q} = 1$ with $p > q$ then $(q+1)d-(n+2) = 0$ which implies that $(q+2)d-(n+2) = d$, and so $h_0^{p-1,q+1}$ is the dimension of the degree d piece of the ring $\mathbb{C}[x]/\text{jacob}(F)$. This in turn is the dimension of the moduli space of hypersurfaces of degree d and dimension n, let us call it r. If $h^{p,q} \neq 0,1$ and $q < \frac{n}{2}-1$ then $0 < (q+1)d-(n+2)$ and hence $d < (q+2)d-(n+2) \leq \frac{n}{2}d-(n+2)$ which by the increasing property of Hodge numbers we get $r < h^{\frac{n}{2}+1,\frac{n}{2}-1}$. One may try to get similar statements for the equivariant part of the cohomology of a hypersurface invariant by a discrete group. "In SGA7 XI 2.5, [see [Del73]] I had checked that for any smooth complete intersection in projective space, the Hodge numbers are increasing up to the middle [...] as the occurrence of the Jacobian ring (in which both moduli and Hodge numbers can be read) remains mysterious to me, I have no clue as to how to generalize your argument [relating the dimension of the moduli with Hodge numbers] to complete intersections", (P. Deligne, personal communication, November 05, 2018). For the original computation of Hodge numbers of smooth hypersurfaces by Hirzebruch see [Hir95].

15.5 Hodge conjecture for the Fermat variety

In this section we state the Hodge conjecture for the Fermat variety. For simplicity, we consider the classical Fermat variety with $d := m_1 = m_2 = \cdots = m_{n+1}$.

Let $X = X_n^d$ be the classical Fermat variety with n an even number, and let $Y = X_{n-1}^d$ be the Fermat variety at infinity, that is, $Y := X \backslash L$. We have the short exact sequence $H_n(L,\mathbb{Z}) \to H_n(X,\mathbb{Z}) \xrightarrow{\tau} H_{n-2}(Y,\mathbb{Z}) \to 0$ and by Lefschetz hyperplane theorem we know that $H_{n-2}(Y,\mathbb{Z}) \cong \mathbb{Z}$. A generator of $H_n(Y,\mathbb{Q}) \cong H_{n-2}(Y,\mathbb{Q})$ is given by the topological class $[Z_\infty]$ of the algebraic cycle $Z_\infty : x_0 = x_1 \cdots = x_{\frac{n}{2}-1} = 0$ in Y. For $\delta \in H_n(X,\mathbb{Z})$, $\tau(\delta)$ is the intersection of δ with Y, and hence, $\tau(\delta) = 0$ if and only if $\langle \delta, [Z_\infty] \rangle = 0$. Let Z be an irreducible subvariety of X of dimension $\frac{n}{2}$.

We define

$$\check{Z} := \frac{(Z_\infty \cdot Z_\infty)Z - (Z \cdot Z_\infty)Z_\infty}{\gcd(Z_\infty \cdot Z_\infty, Z \cdot Z_\infty)}, \tag{15.22}$$

which is characterized by the fact that $\langle \check{Z}, Z_\infty \rangle = 0$. This implies that $\check{Z}$ lies in the kernel of τ, and so its topological class comes from a cycle in L. We fix such a cycle $\delta \in H_n(L, \mathbb{Z})$.

Proposition 15.4 *We have*

$$\int_\delta \omega_\beta = 0, \ \ \forall \beta, \ \ A_\beta < \frac{n}{2}, \ \ A_\beta \notin \mathbb{N}. \tag{15.23}$$

This proposition follows from Theorem 11.3. We use it in order to announce the Hodge conjecture for the Fermat variety.

Conjecture 15.1 *Let $\delta \in H_n(L, \mathbb{Z})$ be a topological cycle with the property* (15.23). *Then there are integers $a, a_1, a_2, \ldots, a_s$ and irreducible algebraic cycles $Z_1, Z_2, \ldots, Z_s$ of dimension $\frac{n}{2}$ of X such that in $H_n(X, \mathbb{Z})$ we have*

$$a \cdot \delta = a_1[Z_1] + a_2 \cdot [Z_2] + \cdots + a_s \cdot [Z_s].$$

15.6 Hodge cycles of the Fermat variety, I

We are going to analyze the Hodge cycles of the Fermat variety in more details. We use Proposition 15.3 and Proposition 15.1 and we obtain an arithmetic interpretation of Hodge cycles which does not involve any topological argument. In the integration formula in Proposition 15.1 the beta factor

$$\frac{(-1)^n}{\prod_{i=1}^{n+1} m_i} \cdot B\left(\frac{\beta_1' + 1}{m_1}, \frac{\beta_2' + 1}{m_2}, \cdots, \frac{\beta_{n+1}' + 1}{m_{n+1}}\right)$$

can be absorbed into the definition of the differential n-form ω_β' and so this, a priori a transcendental number, will not play a role in the definition of a Hodge cycle. Once Hodge cycles are defined it will play an important role for further characterizations of Hodge cycles in Chapter 16.

For a natural number $a \geq 2$ let $I_a := \{0, 1, 2, \ldots, a-2\}$ and for $2 \leq m_i \in \mathbb{N}, \ \ i = 1, 2, \ldots, n+1$ let

$$I := I_{m_1} \times I_{m_2} \times \cdots \times I_{m_{n+1}}. \tag{15.24}$$

We denote the elements of I by $\beta = (\beta_1, \beta_2, \cdots, \beta_{n+1})$. We consider symbols or variables $\delta_\beta, \ \ \omega_\beta, \ \beta \in I$ and define the following $\mathbb{Z}$-modules

$$\begin{aligned} H_n(L, \mathbb{Z}) &:= \text{ The free } \mathbb{Z}\text{-module generated by } \delta_\beta, \ \ \beta \in I, \\ H^n_{\mathrm{dR}}(L) &:= \text{ The free } \mathbb{Z}\text{-module generated by } \omega_\beta, \ \ \beta \in I. \end{aligned}$$

We define

$$\int_{\delta_\beta} \omega_{\beta'} := \prod_{i=1}^{n+1} \left(\zeta_{m_i}^{(\beta_i+1)(\beta_i'+1)} - \zeta_{m_i}^{\beta_i(\beta_i'+1)} \right), \ \beta, \beta' \in I. \tag{15.25}$$

By $\mathbb{Z}$-linearity we define

$$\int_\delta \omega, \quad \delta \in H_n(L, \mathbb{Z}), \ \omega \in H^n_{\mathrm{dR}}(L).$$

Definition 15.1 Let $A_\beta := \sum_{i=1}^{n+1} \frac{\beta_i+1}{m_i}, \ \beta \in I$. An element $\delta \in H_n(L, \mathbb{Z})$ is called an affine Hodge cycle if

$$\int_\delta \omega_\beta = 0,$$

for all $\beta \in I$ with $A_\beta \notin \mathbb{Z}$ and $A_\beta < \frac{n}{2}$.

Here, *affine* refers to the fact that such a Hodge cycle lives in the affine variety L. Trivial examples of such cycles are given in the following definition.

Definition 15.2 We say that a multiple of $\delta \in H_n(L, \mathbb{Z})$ is at infinity if

$$\int_\delta \omega_\beta = 0,$$

for all $\beta \in I$ with $A_\beta \notin \mathbb{N}$.

As far as cycles at infinity and Hodge cycles are concerned, we can use the decomposition

$$\prod_{i=1}^{n+1} \left(\zeta_{m_i}^{(\beta_i+1)(\beta_i'+1)} - \zeta_{m_i}^{\beta_i(\beta_i'+1)} \right) = \prod_{i=1}^{n+1} \zeta_{m_i}^{\beta_i(\beta_i'+1)} \cdot \prod_{i=1}^{n+1} (\zeta_{m_i}^{\beta_i'+1} - 1)$$

and we can use only the first part in their definition. This might be helpful for computing the $\mathbb{Z}$-module of such cycles.

15.7 Intersection form

In order to compute intersection number of two Hodge cycles we need to compute the intersection form in $H_n(L, \mathbb{Z})$. In the topological context this is done in §7.10. The following definition is then inspired by the content of this section.

Definition 15.3 In the freely generated $\mathbb{Z}$-module $H_n(L, \mathbb{Z})$ we consider the bilinear form $\langle \cdot, \cdot \rangle$ given by the following rules:

$$\langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^n \langle \delta_{\beta'}, \delta_\beta \rangle, \quad \forall \beta, \beta' \in I,$$

$$\langle \delta_\beta, \delta_\beta \rangle = (-1)^{\frac{n(n-1)}{2}}(1+(-1)^n), \quad \forall \beta \in I$$

and

$$\langle \delta_\beta, \delta_{\beta'} \rangle = (-1)^{\frac{n(n+1)}{2}}(-1)^{\Sigma_{k=1}^{n+1}\beta'_k - \beta_k}$$

for those $\beta, \beta' \in I$ such that for all $k = 1, 2, \ldots, n+1$ we have $\beta_k \leq \beta'_k \leq \beta_k + 1$ and $\beta \neq \beta'$. In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle \delta_\beta, \delta_{\beta'} \rangle = 0$.

The above bilinear map corresponds to the intersection map of $H_n(L, \mathbb{Z})$.

Proposition 15.5 *For an element $\delta \in H_n(L, \mathbb{Z})$ we have*

$$\int_\delta \omega_\beta = 0, \ \ \forall \beta \in I \quad \textit{with } A_\beta \notin \mathbb{N}$$

if and only if

$$\langle \delta, \delta_\beta \rangle = 0, \ \forall \beta \in I.$$

Proof. One may try to prove the proposition with elementary methods. Our proof involves the geometry behind it. This is namely Proposition 15.1, Proposition 5.10 and the fact that the bilinear map $\langle \cdot, \cdot \rangle$ corresponds to the intersection map of $H_n(L, \mathbb{Z})$. □

For an elementary exposition of Proposition 15.5 see Problem 21.9 in Chapter 21.

15.8 Hodge cycles of the Fermat variety, II

So far, we have worked on the affine variety L. Let us now define the objects related to the compactification X of L. Recall our notations in §5.10. Recall also from Theorem 5.3 that for $m_1 = m_2 = \cdots = m_{n+1}$ the homology group $H_n(X, \mathbb{Z})$ has no torsions, and so, $H_n(X, \mathbb{Z})_{\check{0}} = H_n(X, \mathbb{Z})_0$. Therefore, we can remove $\check{}$ from all our notations below.

$$H_n(L, \mathbb{Z})_{\check{\infty}} := \left\{ \delta \in H_n(L, \mathbb{Z}) \mid \langle \delta, \delta_\beta \rangle = 0, \ \text{ for all } \beta \in I \right\}, \tag{15.26}$$

$$H_n(X, \mathbb{Z})_{\check{0}} := \frac{H_n(L, \mathbb{Z})}{H_n(L, \mathbb{Z})_{\check{\infty}}}. \tag{15.27}$$

Note that in the geometric context, $H_n(X, \mathbb{Z})_{\check{0}}$ is the free part of the $\mathbb{Z}$-module $H_n(X, \mathbb{Z})_0$, and so over rational numbers we have $H_n(X, \mathbb{Q})_{\check{0}} = H_n(X, \mathbb{Q})_0$. We are in a position to define the Hodge cycles of the the compact variety X. These are used in the announcement of the Hodge conjecture.

Definition 15.4 The space of non-torsion n-dimensional primitive Hodge cycles of the generalized Fermat variety X is by definition the $\mathbb{Z}$-module of affine Hodge cycles modulo those at infinity, that is,

$$\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}} := \frac{\left\{\delta \in H_n(L,\mathbb{Z}) \middle| \int_\delta \omega_\beta = 0,\ \ \forall \beta \in I,\ A_\beta < \frac{n}{2},\ A_\beta \notin \mathbb{Z}\right\}}{\left\{\delta \in H_n(L,\mathbb{Z}) \middle| \int_\delta \omega_\beta = 0,\ \ \forall \beta \in I,\ A_\beta \notin \mathbb{Z}\right\}}. \tag{15.28}$$

Using Proposition 15.5, we know that $\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}} \subset H_n(X,\mathbb{Z})_{\check{0}}$. Note that both of these $\mathbb{Z}$-modules are torsion free. This fact is hidden in our $\check{0}$ notation.

15.9 Period matrix

For a systematic computational study of Hodge cycles we will need to put the data of integrals into a matrix.

Definition 15.5 The period matrix is obtained by integrating the differential forms $\omega_{\beta'}$, $A_{\beta'} \notin \mathbb{Z}$ over the vanishing cycles δ_β:

$$\mathbf{P} := \left[\int_{\delta_\beta} \omega_{\beta'}\right]_{\beta,\beta' \in I,\ A_{\beta'} \notin \mathbb{Z}} \tag{15.29}$$

where β, respectively β', is used for indexing rows, respectively columns.

The period matrix above is a $\mu \times \check{\mu}$ matrix, where

$$\begin{aligned} \mu &:= \#I = (m_1-1)(m_2-2)\cdots(m_{n+1}-1), \\ \check{\mu} &:= \#\left\{\beta' \in I,\ A_{\beta'} \notin \mathbb{Z}\right\}, \end{aligned}$$

and so, it is not quadratic. The main reason for defining this period matrix is that we are mainly interested in a compactification X of L, and $\omega_{\beta'}, \beta' \in I,\ \ A_{\beta'} \notin \mathbb{N}$ form a basis of $H_{\mathrm{dR}}(X)_0$. Sometimes, it is useful to define the full period matrix which contains the data of the periods of the affine variety L, see the problems in §21.4 of Chapter 21.

Remark 15.1 There is no canonical way to choose $\check{\mu}$ elements of δ_β's such that they form a basis of $H_n(X,\mathbb{Z})_0$, or even $H_n(X,\mathbb{Q})_0$. That is why in (15.29) we have integrated over all δ_β's. However, in our particular case of the Fermat variety we have a canonical isomorphism

$$\left\{\delta \in H_n(L,\mathbb{Z}) \middle| \int_\delta \omega_\beta = 0,\ \ \forall \beta \in I,\ \ \text{with } A_\beta \in \mathbb{N}\right\} \cong H_n(X,\mathbb{Z})_0 \tag{15.30}$$

given by the inclusion $L \subset X$. It is enough to verify this isomorphism over $\mathbb{Q}$ (note that $H_n(L,\mathbb{Z})$ and $H_n(X,\mathbb{Z})_0$ are free). The fact that $H_n(L,\mathbb{Q})$ and $H^n_{\mathrm{dR}}(L)$ are dual using integration, implies that the above map is injective. The surjectivity follows from the fact that the $\mathbb{C}$-vector space $W_{\mathbb{C}} \subset H^n_{\mathrm{dR}}(L)$ spanned by ω_β, $\beta \in I, A_\beta \in \mathbb{Z}$ is

defined over $\mathbb{Q}$, that is, there is a subspace $W_{\mathbb{Q}} \subset H^n(L,\mathbb{Q})$ such that $W_{\mathbb{C}} = W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. This, in turn, follows from the fact that $W_{\mathbb{C}}$ is invariant under the Galois action $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Here we use the integration formula (15.25).

15.10 Computing Hodge cycles of the Fermat variety

Let us recall the Hodge numbers h_0^i in §15.4:

$$h_0^i := \#\{\beta \in I \mid A_\beta < n+1-i\} = h_0^{n,0} + h_0^{n-1,1} + \cdots + h_0^{i,n-i}.$$

Let φ be the Euler's totient function. Sometimes it is also called Euler's phi function.

Definition 15.6 For a positive integer d, $\varphi(d)$ is the number of positive integers less than or equal to d that are relatively prime to d. For example $\varphi(1) = 1,\ \varphi(2) = 1, \varphi(3) = 2$ etc.

In the following we will take d to be the lowest common multiple of $m_1, m_2, \ldots, m_{n+1}$, $d := [m_1, m_2, \ldots, m_{n+1}]$. The period matrix $\mathbf{P}$ defined in (15.29) has entries in $\mathbb{Z}[\zeta_d]$, where $\zeta_d := e^{\frac{2\pi i}{d}}$ is the d-th primitive root of unity. Note that all ζ_{m_i}'s are by definition in this ring. For computational purposes, we need to split $\mathbf{P}$ into integer valued matrices. For this reason, the following definitions are useful:

$$\mathbf{P}^j := (1 \cdots \mu) \times (1 \cdots h_0^j) \text{ submatrix of } \mathbf{P}, \tag{15.31}$$

$$\mathbf{P}^j = \sum_{i=0}^{\varphi(d)-1} \mathbf{P}_i^j \zeta_d^i, \tag{15.32}$$

$$\tilde{\mathbf{P}}^j := [\mathbf{P}_0^j, \mathbf{P}_0^j, \cdots, \mathbf{P}_{\varphi(d)-1}^j]. \tag{15.33}$$

The matrices $\mathbf{P}_i^j$ have integral entries and the second equality defines them uniquely. The matrix $\tilde{\mathbf{P}}^j$ is the $\mu \times (\varphi(d) \cdot h_0^j)$ matrix obtained by putting $\mathbf{P}_i^j$'s together in a row, that is, the columns of $\tilde{\mathbf{P}}^j$ are built from the columns of $\mathbf{P}_i^j$'s.

For the study of Hodge cycles, we only need the Hodge number $h_0^{\frac{n}{2}+1}$. In order to compute the $\mathbb{Z}$-module of Hodge cycles we first compute the $\mu \times h^{\frac{n}{2}+1}$ period matrix $\mathbf{P}^{\frac{n}{2}+1}$ obtained by integrating a basis of $F^{\frac{n}{2}+1}$ over vanishing cycles. For an affine Hodge cycle $\delta \in H_n(L,\mathbb{Z})$ we write it in the basis $\delta_\beta,\ \beta \in I$

$$\delta = B \cdot [\delta_\beta],$$

where $[\delta_\beta]$ is a $\mu \times 1$ matrix with entries δ_β and B is a $1 \times \mu$ integer-valued matrix, and we identify δ with B. By definition

$$A\mathbf{P}_i^{\frac{n}{2}+1} = 0, \quad i = 0, 1, \cdots, \varphi(d) - 1, \tag{15.34}$$

or equivalently

$$A\tilde{\mathbf{P}}^{\frac{n}{2}+1} = 0. \tag{15.35}$$

We can write (15.28) in the following format

$$\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}} \cong \frac{\left\{A \in \mathrm{Mat}(\mu \times 1, \mathbb{Z}) \mid A\tilde{\mathbf{P}}^{\frac{n}{2}+1} = 0\right\}}{\left\{A \in \mathrm{Mat}(\mu \times 1, \mathbb{Z}) \mid A\tilde{\mathbf{P}}^{0} = 0\right\}}. \tag{15.36}$$

For $d = m_1 = m_2 = \cdots = m_{n+1}$ the rank of the denominator of (15.36) is well-known, see Exercise 15.5. We point out the following statement.

Proposition 15.6 *Let X_n^d be the Fermat variety of degree d and dimension n and assume that n is even. The $\mathbb{Q}$-vector space of n-dimensional Hodge cycles in $H_n(X_n^d, \mathbb{Q})$ is of codimension* $\mathrm{rank}(\tilde{\mathbf{P}}^{\frac{n}{2}+1})$*, that is,*

$$\mathrm{codim}_{H_n(X_n^d,\mathbb{Q})}\mathrm{Hodge}_n(X_n^d, \mathbb{Q}) = \mathrm{rank}(\tilde{\mathbf{P}}^{\frac{n}{2}+1}).$$

Proof. The dimension of the nominator of (15.28) is $(d-1)^{n+1} - \mathrm{rank}(\tilde{\mathbf{P}}^{\frac{n}{2}+1})$. This minus the rank of the denominator, which is computed in (15.61) in Exercise 15.5, is equal to $\dim(\mathrm{Hodge}_n(X,\mathbb{Q})_0)$. Note that

$$\dim H_n(X,\mathbb{Q}) = \dim H_n(X,\mathbb{Q})_0 + 1 = \sum_{i=0}^{n}(-1)^i(d-1)^{n+1-i} + 1.$$

□

By Proposition 15.6 the rank of the matrix $\tilde{\mathbf{P}}^{\frac{n}{2}+1}$ is the codimension of the space of Hodge cycles.

15.11 Dimension of the space of Hodge cycles

We have now all the ingredients in order to compute the dimension of the space of Hodge cycles over rational numbers, and we plan to do this in this section. This number is important because most of the verifications of the Hodge conjecture is done by computing the dimension of the space of homological classes generated by algebraic cycles, and by verifying that both numbers are equal. The same story over integers is a more delicate issue and it rises new challenges even in the case where the Hodge conjecture is well-known. In this chapter we present the result of some computations for Fermat fourfolds.

Proposition 15.7 *The dimension of the space of four dimensional Hodge cycles in the Fermat variety X_4^d for few d is given in the table below. For instance, we read that the dimension of the space of Hodge cycles in a sextic Fermat fourfold is* 1752*.*

Proof. The proposition is proved using Proposition 15.6 and the computer implementation of the matrix $\tilde{\mathbf{P}}^{\frac{n}{2}+1}$, see §20.4. □

d	1	2	3	4	5	6	7	8	9	10
$\dim_{\mathbb{Q}}(\mathrm{Hodge}_4(X_4^d,\mathbb{Q}))$	1	2	21	142	401	1752	1861	5882	?	?
h^{22}	1	2	21	142	581	1752	4333	9332	18153	32662

Table 15.7 The dimension of the space of Hodge cycle in the Fermat fourfold.

By definition we have

$$\dim_{\mathbb{Q}}\mathrm{Hodge}_n(X,\mathbb{Q}) \leq h^{\frac{n}{2},\frac{n}{2}}(X) \tag{15.37}$$

and so it is natural to see when the upper bound is attained. Analyzing Table 15.7 we get the following:

Definition 15.7 An even dimensional variety X has maximal number of Hodge cycles if

$$\dim_{\mathbb{Q}}\mathrm{Hodge}_n(X,\mathbb{Q}) = h^{\frac{n}{2},\frac{n}{2}}(X).$$

This definition is the generalization of ρ-maximal surfaces discussed in §15.12.

Proposition 15.8 *The Fermat fourfold X_4^d for $d = 3,4,6$ has maximal number of Hodge cycles.*

Proof. This follows from Table (15.7). □

The above Proposition for $d = 3,4$ appears in [Bea14] Proposition 11. Again, Shioda's work on the Hodge cycles of Fermat varieties plays an essential role.

For $d \geq 9$ we were not able to compute $\dim_{\mathbb{Q}}\left(\mathrm{Hodge}_4(X_4^d,\mathbb{Q})\right)$. So far, we were discussing the computational aspects of the space of Hodge cycles. It is possible to state some theoretical results too. Here is one example which is inspired by a personal communication with P. Deligne (February 17, 2016).

Theorem 15.1 *The dimension of the space of Hodge cycles for the generalized Fermat variety X satisfies*

$$h^0 - \varphi(d)\left(\frac{h^0 - h^{\frac{n}{2},\frac{n}{2}}}{2}\right) \leq \dim(\mathrm{Hodge}_n(X,\mathbb{Q})) \leq h^{\frac{n}{2},\frac{n}{2}}, \tag{15.38}$$

where $h^0 := \dim H_n(X,\mathbb{Q})$, $h^{\frac{n}{2},\frac{n}{2}}$ is the middle Hodge number of X and φ is the Euler's phi function.

Proof. It it enough to prove the theorem for the space of primitive Hodge cycles in which case h^0 and $h^{\frac{n}{2},\frac{n}{2}}$ are replaced with h_0^0 and $h_0^{\frac{n}{2},\frac{n}{2}}$, respectively.

Proof of the first inequality. Recall our computational approach to Hodge cycles in §15.9. The dimension of the space of affine Hodge cycles reaches its minimum if the columns of $\tilde{\mathbf{P}}^{\frac{n}{2}+1}$ are linearly independent over $\mathbb{Q}$. In this case the codimension of affine Hodge cycles in $H_n(L,\mathbb{Q})$ is $\varphi(d)\cdot h^{\frac{n}{2}+1}$. Since cycles at infinity are also affine Hodge cycles, we conclude that the codimension of primitive Hodge cycles in $H_n(X,\mathbb{Q})_0$ is $\varphi(d)\cdot h^{\frac{n}{2}+1}$.

Proof of the second inequality. It is enough to prove that the $\mathbb{Q}$-vector space generated by the columns of $\tilde{\mathbf{P}}^{\frac{n}{2}+1}$ is of dimension $\geq 2h^{\frac{n}{2}+1}$. This follows from the fact that $\omega_\beta,\ \overline{\omega_\beta},\ \ \beta \in I,\ A_\beta < \frac{n}{2}$ are $\mathbb{C}$-linearly independent in $H^n_{\mathrm{dR}}(L)$. $\square$

Corollary 15.1 *Cubic, quartic and sextic Fermat hypersurfaces in $\mathbb{P}^{n+1}$ for all even n have maximal number of Hodge cycles.*

Proof. For a d with $\varphi(d) = 2$ this follows from Theorem 15.1. It can be easily seen that only $d = 3,4,6$ satisfy this property. $\square$

15.12 A table of Picard numbers for Fermat surfaces

In this section we consider the Fermat surface

$$X : x^d + y^d + z^d + w^d = 0. \tag{15.39}$$

The space of Hodge cycles for surfaces in $\mathbb{P}^3$ is the same as the Picard or Neron-Severi group discussed in §9.3. The dimension of the space of Hodge cycles in this case is known as the Picard number.

For a surface in $\mathbb{P}^3$, the Hodge numbers $h^{20} = h^{02},\ h^{11} = h^{11}_0 - 1$ of X and r are given by:

$$h^{20} = \binom{d-1}{3} = \frac{1}{6}d^3 - d^2 + \frac{11}{6}d - 1, \tag{15.40}$$

$$h^{11} = (d-1)^3 - (d-1)^2 + (d-1) - 2\cdot\binom{d-1}{3} + 1 \tag{15.41}$$

$$= \frac{2}{3}d^3 - 2d^2 + \frac{7}{3}d, \tag{15.42}$$

$$r = \binom{d+3}{3} - 16 = \frac{1}{6}d^3 + d^2 + \frac{11}{6}d - 15,\ \ d \geq 3. \tag{15.43}$$

In particular, the number of integrals used in (9.6) is $\binom{d-1}{3}$. Moreover, the Picard rank of X is given by

$$\rho(X) = \mathrm{rank}(\mathrm{Hodge}_2(X,\mathbb{Z})_0) + 1. \tag{15.44}$$

For $d = 1,2,3$ we have $\rho(X) = h^{11} = h^{11}_0 + 1 = 1,2,7$, respectively. We have

$$\mathrm{NS}(X)_0 \cong \frac{\left\{\delta \in H_n(L,\mathbb{Z}) \middle| \int_\delta \omega_\beta = 0,\ \ \forall\beta \in I,\ A_\beta < 1\right\}}{\left\{\delta \in H_n(L,\mathbb{Z}) \middle| \int_\delta \omega_\beta = 0,\ \ \forall\beta \in I,\ A_\beta \neq 1,2\right\}}.$$

We have used this isomorphism and we have computed the Picard rank of many Fermat surfaces. These are gathered in Table 15.12. The number $\rho_{\mathbb{P}^1}$ is explained

in §16.7. In this table we read that the Picard number of the quintic Fermat surface

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
h^{20}	0	0	0	1	4	10	20	35	56	84	120	165	220	286
h^{11}	1	2	7	20	45	86	147	232	345	490	671	892	1157	1470
ρ	1	2	7	20	37	86	91	176	217	362	271	644	397	806
$\rho_{\mathbb{P}^1}$	1	2	7	20	37	62	91	128	169	218	271	332	397	470
r	0	0	4	19	40	68	104	149	204	270	348	439	544	664

Table 15.8 Picard number of Fermat surfaces.

is 37. This has been explicitly computed in [Moo93] using Aoki and Shioda's work [AS83], [Shi81b]. By definition the Picard rank of a surface X is less than or equal to the Hodge number $h^{11}(X)$.

Definition 15.8 A surface X has maximal Picard number or it is called ρ-maximal if its Picard number $\rho(X)$ is equal to $h^{11}(X)$.

The fact that Fermat surface of degree 4 has maximal Picard number 20 is a well-known fact. From our table (15.12) we can read the following Proposition. It can be derived from Shioda's works as before, however, its first appearance in the literature seems to be due to Beauville in [Bea14].

Proposition 15.9 *A Fermat sextic surface* $X_2^6 : x^6 + y^6 + z^6 + w^6 = 0$ *has the maximal Picard number* 86.

It follows from the explicit formula for the Picard number of the Fermat surface given in [Aok83] that X_2^d for $d \geq 4$ is ρ-maximal if and only if $d = 4,\ 6$.

15.13 Rank of elliptic curves over functions field

The Picard number of elliptically fibered surfaces is related to the rank of the corresponding elliptic curve over a function field of transcendental degree one. We explain this for the example

$$L : x^3 + y^2 + z^d = 1. \tag{15.45}$$

Once we can consider this as a surface in $\mathbb{C}^3$, and then compactify it. In this way we can talk about its Picard number. From a different point of view, we can consider it as an elliptic curve over $\mathbb{C}(z)$, the field of rational functions in z and coefficients in $\mathbb{C}$. In this way we can talk about its rank. Both numbers are related by a formula due to T. Shioda. For this and missing proofs the reader is referred to Shioda's original article [Shi91].

Let $K = \mathsf{k}(C)$ be the function field of a smooth projective curve C over an algebraically closed field k. Let also E be an elliptic curve defined over K. We can define a smooth projective surface X and a map $X \to C$, both defined over k, such that the

generic fiber of f is E. We have a bijection between the set $E(K)$ of K-rational points of E and the set of global sections of f. For $P \in E(K)$, (P) denotes the image curve of $P : C \to X$. We assume that there is at least one singular fiber for f. In this way by the Mordell-Weil theorem $E(K)$ is a finitely generated group. Let

$$T = \mathbb{Z} \cdot (0) + \mathbb{Z} \cdot F + \sum_c T_c, \tag{15.46}$$

where c runs through critical values of f such that $f^{-1}(c)$ is reducible, F is a smooth fiber of $X \to C$, and T_c is the $\mathbb{Z}$-module generated by the irreducible components of $f^{-1}(c)$. It is known that the only relation among the curves in (15.46) is $F =$ sum of the irreducible components of $f^{-1}(c)$. Therefore,

$$\operatorname{rank}(T) = 2 + \sum_c (n_c - 1),$$

where n_c is the number of irreducible components of $f^{-1}(c)$, $\ c \in C$. We have an isomorphism

$$E(K) \to \mathrm{NS}(X)/T, \ \ P \mapsto (P) \tag{15.47}$$

and so

$$\operatorname{rank}(E(K)) = \operatorname{rank}(\mathrm{NS}(X)) - 2 - \sum_c (n_c - 1). \tag{15.48}$$

Let us now consider our main example. In this case $K := \mathbb{C}(\mathbb{P}^1)$ is the field of rational functions in z. We consider (15.45) as an elliptic curve E_d over $\mathbb{C}(z)$. In this case X is any compactification of (15.45) as a surface in $\mathbb{C}^2$. This compactification can be done explicitly, however, for the purpose of our discussion we will not need it, see Exercise 15.9. The critical fibers of f for finite $z \in \mathbb{C}$ are irreducible. The compactification divisor $X \backslash L$ is the sum of (0) and the critical fiber of f over ∞, see Figure 15.1. We use (9.7) and (15.48) and conclude that

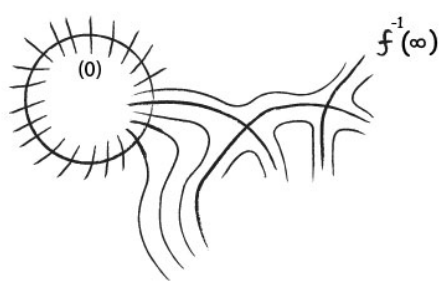

Fig. 15.1 Elliptic fibration with (0) divisor

$$\operatorname{rank}(E_d(\mathbb{C}(z))) = \operatorname{rank}(\mathrm{NS}(X)_0) = \mu_0 - \operatorname{rank}(\tilde{\mathbf{P}}^2), \tag{15.49}$$

where $\tilde{\mathbf{P}}^2$ is defined in Section 15.11 and

$$\mu_0 := \left\{\beta \in I \mid A_\beta \notin \mathbb{Z}\right\}. \tag{15.50}$$

Here are few rank computations. For the computer implementation of this see Chapter 20.

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\operatorname{rank}(E_d)$	2	4	6	8	8	0	6	10	10	0	16	0	2	12	6	0	20	0	14

d	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$\operatorname{rank}(E_d)$	4	2	0	24	8	2	10	6	0	24	0	6	4	2	8	28	0	2	4	14

d	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$\operatorname{rank}(E_d)$	0	8	0	6	18	2	0	24	0	10	4	6	0	20	8	6	4	2	0	48

d	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
$\operatorname{rank}(E_d)$	0	2	10	6	8	8	0	6	4	10	0	36	0	2	12	6	0	8	0	14

Table 15.9 Rank of the elliptic curve $E_d : y^2 + x^3 + z^d = 1$ over $\mathbb{C}(z)$.

Our computations are the beginning of a long story. Shioda in [Shi86, Shi92] proves that the elliptic curve $y^2 + x^3 + z^d - 1,\ \ d \geq 2$ over $\mathbb{C}(z)$ has rank ≤ 68. The equality happens if and only d is divisible by 360. We have the following which has been appeared in [CMT00] Proposition 4.2.

Proposition 15.10 *Let d be a positive multiple of 72. The elliptic curve given by $y^2 + x^3 + z^d - 1,\ \ d \geq 2$ over $\mathbb{C}(z)$ has rank ≥ 36.*

Proof. This follows from

$$d_1 \mid d_2 \text{ implies } \operatorname{rank}(E_{d_1}) \leq \operatorname{rank}(E_{d_2})$$

and $\operatorname{rank}(E_{72}) = 36$. □

A full classification of the Mordell-Weil lattice of the elliptic curve (15.45) is due to H. Usui in his seminal work on Mordell-Weil lattices. In the main theorem of [Usu08] he gives a complete classification of the Mordell-Weil lattices of E_d. This confirms our computations of $\operatorname{rank}(E_d)$. Using a computer implementation of Hodge cycles, the author was also able to verify

$$\operatorname{rank}(E_{90}) = 36,\ \operatorname{rank}(E_{120}) = 56,\ \operatorname{rank}(E_{180}) = 60,\ \ \operatorname{rank}(E_{360}) = 68.$$

see Chapter 20. One can also apply our method to other families of elliptic curves over $\mathbb{C}(z)$. Few examples are

$$x^3 + y^3 + z^d = 1,\ \ x^4 + y^2 + z^d = 1.$$

For a table of such elliptic curves see for instance [Hei12]. In this article one can also find the maximal rank obtained by many families of elliptic curves.

15.14 Computing a basis of Hodge cycles using Gaussian elimination

We are mainly motivated by the computational Hodge conjecture, that is, if we are given an explicit Hodge cycle δ, we would like to find an explicit algebraic cycle whose homology class is δ. For this reason we would like to find a *good* basis of $\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}}$. Note that by our definition this $\mathbb{Z}$-module has no torsion. Here, we do not have a precise description of a good basis, however, we may suspect that 1. It must be easy to verify the Hodge conjecture for its elements. 2. Its elements must be the homology class of irreducible subvarieties of X etc. Since all these properties involve the verification of computational Hodge conjecture, it is almost impossible to find such a basis, however, in Chapter 16 we will define an invariant ξ of Hodge cycles which will give us a precise meaning of a good basis. In this section we explain how to obtain a basis of the $\mathbb{Q}$-vector space $\mathrm{Hodge}_n(X,\mathbb{Q})_0$. In order to do this we use the Gaussian elimination which is implemented in the procedure `gaussred-pivot` in Singular. We will use the notation of the help section of this procedure. For examples and further details see Chapter 20.

Let us define

$$\mathtt{A} := \tilde{\mathbf{P}}^{\frac{n}{2}+1}. \tag{15.51}$$

Note that the Gaussian elimination freely uses divisions over integers and so it works only over rational numbers. We get

$$\mathtt{P}\cdot\mathtt{A} = \mathtt{U}\cdot\mathtt{S}, \tag{15.52}$$

where $\mathtt{S}$ is a row reduced matrix, $\mathtt{P}$ is a permutation matrix and $\mathtt{U}$ is a normalized lower triangular matrix. Let $\mathtt{m} := \mathrm{rank}(\mathtt{A})$ be the rank of the matrix $\mathtt{A}$. If $\mathtt{m} = \mu$ then there is no affine Hodge cycle or cycle at infinity. Note that $\mathtt{A}$ and $\mathtt{S}$ are $\mu \times (\varphi(d)\cdot h^{\frac{n}{2}+1})$ matrices and this situation may occur.

The first $\mathtt{m}$ rows of $\mathtt{S}$ are linearly independent, and the next rows are zero. Let $\mathtt{Y}$ be the $(\mu-\mathtt{m})\times\mu$ matrix which is obtained by joining the $(\mu-\mathtt{m})\times\mathtt{m}$ zero matrix with the $(\mu-\mathtt{m})\times(\mu-\mathtt{m})$ identity matrix in a row. The rows of $\mathtt{Y}$ form a basis for the space of vectors $\mathtt{B}$ such that $\mathtt{B}\cdot\mathtt{S}=0$. Therefore, the rows of

$$\mathtt{X} := \mathtt{Y}\mathtt{U}^{-1}\mathtt{P}$$

form a basis of $1\times\mu$ matrices $\mathtt{B}$ such that $\mathtt{B}\cdot\mathtt{A}=\mathtt{0}$.

In order to study integral Hodge cycles, we need to find a basis of the $\mathbb{Z}$-module of $1\times\mu$ integer valued matrices which are perpendicular to all columns of the matrix $\mathtt{A}$ (15.51). For this purpose the Gaussian elimination does not work, and instead we have to use another algorithm which writes $\mathtt{A}$ in the Smith normal form. Recall that

$\mathtt{A}$ is an integer-valued $\mu \times \mu_1$ matrix, where $\mu_1 := \varphi(d) \cdot h^{\frac{n}{2}+1}$. There exist $\mu \times \mu$ and $\mu_1 \times \mu_1$ integer-valued matrices $\mathtt{U}$ and $\mathtt{T}$ such that $\det(\mathtt{U}) = \pm 1$, $\det(\mathtt{T}) = \pm 1$ and

$$\mathtt{U} \cdot \mathtt{A} \cdot \mathtt{T} = \mathtt{S}$$

where $\mathtt{S}$ is the diagonal matrix

$$\mathtt{S} := \begin{pmatrix} a_1 & 0 & 0 & \cdots & & & & \\ 0 & a_2 & 0 & \cdots & & & & \\ 0 & 0 & a_3 & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ & & & & a_{\mathtt{m}} & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \end{pmatrix}.$$

Here, $\mathtt{m}$ is the rank of the matrix $\mathtt{A}$ and the natural numbers $a_i \in \mathbb{N}$ satisfy $a_i \mid a_{i+1}$ for all $i < r$. The matrix $\mathtt{S}$ is called the Smith normal form of $\mathtt{A}$. The integers a_i are unique and are called the elementary divisors. They can be computed directly from the matrix $\mathtt{A}$ using

$$a_i = \frac{d_i(A)}{d_{i-1}(A)}$$

where $d_i(\mathtt{A})$ (called i-th determinant divisor) equals the greatest common divisor of all $i \times i$ minors of the matrix $\mathtt{A}$. By definition $d_0(\mathtt{A}) = 1$. Let $\mathtt{Y}$ be as before. The rows of

$$\mathtt{X} := \mathtt{Y} \cdot \mathtt{U} \tag{15.53}$$

form a basis of the integer-valued $1 \times \mu$ matrices $\mathtt{B}$ such that $\mathtt{B} \cdot \mathtt{A} = \mathtt{0}$. The primes dividing the integers a_i might have some interpretation in terms of the étale cohomology of the Fermat variety. This and many other challenging problems are left to the reader, see Exercise 15.6.

15.15 Choosing a basis of homology

Let us now look at the classical Fermat variety $X = X_n^d$ and its hyperplane section Y at infinity. We have the exact sequence:

$$0 \to H_{n+1}(X,\mathbb{Z}) \to H_{n-1}(Y,\mathbb{Z}) \to H_n(L,\mathbb{Z}) \to H_n(X,\mathbb{Z}) \to H_{n-2}(Y,\mathbb{Z}) \to 0$$

and we define

$$\begin{aligned} H_n(X)_0 &:= \ker(H_n(X,\mathbb{Z}) \to H_{n-2}(Y,\mathbb{Z})), \\ H_{n-1}(Y)_0 &:= \operatorname{coker}(H_{n+1}(X,\mathbb{Z}) \to H_{n-1}(Y,\mathbb{Z})), \end{aligned}$$

which gives us

$$0 \to H_{n-1}(Y,\mathbb{Z})_0 \to H_n(L,\mathbb{Z}) \to H_n(X,\mathbb{Z})_0 \to 0. \tag{15.54}$$

We have a basis $\delta_\beta,\ \beta \in I$ for $H_n(L,\mathbb{Z})$ and we would like to find a basis of $H_n(X,\mathbb{Z})_0$. From (15.54) we can prove by induction that the rank of $H_n(X,\mathbb{Z})_0$ is

$$\begin{aligned} \check{\mu} &:= \mathrm{rank} H_n(X,\mathbb{Z})_0 \\ &= (d-1)^{n+1} - (d-1)^n + (d-1)^{n-1} - \cdots + (-1)^n(d-1). \end{aligned} \tag{15.55}$$

For the general Fermat variety, we take any compactification X of L and define $H_n(X,\mathbb{Z})_0$ to be the image of $H_n(L,\mathbb{Z})$ in $H_n(X,\mathbb{Z})$ induced by the inclusion $L \subset X$. In de Rham cohomology level we define $H^n_{\mathrm{dR}}(X)_0$ to be the subspace of $H^n_{\mathrm{dR}}(X)$ generated by $\omega_\beta,\ \beta \notin \mathbb{N}$. In this way we define

$$\check{\mu} := \dim(H^n_{\mathrm{dR}}(X)_0) = \#\left\{\beta \in I \mid A_\beta \notin \mathbb{N}\right\}. \tag{15.56}$$

This is the same definition as in (15.55) for the case $m_1 = m_2 = \cdots = m_{n+1}$.

Proposition 15.11 *There exits a set of $\check{\mu}$ vanishing cycles among $\delta_\beta,\ \beta \in I$ forming a basis for $H_n(X,\mathbb{Q})_0$.*

Proof. The proposition follows from the facts that the map $H_n(L,\mathbb{Z}) \to H_n(X,\mathbb{Z})_0$ induced by the inclusion $L \subset X$ is surjective and $\delta_\beta,\ \beta \in I$ form a basis of $H_n(L,\mathbb{Z})$. □

Let A be a submodule of $H_n(X,\mathbb{Z})_0$ generated by $\check{\mu}$ vanishing cycles among $\delta_\beta,\ \beta \in I$. In Proposition 15.11 we have claimed that for some A, $H_n(X,\mathbb{Z})_0/A$ is a torsion group, and hence, it is finite. We may try to minimize the cardinality of this quotient. For this reason it seems interesting to study the number

$$\mathsf{M}(X) := \text{minimum}\left\{ \#\frac{H_n(X,\mathbb{Z})_0}{\mathbb{Z}\delta_1 + \mathbb{Z}\delta_2 + \cdots + \mathbb{Z}\delta_{\check{\mu}}} \,\middle|\, \{1,2,\ldots,\check{\mu}\} \subset I \right\}. \tag{15.57}$$

Note that by abuse of notation, we have reindexed a $\check{\mu}$-tuple of elements in I. In the best situation, we may want to prove Proposition 15.11 over the ring of integers, that is we prove that $\mathsf{M}(X) = 1$. This does not seem to be the case. A computer assisted investigation of the number M is formulated in Exercise 15.7.

There are two computational methods in order to find the basis of Proposition 15.11. Let us randomly choose the vanishing cycles $\delta_1, \delta_2, \ldots, \delta_{\check{\mu}}$ among $\delta_\beta,\ \beta \in I$.
1. We prove that the intersection matrix in this basis is nondegenerated, that is,

$$\begin{vmatrix} \langle\delta_1,\delta_1\rangle, & \langle\delta_1,\delta_2\rangle, & \cdots, & \langle\delta_1,\delta_{\check{\mu}}\rangle \\ \langle\delta_2,\delta_1\rangle, & \langle\delta_2,\delta_2\rangle, & \cdots, & \langle\delta_2,\delta_{\check{\mu}}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\delta_{\check{\mu}},\delta_1\rangle, & \langle\delta_{\check{\mu}},\delta_2\rangle, & \cdots, & \langle\delta_{\check{\mu}},\delta_{\check{\mu}}\rangle \end{vmatrix} \neq 0. \tag{15.58}$$

Since the intersection form in $H_n(X,\mathbb{Q})_0$ is nondegenerated, this implies that δ_i's form a basis of $H_n(X,\mathbb{Q})_0$. 2. We prove that the period matrix of $H_n(X,\mathbb{Q})_0$ has a non-zero determinant, that is,

$$\begin{vmatrix} \int_{\delta_1}\omega_1, & \int_{\delta_1}\omega_2, & \cdots, & \int_{\delta_1}\omega_{\check{\mu}} \\ \int_{\delta_2}\omega_1, & \int_{\delta_2}\omega_2, & \cdots, & \int_{\delta_2}\omega_{\check{\mu}} \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\delta_{\check{\mu}}}\omega_1, & \int_{\delta_{\check{\mu}}}\omega_2, & \cdots, & \int_{\delta_{\check{\mu}}}\omega_{\check{\mu}} \end{vmatrix} \neq 0, \tag{15.59}$$

where $\omega_j, \quad j = 1,2,\ldots,\check{\mu}$ form a basis of $H^n_{\mathrm{dR}}(X)_0$. Both methods for particular values of d and n can be performed by computer, see Exercise 15.7.

15.16 An explicit Hodge cycle $\check{\delta}$

So far, we know two different ways of writting Hodge cycles of the Fermat variety. First, we can write a Hodge cycle as a linear combination of vanishing cycles. This presentation of a Hodge cycle is not unique and it is defined up to addition of cycles at infinity. Second, we can identify a Hodge cycle δ with the periods of $H^{\frac{n}{2},\frac{n}{2}}$ over δ. In both presentation of a Hodge cycle we do not see why the cycle is Hodge, that is, we have just a bunch of integers or elements in $\mathbb{Q}(\zeta_d)$ which are used to verify the property of being Hodge computationally. Moreover, in both methods we can compute a basis of the $\mathbb{Z}$-module of Hodge cycles, but we do not have any closed formula for particular Hodge cycles. In this section, we give a third presentation of a Hodge cycle, this is through its intersection with other vanishing cycles. Surprisingly, for some Hodge cycles we get closed formulas.

Let us consider a cycle $\delta \in H_n(L,\mathbb{Z})$ such that we know its intersection with the basis $\delta_\beta, \beta \in I$ of $H_n(L,\mathbb{Z})$. We consider its image in $H_n(X,\mathbb{Z})_0$, that we denote it with the same letter δ, and we would like to know its periods. We choose $\check{\mu} := \mathrm{rank}(H_n(X,\mathbb{Z})_0)$ elements among δ_β's, let us say $\delta_i,\ i = 1,2,\cdots,\check{\mu}$, such that they form a basis of $H_n(X,\mathbb{Q})_0$. Let also $\omega_1,\omega_2,\cdots,\omega_{\check{\mu}}$ be a basis $H^n_{\mathrm{dR}}(X)_0$. The following proposition enables us to compute periods of a cycle by using its intersection data with other vanishing cycles.

Proposition 15.12 *The periods of $\delta \in H_n(L,\mathbb{Z})$ are given by the formula*

$$\left[\int_\delta \omega_1,\ \int_\delta \omega_2,\ \cdots,\ \int_\delta \omega_{\check{\mu}}\right] = \left[\langle\delta_1,\delta\rangle,\ \langle\delta_2,\delta\rangle,\ \cdots,\ \langle\delta_{\check{\mu}},\delta\rangle\right].$$

$$\begin{bmatrix} \langle\delta_1,\delta_1\rangle, & \langle\delta_1,\delta_2\rangle, & \cdots, & \langle\delta_1,\delta_{\check{\mu}}\rangle \\ \langle\delta_2,\delta_1\rangle, & \langle\delta_2,\delta_2\rangle, & \cdots, & \langle\delta_2,\delta_{\check{\mu}}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\delta_{\check{\mu}},\delta_1\rangle, & \langle\delta_{\check{\mu}},\delta_2\rangle, & \cdots, & \langle\delta_{\check{\mu}},\delta_{\check{\mu}}\rangle \end{bmatrix}^{-\mathsf{tr}} \begin{bmatrix} \int_{\delta_1}\omega_1, & \int_{\delta_1}\omega_2, & \cdots, & \int_{\delta_1}\omega_{\check{\mu}} \\ \int_{\delta_2}\omega_1, & \int_{\delta_2}\omega_2, & \cdots, & \int_{\delta_2}\omega_{\check{\mu}} \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\delta_{\check{\mu}}}\omega_1, & \int_{\delta_{\check{\mu}}}\omega_2, & \cdots, & \int_{\delta_{\check{\mu}}}\omega_{\check{\mu}} \end{bmatrix}.$$

Proof. Let $\Delta := \begin{bmatrix}\delta_1 & \delta_2 & \cdots & \delta_{\mu_1}\end{bmatrix}$ and $\check{\mathbf{P}}_1 = \langle\Delta,\delta\rangle\langle\Delta^{\mathsf{tr}},\Delta\rangle^{-\mathsf{tr}}\check{\mathbf{P}}$ be the above equality. We write $\delta = A\cdot\Delta^{\mathsf{tr}}$, where A is a $1\times\check{\mu}$ matrix with entries in $\mathbb{Z}$. We have $\langle\Delta,\delta\rangle = A\cdot\langle\Delta,\Delta^{\mathsf{tr}}\rangle$ and so $A = \langle\Delta,\delta\rangle\langle\Delta,\Delta^{\mathsf{tr}}\rangle^{-1}$. □

Proposition 15.13 (Conjecture) Let $d := m_1 = m_2 = \cdots = m_{n+1}$ be an even number. We have

1. There is a cycle $\check{\delta}\in H_n(X_n^d,\mathbb{Z})_0$ such that

$$\langle\check{\delta},\delta_\beta\rangle = \begin{cases} (-1)^{\sum_{i=1}^{n+1}\beta_i} & \text{if } \beta_1 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. The cycle $\check{\delta}$ is a Hodge cycle.
3. For any $\beta\in I$ with $A_\beta\notin\mathbb{Z}$ we have

$$\int_{\check{\delta}}\omega_\beta = \begin{cases} (-1)^{\frac{n}{2}-1}d^nB_\beta & \text{if } \beta_2=\beta_3=\cdots=\beta_{n+1}=\frac{d}{2}-1 \\ 0 & \text{otherwise.} \end{cases} \tag{15.60}$$

Proof. Note that the intersection numbers of a cycle $\check{\delta}$ with $\check{\mu}$ cycles selected from $\delta_\beta,\ \beta\in I$, determines $\check{\delta}$ uniquely, and so it is not clear that this $\check{\delta}$ must have the given intersection number with the rest of δ_β's. That is why the first part of the proposition is not trivial. We have just verified the proposition for many values of d and n. For the computer code used in this verification see §20.9. The general statement must not be difficult to prove by theoretical methods and it is left as a challenging conjecture to the reader. □

15.17 Exercises

15.1. Prove the integral computations (15.3) and (15.4).

15.2. Consider the affine quadratic Fermat surface $L: x^2+y^2+z^2-1=0$. Its second homology is generated by the oriented sphere $\delta = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$. We have also two types of algebraic cycles in the compactification X of L, namely, the curve at infinity $C_\infty := \{[x;y;z]\in\mathbb{P}^2 \mid x^2+y^2+z^2=0\}\subset X$ and the curve C given by a hyperplane section in $\mathbb{P}^3$, for instance take the intersection of X with $z=0$. Describe all the relations between the homology classes of these curves and δ.

15.3. Recall the notations introduced in Exercise 4.2. Show that δ_β's form a basis of the $\mathbb{Z}$-module $H_n(L,\mathbb{Z})$. Hint: For ε a very small number and t near to 1, consider the family of varieties

$$L_{\varepsilon,t} : x_1^{m_1} - m_1\varepsilon x_1 + x_2^{m_2} - m_2\varepsilon x_2 + \cdots + x_{n+1}^{m_{n+1}} - m_{n+1}\varepsilon x_{n+1} = t.$$

Show that the monodromy $\delta_{\beta,t,\varepsilon} \in H_n(L_{\varepsilon,t},\mathbb{Z})$ of δ_β, form a distinguished set of vanishing cycles. Then use Theorem 7.2.

15.4. Prove Proposition 15.5 using elementary methods. At least this can be done by computer for particular examples of $m_1, m_2, \ldots, m_{n+1}$. For an elementary presentation of this proposition see Problem 21.9 in Chapter 21.

15.5. For $d = m_1 = m_2 = \cdots = m_{n+1}$ find an elementary proof for the following affirmation:

$$\operatorname{rank}\left\{A \in \operatorname{Mat}(\mu \times 1, \mathbb{Z}) \mid A\tilde{\mathbf{P}}^0 = 0\right\} = \sum_{i=0}^{n-1} (-1)^i (d-1)^{n-i}. \tag{15.61}$$

A geometric proof of this is as follows: the $\mathbb{Z}$-module in the left hand side of (15.61) corresponds to the $\mathbb{Z}$-module of cycles at infinity in $H_n(L,\mathbb{Z})$. Let $Y := X_{n-1}^d$ be the hyperplane section of $X := X_n^d$ obtained by $x_0 = 0$. The mentioned $\mathbb{Z}$-module is the image of $H_{n-1}(Y,\mathbb{Z})$ under the map σ in (5.18).

15.6 (Computing a basis of Hodge cycles using Smith algorithm). The Smith normal form is implemented in Mathematica under the name `SmithDecomposition`. For the Fermat variety X_4^6 use the algorithm developed in this chapter to compute $\tilde{\mathbf{P}}^{\frac{n}{2}+1}$ and its Smith normal form.

15.7. For particular values of n and d, for instance $n = 2,4$ and $d = 4,5$, compute the number $\mathsf{M}(X)$ defined (15.57). To do this, we have to find all $\mathbb{Z}$-linear relations between $\delta_\beta, \beta \in I$ in $H_n(X,\mathbb{Z})_0$. An element $\delta := \sum_{\beta\in I} n_\beta \delta_\beta$ is zero in $H_n(X,\mathbb{Z})_0$ if and only if $\langle \delta, \delta_\beta \rangle = 0$ for all $\beta \in I$, or equivalently $\displaystyle\int_\delta \omega = 0$ for all $\omega \in H^n_{\mathrm{dR}}(X)_0$. We also define $\check{\mathsf{M}}(X)$ to be the minimum of the absolute value of the determinant (15.58). Is there any relation between $\mathsf{M}(X)$ and $\check{\mathsf{M}}(X)$?

15.8. Theorem 15.1 can be stated for all generalized Fermat varieties (15.9) with $m_1, m_2, \ldots, m_{n+1} \in \{2,3,4,6\}$. Among this class find an example of the Fermat variety with Hodge numbers of the form $1,a,b,a,1$ and with a minimum value for a and b. So far, our record is for Fermat cubic tenfold which has the Hodge numbers 1, 220, 925,220,1. This appears in Table 15.5.

15.9. Describe a smooth compactification of the affine surface $L : x^3 + y^2 + z^d = 1$.

15.10. * Let us consider the curve $C : x^{m_1} + y^{m_2} + z^{m_3} = 1$ defined over the function field $\mathbb{C}(z)$. Describe the decomposition of the Jacobian variety of C over $\mathbb{C}(z)$. Is there an elliptic curve defined over $\mathbb{C}(z)$ in this decomposition? Can you compute

its rank? For a decomposition of the Jacobian variety of the Fermat curve over $\bar{\mathbb{Q}}$ see the Koblitz's article [Kob78].

15.11. We have

1. For natural numbers $m_1, m_2, \ldots, m_{n+1}$ all ≥ 2 the condition
$$[\mathbb{Q}(\zeta_{m_1}, \zeta_{m_2}, \ldots, \zeta_{m_{n+1}}), \mathbb{Q}] = (m_1 - 1)(m_2 - 1) \cdots (m_{n+1} - 1) \tag{15.62}$$
is satisfied if and only if all m_i's are prime numbers.
2. If (15.62) is satisfied and $\sum_{i=1}^{n+1} \frac{1}{m_i} < 1$ then there does not exist a non-zero $\delta \in H_n(X, \mathbb{Q})_0$ and $\beta \in I$ such that $\displaystyle\int_\delta \omega_\beta = 0$.
3. In particular, there does not exist a non zero Hodge cycle, and also, there does not exist a cycle at infinity and so
$$\forall \beta \in I, \;\; A_\beta \notin \mathbb{N}.$$
4. The elliptic curve $E_d : x^3 + y^2 + z^d = 1$ over $\mathbb{C}(z)$, with d a prime number ≥ 7, is of rank zero. Why this statement is false for primes $2, 3, 5$, see Table 15.9. Note that $2, 3, 5$ are the only primes of $360 = 2^3 3^2 5$ and $\operatorname{rank}(E_{360}) = 68$ is the maximum value of $\operatorname{rank}(E_d)$.

15.12. For small values of d, for instance $d = 2, 3, 4, 5$, find some $\mathbb{C}(z)$-rational points on the elliptic curve $y^2 + x^3 + z^d = 1$ over $\mathbb{C}(z)$. Using Table 15.9 we know that $\operatorname{rank}(E_2) = 2$, $\operatorname{rank}(E_3) = 4$, $\operatorname{rank}(E_4) = 6$, $\operatorname{rank}(E_5) = 8$.

15.13. Compute the dimension of the space of Hodge cycles of the variety
$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^d = 1$$
for small values of d. For d a primes number find a closed formula for such a number. Find some algebraic cycles of this variety using the factorization of $x_1^2 + x_2^2$, $x_3^3 + x_4^3$ and $x_5^d - 1$. For d a prime number, are these algebraic cycles enough in order to prove the Hodge conjecture?

15.14. Let $L_n^d \subset \mathbb{C}^{n+1}$ be the classical affine Fermat variety given by the polynomial $x_1^d + \cdots + x_{n+1}^d = 1$ in the affine chart $(x_1, x_2, \ldots, x_{n+1})$. For $d = d_1 d_2, \;\; d_i \geq 2$ we have the canonical maps:
$$\pi_i, \; L_n^{d_1 d_2} \to L_n^{d_i}, \;\; (x_1, x_2, \ldots, x_{n+1}) \mapsto (x_1^{d_{3-i}}, x_2^{d_{3-i}}, \ldots, x_{n+1}^{d_{3-i}}), \;\; i = 1, 2. \tag{15.63}$$

1. Compute the kernel A of the induced maps $H_n(L_n^{d_1 d_2}, \mathbb{Z}) \to H_n(L_n^{d_i}, \mathbb{Z})$.
2. Compute the induced maps in $H_{\mathrm{dR}}^n(L_n^{d_i}) \to H_{\mathrm{dR}}^n(L_n^{d_1 d_2})$ and then the image B of the map
$$H_{\mathrm{dR}}^n(L_n^{d_1}) \otimes_{\mathbb{C}} H_{\mathrm{dR}}^n(L_n^{d_2}) \to H_{\mathrm{dR}}^n(L_n^{d_1 d_2}).$$
3. Show that the integration of the elements of B over the elements of A is zero.

4. * Can the vanishing integrals in the previous item be used in order to find a Hodge cycle in $L_n^{d_1 d_2}$?

15.15. * Prove or disprove: for fixed values of natural numbers $m_1, m_2, \ldots, m_n \geq 2$, n even, there is a natural number N depending only on these numbers such that the dimension of $\mathrm{Hodge}_n(X, \mathbb{Q})_0$ is less than or equals to N for all values of $m_{n+1} \geq 2$. Here, X is the projective Fermat variety given by $x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 1$ in an affine chart. Note that for $n = 2,\ m_1 = 2,\ m_3 = 3$ the number N is equal to 68 and this is the rank record of elliptic curves over function fields obtained by Shioda in [Shi86, Shi92].

15.16. Find an $(n+1)$-tuple $m_1 \leq m_2 \leq \ldots \leq m_{n+1}$, all natural numbers ≥ 2, such that

$$0 < \frac{n}{2} - \sum_{i=1}^{n+1} \frac{1}{m_i} \leq \frac{1}{m_{n+1}}$$

and $(m_1 - 1)(m_2 - 1) \cdots (m_{n+1} - 1)$ is the minimum possible. How about the same problem replacing $\frac{n}{2}$ with $\frac{n}{2} - 1$ in the above inequalities. The first problem gives us Hodge numbers $\cdots, 0, 1, a, 1, 0, \cdots$ with minimum middle Hodge number a and the second problem gives us Hodge numbers $\cdots, 0, 1, b, a, b, 1, 0, \cdots$ with minimum $a + 2b$.

15.17 (Hodge numbers after group action). It is always desirable to work with small Hodge numbers. One way to get this, is by using group actions. Let $X = X_n^d$ be the classical Fermat variety of dimension n and degree d. The group S_{n+2} of permutations in $n+2$ points acts in a natural way on X (by permuting the variables) and hence on $H^n_{\mathrm{dR}}(X)_0$. It leaves the Hodge filtration invariant and hence it is natural to consider the invariant part of $H^n_{\mathrm{dR}}(X)_0$

$$\{\omega \in H^n_{\mathrm{dR}}(X)_0 \ \mid \ \sigma^* \omega = \omega \quad \forall \sigma \in \mathsf{S}_{n+2}\}$$

and the induced Hodge filtration. Compute the dimension of these Hodge filtrations, and hence, the corresponding Hodge numbers in few cases. For instance, show that for $(n, d) = (4, 6)$ we get the Hodge numbers $1, 9, 18, 9, 1$ and for $(n, d) = (10, 3)$ we get $0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0$. Working with smaller Hodge numbers is proposed by P. Deligne, (personal communication, February 20, 2016).

Chapter 16
Periods of Hodge cycles of Fermat varieties

Grothendieck liked to have things in their natural generality; to have an understanding of the whole story. Serre appreciates this, but he prefers beautiful special cases [...] Serre had a much wider mathematical culture than Grothendieck. In case of need, Grothendieck redid everything for himself, while Serre could tell people to look at this or that in the literature. Grothendieck read extremely little; his contact with classical Italian geometry came basically through Serre and Dieudonné, (P. Deligne in [RS14] page 180).

16.1 Introduction

Let X be a smooth projective variety over $\mathsf{k} := \bar{\mathbb{Q}} \subset \mathbb{C}$ and $Z \subset X$ be an algebraic cycle of dimension $\frac{m}{2}$. Periods of the form

$$\int_Z \omega, \quad \omega \in H^m_{\mathrm{dR}}(X) \tag{16.1}$$

first appeared in the work of P. Deligne in [DMOS82, Del81]. He used these periods in order to define the notion of an absolute Hodge cycle which turned out to be an important development in direction of Grothendieck's theory of motives. These periods multiplied with the transcendental factor $(2\pi i)^{-\frac{m}{2}}$, which is usually called the Tate twist, is an algebraic number. For an example of this see Legendre relation between elliptic integrals explained in §2.3. The algebraicity statement in the case of Hodge cycles is a direct consequence of the Hodge conjecture. The first information these numbers encode is the field of definition of the conjecturally existing algebraic cycles Z. In [Mov17b] the author noticed that these numbers encode more information of the algebraic cycle Z. This is namely the Zariski tangent space of the deformation space of (X,Z) within the moduli space of X itself. In view of Theorem 13.2 these periods actually determine the defining equations of such a deformation space. In this chapter we would like to exploit all these ideas for the main protago-

nist of the present text. This is namely the variety X given in an affine chart by the generalized Fermat variety $x_1^{m_1}+x_2^{m_{n+1}}\cdots+x_{n+1}^{m_{n+1}}-1=0$. We define an invariant of Hodge cycles which we call it the ξ-invariant. This is the codimension of the Zariski tangent space explained earlier. We will use many notations introduced in Chapter 15 and the reader may go back and forth between this and the present chapter. Most of the discussion in this chapter can be rewritten for an arbitrary homogeneous tame polynomial $g\in\mathbb{C}[x]$. However, for simplicity we stick to the generalized Fermat variety as above, or even the classical Fermat variety with $d=m_1=m_2=\cdots=m_{n+1}$.

16.2 Computing periods of Hodge cycles

In §15.14 we gave explicit expressions of affine Hodge cycles in terms of vanishing cycles. Apart from computing a basis and dimension of the space of Hodge cycles, we do not have any application for this way of writing Hodge cycles. Moreover, if one uses vanishing cycles to describe Hodge cycles of the compact Fermat variety, then this is done up to addition of cycles at infinity. This arises from the denominator of (15.28). This situation can be remedied using Remark 15.1, however, we have not exploited this idea.

An alternative way to express Hodge cycles is through their periods. In theoretical terms this can be interpreted as considering Poincaré dual of Hodge cycles and writing them in a basis of the underlying cohomology.

Proposition 16.1 *Let X be the generalized Fermat variety. The following map is an injection*

$$\begin{aligned}\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}} &\to \mathbb{C}^{h_0^{\frac{n}{2},\frac{n}{2}}} \\ \delta &\mapsto \left(\int_\delta \omega_\beta,\ \beta\in I,\ \frac{n}{2}<A_\beta<\frac{n}{2}+1\right).\end{aligned} \tag{16.2}$$

Proof. Our proof uses the topology behind the Fermat variety, and so, it is not as elementary as the announcement of the proposition. Recall that we are working in the abstract context of §15.6 in which we do not need to know the topology of the Fermat variety. If we have an element δ in the kernel of the above map, then we have

$$\int_\delta \omega_\beta=0,\quad \forall\beta\in I,\ \frac{n}{2}<A_\beta<\frac{n}{2}+1.$$

We consider δ as a cycle in $H_n(X,\mathbb{Z})_{\check{0}}$ and we conclude that the integration of the $\frac{n}{2}$-th piece $F^{\frac{n}{2}}$ of the Hodge filtration of $H^n_{\mathrm{dR}}(X)_0$ over δ is zero. It follows from (15.10) that $F^{\frac{n}{2}}$ and its complex conjugation generate the whole $H^n_{\mathrm{dR}}(X)_0$. Since δ is defined over integers (we only need that δ is defined over real numbers), we conclude that the integration of $H^n_{\mathrm{dR}}(X)_0$ over δ is zero, and so, δ is zero in $H_n(X,\mathbb{Z})_{\check{0}}$. This implies that δ as an element in $H_n(L,\mathbb{Z})$ is a multiple of a cycle at infinity. $\square$

Recall the period matrix $\mathbf{P}$, its submatrices $\mathbf{P}^j$ defined in (15.29) and the computation of the matrix $\mathtt{X}$ in §15.14. The matrix $\mathbf{P}^{\frac{n}{2}+1}$ is used to define Hodge cycles. For computing the period of Hodge cycles we need $\mathbf{P}^{\frac{n}{2}}$. We do the multiplication $\mathtt{X}\cdot\mathbf{P}^{\frac{n}{2}}$ and define the matrix

$$\mathtt{H} := \mu \times (h^{\frac{n}{2}+1}\cdots h^{\frac{n}{2}}) \text{ submatrix of } \mathtt{X}\cdot\mathbf{P}^{\frac{n}{2}}, \tag{16.3}$$

whose rows are periods of Hodge cycles. Note that by definition of $\mathtt{X}$, all other columns of $\mathtt{X}\cdot\mathbf{P}^{\frac{n}{2}}$ are automatically zero. We rewrite Proposition 16.1 in the following way:

Proposition 16.2 *Let X be the generalized Fermat variety. We have a canonical isomorphism between*

$$\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}} \cong \textit{The } \mathbb{Z}\textit{-module genrated by the rows of } \mathtt{H}\,. \tag{16.4}$$

Proof. The proof follows from the exact sequence (5.18) and Proposition 16.1. □

The periods of cycles at infinity (which are affine Hodge cycles) are automatically zero, and hence, the multiplication $\mathtt{X}\cdot\mathbf{P}^{\frac{n}{2}}$ kills them. However, this produces many $\mathbb{Z}$-linear relations between the rows of $\mathtt{H}$. Each linear relation arises from a cycle at infinity in $H_n(L,\mathbb{Z})$.

16.3 The beta factors

Let $\omega_\beta,\ \beta\in I$ be the differential form defined in (15.6) and let

$$B_\beta := (2\pi i)^{-\frac{n}{2}} B\left(\frac{\beta_1+1}{m_1},\frac{\beta_2+1}{m_2},\cdots,\frac{\beta_{n+1}+1}{m_{n+1}}\right) \tag{16.5}$$

$$= (2\pi i)^{-\frac{n}{2}-1}\frac{e^{\pi i(A_\beta-\frac{n}{2})}-e^{-\pi i(A_\beta-\frac{n}{2})}}{(A_\beta-\frac{n}{2})_{\frac{n}{2}}}\prod_{i=0}^{n+1}\Gamma\left(\frac{\beta_i+1}{m_i}\right). \tag{16.6}$$

In the second way of writing B_β, we have $\frac{\beta_0+1}{m_0} := \frac{n}{2}+1-A_\beta$. We define $\mathbf{B}$ to be the diagonal $\check{\mu}\times\check{\mu}$-matrix so that in its (β,β)-entry we have B_β. The full period matrix is defined in the following way

$$\check{\mathbf{P}} := \left[\int_{\delta_\beta}\omega_{\beta'}\right]_{\beta,\beta'\in I} = \mathbf{P}\cdot\mathbf{B} \tag{16.7}$$

In this new period matrix we have included the contribution of the B-factors.

Theorem 16.1 (P. Deligne, [DMOS82], Theorem 7.15) *Let $\delta\in H_n(X,\mathbb{Z})_{\check{0}}$ be a Hodge cycle then for all $\beta\in I$ with $\frac{n}{2}<A_\beta<\frac{n}{2}+1$ we have*

$$\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_{\delta}\omega_{\beta}\in\bar{\mathbb{Q}}. \tag{16.8}$$

In particular, if for such a $\beta\in I$, B_{β} in (16.5) *is not an algebraic number then*

$$\int_{\delta}\omega_{\beta}=0.$$

The second part of the above theorem says that for a Hodge cycle in the generalized Fermat variety, apart from vanishing of integrals on δ due to the $F^{\frac{n}{2}+1}$ piece of the Hodge filtration, we have other vanishing of integrals over δ due to transcendental B-factors. Theorem 16.1 is due to P. Deligne in [DMOS82]. It uses the fact that the motive of a Fermat variety is contained in the category of motives generated by abelian varieties, and therefore, any Hodge cycle in a Fermat variety is absolute. It is beyond the main objective of the present text to explain all these. Instead, we remark that the proof of Theorem 16.1 reduces to an elementary problem as follows.

Proof. Let us recall the computation of periods in Proposition 15.1. We define $\check{\omega}_{\beta}:=B_{\beta}^{-1}\omega_{\beta}$ which has periods in $\mathbb{Q}(\zeta_d)$. The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})\cong(\frac{\mathbb{Z}}{d\mathbb{Z}})^{\times}$ acts in the $\mathbb{Q}(\zeta_d)$-vector space $\check{H}^n_{\mathrm{dR}}(X)_0$ generated by $\check{\omega}_{\beta}$'s in a natural way:

$$\int_{\delta}\sigma(\check{\omega}_{\beta}):=\sigma\left(\int_{\delta}\check{\omega}_{\beta}\right),\ \ \beta\in I,\ \delta\in H_n(X,\mathbb{Q})_0. \tag{16.9}$$

It turns out that $\sigma(\check{\omega}_{\beta})=\check{\omega}_{p*\beta}$, where $p*\beta$ is defined in the following way. We identify an element of the Galois group with $p\in(\frac{\mathbb{Z}}{d\mathbb{Z}})^{\times}$. The corresponding element of the Galois group is given by $\zeta_d\mapsto\zeta_d^p$. We have

$$p*\beta:=\left([p\cdot(\beta_1+1)]_1-1,[p\cdot(\beta_2+1)]_2-1,\cdots,[p\cdot(\beta_{n+1}+1)]_{n+1}-1\right). \tag{16.10}$$

Here, for an integer r and $i=1,2,\ldots,n+1$ by $[r]_i$ we denote the unique positive integer with $0\leq[r]<m_i$. The proof of the theorem is reduced to Problem 21.10 items 3 and 2 in Chapter 21. $\square$

We have a distinction of $\beta\in I$, $\frac{n}{2}<A_{\beta}<\frac{n}{2}+1$ based on whether B_{β} is an algebraic number or not. Therefore, we define

$$I^{\frac{n}{2},\frac{n}{2}}_{\mathrm{alg}}:=\left\{\beta\in I\,\middle|\,\frac{n}{2}<A_{\beta}<\frac{n}{2}+1,\ \ B_{\beta}\in\bar{\mathbb{Q}}\right\},$$
$$I^{\frac{n}{2},\frac{n}{2}}_{\mathrm{tra}}:=\left\{\beta\in I\,\middle|\,\frac{n}{2}<A_{\beta}<\frac{n}{2}+1,\ \ B_{\beta}\notin\bar{\mathbb{Q}}\right\}.$$

There are three well-known formulas involving the Gamma function which may be used in order to check whether $B_{\beta}\in\bar{\mathbb{Q}}$ or not. These are

$$\Gamma(z+1)=z\cdot\Gamma(z), \tag{16.11}$$

Euler's reflection formula

$$\frac{1}{2\pi i}\Gamma(z)\Gamma(1-z) = \frac{1}{e^{\pi i z} - e^{-\pi i z}}, \tag{16.12}$$

and the Gauss multiplication relation:

$$\frac{1}{(2\pi i)^{\frac{m-1}{2}}}\frac{1}{\Gamma(m\cdot z)}\prod_{k=0}^{m-1}\Gamma(z+\frac{k}{m}) = m^{\frac{1}{2}-m\cdot z} i^{-\frac{m-1}{2}}, \tag{16.13}$$

for $z \in \mathbb{C}$. We call these the standard relations of the Γ function. The affirmation of a number being algebraic or transcendental, and hence determining the sets $I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}$ and $I_{\text{tra}}^{\frac{n}{2},\frac{n}{2}}$, is a hard open problem in transcendental number theory. The proof of Theorem 16.1 and the corresponding Problem 21.10 in Chapter 21 suggest to redefine:

$$I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}} := \left\{\beta \in I \,\middle|\, \frac{n}{2} < A_{p*\beta} < \frac{n}{2}+1,\ \forall p \in (\frac{\mathbb{Z}}{d\mathbb{Z}})^{\times}\right\}, \tag{16.14}$$

$$I_{\text{tra}}^{\frac{n}{2},\frac{n}{2}} := \left\{\beta \in I \,\middle|\, \frac{n}{2} < A_{\beta} < \frac{n}{2}+1,\ \beta \notin I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}\right\}. \tag{16.15}$$

The new $I_{\text{tra}}^{\frac{n}{2},\frac{n}{2}}$ might be bigger than the old one and it has the advantage that it is possible to prove the following without proving an extremely more difficult problem that B_β is a transcendental number.

Proposition 16.3 *For a Hodge cycle $\delta \in \text{Hodge}_n(X,\mathbb{Z})$ the periods of ω_β, $\beta \in I_{\text{tra}}^{\frac{n}{2},\frac{n}{2}}$ over δ are zero.*

Proof. If $\beta \in I_{\text{tra}}^{\frac{n}{2},\frac{n}{2}}$ then there is a $p \in (\frac{\mathbb{Z}}{d\mathbb{Z}})^{\times}$ such that $A_{p*\beta} < \frac{n}{2}$. The affirmation follows from the definition of a Hodge cycle. □

For a Hodge cycle $\delta \in \text{Hodge}_n(X,\mathbb{Z})$ the periods of ω_β, $\beta \in I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}$ over δ are in an abelian extension of the cyclotomic field $\mathbb{Q}(\zeta_d)$. This statement is due to P. Deligne in [DMOS82], Proposition 7.2 and Theorem 7.15 (a). A proof can be given after analyzing the final step of the proof of Theorem 16.1 in which we have used Problem 21.10 items 3 and 2 in Chapter 21. Class field theory studies such extensions.

We are going to discuss a conjecture (Conjecture 16.1) which together with Problem 21.10 in Chapter 21 implies that the new definitions of $I_{\text{tra}}^{\frac{n}{2},\frac{n}{2}}$, $I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}$ are the same as the old ones. We have used Griffiths' theorem on the cohomology of hypersurfaces (Theorem 11.3) in order to write down an explicit basis for the de Rham cohomology of the Fermat variety and compatible with the Hodge filtration, we have computed the integrals of such a basis over topological cycles (Proposition 15.1), and then we have computed the periods of Hodge cycles. If we assume the Hodge conjecture for Fermat variety then it follows that such periods are algebraic up to a power of $2\pi\sqrt{-1}$ factor. This amounts to the fact that either such a period is zero or the corresponding beta factor is algebraic. This has been proved in Theorem 16.1

independent of the Hodge conjecture. P. Deligne has conjectured that the algebraicity statement must follow from the standard relations (16.11), (16.12) and (16.13) of the Γ function. This has been proved by A. Ogus and N. Koblitz in the appendix of Deligne's article [Del79] which is reformulated in Problem 21.10, Chapter 21. This ends up naturally to the fact that Hodge cycles of the Fermat variety are absolute. It is now natural to state the following conjecture:

Conjecture 16.1 *Any multiplicative dependence relation of the form*

$$(2\pi i)^{\frac{m}{2}}\Gamma(z_1)^{m_1}\Gamma(z_2)^{m_2}\cdots\Gamma(z_k)^{m_k}\in\bar{\mathbb{Q}} \tag{16.16}$$

with $m, m_1, \cdots, m_k \in \mathbb{Z}$ *and* $z_1, z_2, \cdots, z_k \in \mathbb{Q}$*, is a consequence of the standard relations* (16.11),(16.12) *and* (16.13).

In our Hodge theoretic derivation of Conjecture 16.1 it is natural to attribute it to P. Deligne, see for instance the appendix of N. Koblitz and A. Ogus in [Del79] and N. Koblitz' article [Kob78]. However, the history of Conjecture 16.1 in transcendental number theory is even older. In [Wal06] page 445 this conjecture is attributed to D. Rohrlich and it is mentioned that there is a generalization of this due to S. Lang, see [Lan78] page 40-03. See also [Wal16] for an up-to-date account on this. " I thought of this conjecture in the late 1970's and mentioned it to several people including Dick Gross and Serge Lang. Although I arrived at the conjecture independently, I do not want to claim it as my conjecture, because some people seem to regard it as a folk conjecture of many years earlier," (D. Rohrlich, personal communication, November 26, 2016). In N. Schappacher's monograph [Sch88] page 110, Conjecture 16.1 is referred as "folklore conjecture", whereas in page 136 it is referred as "Rohrlich conjecture". After all these, it is natural to call Conjecture 16.1 as Deligne-Lang-Rohrlich conjecture as the author (and many others involved in this conjecture) is not aware of any other appearance of this apart from those mentioned above.

In general, it is hard to prove that a given number is transcendental. This also applies to products of values of the Γ function at rational numbers. J. Wolfart in [Wol83] has used a theorem of A. Baker and has proved that a certain product of exponential of logarithmic derivatives of Γ values is transcendental. J. Wolfart and G. Wüstholz' in [WW85] page 6 have mentioned the contribution of P. Deligne, A. Ogus and N. Koblitz, as we discussed earlier, and have proved that certain products of Γ values at rational numbers are transcendental. One might expect that Conjecture 16.1 follows from more general conjectures on periods, such as Grothendieck's period conjecture or Kontsevich-Zagier conjecture, see [BC16, KZ01] and the references therein. One of the referees of the present text mentioned that this is indeed the case, see Chapter 24 of [And04]. It is beyond the scope of the present text to go into further details of this. For a relation between multiplicative relations of Γ values and the Gauss sum see also [Aok13].

16.4 An invariant of Hodge cycles

In this section we present an invariant of Hodge cycles which gives us a good criterion in order to distinguish them among each other. It was originally computed in [Mov17b] for Hodge cycles of the classical Fermat variety, using the so-called infinitesimal variation of Hodge structures, see [CGGH83].

Let us define

$$\begin{aligned} I^{\frac{n}{2}+1,\frac{n}{2}-1} &:= \left\{ j \in I \middle| \frac{n}{2} - 1 < A_j < \frac{n}{2} \right\}, \\ I^{\frac{n}{2},\frac{n}{2}} &:= \left\{ \beta \in I \middle| \frac{n}{2} < A_\beta < \frac{n}{2} + 1 \right\}, \\ I_d &:= \left\{ i \in I \middle| \sum_{k=0}^{n+1} \frac{1}{m_k} < A_i < 1 + \sum_{k=0}^{n+1} \frac{1}{m_k} \right\}, \quad m_0 := d. \end{aligned}$$

Note that we have used the multi-index i for the elements of I_d and j for $I^{\frac{n}{2}+1,\frac{n}{2}-1}$. The cardinality of these sets can be computed in a closed formula in the classical Fermat case, see Exercise 16.2. Throughout this section we assume that n is an even number and $I^{\frac{n}{2}+1,\frac{n}{2}-1}$ is not an empty set. This is equivalent to:

$$2\sum_{i=0}^{n+1} \frac{1}{m_i} \leq n, \quad \text{where } m_0 := d. \tag{16.17}$$

In the case of classical Fermat variety this is $d \geq 2 + \frac{4}{n}$. It is worth to highlight that if (16.17) is not satisfied then all the n-dimensional cycles of X are automatically Hodge, see Exercise 16.7.

Definition 16.1 Consider a collection of numbers p_β indexed by $\beta \in I^{\frac{n}{2},\frac{n}{2}}$. For any other β which is not in the set $I^{\frac{n}{2},\frac{n}{2}}$, p_β by definition is zero. Let $[\mathsf{p}_{i+j}]$ be the matrix whose rows and columns are indexed by $i \in I^{\frac{n}{2}+1,\frac{n}{2}-1}$ and $j \in I_d$, respectively, and in its (i,j) entry we have p_{i+j}.

We start this section with the following elementary problem.

Proposition 16.4 *For the case $d := m_1 = m_2 = \cdots = m_{n+1}$ and under the assumption* (16.17)*, if all the numbers $\mathsf{p}_\beta, \beta \in I^{\frac{n}{2},\frac{n}{2}}$ are not simultaneously zero then*

$$\binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2 \leq \operatorname{rank}([\mathsf{p}_{i+j}]) \leq \begin{cases} \#I^{\frac{n}{2}+1,\frac{n}{2}-1} = \binom{\frac{n}{2}d-1}{n+1} & \text{if } d < \frac{2(n+1)}{n-2}, \\ \#I^{\frac{n}{2}+1,\frac{n}{2}-1} = \binom{d+n}{n+1} - (n+2) & \text{if } d = \frac{2(n+1)}{n-2}, \\ \#I_d = \binom{d+n+1}{n+1} - (n+2)^2 & \text{if } d > \frac{2(n+1)}{n-2}. \end{cases} \tag{16.18}$$

The lower and upper bounds are sharp, that is, there are complex numbers p_β such that the lower (resp. upper) bound is attained.

For $n = 2$ only the first case for the upper bound occurs. In this case

$$d - 3 \leq \operatorname{rank}([\mathsf{p}_{i+j}]) \leq \binom{d-1}{3}$$

which is the well-known range for the codimension of the components of the Noether-Lefschetz locus, see §16.5. The second case for the upper bound only occurs for $n = 4, 8$. For $(n,d) = (4,5)$ and $(8,3)$ the upper bound is $246, 45$, respectively.

Proof. The proposition is an elementary high school problem and it is left to the reader. For the discussion of the lower bound see §21.3. Note that the upper bound is the minimum of $\#I_d$ and $\#I^{\frac{n}{2}+1,\frac{n}{2}-1}$. This is a canonical upper bound for the rank of a matrix. Using (16.17) we know that

$$\frac{n}{2}d - n - 2 \text{ and } d < \frac{(n+2)(d-2)+1}{2}$$

and so $(\frac{n}{2}d - n - 2) \bullet d$ then $\#I_d \bullet \#I^{\frac{n}{2}+1,\frac{n}{2}-1}$, where $\bullet$ is one of $<, >$ or $=$. We have distinguished three cases $(\frac{n}{2}d - n - 2) \bullet (d-1)$, where $\bullet$ is as before. $\square$

We are going to interpret p_i's as periods of Hodge cycles. Let $g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$.

Definition 16.2 Let X be the generalized Fermat variety. For a Hodge cycle $\delta \in H_n(X,\mathbb{Z})_0$ we define

$$\begin{aligned}\mathsf{p}_\beta = \mathsf{p}_\beta(\delta) &:= (2\pi i)^{-\frac{n}{2}} \int_\delta \operatorname{Resi}\left(\frac{x^\beta dx}{(g-1)^{\frac{n}{2}+1}}\right) \\ &= (2\pi i)^{-\frac{n}{2}} \frac{(A_\beta - \frac{n}{2})_{\frac{n}{2}}}{\frac{n}{2}!} \int_\delta \omega_\beta, \qquad \beta \in I^{\frac{n}{2},\frac{n}{2}}.\end{aligned}$$

We define

$$\xi(\delta) := \operatorname{rank}[\mathsf{p}_{i+j}(\delta)]. \tag{16.19}$$

and call it the ξ-invariant of the Hodge cycle δ.

16.5 The geometric interpretation of ξ

In this section we explain the geometric meaning of the number $\xi(\delta)$. Our discussion in this section can be reformulated for any homogeneous tame polynomial $g \in \mathbb{C}[x]$. Let $\{X_t\}_{t\in\mathsf{T}}$ be a family of hypersurfaces $X_t \subset \mathbb{P}^{(1,\nu)}$, $t \in \mathsf{T}$ such that in the affine chart $x_0 = 1$, X_t is given by:

$$g - 1 + \sum_{i\in I_d} t_i x^i = 0,$$

where t_i's are parameters. Here, I_d can be any set of exponents such that the monomials $x^i,\ \ i \in I_d$ are of degree $\leq d$. For us T is the affine space with the coordinate system $t = (t_i,\ i \in I_d)$. We work in a small neighborhood $(\mathsf{T},0)$ of 0 in the usual topology of T, therefore, $(\mathsf{T},0)$ is simply connected and so any two paths connecting 0 to another point are homotopic to each other. For a Hodge cycle $\delta \in \mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}}$, let $\delta_t \in H_n(X_t,\mathbb{Z})_{\check{0}}$ be the monodromy of δ to the hypersurface X_t. We consider sections $\omega_1,\omega_2,\ldots,\omega_a$ of the cohomology bundle $H^n_{\mathrm{dR}}(X_t),\ t \in (\mathsf{T},0)$ such that for any $t \in (\mathsf{T},0)$ they form a basis of the $F^{\frac{n}{2}+1}$-piece of the Hodge filtration of $H^n_{\mathrm{dR}}(X_t)_0$. Let $\mathscr{O}_{\mathsf{T},0}$ be the ring of holomorphic functions in a neighborhood of 0 in T. We have the elements

$$\int_{\delta_t} \omega_i \in \mathscr{O}_{\mathsf{T},0},\ \ i = 1,2,\ldots,a.$$

Definition 16.3 The (analytic) Hodge locus passing through 0 and corresponding to δ is the analytic variety

$$V_\delta := \left\{\ t \in (\mathsf{T},0)\ \middle|\ \int_{\delta_t}\omega_1 = \int_{\delta_t}\omega_2 = \cdots = \int_{\delta_t}\omega_a = 0\ \right\}. \tag{16.20}$$

We consider it as an analytic scheme with

$$\mathscr{O}_{V_\delta} := \mathscr{O}_{\mathsf{T},0} \Big/ \left\langle \int_{\delta_t}\omega_1, \int_{\delta_t}\omega_2, \cdots, \int_{\delta_t}\omega_a \right\rangle. \tag{16.21}$$

In the two dimensional case, that is $\dim(X_t) = 2$, the Hodge locus is usually called Noether-Lefschetz locus.

The Hodge locus is given by the vanishing of $a = h^{\frac{n}{2}+1}$ holomorphic functions in t. By definition of a Hodge cycle, we already know that 0 is a point of this variety. This is a local analytic, not necessarily irreducible, subset of T and by a deep theorem of Cattani-Deligne-Kaplan in[CDK95] we know that it is algebraic, in the sense that it is an open subset of an algebraic subvariety of T. This is also a consequence of the Hodge conjecture. The Hodge locus $\mathrm{Ho}_n(X/\mathsf{T})$ in T is the union of all such local loci defined as before. A general version of the above definition independent of the point $0 \in \mathsf{T}$, can be given in the following way. In view of the Cattani-Deligne-Kaplan theorem we only consider the algebraic context.

Definition 16.4 An irreducible component H of the Hodge locus $\mathrm{Ho}_n(X/\mathsf{T})$ is any irreducible closed subvariety of T with a continuous family of Hodge cycles $\delta_t \in H_n(X_t,\mathbb{Z}),\ t \in H$ such that for points t in a Zariski open subset of H, the monodromy of δ_t to a point in a neighborhood (in the usual topology) of t and outside H, is no more a Hodge cycle.

Theorem 16.2 *The Zariski tangent space of the Hodge locus passing through the Fermat point* 0 *and corresponding to the Hodge cycle* $\delta \in \mathrm{Hodge}_n(X^d_n,\mathbb{Z})_{\check{0}}$ *is given by*

$$\ker([\mathsf{p}_{i+j}]) := \left\{ v \in \mathbb{C}^{\#I_d} \quad \middle| \quad [\mathsf{p}_{i+j}] \cdot v = 0 \right\},$$

and hence, it is of codimension $\xi(\delta)$.

We have a reason to use the scheme theoretic language in (16.21), this is namely, in Theorem 16.2 we have used the Zariski tangent space of the Hodge locus as an analytic scheme and not as the set (16.20). Recall that for a coordinate function x of $\mathbb{C}$ the Zariski tangent spaces of $\mathrm{Spec}(\langle x \rangle)$ and $\mathrm{Spec}(\langle x^2 \rangle)$ are respectively 0 and $\mathbb{C}$, but the corresponding zero set in both cases is $\{0\}$.

The origin of Theorem 16.2 lies in the so called infinitesimal variation of Hodge structures (IVHS) of families of projective varieties developed by P. Griffiths and his collaborators, see [CGGH83]. Voisin in [Voi03] 5.3.3 relates IVHS with the Zariski tangent space of the Hodge locus. For Fermat varieties one has to do further computations to get Theorem 16.2. This is first done in [Mov17b]. Below, we give an elementary proof of Theorem 16.2. One of the main objectives of the present chapter is to use ξ and study Hodge cycles, independent of the fact that whether they are algebraic or not.

Proof. We consider sections

$$\omega_1, \omega_2, \ldots, \omega_c,$$

$$\check{\omega}_1, \check{\omega}_2, \cdots, \check{\omega}_a,$$

$$\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_b$$

of the cohomology bundle $H^n_{\mathrm{dR}}(X_t)_0,\ t \in (\mathsf{T}, 0)$ such that for $t \in (\mathsf{T}, 0)$ the first, second and third group form a basis of the $F^{\frac{n}{2}+2}$, $F^{\frac{n}{2}+1}/F^{\frac{n}{2}+2}$ and $F^{\frac{n}{2}}/F^{\frac{n}{2}+1}$. More importantly, these sections have to be part of the Griffiths-Steenbrink basis of the Gauss-Manin system as in Definition 11.7.

Let $f_i, \check{f}_i, \hat{f}_i$ be the integral of these differential forms over δ_t. These are holomorphic functions in a neighborhood of 0 in T. By Griffiths transversality theorem $df_i, i = 1, 2, \cdots, c$ can be written as a linear combination of $f_i, \check{f}_i$'s with coefficients which are holomorphic differential 1-forms, and hence, restricted to V_δ they are automatically zero. Therefore, the Zariski tangent space of the Hodge locus passing through 0 and corresponding to δ_0 is the kernel of the linear part of $[d\check{f}_1, d\check{f}_2, \cdots, d\check{f}_a]^{\mathsf{tr}}$. The assertion follows from the computation of the linear part of $\check{f}_i$'s in Theorem 13.2. $\square$

For the classical presentation of Hodge locus in the literature, as well as a complete overview of results in this direction, the reader is referred to Voisin's article [Voi13b].

16.6 ξ-invariant of $\check{\delta}$

Recall the Hodge cycle $\check{\delta}$ introduced in §15.16. In this section we would like to compute its ξ invariant. The B-factors of the periods of the differential forms in

(15.60) are

$$B_\beta := (2\pi i)^{-\frac{n}{2}} B(\frac{\beta_1+1}{d}, \frac{1}{2}, \cdots, \frac{1}{2}) = (2\pi i)^{-\frac{n}{2}} \Gamma\left(\frac{1}{2}\right)^n \cdot \left(\frac{\beta_1+1}{d}\right)^{-1}_{\frac{n}{2}}$$
$$= (2i)^{-\frac{n}{2}} \left(\frac{\beta_1+1}{d}\right)^{-1}_{\frac{n}{2}} = (2i)^{-\frac{n}{2}} \left(A_\beta - \frac{n}{2}\right)^{-1}_{\frac{n}{2}}$$

We have also

$$(2\pi i)^{-\frac{n}{2}} \int_{\delta_{\beta'}} \mathrm{Resi}\left(\frac{x^\beta dx}{(g-1)^{\frac{n}{2}+1}}\right) = \frac{(A_\beta - \frac{n}{2})_{\frac{n}{2}}}{\frac{n}{2}!} \int_{\delta_{\beta'}} \omega_\beta.$$

This means that $B_\beta$factors will not play any role in the definition of $\mathsf{p}_\beta(\check{\delta})$ and only the algebraic part of the periods computed in Proposition 15.1 matters for the definition of ξ. We have collected our results in the table below. The computer codes of this computation are explained in §20.9.

$n\backslash d$	2	4	6	8	10	12	14	16	18	20	22
2	*	1	6	13	22	33					
4	*	11	71								
6											

Table 16.1 The ξ-invariant of the Hodge cycle $\check{\delta}$

16.7 Linear Hodge cycles

Let

$$I^{\frac{n}{2},\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}} := \left\{ \beta \in I \,\middle|\, \frac{\beta_i+1}{m_i} + \frac{\beta_{\sigma(i)}+1}{m_{\sigma(i)}} = 1, \;\; i = 0,1,2,\ldots,n+1, \;\; \text{for some } \sigma \right\}, \tag{16.22}$$

where σ is a permutation of $0,1,\ldots,n+1$ without fixed point and with σ^2 being identity. Here, $\frac{\beta_0+1}{m_0} := \frac{n}{2}+1-A_\beta$ arises from the projectivization of L. The reason for this definition will be explained in §17.4.

Definition 16.5 The $\mathbb{Z}$-module of linear Hodge cycles is

$$\mathrm{Hodge}_n(X,\mathbb{Z})_{\mathbb{P}^{\frac{n}{2}}} := \left\{ \delta \in \mathrm{Hodge}_n(X,\mathbb{Z}) \,\middle|\, \int_\delta \omega_\beta = 0, \;\; \forall \beta \in I^{\frac{n}{2},\frac{n}{2}} \backslash I^{\frac{n}{2},\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}} \right\}. \tag{16.23}$$

The B_β factor of the elements of $I^{\frac{n}{2},\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}}$ can be calculated easily from the Euler's reflection formula (16.12):

$$B_\beta = \left((A_\beta - \frac{n}{2})_{\frac{n}{2}} \prod_{j\in A} \left(\zeta_{2m_j}^{\beta_j+1} + \zeta_{2m_{\sigma(j)}}^{\beta_{\sigma(j)}+1} \right) \right)^{-1}, \tag{16.24}$$

where A is a set of cardinality $\frac{n}{2}$ such that $A \cup \sigma(A) \cup \{0, \sigma(0)\} = \{0, 1, \ldots, n+1\}$. Combining the equality (16.24) and Proposition 15.1 we have the following formula:

$$(2\pi i)^{-\frac{n}{2}} \int_{\delta_{\beta'}} \mathrm{Resi} \left(\frac{x^\beta dx}{(g-1)^{\frac{n}{2}+1}} \right) = \frac{(-1)^n}{\frac{n}{2}! \prod_{i=1}^{n+1} m_i} \frac{\prod_{i=1}^{n+1} \left(\zeta_{m_i}^{(\beta_i'+1)(\beta_i+1)} - \zeta_{m_i}^{\beta_i'(\beta_i+1)} \right)}{\prod_{j\in A} \left(\zeta_{2m_j}^{\beta_j+1} + \zeta_{2m_{\sigma(j)}}^{\beta_{\sigma(j)}+1} \right)} \tag{16.25}$$

for $\beta \in I^{\frac{n}{2},\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}}$. This will be useful for computing the ξ-invariant of Hodge cycles, see Definition 16.2.

Proposition 16.5 *The $\mathbb{Z}$-module of primitive linear Hodge cycles in* $\mathrm{Hodge}_4(X_4^6, \mathbb{Z})_{\check{0}}$ *is of rank* 1001*, and so,*

$$\dim_{\mathbb{Q}} \mathrm{Hodge}_n(X, \mathbb{Q})_{\mathbb{P}^{\frac{n}{2}}} = 1002.$$

Proof. This follows from from our computer implementation of the space of linear Hodge cycles, see §20.7. □

In Table 15.12 we have used the notation $\rho_{\mathbb{P}^1} := \dim_{\mathbb{Q}} \mathrm{Hodge}_2(X, \mathbb{Q})_{\mathbb{P}^1}$.

16.8 General Hodge cycles

Let X be the generalized Fermat variety. We have a finitely generated free $\mathbb{Z}$-module $\mathrm{Hodge}_n(X, \mathbb{Z})_{\check{0}}$ and the ξ-map

$$\xi : \mathrm{Hodge}_n(X, \mathbb{Z})_{\check{0}} \to \mathbb{N},$$

which is not $\mathbb{Z}$-linear. It satisfies $\xi(a\delta) = \xi(\delta),\ a \in \mathbb{Z}$ and we do not have any comparison of the numbers $\xi(\delta_1 + \delta_2),\ \xi(\delta_1)$ and $\xi(\delta_2)$ for $\delta_1, \delta_2 \in \mathrm{Hodge}_n(X_n^d, \mathbb{Z})_{\check{0}}$.

Definition 16.6 Let A be a finitely generated free $\mathbb{Z}$-module and let B a subset of A. We say that B is Zariski dense in A if B is Zariski dense in the affine variety $A \otimes_{\mathbb{Z}} \mathbb{C}$. By a general element of A we mean an element in some Zariski dense subset of A.

Definition 16.7 A Hodge cycle δ such that the upper bound in (16.18) is attained is called a general Hodge cycle. If the upper bound is not attained then it is called a special Hodge cycles.

The following conjecture justifies the previous definitions.

Conjecture 16.2 *A generalized Fermat variety X with $2\sum_{i=0}^{n+1}\frac{1}{m_i}\leq n$ has always a general Hodge cycle, and so, the set of Hodge cycle of X such that the upper bound in* (16.18) *is attained is Zariski dense in* $\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}}$.

The upper bound in (16.18) is the canonical upper bound for the rank of a matrix. This is the minimum of the number of rows and columns of $[\mathsf{p}_{i+j}]$. Since $\mathrm{Hodge}_n(X,\mathbb{Q})_0$ is a vector space, in order to prove Conjecture 16.2 it is enough to find at least one Hodge cycle δ with $\xi(\delta)$ which is the upper bound in (16.18). For particular values of $m_1,m_2,\ldots,m_{n+1}$ one can compute $\mathrm{Hodge}_n(X,\mathbb{Q})_0$ and any random choice of δ is most probably a good candidate for this. In general, I do not know how to pick up such a cycle. Note that our computations in §16.6 tells us that the cycle $\check{\delta}$ is not general.

We now consider the two dimensional classical Fermat variety

$$X = X_2^d: \quad x^d + y^d + z^d + w^d = 0.$$

The origin of Definition 16.7 comes from the notion of general and special componenents of the Noether-Lefschetz locus (Hodge locus) in this case, see [CHM88, Voi88]. Recall that for $d = 1,2,3$ Hodge cycles generate the whole homology group of any smooth surface $X \subset \mathbb{P}^3$ of degree d, that is

$$\mathrm{Hodge}_2(X,\mathbb{Z}) = H_2(X,\mathbb{Z}), \quad d = 1,2,3. \tag{16.26}$$

For a detailed study of these cases see Exercise 16.3. Recall that in order to define the matrix $[\mathsf{p}_{i+j}]$, and hence the ξ-invariant, of Hodge cycles we need to assume that $d \geq 2 + \frac{4}{2} = 4$. We know that $[\mathsf{p}_{i+j}]$ is a $r \times a$ matrix, where

$$a := \#I^{2,0} = \binom{d-1}{3}, \qquad r := \#I_d = \binom{d+3}{3} - 16.$$

A Hodge cycle with

$$\xi(\delta) = \binom{d-1}{3} \tag{16.27}$$

is called general and those with

$$d-3 \leq \xi(\delta) < \binom{d-1}{3} \tag{16.28}$$

are called special Hodge cycles. Note that $a < r$ and so a is the canonical upper bound for ξ. Later we will need to know for which values of d, $2a > r$. This equality is valid for $d \geq 18$ and no other small values of d.

Proposition 16.6 *For $4 \leq d \leq 8$, the Fermat surface of degree d has a general Hodge cycle and so Conjecture 16.2 is true in these cases.*

Proof. The proof uses the algorithms introduced in Chapter 15. We only need to find at least one general Hodge cycle. We find a basis of Hodge cycles as it is explained

in §15.14 and then we compute a matrix H whose rows are periods of Hodge cycles. We choose the first row of H which corresponds to the periods of a Hodge cycle let us say δ. We then construct the corresponding $[\mathsf{p}_{i+j}]$ matrix and compute its rank. For all the cases of d announced in the proposition, $[\mathsf{p}_{i+j}]$ has the maximal rank and so δ is general. The upper bound for d is just the limit of our computer device, and it might be improved if we use a better one. For some details of our computations see §20.8. □

Let us discuss the case $n \geq 4$, $d > \frac{2(n+1)}{n-2}$. Conjecture 16.2 claims that Hodge cycles with

$$\xi(\delta) = \binom{d+n+1}{n+1} - (n+2)^2 \tag{16.29}$$

are Zariski dense in $\mathrm{Hodge}_n(X,\mathbb{Z})_{\check{0}}$. This number is the dimension of the moduli space of hypersurfaces of dimension n and degree d. This implies that most of the Hodge cycles of the Fermat variety cannot be deformed. As far as Conjecture 16.2 is not proved in general, random choices of Fermat varieties and proving it by a computer would be a good alternative. Here is one example which is within the limit of our computers.

Proposition 16.7 *Conjecture 16.2 is true for the Fermat sextic fourfold*

$$X_4^6 : x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 = 0. \tag{16.30}$$

Proof. In this case the matrix $[\mathsf{p}_{i+j}]$ is a quadratic matrix of dimension 426. It is enough to find a Hodge cycle $\delta \in \mathrm{Hodge}_n(X_4^6,\mathbb{Z})_{\check{0}}$ such that

$$\det([\mathsf{p}_{i+j}]) \neq 0.$$

For some details of our computations see §20.8. □

16.9 Periods of join Hodge cycles

In this section we give an application of Proposition 13.11 and Theorem 16.1.

Proposition 16.8 *Let L be the smooth curve (resp. surface) given by $y^d - f(x) = 0$ (resp. $z^d - f(x,y) = 0$) in an affine chart $(x,y) \in \mathbb{C}^2$ (resp. $(x,y,z) \in \mathbb{C}^3$), where f is a degree d polynomial in x (resp. x,y) and with coefficients in $\mathbb{Q}$. In the surface case, we further assume that the last homogeneous piece of $f(x,y)$ is a product of distinct lines. The following periods/multiple integrals*

$$\frac{1}{2\pi\sqrt{-1}} \frac{\Gamma(\frac{1}{d})\Gamma(\frac{d-2}{d})}{\Gamma(\frac{d-1}{d})} \int_\delta \frac{x^i y^{dk-2} dx \wedge dy}{(y^d - f(x))^{k+1}},\ i,k-1 \in \mathbb{N}_0,\ i < d = 3,4,6, \tag{16.31}$$

for all $\delta \in H_2(\mathbb{C}^2 - L, \mathbb{Q})$ (resp.

$$\frac{1}{2\pi\sqrt{-1}}\int_{\delta}\frac{x^i y^j z^{dk-3}dx\wedge dy\wedge dz}{(z^d-f(x,y))^{k+1}},\ i,j,k-1\in\mathbb{N}_0,\ i+j<d-1,\ d=3,4, \quad (16.32)$$

for all $\delta\in H_3(\mathbb{C}^3-L,\mathbb{Z})$*) are algebraic numbers.*

Proof. It is enough to prove Proposition 16.8 for $k=1$. For this we use (11.1) and (11.2) for tame polynomials of the form $f:=y^d-\tilde{f}(x_1,x_2,\dots,x_{n+1})$:

$$\begin{aligned}\frac{x^\beta y^k dx\wedge dy}{f^i} &= \frac{1}{d}\frac{x^\beta y^{k-d+1}dx\wedge df}{f^i}\\ &= \frac{-1}{d\cdot(i-1)}\frac{d(x^\beta y^{k-d+1}dx)}{f^{i-1}}\\ &= \frac{(-1)^n}{d\cdot(i-1)(k-d+1)}\frac{x^\beta y^{k-d}dx\wedge dy}{f^{i-1}} \quad \text{in } \mathsf{M}\end{aligned}$$

for $k\neq d-1$ and it is zero for $k=d-1$. Despite this, we proceed the proof for an arbitrary k. Recall the terminology in Proposition 13.11. We will only consider the case where all the variables x_i and y_i's have weight one, $g=y_1^d+y_2^d+\cdots+y_{m+1}^d$, f is a polynomial of degree $d\geq 3$ defined over $\mathbb{Q}$, and $n+m+1$ is an even number. Let δ_1 be a vanishing cycle for f and δ_2 an arbitrary cycle from

$$\left\{\delta_2\in H_m(\{g=1\},\mathbb{Q})\,\middle|\,\int_{\delta_2}\frac{y^{\beta_2}dy}{dg}=0,\ \forall\beta_2\in\mathbb{N}_0^{m+1},\ A_{\beta_2}<\frac{n+m+1}{2}-\frac{n+1}{d}\right\}. \quad (16.33)$$

The following

$$\delta_1 *_{t.} \delta_2\in H_{n+m+1}(\{f-g=0\},\mathbb{Q})$$

is a Hodge cycle. This is a direct consequence of the definition of a Hodge cycle and Proposition 13.11. Note that we have only defined the join of vanishing cycles and for an arbitrary $\delta_2\in H_m(\{g=1\},\mathbb{Q})$ we first write this in terms of a distinguished set of vanishing cycles δ_i^g, $i=1,2,\dots,\mu_g$, that is $\delta_2=\sum_{i=1}^{\mu_g}n_i\delta_i^g$, and define $\delta_1 *_{t.} \delta_2=\sum_{i=1}^{\mu_g}n_i\delta_1 *_{t.} \delta_i^g$, where $t.$'s are the corresponding distinguished set of paths from the critical value of f corresponding to δ_1 to 0.

The equality (15.10) implies that if $\frac{n+m+1}{2}-\frac{n+1}{d}\geq[\frac{m}{2}]+1$ then (16.33) consists of cycles at infinity and hence it gives a zero cycle in the compactification of $\{g=1\}$. Otherwise, we have necessarily $n=0$ and $n=1$. Let K be the image of (16.33) in the m-th homology of the compactified $\{g=1\}$. Next we observe $K\neq 0$ for $(n,d)=(0,3),(0,4),(0,6),(1,3),(1,4)$. We do this by computer, see §20.12. This seems to be an 'if and only if' statement. Note that for $n=0$ the differential forms $\frac{y^{\beta_2}dy}{dg}$, $A_{\beta_2}=\frac{m+1}{2}-\frac{1}{d}$ are missing in (16.33). If they were there, then we could conclude that δ_2 is a cycle at infinity. For $n=1$ the differential forms $\frac{y^{\beta_2}dy}{dg}$, $A_{\beta_2}=\frac{m}{2}+1-\frac{2}{d}$, $\frac{m}{2}+1-\frac{1}{d}$ are missing in (16.33). In the last step of the proof we apply Theorem 16.1 and conclude that

$$\frac{1}{(2\pi i)^{\frac{n+m+1}{2}}}\int_{\sigma(\delta_1 *_{t.} \delta_2)}\frac{x^{\beta_1}y^{\beta_2}dx\wedge dz}{(f-g)^{\frac{m+n+1}{2}+1}}\in\bar{\mathbb{Q}},\ \ A_{\beta_1}+A_{\beta_2}<\frac{m+n+1}{2}+1,$$

provided that $\delta_1 *_{t.} \delta_2$ is supported in an algebraic cycle (or it is an absolute Hodge cycle). Since we do not know this, we will proceed with the case $(n,m)=(0,1),(1,0)$ for which the Lefschetz $(1,1)$ theorem provides us with such an algebraic cycle. We use the above algebraicity for $n=0,\ \ \frac{\beta_1+1}{d}<1+\frac{1}{d},\ \ A_{\beta_2}=\frac{m+n+1}{2}-\frac{1}{d}$ and $n=1,\ \ \frac{\beta_1+1}{d}+\frac{\beta_2+1}{d}<1+\frac{2}{d},\ \ A_{\beta_2}=\frac{m+n+1}{2}-\frac{2}{d}$, see §20.12. We find $\int_{\delta_2}\frac{y^{\beta_2}dy}{dg}\neq 0$ and so by Proposition 13.11 we have

$$\frac{B_{\beta_2}}{(2\pi i)^{\frac{n+m+1}{2}}}\int_{\sigma(\delta_1 *_{t.} \delta_3)}\frac{x^{\beta_1}z^{\beta_3}dx\wedge dz}{(f-z^d)^{\frac{m+n+1}{2}+1}}\in\bar{\mathbb{Q}}, \tag{16.34}$$

for $n=0,\frac{\beta_1+1}{d}<1+\frac{1}{d},\ \ \frac{\beta_3+1}{d}=\frac{m+1}{2}-\frac{1}{d}$ and $n=1,\frac{\beta_1+1}{d}+\frac{\beta_2+1}{d}<1+\frac{2}{d},\ \ \frac{\beta_3+1}{d}=\frac{m+1}{2}-\frac{2}{d}$. We have

$$\frac{B_{\beta_2}}{(2\pi i)^{\frac{n+m+1}{2}}}=\begin{cases}\frac{1}{2\pi i}\frac{\Gamma(\frac{1}{d})\Gamma(\frac{d-2}{d})}{\Gamma(\frac{d-1}{d})}, & n=0\\ \frac{1}{2\pi i}, & n=1\end{cases}$$

which finishes the proof. Note that for the first equality we take $m=1$ and $\beta_2=(0,d-3)$ and for $d=6$ we have two options for $\beta_2=(0,3),\ (1,2)$ with $A_{\beta_2}=1-\frac{1}{d}$. The quotient of the corresponding (16.34) must be an algebraic number which is actually the case $\frac{\Gamma(\frac{2}{6})\Gamma(\frac{3}{6})}{\Gamma(\frac{1}{6})\Gamma(\frac{4}{6})}=2^{-\frac{2}{3}}$. $\square$

In the case $L: y^3-f(x),\ \ \deg(f)=3$, we have an affine equation for a family of isotrivial elliptic curves X, that is, this family of elliptic curves has constant j-function. This is because any elliptic curve in this family has the multiplication by ζ_3 in y variable with three fixed points $(0,a)$, a a root of f. Taking one of these fixed point as the neutral element of X, we conclude that $X\cong\mathbb{C}/(\mathbb{Z}e^{\frac{2\pi i}{3}}+\mathbb{Z})$ and its j-invariant is zero. It follows that any integral $\int_\delta\omega,\ \ \omega\in H^1_{\mathrm{dR}}(X)$ in this case is algebraic. Note that Proposition 16.8 only implies the algebraicity of some of these integrals. Note also that in this case (16.31) is a particular case of Chowla-Selberg theorem, see Exercise 2.6 in Chapter 2. There have been generalizations of Chowla-Selberg theorem for differential forms in the first de Rham cohomology of *CM* abelian varieties by P. Deligne, see [Gro79, Theorem 4] and abelian varieties with multiplication with an order in an imaginary quadratic fields by D. Gross in [Gro78, Theorem 3]. For arbitrary Hodge structures this has been generalized and is known under Deligne-Gross conjecture, see [Fre17, MR04, Col93]. It seems to the author that in the decomposition of the Jacobian of curves involved in (16.31) we can find an isotrivial families of abelian varieties producing the statement (16.31). The statement (16.32) for $d=3$ follows from the fact that cubic surfaces have Hodge numbers $0,7,0$, and hence all the homological two cycles are Hodge. For $d=4$

it follows from the fact that the underlying surface and differential forms can be projected to the surface $z^2 = f(x,y)$ which has Hodge numbers $0,8,0$.

16.10 Computing periods of hypersurfaces (by Emre Sertöz)

We will demonstrate the validity of Proposition 16.8 on an example using numerical computation of periods. The user is invited to try many more examples using `PeriodSuite` [i] in [Ser19]. This package is operated through Magma [BCP97] and uses additional functionalities from SageMath [S$^+$20].

Let $f = y^4 + x^4 + z^4 - y^3w - 2w^4$, which defines a smooth quartic surface X in $\mathbb{P}^3$. In the algebraic de Rham cohomology $H^2_{\mathrm{dR}}(X/\mathbb{Q})$ we consider the form defined by

$$\omega := \operatorname{res}\frac{zw^3\Omega}{f^2},$$

where $\Omega = xdy\wedge dz\wedge dw - ydx\wedge dz\wedge dw + zdx\wedge dy\wedge dw - wdx\wedge dy\wedge dz$. For some basis $\gamma_1,\dots,\gamma_{22} \in H_2(X(\mathbb{C}),\mathbb{Z})$ of integral homology, we would like to approximate the period integrals

$$\frac{1}{2\pi i}\int_{\gamma_i}\omega, \quad \forall i = 1,\dots,22. \tag{16.35}$$

The command

```
X:=PeriodHomotopy(f : precision:=3000, bound_pole_order:=false)
```

will compute all periods

$$\int_{\gamma_i}\omega_j, \quad i,j = 1,\dots,22$$

for an explicit basis $\omega_1,\dots,\omega_{22} \in H_{\mathrm{dR}}(X/\mathbb{Q})$ and record it in X. The following commands compute the coordinates of ω in $\{\omega_j\}_{j=1}^{22}$:

```
B:=ComputeCohomologyBasis(f);
coords:=CoordinatesInCohomology(f,B,[zw^3]);
```

In this case, it turns out that $\omega = \omega_3$. But we will continue without using this special feature. The following commands rearrange the periods of ω_j's and divide by $2\pi i$ to obtain the periods (16.35):

```
CC:=CoefficientRing(Parent(X`periods));
P:=ChangeRing(coords,CC)*X`periods;
P:=(2*Pi(CC)*CC.1)^-1*P;
```

Using one of many standard techniques, we can now recover the minimal polynomials that are satisfied by the periods (16.35). We use the following command in `PeriodSuite`:

```
polys:=MinimalPolynomials(Eltseq(P) : precision:=X`precision);
```

[i] `https://github.com/emresertoz/PeriodSuite`

This computation is necessarily heuristic, since we only have approximations of numbers which we believe to be algebraic. Nevertheless, the results are convincing. With 3000 digits of precision, we find that all 22 periods (16.35) minimally satisfy the same polynomial:

$$\begin{aligned} p(t) := {} & 763855162569648738446345428774562447410940405261741783529487038796110233 6t^{24} \\ & - 176072012256614292793664107357724188448812747896913920t^{16} \\ & + 781709025217155728625767967082569836317900 80t^{12} - 608780728714390532703227593359360t^{8} \\ & + 360374814005636125163 52t^{4} - 282429536481. \end{aligned}$$

The number field $K = \mathbb{Q}[t]/(p(t))$ can also be represented by the polynomial

$$t^{24} - 8t^{20} + 200t^{16} + 1340t^{12} - 30880t^{8} + 100880t^{4} - 2156,$$

which was found using `OptimizedRepresentation` in Magma. The simple nature of the answers, when compared to the number of digits used, suggests that the periods (16.35) are indeed algebraic. This is, of course, the implication of Proposition 16.8.

16.11 Exercises

16.1. The compactification of the affine variety L in (15.2) is done in the following way:

$$Y := \mathbb{P}\{x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 0\} \subset \mathbb{P}^{\nu_1, \nu_2, \ldots, \nu_{n+1}}, \tag{16.36}$$

$$X := \mathbb{P}\{x_0^{m_0} + x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} = 0\} \subset \mathbb{P}^{\nu_0, \nu_1, \ldots, \nu_{n+1}}, \tag{16.37}$$

where $\nu_0 := 1$ and m_0 is the smallest common multiple of m_i's. The variety X is the union of the open set L and the codimension one subvariety Y. In the case $m_1 = m_2 = \cdots = m_{n+1}$, X is just obtained by the usual compactification of $\mathbb{C}^{n+1}$ in $\mathbb{P}^{n+1}$, and so, Y is smooth. However, in general Y and X are singular. A desingularization of X will replace Y with a union of divisors. Discuss this for particular examples of m_i's. Note that the singularities of X lies in Y.

16.2. For natural numbers N, n and d let us define

$$I_N := \left\{ (i_0, i_1, \ldots, i_{n+1}) \in \mathbb{Z}^{n+2} \mid 0 \le i_e \le d-2, \;\; i_0 + i_1 + \cdots + i_{n+1} = N \right\}. \tag{16.38}$$

Show that

$$\#I_N = \binom{N+n+1}{n+1} \qquad \text{for } N \le d-2, \tag{16.39}$$

$$\#I_{d-1} = \binom{d+n}{n+1} - (n+2), \tag{16.40}$$

$$\#I_d = \binom{d+n+1}{n+1} - (n+2)^2, \;\; d \ge 3. \tag{16.41}$$

Hint: Use the geometric series! The sets $I_{\frac{n}{2}d-n-2}$, $I_{(\frac{n}{2}+1)d-n-2}$ and I_d are in one to one bijection with the sets $I^{\frac{n}{2}+1,\frac{n}{2}-1}, I^{\frac{n}{2},\frac{n}{2}}$ and I_d defined in §16.4. These bijections are simply obtained by forgetting i_0.

16.3. Describe Hodge cycles of a smooth surface $X \subset \mathbb{P}^3$ of degree $d = 1,2,3$.

16.4. Prove Proposition 16.1 using elementary methods. At least this can be done by computer for particular examples of $m_1, m_2, \ldots, m_{n+1}$.

16.5. Let $H_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}$ be the $\mathbb{C}$-vector space generated by ω_β, $\beta \in I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}$ defined in (16.14). The integration bilinear map

$$\text{Hodge}_n(X,\mathbb{Q})_0 \times H_{\text{alg}}^{\frac{n}{2},\frac{n}{2}} \to \mathbb{C}$$

is non-degenerate and so

$$\dim_{\mathbb{Q}}(\text{Hodge}_n(X,\mathbb{Q})_0) = \#I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}.$$

This together with Proposition 16.3 implies that the set of Poincaré duals δ^{pd}, $\delta \in \text{Hodge}_n(X,\mathbb{Q})_0$ are in $H_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}$ and they generate it as a $\mathbb{C}$-vector space. This is the classical presentation of Hodge cycles in [Shi79a] and the references therein. Hint: The $\mathbb{Q}$-vector space generated by ω_β/B_β, $\beta \in I_{\text{alg}}^{\frac{n}{2},\frac{n}{2}}$ is invariant under the action of the Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and hence, it is a subspace of of $H^n(X,\mathbb{Q})_0$.

16.6. Let $H_{\mathbb{P}^{\frac{n}{2}}}^{\frac{n}{2},\frac{n}{2}}$ be the $\mathbb{C}$ vector space generated by ω_β, $\beta \in I_{\mathbb{P}^{\frac{n}{2}}}^{\frac{n}{2},\frac{n}{2}}$ defined in (16.22). The integration bilinear map

$$\text{Hodge}_n(X,\mathbb{Q})_{\mathbb{P}^{\frac{n}{2}},0} \times H_{\mathbb{P}^{\frac{n}{2}}}^{\frac{n}{2},\frac{n}{2}} \to \mathbb{C}$$

is non-degenerate and so

$$\dim_{\mathbb{Q}}(\text{Hodge}_n(X,\mathbb{Q})_{\mathbb{P}^{\frac{n}{2}}}) = \#I_{\mathbb{P}^{\frac{n}{2}}}^{\frac{n}{2},\frac{n}{2}} + 1.$$

In other words

$$\text{Hodge}_n(X,\mathbb{Q})_{\mathbb{P}^{\frac{n}{2}},0}^{\mathsf{pd}} = H_{\mathbb{P}^{\frac{n}{2}}}^{\frac{n}{2},\frac{n}{2}}$$

where pd is the Poincaré dual.

16.7. Let us consider natural numbers $2 \leq m_i \in \mathbb{N},\ i = 1,2\ldots,$ with n even and $n < 2\sum_{i=0}^{n+1} \frac{1}{m_i}$. Let also consider any tame polynomial f with a non-zero discriminant and with the last homogeneous piece $x^{m_1} + x^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$. We denote by X the projective variety which is a compactification of $f = 0$. Show that all the cycles in $H_n(X,\mathbb{Z})_0$ are Hodge. Hint: All the Hodge numbers $h^{i,n-i}$ of X are zero except the middle one $h^{\frac{n}{2},\frac{n}{2}}$. The most famous example of this situation is the case of cubic surfaces $m_1 = m_2 = m_3 = 3$. Can you prove the Hodge conjecture for this class of projective varieties?

16.8 (Modular curves). A modular curve is an example of Hodge locus with many applications in number theory. Let E_1 and E_2 be two elliptic curves (curves of genus one) over complex numbers. We say that E_1 is isogenous to E_2 if there is a non-constant morphism $f : E_1 \to E_2$ of algebraic curves. We also say that f is the isogeny between E_1 and E_2. Let us consider a one parameter family of elliptic curves, for instance

$$E_z\colon y^2 + xy - x^3 + \frac{36}{z-1728}x + \frac{1}{z-1728} = 0, \quad z \in T := \mathbb{C}\backslash\{0,1728\} \tag{16.42}$$

or

$$E_z : y^2 - 4x^3 + 12x - 4(2-4z) = 0, \quad z \in T := \mathbb{C}\backslash\{0,1\}. \tag{16.43}$$

Note that the j-invariant of the elliptic curve E_z in (16.42) is z.

1. Show that the set

$$M := \{(z_1,z_2) \in T \times T \ \mid\ E_{z_1} \text{ is isogenous to } E_{z_2}\}$$

   is a Hodge locus. Hint: The graph of an isogeny is an algebraic cycle. Write down the Hodge filtration of the product of two elliptic curves.
2. Show that M is an enumerable union of algebraic curves defined over $\mathbb{Q}$. What is the degree of these curves in the first and second coordinates?

16.9 (A non-reduced component of the Noether-Lefschetz locus). * This is Exercise 2, page 154 of [Voi03] adapted to the context of the present text. Let T be the parameter space of smooth surfaces $X \subset \mathbb{P}^3$ of degree d, and let $\check{\mathsf{T}} \subset \mathsf{T}$ be the subvariety parameterizing surfaces $X_t,\ t \in \check{\mathsf{T}}$ with two lines $\mathbb{P}^1_1,\ \mathbb{P}^1_2$ intersecting each other in a point. For

$$\delta := a[\mathbb{P}^1_1] + b[\mathbb{P}^1_2] \in H_2(X_t,\mathbb{Z}),\ \ a,b, \in \mathbb{N}_0$$

let V_δ be the Noether-Lefschetz locus passing through t and corresponding to δ. For $d = 5$, $ab \neq 0$ and $a \neq b$ show that

1. $\operatorname{rank}[\mathsf{p}_{i+j}(\delta)] = 3$.
2. The analytic set underlying V_δ is $\check{\mathsf{T}}$.
3. Show that the codimension of $\check{\mathsf{T}}$ in T is four. Conclude that V_δ is not reduced.

Chapter 17
Algebraic cycles of the Fermat variety

But the whole program [Grothendieck's program on how to prove the Weil conjectures] relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.... For the proposed definition [of Grothendieck on a category of pure motives] to be viable, one needs the existence of "enough" algebraic cycles. On this question almost no progress has been made, (P. Deligne in [RS14] page 181, 182)....la construction de cycles algébriques intéressants, les progrès ont été maigres, (P. Deligne in [Del94], page 143).

17.1 Introduction

What makes the Hodge conjecture difficult is our poor understanding of algebraic cycles of a given variety. The situation is even worse when one wants to construct them explicitly. For instance, explicit set of generators for the Picard (Neron-Severi) group of the Fermat surface of degree 12 is being worked out by N. Aoki and it is announced in [Aok15], for more details see Exercise 9.2. The situation can be compared to solving Diophantine equations over rational numbers, despite the fact that one looks for algebraic cycles defined over complex numbers. The similarity between the two contexts can be rigorously stated for curves over function fields and surfaces as we did it in §15.13.

Once a sufficient number of algebraic cycles is constructed and the dimension of the space of Hodge cycles is computed, one can use Theorem 8.3 and Theorem 8.4 in order to verify the Hodge conjecture. For Fermat varieties X_n^d the latter is done in Chapter 15. In this chapter we analyze two classes of algebraic cycles for Fermat varieties: linear projective spaces $\mathbb{P}^{\frac{n}{2}} \subset X_n^d$ and Aoki-Shioda algebraic cycles. The main objective is to prepare the ground for the classification of all dimensions n and degrees d such that these algebraic cycles are enough to prove the Hodge conjecture and its refined version introduced in §18.2. In the literature, there is no complete classification of algebraic cycles of X_n^d. New algebraic cycle for X_n^d, which might

be obtained by some high school polynomial identities similar to the case of Aoki-Shioda cycles, are necessary for the full proof of the (refined) Hodge conjecture for all n and d.

17.2 Trivial algebraic cycles

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of even dimension. A trivial algebraic cycle $Z_\infty \subset X \subset \mathbb{P}^{n+1}$ is obtained by an intersection of a linear $\mathbb{P}^{\frac{n}{2}+1} \subset \mathbb{P}^{n+1}$ with X. For instance,

$$Z_\infty : \ \{x_0 = x_2 = \cdots = x_{n-4} = x_{n-2} = 0\} \cap X.$$

All these cycles are homologous to each other and so the homology class $[Z_\infty] \in H_n(X, \mathbb{Z})$ does not depend on the choice of $\mathbb{P}^{\frac{n}{2}+1}$. It is also common to say that the algebraic cycle Z_∞ is induced by the polarization $X \subset \mathbb{P}^{n+1}$. We have

$$Z_\infty \cdot Z_\infty = d$$

which follows from the fact that the intersection of two general $\mathbb{P}^{\frac{n}{2}+1}$ in $\mathbb{P}^{n+1}$ is a line $\mathbb{P}^1$, and this intersects X in d-points. We will frequently use the notation introduced in Definition 8.7.

17.3 Automorphism group of the Fermat variety

The Fermat variety has many automorphisms which make it distinguished among other hypersurfaces. Throughout the book we have tried not to use such automorphisms because we would like to keep open further generalizations of many results for an arbitrary hypersurface.

The first group which acts on X_n^d is the group S_{n+2} of all permutations in $n+2$ point $\{0, 1, \ldots, n+1\}$. An element in $b \in \mathsf{S}_{n+2}$ acts on X_n^d by permuting the coordinates:

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto (x_{b_0}, x_{b_1}, \ldots, x_{b_{n+1}}).$$

Multiplication of the coordinates by d-th roots of unity provides other automorphisms of the Fermat variety. Let

$$\mu_d^{n+2}/\mu_d := \underbrace{\mu_d \times \mu_d \times \cdots \times \mu_d}_{(n+2)-\text{ times}}/\mathrm{diag}(\mu_d), \tag{17.1}$$

where

$$\mu_d := \{1, \zeta_d, \ldots, \zeta_d^{d-1}\} \tag{17.2}$$

is the group of d-th roots of unity and $\mathrm{diag}(\mu_d)$ is the image of the diagonal map

$$\mu_d \to \mu_d^{n+2}, \;\; \zeta \mapsto (\zeta, \zeta, \cdots, \zeta).$$

The group μ_d^{n+2}/μ_d acts on X_n^d by multiplication of coordinates:

$$(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}), (x_0, x_1, \ldots, x_{n+1}) \mapsto (\zeta_0 x_0, \zeta_1 x_1, \ldots, \zeta_{n+1} x_{n+1}). \tag{17.3}$$

In summary, let us define the free product group

$$\mathsf{G}_n^d := \left(\mu_d^{n+2}/\mu_d\right) * \mathsf{S}_{n+2}, \tag{17.4}$$

which acts on the Fermat variety X_n^d.

17.4 Linear projective cycles

The very special format of the Fermat variety gives us the algebraic cycle $\mathbb{P}^{\frac{n}{2}}$ inside X:

$$\mathbb{P}^{\frac{n}{2}} : x_0 - \zeta_{2d} x_1 = x_2 - \zeta_{2d} x_3 = x_4 - \zeta_{2d} x_5 = \cdots = x_n - \zeta_{2d} x_{n+1} = 0.$$

All other linear projective cycles $\mathbb{P}^{\frac{n}{2}} \subset X_n^d$ are obtained by the action of μ_d^{n+2} and S_{n+2} on the cycle $\mathbb{P}^{\frac{n}{2}}$. We define

$$\mathbb{P}^{\frac{n}{2}}_{a,b} : b^{-1} \circ a^{-1}(\mathbb{P}^{\frac{n}{2}}), \tag{17.5}$$

$$a \in \mu_d^{n+2}, \;\; b \in \mathsf{S}_{n+2},$$

that is $\mathbb{P}^{\frac{n}{2}}_{a,b}$ is the pull-back of $\mathbb{P}^{\frac{n}{2}}$ by a and then by b. For

$$a = (\zeta_d^{a_0}, \zeta_d^{a_1}, \ldots, \zeta_d^{a_{n+1}}), \;\; b = (b_0, b_1, \ldots, b_{n+1})$$

we have:

$$\mathbb{P}^{\frac{n}{2}}_{a,b} : \begin{cases} x_{b_0} - \zeta_{2d}^{1+2a_1-2a_0} x_{b_1} = 0, \\ x_{b_2} - \zeta_{2d}^{1+2a_3-2a_2} x_{b_3} = 0, \\ x_{b_4} - \zeta_{2d}^{1+2a_5-2a_4} x_{b_5} = 0, \\ \cdots \\ x_{b_n} - \zeta_{2d}^{1+2a_{n+1}-2a_n} x_{b_{n+1}} = 0. \end{cases} \tag{17.6}$$

Let $\mathrm{Stab}(\mathbb{P}^{\frac{n}{2}}) \subset \mathsf{G}_n^d$ be the stabilizer of $\mathbb{P}^{\frac{n}{2}}$. Such linear cycles are indexed by the elements of $\mathrm{Stab}(\mathbb{P}^{\frac{n}{2}}) \backslash \mathsf{G}_n^d$. Note that

$$\mathbb{P}^{\frac{n}{2}} \cdot Z_\infty = 1 \tag{17.7}$$

and so

$$\check{\mathbb{P}}^{\frac{n}{2}} = d \cdot \mathbb{P}^{\frac{n}{2}} - Z_\infty. \tag{17.8}$$

which is valid for all $\mathbb{P}^{\frac{n}{2}} = \mathbb{P}^{\frac{n}{2}}_{a,b}$.

Proposition 17.1 *We have*

$$P(n,d) := \#\left(\mathrm{Stab}(\mathbb{P}^{\frac{n}{2}})\backslash \mathsf{G}_n^d\right) = (n+1)\cdot(n-1)\cdots 3\cdot 1\cdot d^{\frac{n}{2}+1} \tag{17.9}$$

and so we have this number of linear cycles in the Fermat variety X_n^d.

For $n=2$ and $d=3$, the number in (17.9) is 27. This is the well-known number of lines inside a smooth cubic surface.

Proof. We choose representatives in the quotient group in (17.9) and count them. Let $b \in \mathsf{S}_{n+2}$ and $a \in \mu_d^{n+2}/\mu_d$, where a,b are chosen as follows. We have $b_0 = 0$ and for i an even number b_i is the smallest number in $\{0,1,\dots,n+1\}\backslash\{b_0,b_1,b_2,\dots,b_{i-1}\}$. The number of such $b \in \mathsf{S}_{n+2}$ is $(n+1)\cdot(n-1)\cdots 3\cdot 1$. The element $a \in \mu_d^{n+2}/\mu_d$ depends on b. It does not change x_{b_i}, i even, and it multiplies x_{b_i}, i odd with a d-th root of unity. For fixed b there are $d^{\frac{n}{2}+1}$ number of such a's. □

Proposition 17.2 *For $d \geq \frac{n}{2}+1$ and $n \geq 2$, any linear projective space $\mathbb{P}^{\frac{n}{2}}$ inside the Fermat variety X_n^d is of the form $\mathbb{P}^{\frac{n}{2}}_{a,b}$ in* (17.6).

Proof. Let $m := \frac{n}{2}$. Without loss of generality we can assume $\mathbb{P}^m$ is given by the homogeneous linear equations $f_0 = f_1 = \cdots = f_m = 0$, where $f_i = x_{m+i+1} - g_i(x)$ and g_i is in the vector space $\mathbb{C}[x]_1$ of homogeneous linear polynomials in $x := (x_0, x_1, \cdots, x_m)$. It is enough to show that g_i depends only on one variable. Since $\mathbb{P}^m \subset X_n^d$ we get the identity $x_0^d + x_1^d + \cdots + x_m^d + g_0^d + g_1^d + \cdots + g_m^d = 0$. We write this as

$$x_0^d + x_1^d + \cdots + x_m^d + h_1^d + h_2^d + \cdots + h_p^d = 0, \text{ where } h_i \in \mathbb{C}[x]_1,\ h_i \neq 0,\ \ p \leq m \tag{17.10}$$

and h_i's are distinct in the projectivization of $\mathbb{C}[x]_1$. If the desired statement for g_i's is not valid, then there is an h_i which has non-zero coefficients in two variables of x, say x_0, x_1. The derivation of (17.10) with respect to x_0 and x_1 gives us an equation of type

$$a_1^b + a_2^b + \cdots + a_k^b = 0, \quad \text{where } b := d-2,\ \ k \leq m,\ \ a_i \in \mathbb{C}[x]_1,\ a_i \neq 0. \tag{17.11}$$

and a_i's are distinct in the projectivization of $\mathbb{C}[x]_1$. By a linear change of coordinates, we can take a_k and a_{k-1} as independent variables. Derivation with respect to a_{k-1} will kill a_k and we get a similar identity (17.11) with smaller k and with b replaced with $b-1$. This process after at most $m-2$ steps will result in (17.11) with $k=2$ which implies that a_1 and a_2 are equal in the projectivization of $\mathbb{C}[x]_1$ which is a contradiction. Note that the exponent b must alway remain ≥ 2 and in the last step it can be 1. This forces us to assume $d-2-(m-2) \geq 1$. □

We would like to study the $\mathbb{Z}$-module generated by the homology classes of all linear $\mathbb{P}^{\frac{n}{2}}$'s inside X, and in this way justify the definition of $\mathrm{Hodge}_n(X,\mathbb{Z})_{\mathbb{P}^{\frac{n}{2}}}$ in (16.22).

Proposition 17.3 *The $\mathbb{Q}$-vector space* $\mathrm{Hodge}_n(X,\mathbb{Q})_{\mathbb{P}^{\frac{n}{2}}}$ *is generated by the homology classes of all $\mathbb{P}^{\frac{n}{2}}$'s inside X.*

We can use Poincaré duality and then derive this proposition from Shioda's work [Shi79a]. There is an alternative way using a direct computation of periods of linear cycles $\mathbb{P}^{\frac{n}{2}}$. This is done in [MV19] and it involves the algebraic de Rham cohomology of arbitrary projective varieties which is beyond the scope of the present text. For some details see §18.2. We have a sequence of inclusions

$$\begin{array}{ccc} \sum_{a,b}\mathbb{Z}[\mathbb{P}^{\frac{n}{2}}_{a,b}] & \subset & \mathrm{Hodge}_n(X,\mathbb{Z})_{\mathbb{P}^{\frac{n}{2}}} \\ \cap & & \cap \\ \mathrm{Hodge}_n(X,\mathbb{Z})_{\mathrm{alg}} & \subset & \mathrm{Hodge}_n(X,\mathbb{Z}), \end{array}$$

By Proposition 17.3, the quotient in the first inclusion is a torsion group. It is natural to ask whether it is an equality or not.

Conjecture 17.1 (Integral Hodge conjecture for linear cycles) *The $\mathbb{Z}$-module of Hodge cycles* $\mathrm{Hodge}_n(X,\mathbb{Z})_{\mathbb{P}^{\frac{n}{2}}}$ *is generated by the homology class of linear cycles.*

Similar to Lefschetz $(1,1)$ theorem one may expect that this conjecture is true at least in the two dimensional case $n=2$. Recall that in this case $\mathrm{Hodge}_2(X,\mathbb{Z})$ is equal to the Neron-Severi group $\mathrm{NS}(X)$. For some result in this direction see 17.7. For some developements in direction of Conjecture 17.1 see [AMV19].

17.5 Aoki-Shioda algebraic cycles

Let us consider variables $y_1,y_2,\dots,y_s$ with s an odd number and the corresponding classical symmetric polynomials

$$\begin{aligned} f_1(y) &:= \sum_i y_i, \\ f_2(y) &:= \sum_{i<j} y_i y_j, \\ f_3(y) &:= \sum_{i<j<k} y_i y_j y_k, \\ &\vdots \\ f_s(y) &:= y_1 y_2 \cdots y_s. \end{aligned} \tag{17.12}$$

We have

$$\sum_{i=1}^{s} y_i^s = f_1 g_1 + \cdots + f_{[\frac{s}{2}]} g_{[\frac{s}{2}]} + (-1)^{s+1}\cdot s\cdot f_s \tag{17.13}$$

for some symmetric polynomials g_j in y_i's. Here, $[\frac{s}{2}]$ is the integer part of $\frac{s}{2}$. For the proof of this we observe that the left hand side of (17.13) is symmetric in y_i's and so

it can be written as a polynomial P in f_i's. We consider the weights weight$(f_i) := i$ and we know that such a polynomial is of degree s. The variables $f_{[\frac{s}{2}]+1}, \cdots, f_{s-1}$ which appear in the expression of P are absorbed in $g_1, \cdots, g_{[\frac{s}{2}]}$ and the equality (17.13) follows. For a complete description of this equality see Exercise 17.5.

Now we use the equality (17.13) in order to produce new algebraic cycles in the classical Fermat variety X_n^d of degree d and dimension n with

$$s \leq n+1, \quad s \neq n, \quad s \mid d.$$

We write

$$x_0^d + x_1^d + \cdots + x_s^d + \cdots = x_0^d + (-1)^{s+1} s (x_1 x_2 \cdots x_s)^{\frac{d}{s}} + f_1 g_1 + \cdots + f_{[\frac{s}{2}]} g_{[\frac{s}{2}]} + \cdots,$$

where f_i, g_i are symmetric functions as in (17.13) in $y_i = x_i^{\frac{d}{s}}$, $i = 1, 2, \ldots, s$. We write f_i's in terms of the symmetric functions $\sum_{i=1}^s y_i^k$, $k = 1, 2, \ldots, [\frac{s}{2}]$ and find the following algebraic cycle of dimension $\frac{n}{2}$ in X_n^d:

$$Z^s : \begin{cases} x_1^{\frac{d}{s}k} + x_2^{\frac{d}{s}k} + \cdots + x_s^{\frac{d}{s}k} = 0, & k = 1, 2, \ldots, [\frac{s}{2}], \\ x_0^s - \sqrt[d/s]{(-1)^s s} \cdot x_1 x_2 \cdots x_s = 0, & \\ x_{s+1}^d + \cdots + x_{n+1}^d = 0, & \\ x_i = 0, & i = s+1, s+2, \cdots, s + \frac{n}{2} - [\frac{s}{2}]. \end{cases} \tag{17.14}$$

For $s = n+1$ the last two lines in the definition of Z^s are empty and for $s = n-1$ it consists of $x_n^d + x_{n+1}^d = 0$ (here we need to discard the case $s = n$). What we need for the last two lines is an algebraic cycle of codimension $\frac{n}{2} - [\frac{s}{2}]$ in the Fermat variety $x_{s+1}^d + \cdots + x_{n+1}^d = 0$. We call Z^s the Aoki-Shioda cycles as it appears in the works of N. Aoki and T. Shioda, see [Aok87]. Note that if $A_1, A_2, \cdots, A_{\frac{n}{2}+1}$ are defining polynomials of Z^s as above then

$$x_0^d + x_1^d + \cdots + x_{n+1}^d = A_1 B_1 + \cdots + A_{\frac{n}{2}+1} B_{\frac{n}{2}+1}$$

for some polynomials B_i. Moreover, Z^s is a complete intersection of type

$$(\frac{d}{s}, 2\frac{d}{s}, \cdots, [\frac{s}{2}]\frac{d}{s}, s, \underline{d, 1, \ldots, 1}).$$

where for $s = n+1$ the underline is empty, for $s = n-1$ it consists of d and for $s < n-1$ it consists of d and $\frac{n}{2} - [\frac{s}{2}]$ 1's. For the Fermat surface and $s = 3$, the construction of Aoki-Shioda cycles is basically done using the following high school algebra identity:

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\zeta_3 y + \zeta_3^2 z)(x + \zeta_3^2 y + \zeta_3 z).$$

An algebraic geometer would easily interpret this as the degeneration of an elliptic curve in $\mathbb{P}^2$ into three distinct lines. Assume that the degree d is divisible by 3, the Aoki-Shioda cycle in this case is given by:

$$Z^3 : \begin{cases} x_1^{\frac{d}{3}} + x_2^{\frac{d}{3}} + x_3^{\frac{d}{3}} = 0, \\ x_0^3 - \sqrt[d/3]{-3}x_1x_2x_3 = 0. \end{cases} \tag{17.15}$$

Let $\mathrm{Stab}(Z^s)$ be the stabilizer of Z^s under the action of G_n^d. The Aoki-Shioda algebraic cycles are indexed by the elements of $\mathrm{Stab}(Z^s)\backslash\mathsf{G}_n^d$.

$$Z_{\mathsf{g}}^s := \mathsf{g}^{-1}(Z^s), \;\; \mathsf{g} \in \mathrm{Stab}(Z^s)\backslash\mathsf{G}_n^d. \tag{17.16}$$

17.6 Adjunction formula

We invested a good portion of the present text on computing the intersection numbers of vanishing and Hodge cycles, and the main reason for this is that we want to investigate the space of Hodge cycles as a lattice. The same reasoning also works for algebraic cycles, and so, in this chapter we introduce the main tool for computing intersection numbers of algebraic cycles. This is namely the adjunction formula. This together with Theorem 8.4, and the computation of the dimension of the space of Hodge cycles in Chapter 15 will give us computational proofs of the Hodge conjecture for the Fermat variety X_n^d with particular values of d and n.

Throughout the present text we have avoided to use the machinery of line and vector bundles, sheaves, Chern classes etc. The content of the present chapter is one of the few places in this book such that the usage of such a machinery is indispensable. Author's favorite source for this is Bott and Tu's book [BT82], even though it is not written in the framework of Algebraic Geometry.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and $Z \subset X$ be a smooth subvariety of dimension $n+1-r$. We have the following short exact sequences of vector bundles:

$$0 \to \mathbf{T}_Z \to \mathbf{T}_X\Big|_Z \to \mathbf{N}_{Z\subset X} \to 0, \tag{17.17}$$

$$0 \to \mathbf{T}_X\Big|_Z \to \mathbf{T}_{\mathbb{P}^{n+1}}\Big|_Z \to \mathbf{N}_{X\subset\mathbb{P}^{n+1}}\Big|_Z \to 0. \tag{17.18}$$

Here, $\mathbf{T}$ is the tangent bundle of the underlying variety and $\mathbf{N}$ is the normal bundle of the underlying subvariety of a variety. We know that $\mathbf{N}_{X\subset\mathbb{P}^{n+1}} = \mathscr{O}_X(d)$ extends to $\mathbb{P}^{n+1}$. Since the Chern class behaves canonically under restriction of vector bundles, we have

$$\begin{aligned} \mathrm{cl}(\mathbf{N}_{Z\subset X}) &= \frac{\mathrm{cl}(\mathbf{T_X}\big|_Z)}{\mathrm{cl}(\mathbf{T_Z})} \\ &= \frac{\mathrm{cl}(\mathbf{T}_{\mathbb{P}^{\mathbf{n+1}}})}{\mathrm{cl}(\mathbf{T_Z})\mathrm{cl}(\mathscr{O}_X(d))} \\ &= \frac{(1+d_1\theta)(1+d_2\theta_2)\cdots(1+d_r\theta)}{1+d\theta}, \end{aligned}$$

where θ is the generator of $H^2(\mathbb{P}^{n+1},\mathbb{Z})$ induced by the canonical orientation of $\mathbb{P}^{n+1}$. In the last equality, we have assumed that Z is a complete intersection of type $(d_1,d_2,\dots,d_r)$ in $\mathbb{P}^{n+1}$. We have used the following well-known results:

$$\begin{aligned}
\mathrm{cl}(\mathbf{T}_{\mathbb{P}^{n+1}}) &= (1+\theta)^{n+2},\\
\mathrm{cl}(\mathscr{O}_X(d)) &= (1+d\theta),\\
\mathrm{cl}(\mathbf{T}_Z) &= \frac{(1+\theta)^{n+2}}{(1+d_1\theta)(1+d_2\theta)\cdots(1+d_r\theta)}.
\end{aligned}$$

Proposition 17.4 *A complete intersection Z of type $(d_1,d_2,\dots,d_{\frac{n}{2}+1})$ inside a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ has the self-intersection*

$$Z\cdot Z := d_1d_2\cdots d_{\frac{n}{2}+1}\left(\frac{d_1d_2\cdots d_{\frac{n}{2}+1}-(d_1-d)(d_2-d)\cdots(d_{\frac{n}{2}+1}-d)}{d}\right). \quad (17.19)$$

In particular,

$$\begin{aligned}
&Z\cdot Z = d_1d_2(d_1+d_2-d) \qquad \textit{for } n=2,\\
&Z\cdot Z = d_1d_2d_3(d^2-(d_1+d_2+d_3)d-d_1d_2-d_1d_3-d_2d_3) \quad \textit{for} \quad n=4,
\end{aligned}$$

and for a linear projective space $\mathbb{P}^{\frac{n}{2}}\subset X$ we have

$$\mathbb{P}^{\frac{n}{2}}\cdot\mathbb{P}^{\frac{n}{2}} = \frac{1-(-d+1)^{\frac{n}{2}+1}}{d}. \quad (17.20)$$

Proof. The integration of $\theta^{\frac{n}{2}+1}$ over Z is $d_1d_2\cdots d_{\frac{n}{2}+1}$. The second factor in (17.19) is the coefficient of $\theta^{\frac{n}{2}+1}$ in $\mathrm{cl}(\mathbf{N}_{Z\subset X})$. $\square$

Let us consider two linear projective spaces $\mathbb{P}_i^{\frac{n}{2}}$, $i=1,2$ in a hypersurface $X\subset\mathbb{P}^{n+1}$. By linear transformations of $\mathbb{P}^{n+1}$ we can assume that $\mathbb{P}_i^{\frac{n}{2}}$'s are given by

$$\begin{aligned}
&\mathbb{P}_1^{\frac{n}{2}} : x_0=x_1=\cdots=x_a=x_{a+1}=\cdots=x_{\frac{n}{2}}=0,\\
&\mathbb{P}_2^{\frac{n}{2}} : x_0=x_1=\cdots=x_a=x_{\frac{n}{2}+1}=\cdots=x_{n-a}=0,
\end{aligned}$$

where $a\le\frac{n}{2}$. The intersection set of $\mathbb{P}_1^{\frac{n}{2}}$ and $\mathbb{P}_2^{\frac{n}{2}}$ is given by

$$\mathbb{P}_1^{\frac{n}{2}}\cap\mathbb{P}_2^{\frac{n}{2}} : x_0=x_1=\cdots=x_{n-a}=0$$

which is of dimension a. We would like to compute the intersection number of these algebraic cycles inside X.

Proposition 17.5 *Assume that the set theoretical intersection of two linear* $\mathbb{P}_i^{\frac{n}{2}}$, $i = 1,2$ *in a smooth hypersurface* X *is a projective space* $\mathbb{P}^a$ *of dimension* a. *Then*

$$\mathbb{P}_1^{\frac{n}{2}} \cdot \mathbb{P}_2^{\frac{n}{2}} = \frac{1-(-d+1)^{a+1}}{d} \tag{17.21}$$

and so

$$\check{\mathbb{P}}_1^{\frac{n}{2}} \cdot \check{\mathbb{P}}_2^{\frac{n}{2}} = -d \cdot (-d+1)^{a+1}. \tag{17.22}$$

Proof. For $a = \frac{n}{2}$ the proposition is (17.20). For $a = \frac{n}{2} - 1$ the idea is as follows: we know that X is given by

$$f := x_0 f_0 + x_1 f_1 + \cdots + x_{\frac{n}{2}-1} f_{\frac{n}{2}-1} + x_{\frac{n}{2}} x_{\frac{n}{2}+1} f_{\frac{n}{2}}$$

We deform this one and use the self-intersection of a complete intersection Z of type $(1,1,\ldots,1,2)$. The desired number is

$$\frac{1}{2}(Z \cdot Z - \mathbb{P}_1^{\frac{n}{2}} \cdot \mathbb{P}_1^{\frac{n}{2}} - \mathbb{P}_2^{\frac{n}{2}} \cdot \mathbb{P}_2^{\frac{n}{2}})$$

which implies (17.21) in this case. For arbitrary a, the argument is as follows. Let N_a be the intersection number in (17.21). We take a complete intersection Z of type

$$\underbrace{2,2,\ldots,2}_{\frac{n}{2}-a \text{ times}}, \underbrace{1,1,\ldots,1}_{a+1 \text{ times}}$$

and let it generate into linear cycles. Without loss of generality, we can assume that this is given by

$$Z := x_0 x_1 = x_2 x_3 = \cdots = x_{n-2a-2} x_{n-2a-1} = x_{n-2a} = x_{x_{n-2a+1}} = \cdots = x_{n-a} = 0.$$

We have

$$2^{\frac{n}{2}-a} \frac{2^{\frac{n}{2}-a} - (1-d)^{a+1}(2-d)^{\frac{n}{2}-a}}{d} = \langle Z, Z \rangle = \sum_{x=a}^{\frac{n}{2}} 2^{\frac{n}{2}-a} \cdot N_x \cdot \binom{\frac{n}{2}-a}{\frac{n}{2}-x}. \tag{17.23}$$

This implies (17.21). The affirmation on primitive cycles follows immediately from the definition of $\check{\mathbb{P}}^{\frac{n}{2}}$ and the equalities $Z_\infty \cdot Z_\infty = d$, $\mathbb{P}^{\frac{n}{2}} \cdot Z_\infty = 1$. $\square$

17.7 Hodge conjecture for Fermat variety

Let us first state the well-known cases of the Hodge conjecture for the Fermat variety:

Theorem 17.1 ([Ran81], [Shi79a], [AS83]) *Suppose that either* d *is a prime number or* $d = 4$ *or* d *is relatively prime with* $(n+1)!$. *Then* $\mathrm{Hodge}_n(X_n^d, \mathbb{Q})$ *is generated*

by the homology classes of the linear cycles $\mathbb{P}^{\frac{n}{2}}$, and in particular, the Hodge conjecture for X_n^d is true.

For a generalization of this result using both linear cycles and Aoki-Shioda cycles see [Aok87]. The study of the integral Hodge conjecture, when it is true and when not, is another difficult problem in Hodge theory.

Theorem 17.2 (Schuett, Shioda, van Luijk, [SSvL10]) *If $d \leq 100$ and $(d,6) = 1$, then the Neron-Severi group of the Fermat surface of degree d is generated by lines.*

A more complete version of this theorem is the following.

Theorem 17.3 (Degtyarev, [Deg15]) *If $d \leq 4$ or $\gcd(d,6) = 1$ then the Neron-Severi group of the Fermat surface of degree d is generated by lines.*

We are not interested to prove the above theorems in such a generality. Instead, as the reader may have noticed, we have gathered all the tools to check the Hodge conjecture for Fermat varieties using linear and Aoki-Shioda cycles, for explicit examples of n and d and as far as our computer carries the computations. We just need to compute $s := \dim\mathrm{Hodge}_n(X_n^d, \mathbb{Q})_0$ as in §15.10, pick up s primitive algebraic cycles of X_n^d among the linear and Aoki-Shioda cycles, see §17.4 and §17.5, compute the corresponding intersection matrix and verify that its determinant is not zero. By Theorem 8.4 this computation implies that the Hodge conjecture for X_n^d is true. By the equality (8.13) we also compute the number N, for which we have the integral Hodge conjecture with coefficients in $\frac{1}{N}\mathbb{Z}$. For instance, the proof of Theorem 17.1 is reduced to the statement that the number $s(n,d)$ in Problem 21.12 in Chapter 21, is equal to the dimension of the space of primitive Hodge cycles computed in Chapter 15, see also Problem 21.13.

17.8 Some conjectural components of the Hodge locus

In this section we are mainly interested to identify, at least conjecturally, some irreducible components of the Hodge locus defined in §16.5 and compute their codimensions. We work with complete intersection algebraic cycles inside smooth hypersurfaces in the usual projective space $\mathbb{P}^{n+1}$, that is, the weights of the variables are equal to one. For further developments of the content of the present section, many arguments of commutative algebra, such as syzygies, might play an important role. For this the reader is referred to Eisenbud's books [Eis95, Eis05].

For a projective variety $X \subset \mathbb{P}^{n+1}$, the most simple algebraic cycles in X that we know are intersections of X with projective subspaces of $\mathbb{P}^{n+1}$. The next class of algebraic cycles are those which are complete intersections inside $\mathbb{P}^{n+1}$ and are inside X. This includes the linear cycles $\mathbb{P}^{\frac{n}{2}}_{\mathbf{g}}$ and the Aoki-Shioda cycles $Z^s_{\mathbf{g}}$ introduced in §17.4 §17.5. Let us explain this in the case of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ given by the homogeneous polynomial $f \in \mathsf{k}[x]_d$.

Definition 17.1 A projective variety $Z \subset \mathbb{P}^{n+1}$ is called an algebraic complete intersection of type $(d_1, d_2, \dots, d_s)$ if it is of codimension s and its ideal is generate by homogeneous polynomials $f_i \in \mathsf{k}[x]_{d_i}, \ \ i = 1, 2, \dots, s$.

If the hypersurface X given by the homogeneous polynomial f contains a complete intersection Z as above then for some $b \in \mathbb{N}$, f^b is in the ideal generated by f_i's. In this chapter we only consider the case $b = 1$, that is

$$f = f_1 f_{s+1} + f_2 f_{s+2} + \cdots + f_s f_{2s} \tag{17.24}$$

for some homogeneous polynomials $f_{s+i} \in \mathsf{k}[x]_{d-d_i}, \ \ i = 1, 2, \dots, s$. Notice that such a hypersurface contains more algebraic cycles $A_1 = A_2 = \cdots = A_s = 0$, where A_i is one of f_i or f_{s+i}'s.

Let us consider a sequence of homogeneous polynomials $A_i \in \mathsf{k}[x]_{a_i}, \quad i = 1, 2, \dots, s$ and assume that this is a regular sequence in the ring $\mathsf{k}[x]$, see §10.5. For homogeneous polynomials $B_i \in \mathsf{k}[x]_{a-a_i}$ for some $a \in \mathbb{N}$ which is bigger than all a_i's, if

$$A_1 B_1 + A_2 B_2 + \cdots + A_s B_s = 0 \tag{17.25}$$

then there are homogeneous polynomials $C_{ij} \in \mathsf{k}[x]_{a-a_i-a_j}$ such that

$$B_i = \sum_{j=1}^{s} A_j C_{ji}, \ \ i = 1, 2, \dots, s, \tag{17.26a}$$

$$C_{ij} = -C_{ji}, \ i, j = 1, 2, \dots, s. \tag{17.26b}$$

Note the the polynomials B_i in (17.26) satisfy the identity (17.25). In general we have the Koszul complex:

$$0 \to \mathsf{k}[x]^{\binom{s}{s}} \to \mathsf{k}[x]^{\binom{s}{s-1}} \to \cdots \to \mathsf{k}[x]^{\binom{s}{2}} \to \mathsf{k}[x]^{\binom{s}{1}} \to \langle A_1, A_2, \dots, A_s \rangle \to 0 \tag{17.27}$$

From this we can compute the dimension of the vector space of B_i polynomials with the property (17.25). We will need this in the following statement.

Proposition 17.6 *For $s \leq \frac{n}{2} + 1$, the space of homogeneous polynomials f of the form*

$$f = f_1 f_{s+1} + f_2 f_{s+2} + \cdots + f_s f_{2s}, \ \ f_i \in \mathsf{k}[x]_{d_i}, \ f_{s+i} \in \mathsf{k}[x]_{d-d_i} \tag{17.28}$$

has codimension

$$\binom{n+1+d}{n+1} - \sum_{k=1}^{2s} (-1)^{k-1} \sum_{a_{i_1}+a_{i_2}+\cdots+a_{i_k} \leq d} \binom{n+1+d-a_{i_1}-a_{i_2}-\cdots-a_{i_k}}{n+1} \tag{17.29}$$

in $\mathbb{C}[x]_d$, where $(a_1, a_2, \dots, a_{2s}) = (d_1, d_2, \dots, d_s, d-d_1, d-d_2, \dots, d-d_s)$ and the second sum runs through all k elements (without order) of $a_i, \ \ i = 1, 2, \dots, 2s$.

Proof. We consider T the parameter space of homogeneous polynomials of degree d in x. This is an affine variety over k and in this way the space $\mathsf{T}_{\underline{d}}$ of homogeneous

polynomials f of the form (17.28) is an affine subvariety of T. For a smooth point 0 of $\mathsf{T}_{\underline{d}}$ representing the polynomial f in (17.28), the tangent space of $\mathsf{T}_{\underline{d}}$ at 0 is given by:

$$\left\{ g_1 f_1 + g_2 f_2 + \cdots + g_{2s} f_{2s} \quad | \quad g_i \in \mathsf{k}[x]_{d-a_i} \right\}. \tag{17.30}$$

We need to compute the dimension of this vector space for generic choice of f_i. Since $2s \leq n+2$ we can assume that $f_1, f_2, \cdots, f_{2s}$ is a regular sequence, see Exercise 17.8. Therefore, this computation is an easy consequence of the Koszul complex 17.27. $\square$

The assertion of Proposition 17.6 can be stated for non-smooth points $0 \in \mathsf{T}_{\underline{d}}$ provided that $(\mathsf{T}_{\underline{d}}, 0)$ is a union of smooth local analytic varieties. In this case, we are computing the tangent space of the branches of $(\mathsf{T}_{\underline{d}}, 0)$. The most well-known example in Proposition 17.6 is the case of linear cycles $\mathbb{P}^{\frac{n}{2}}$ inside hypersurfaces, that is, the case $\underline{d} = \underline{1} = (1, 1, \ldots, 1)$. In this case the number (17.29) turns out to be

$$\operatorname{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{1}}) = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2. \tag{17.31}$$

Note that we have used the following identity:

$$\sum_{a=0}^{s} (-1)^a \binom{2s-1+d-a}{2s-1} \binom{s}{a} = \binom{d+s-1}{d}. \tag{17.32}$$

In Table 17.1, Table 17.2 and Table 17.3 we have gathered the data of codimensions in Proposition 17.6 in the case of quartic, quintic and sextic fourfolds in $\mathbb{P}^5$, respectively. For the computer implementation of the number (17.29) see §20.10. Some examples of the number (17.29) might be useful for future investigations, for instance for the verification of the refined Hodge conjecture for Fermat varieties. The case of surfaces of degree d and curves which are complete intersections of type (d_1, d_2), $d_1 \leq d_2 \leq \frac{d}{2}$ is:

$$\begin{aligned} \operatorname{codim}_{\mathsf{T}}(\mathsf{T}_{d_1,d_2}) = & \binom{3+d}{3} - \binom{3+d-d_1}{3} - \binom{3+d-d_2}{3} \\ & - \binom{3+d_1}{3} - \binom{3+d_2}{3} + \binom{3+d-d_1-d_2}{3} + 2 + \binom{3+d_2-d_1}{3} + \varepsilon. \\ = & \, d_1 d_2 d + \frac{1}{3}(-d_1^3 - 3d_1 d_2^2 - 11 d_1 + 3) + \varepsilon \end{aligned}$$

where $\varepsilon = 1$ if $d_1 = d_2 = \frac{d}{2}$ and $= 0$ otherwise, see Table 17.4 for the value of this number for small d's.

For d an even number the case of complete intersections of type $\underline{\frac{d}{2}} := (\frac{d}{2}, \frac{d}{2}, \cdots, \frac{d}{2})$ is:

$$\operatorname{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{\frac{d}{2}}}) = \binom{n+1+d}{n+1} - (n+2)\binom{n+1+\frac{d}{2}}{n+1} + \binom{n+2}{2}.$$

This is the biggest codimension among all $\operatorname{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$.

(d_1,d_2,d_3)	$\mathrm{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$	(d_1,d_2,d_3)	$\mathrm{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$	(d_1,d_2,d_3)	$\mathrm{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$
$(1,1,1)$	6	$(1,1,2)$	8	$(1,1,3)$	6
$(1,2,2)$	11	$(2,2,2)$	15		

Table 17.1 Codimension of the loci of complete intersection algebraic cycles for quartic fourfolds

(d_1,d_2,d_3)	$\mathrm{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$	(d_1,d_2,d_3)	$\mathrm{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$
$(1,1,1)$	12	$(1,1,2)$	19
$(1,2,2)$	30	$(1,2,3)$	30
$(2,2,3)$	48	$(2,2,2)$	48

Table 17.2 Codimension of the loci of complete intersection algebraic cycles for quintic fourfolds

(d_1,d_2,d_3)	$\mathrm{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$	(d_1,d_2,d_3)	$\mathrm{codim}_{\mathsf{T}}(\mathsf{T}_{\underline{d}})$
$(1,1,1)$	19	$(1,3,3)$	71
$(1,1,2)$	32	$(2,2,2)$	92
$(1,1,3)$	37	$(2,2,3)$	106
$(1,2,2)$	54	$(2,3,3)$	122
$(1,2,3)$	62	$(3,3,3)$	141

Table 17.3 Codimension of the loci of complete intersection algebraic cycles for sextic fourfolds

$d\backslash(d_1,d_2)$	$(1,1)$	$(1,2)$	$(2,2)$	$(1,3)$	$(2,3)$	$(3,3)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	$\binom{d-1}{3}$
4	1	1	1													1
5	2	3	4													4
6	3	5	8	6	9	10										10
7	4	7	12	9	15	18										20
8	5	9	16	12	21	27	13	23	29	31						35
9	6	11	20	15	27	36	17	31	41	46						56
10	7	13	24	18	33	45	21	39	53	62	22	41	56	65	68	84

Table 17.4 Codimension of the loci of complete intersection algebraic cycles for surfaces

Another interesting way to construct some conjectural components of the Hodge locus is through pull-back. Let f (resp. $F_0, F_1, \cdots, F_{n+1}$) be homogeneous polynomials of degree d (resp. s). Let also

$$\begin{aligned} F \ &: \ \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n+1}, \\ F(x) &= [F_0(x) : F_1(x) : \cdots : F_{n+1}(x)]. \end{aligned}$$

For a hypersurface $X \subset \mathbb{P}^{n+1}$ given by $f = 0$ and an algebraic cycle $Z \subset X$ of dimension $\frac{n}{2}$, we have the pull-back hypersurface $\tilde{X} \subset \mathbb{P}^{n+1}$ given by $f(F_0, F_1, \cdots, F_{n+1}) = 0$ and the pull-back algebraic cycle. In this way we can consider pull-backs of components of the Hodge locus in the parameter space of degree d hypersurfaces and get some conjectural components of the Hodge locus in the parameter space of hypersurfaces of degree ds. In [Voi91] Voisin has used this idea to construct a counter example to a conjecture of J. Harris, without formulating or proving any conjectural statement as above.

17.9 Intersections of components of the Hodge loci

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface given by the homogeneous polynomial f and Z be a complete intersection algebraic cycle of codimension $s-1$ in X as in (17.24). We denote by 0 the point in T representing X. Let V_Z be the local analytic and irreducible branch of $(\mathsf{T}_{\underline{d}}, 0)$ corresponding to deformations of (X,Z). For generic choice of (X,Z), the tangent space of V_Z at 0 is given by:

$$\mathbf{T}_0 V_Z \cong \langle f_1, f_2, \cdots, f_{2s}\rangle_d := \left\{ p_1 f_1 + p_2 f_2 + \cdots + p_{2s} f_{2s} \quad | \quad p_i \in \mathsf{k}[x]_{d-a_i} \right\}, \tag{17.33}$$

where $a_i := \deg(f_i)$. For arbitrary (X,Z) the right hand side of (17.33) is the image of the tangent space of $\mathsf{k}[x]_{a_1} \times \mathsf{k}[x]_{a_2} \times \cdots \mathsf{k}[x]_{a_{2s}}$ at a point and under the map which sends $f_1, f_2, \ldots, f_{2s}$ to $f_1 f_{s+1} + \cdots + f_s f_{2s}$. Therefore, it might be a proper subset of the Zariski tangent space of V_Z at 0. For a sequence of natural numbers $\underline{a} = (a_1, \ldots, a_{2s})$ let us define

$$\mathsf{C}_{\underline{a}} = \binom{n+1+d}{n+1} - \sum_{k=1}^{2s} (-1)^{k-1} \sum_{a_{i_1}+a_{i_2}+\cdots+a_{i_k} \le d} \binom{n+1+d-a_{i_1}-a_{i_2}-\cdots-a_{i_k}}{n+1} \tag{17.34}$$

where the second sum runs through all k elements (without order) of $a_i, \quad i = 1, 2, \ldots, 2s$. For the computer implementation of this number see §20.10. Now, let us consider two complete intersection algebraic cycle Z and $\hat{Z}$ in X.

Proposition 17.7 *We have*

$$\begin{aligned} &\operatorname{codim}\left(\langle f_1, f_2, \cdots, f_{2s}\rangle_d \cap \langle \hat{f}_1, \hat{f}_2, \cdots, \hat{f}_{2s}\rangle_d\right) = \\ &\operatorname{codim}\left(\langle f_1, f_2, \cdots, f_{2s}\rangle_d\right) + \operatorname{codim}\left(\langle \hat{f}_1, \hat{f}_2, \cdots, \hat{f}_{2s}\rangle_d\right) \\ &-\operatorname{codim}\left(\langle f_1, f_2, \cdots, f_{2s}, \hat{f}_1, \hat{f}_2, \cdots, \hat{f}_{2s}\rangle_d\right). \end{aligned} \tag{17.35}$$

If the sequence $f_1, f_2, \cdots, f_{2s}, \hat{f}_1, \hat{f}_2, \cdots, \hat{f}_{2s}$ is regular then the right hand side of (17.35) *is the number*

$$\mathsf{C}_{\underline{a}} + \mathsf{C}_{\underline{\hat{a}}} - \mathsf{C}_{\underline{a},\underline{\hat{a}}}.$$

Proof. Let

$$\begin{aligned} W &:= \mathsf{k}[x]_{d-a_1} \times \mathsf{k}[x]_{d-a_2} \times \cdots \times \mathsf{k}[x]_{d-a_{2s}}, \\ \hat{W} &:= \mathsf{k}[x]_{d-\hat{a}_1} \times \mathsf{k}[x]_{d-\hat{a}_2} \times \cdots \times \mathsf{k}[x]_{d-\hat{a}_{2s}}, \\ A &:= \langle f_1, f_2, \cdots, f_{2s}\rangle_d, \\ \hat{A} &:= \langle \hat{f}_1, \hat{f}_2, \cdots, \hat{f}_{2s}\rangle_d, \\ B &:= \langle f_1, f_2, \cdots, f_{2s}, \hat{f}_1, \hat{f}_2, \cdots, \hat{f}_{2s}\rangle_d. \end{aligned}$$

We consider the kernel $\ker(\alpha)$ of the linear map $\alpha : W \times \hat{W} \to \mathsf{k}[x]_d$ sending $p_1, p_2, \cdots, p_{2s}, \hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{2s}$ to $\sum_{i=1}^{2s} p_i f_i - \sum_{i=1}^{2s} \hat{p}_i \hat{f}_i$. On $\ker(\alpha)$ we also consider the map β which sends $p_1, p_2, \cdots, p_{2s}, \hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{2s}$ to $\sum_{i=1}^{2s} p_i f_i$. We have

$$\begin{aligned}
\operatorname{codim}(A\cap\hat{A}) &= \operatorname{codim}(\operatorname{Im}(\beta))\\
&= \dim k[x]_d - \dim(\ker(\alpha)) + \dim(\ker(\beta))\\
&= \dim k[x]_d - \dim(W) - \dim(\hat{W}) + \dim(\operatorname{Im}(\alpha)) + \dim(\ker(\beta))\\
&= \dim k[x]_d - \dim(A) - \dim(\hat{A}) + \dim(\operatorname{Im}(\alpha))\\
&= \operatorname{codim}(A) + \operatorname{codim}(\hat{A}) - \operatorname{codim}(B)
\end{aligned}$$

The proof of the last part of the proposition is the same as the proof of Proposition 17.6. □

Let us assume that the homogeneous polynomial f is of the following form:

$$f = f_1 f_{s+1} + f_2 f_{s+1} + \cdots + f_l f_{s+l} + f_{l+1} f_{s+l+1} g_1 + \cdots + f_s f_{2s} g_{s-l}. \tag{17.36}$$

We have the algebraic cycles:

$$\begin{aligned}
Z &: f_1 = f_2 = \cdots = f_s = 0,\\
\hat{Z} &: f_1 = f_2 = \cdots = f_l = f_{s+l+1} = \cdots = f_{2s} = 0.
\end{aligned}$$

In this case the last term in the the equality (17.35) is the codimension of $\langle f_1, f_2, \cdots, f_{2s}\rangle_d$. Let $a_i = \deg(f_i)$ and $b_i = \deg(g_i)$ and

$$\begin{aligned}
a &:= a_1, a_2, \cdots, a_{s+l}, a_{s+l+1} + b_1, \cdots, a_{2s} + b_{s-l},\\
\hat{a} &:= a_1, a_2, \cdots, a_l, a_{l+1} + b_1, \cdots, a_s + b_{s-l}, a_{s+1}, \cdots, a_{2s},\\
a * \hat{a} &:= a_1, a_2, \ldots, a_{2s}.
\end{aligned}$$

Proposition 17.8 *If $s \le \frac{n}{2} + 1$ then for generic Z and $\hat{Z}$ as above we have*

$$\operatorname{codim}\left(\mathbf{T}_0 V_Z \cap \mathbf{T}_0 V_{\hat{Z}}\right) = \mathsf{C}_a + \mathsf{C}_{\hat{a}} - \mathsf{C}_{a*\hat{a}}.$$

In particular, V_Z intersects $V_{\hat{Z}}$ at 0 transversely if $\mathsf{C}_{a*\hat{a}} = 0$.

Proof. For $s \le \frac{n}{2} + 1$ and for generic choice of f_i's and g_i's, all the sequences of polynomials in Proposition 17.7 are regular, and so, this is a direct consequence of the second part of Proposition 17.7. □

In general, $\mathbf{T}_0(V_Z \cap V_{\hat{Z}}) \subset \mathbf{T}_0 V_Z \cap \mathbf{T}_0 V_{\hat{Z}}$ and the equality may not happen. We are going to discuss one example in which we have the equality. In this example we will also see that $V_Z \cap V_{\hat{Z}}$ is larger than the space of polynomials (17.36). By abuse of notation we write

$$a^b := \underbrace{a, a, \cdots, a}_{b \text{ times}}.$$

Hopefully, there will no confusion with the exponential a^b. By our convention, the projective space $\mathbb{P}^{-1}$ means the empty set.

Proposition 17.9 *Let $\mathbb{P}^{\frac{n}{2}}, \hat{\mathbb{P}}^{\frac{n}{2}}$ be two linear algebraic cycles in a smooth hypersurface of dimension n and degree d, and with the intersection $\mathbb{P}^m$. We have $\mathbf{T}_0\left(V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\hat{\mathbb{P}}^{\frac{n}{2}}}\right) = \mathbf{T}_0 V_{\mathbb{P}^{\frac{n}{2}}} \cap \mathbf{T}_0 V_{\hat{\mathbb{P}}^{\frac{n}{2}}}$ and*

$$\operatorname{codim}(V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\hat{\mathbb{P}}^{\frac{n}{2}}}) = 2\mathsf{C}_{1^{\frac{n}{2}+1},(d-1)^{\frac{n}{2}+1}} - \mathsf{C}_{1^{n-m+1},(d-1)^{m+1}}. \tag{17.37}$$

In particular, if $\mathbb{P}^{\frac{n}{2}}$ does not intersect $\hat{\mathbb{P}}^{\frac{n}{2}}$ then $V_{\mathbb{P}^{\frac{n}{2}}}$ intersects $V_{\hat{\mathbb{P}}^{\frac{n}{2}}}$ transversely.

Proof. In Proposition 17.8 with $s := \frac{n}{2}+1$ we have already proved that the codimension of $\mathbf{T}_0 V_{\mathbb{P}^{\frac{n}{2}}} \cap \mathbf{T}_0 V_{\hat{\mathbb{P}}^{\frac{n}{2}}}$ is the number in the right hand side of (17.37). Note that $\mathsf{C}_{1^{n+2}} = 0$, which follows from

$$\sum_{k=0}^{\min\{d,n+2\}} (-1)^k \binom{n+2}{k}\binom{n+1+d-k}{n+1} = 0. \tag{17.38}$$

Let $l := m+1$ and $s := \frac{n}{2}+1$. To finish, we prove that the codimension of $V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\hat{\mathbb{P}}^{\frac{n}{2}}}$ is

$$\begin{aligned}
&2\mathsf{C}_{1^s,(d-1)^s} - \mathsf{C}_{1^{2s-l},(d-1)^l} = \\
&\binom{n+1+d}{d} - \sum_{j=1}^{l}\binom{n+1-j+d}{d-1} - \sum_{j=1}^{s-l}\sum_{i=1}^{s-l}\binom{n+1-l-i-j+d}{d-2} && (17.39)\\
&-(n+2-2s+l)(2s-l) - 2s(s-l). && (17.40)
\end{aligned}$$

By a linear change of coordinates in $\mathbb{P}^{n+1}$ we assume that

$$\begin{aligned}
\mathbb{P}^{\frac{n}{2}} &: x_1 = x_2 = \cdots = x_l = x_{l+1} = x_{l+2} = \cdots = x_s = 0,\\
\hat{\mathbb{P}}^{\frac{n}{2}} &: x_1 = x_2 = \cdots = x_l = x_{s+l+1} = x_{s+l+2} = \cdots = x_{2s} = 0.
\end{aligned}$$

Since $f = 0$ contains $\mathbb{P}^{\frac{n}{2}}$ and $\hat{\mathbb{P}}^{\frac{n}{2}}$, the polynomial f can be written as

$$f = \sum_{j=1}^{l} x_j f_{s+j} + \sum_{j=1}^{s-l} x_{l+j}\left(\sum_{i=1}^{s-l} x_{s+l+i} g_{ij}\right).$$

We can assume that f_{s+j} does not depend on the variables $x_1, x_2, \ldots, x_{j-1}$ and g_{ij} does not depend on $x_k,\ k = 1,2,\cdots,l,\ k = l+1, l+2, \cdots, l+j-1,\ \ k = s+l+1, s+l+2, \cdots, s+l+i-1$. Computing the dimension of all these polynomials we reach the sum and double sum in (17.39). The number (17.40) is the dimension of the Grassmannian of two codimension s vector spaces V_1 and V_2 inside a $n+2$ dimensional vector space V such that the codimension of $V_1 \cap V_2$ in V is $2s-l$. $\square$

Note that for a linear cycle $\mathbb{P}^{\frac{n}{2}}$ in the Fermat variety of degree $d \geq 3$ we have

$$\operatorname{codim}(V_{\mathbb{P}^{\frac{n}{2}}}) = \mathsf{C}_{1^{\frac{n}{2}+1}(d-1)^{\frac{n}{2}+1}} = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2.$$

17.10 Exercises

17.1. Let us consider the classical Fermat surface

$$X_2^d : x^d + y^d + z^d + w^d = 0$$

of degree d. It contains three groups of d^2 lines. Each group is obtained by factorization of the terms in parenthesis in $(x^d+y^d)+(z^d+w^d)$, $(x^d+z^d)+(y^d+w^d)$ and $(x^d+w^d)+(z^d+y^d)$, respectively. Take one of these groups, for instance the first one, and write $x^d+y^d = f_1 g_2$ and $z^d+w^d = -f_2 g_1$. This implies that the algebraic cycle $f_1 = f_2 = 0$ is equal to $g_1 = g_2 = 0$ in $H_2(X_2^d, \mathbb{Z})$, and thus we get many $\mathbb{Z}$-linear relations among $\mathbb{P}^1$'s. For $d = 2,3,4,\cdots$, compute the rank of the $\mathbb{Z}$-module generated by $3d^2$ lines modulo such relations.

17.2. Let $f = f_1 f_{s+1} + f_2 f_{s+2} + \cdots + f_s f_{2s},\ f_i \in \mathbb{C}[x]_{d_i},\ \ f_{s+i} \in \mathbb{C}[x]_{d-d_i}$. We have 2^s algebraic cycles $Z_j : g_1 = g_2 = \cdots = g_s = 0$, where $g_i \in \{f_i, f_{s+i}\}$, together with $Z_\infty := \mathbb{P}^{n+2-s} \cap X$ inside the hypersurface $X \subset \mathbb{P}^{n+1}$ given by $f = 0$. In the homology $H_{2(n+1-s)}(X, \mathbb{Z})$ we have $s \cdot 2^{s-1}$ linear relations between these algebraic cycles, for instance $\{f_1 = f_2 = \cdots = f_{s-1} = f_s = 0\} + \{f_1 = f_2 = \cdots = f_{s-1} = f_{2s} = 0\} = d_1 d_2 \cdots d_{s-1} Z_\infty$. Find a basis of the $\mathbb{Z}$-module generated by Z_j's modulo these relations. Hint: modulo Z_∞ any two cycles Z_j and $Z_{j'}$ are equal up to sign.

17.3. For the Fermat surface $X_2^3 : x^3 + y^3 + z^3 + w^3 = 0$ we have 27 lines $\mathbb{P}^1$.

1. Use 17.20 and (17.21) and write down the 27×27 intersection matrix of these 27 lines.
2. Find seven lines such that the corresponding 7×7 intersection matrix has determinant ± 1. Conclude that such seven lines generate $H_2(X, \mathbb{Z})$.
3. Any smooth cubic in $\mathbb{P}^3$ is obtained by a blow-up of $\mathbb{P}^2$ in 6 points (see [Har77] Chapter V Section 4). Use this interpretation and find a basis of $H_2(X, \mathbb{Z})$ generated by lines. Compare your result with the previous one.

17.4. For d an even number let us consider the following topological cycle in the affine Fermat variety L:

$$\delta := \{x \in \mathbb{R}^{n+1} \mid x_1^d + x_2^d + \cdots + x_{n+1}^d = 1\}$$

with the orientation induced from $\mathbb{R}^{n+1}$. Do the following set of vanishing cycles $a_* \delta,\ \ a \in \mu_d^{n+1}$ form a basis of the $\mathbb{Z}$-module $H_n(L, \mathbb{Z})$?

17.5. Let $d \in \mathbb{N}$ and $y_1, y_2, \cdots, y_s$ be s variables. Show that

$$y_1^d + y_2^d + \cdots + y_s^d = \tag{17.41}$$

$$(-1)^d \cdot d \cdot \sum_{\sum_{i=1}^s i e_i = d,\ e_i \in \mathbb{N}_0} (-1)^{\sum_{i=1}^s e_i} \frac{(-1+\sum_{i=1}^s e_i)!}{e_1! e_2! \cdots e_s!} f_1^{e_1} f_2^{e_2} \cdots f_s^{e_s}$$

where $f_1, f_2, \cdots, f_s$ are the symmetric polynomials (17.12). Source [Aok87] Lemma 3.2.

17.6. Compute the intersection numbers of the Aoki-Shioda cycle Z^s with Z_∞ and $\mathbb{P}^{\frac{n}{2}}$.

17.7. Verify Hodge index theorem (Theorem 8.6) for examples of $\mathbb{Z}$-linear combination of linear cycles $\mathbb{P}^{\frac{n}{2}}_{\mathsf{g}}$, $\mathsf{g} \in \mathsf{G}^n_d$.

17.8. For a polynomial ring $\mathsf{k}[x_1, x_2, \ldots, x_n]$ over a field and natural numbers d_1, d_2, $\ldots, d_n \in \mathbb{N}$, show that we have always a regular sequence $A_1, A_2, \ldots, A_n \in \mathsf{k}[x]$ with $\deg(A_i) = d_i$. Moreover, in the parameter space of A_i's the property of being a regular sequence is Zariski dense. For some hints see the references used in §10.5.

17.9. Let T be the parameter space of pairs of polynomials $a, b \in \mathbb{C}[z]$ with $\deg(a) \leq 4d$, $\deg(b) \leq 6d$, $d \geq 3$. Let also X_t, $t \in \mathsf{T}$ be the family of projective surfaces given in an affine chart by the tame polynomial:

$$y^2 + x^3 + a(z)x + b(z) = 0 \tag{17.42}$$

Projecting X_t to the z coordinate, it becomes an elliptically fibered surface. We denote by $0 \in \mathsf{T}$ the point corresponding to the Fermat surface $y^2 + x^3 + z^{6d} - 1 = 0$ and call it the Fermat point. Show that any component of the Noether-Lefschetz locus in $(\mathsf{T}, 0)$ passing through 0 is smooth, reduced and of codimension $h^{20}(X_t) := d - 1$. Hint: Show that the rank of the matrix $[\mathsf{p}_{i+j}]$ defined in §16.4 is always $d - 1$. The Picard rank of a generic X_t is two and there is only one component of the Noether-Lefschetz locus which has codimension strictly less than $d - 1$, see [Cox90]. This does not pass through the Fermat point. For a generic X_t, the only rational curves on X_t are the zero section and components of singular fibers, see [Ulm14]. This statement does not follow from our Hodge theoretic arguments.

17.10. Classify all $n, d, \underline{a} := (a_1, a_2, \ldots, a_k)$, $k \leq n + 2$ such that the number $\mathsf{C}_{\underline{a}}$ in (17.34) is zero. For instance, for $n = 2$, $d = 3$ and $\underline{a} = (1, 1, 2, 2)$ we have $\mathsf{C}_{\underline{a}} = 0$, from which we can derive the well-known fact that all 27 lines of the Fermat cubic surface are survived in smooth cubic surfaces. For $n = 3, d = 5$ and $\underline{a} = (1, 1, 1, 4, 4, 4)$ we have also $\mathsf{C}_{\underline{a}} = 0$ (a general smooth quintic threefold has 2875 lines). However, this case is not covered in Proposition 17.6. Can you explain this? Discuss Proposition 17.6 for arbitrary s.

17.11. Show that any linear algebraic cycle $\mathbb{P}^{\frac{n}{2}}$ (complete intersection of type $\underbrace{1, 1, \ldots, 1}_{(\frac{n}{2}+1)-\text{ times}}$) inside the Fermat variety X^d_n is necessarily of the form (17.6).

Chapter 18
Why should one compute periods of algebraic cycles?

If you have invented a good strategy of proof which includes however an extensive search or long formal calculations, and afterwards you have written a program implementing this search, it's perfectly OK. But computer assisted proofs, as well as computer unassisted ones, can be good or bad. A good proof is a proof that makes us wiser, (Y. Manin in The Berlin Intelligencer, 1998, p. 16-19).

18.1 Introduction

A quick answer to the question of the title is the following: if we compute such numbers, put them inside a certain matrix and compute its rank, then either we will be able to verify the Hodge conjecture for deformed Hodge cycles, or more interestingly, we will find a right place to look for counterexamples for the Hodge conjecture. In direction of the second situation, we collect evidences to Conjecture 18.1, and for the first situation we prove Theorem 18.1. In the present text all homologies with $\mathbb{Z}$ coefficients are up to torsion and all varieties are defined over complex numbers. Let n be an even number. For an integer $-1 \leq m \leq \frac{n}{2}$ let $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}} \subset \mathbb{P}^{n+1}$ be projective spaces given by:

$$\mathbb{P}^{\frac{n}{2}} : \begin{cases} x_0 - \zeta_{2d} x_1 = 0, \\ x_2 - \zeta_{2d} x_3 = 0, \\ x_4 - \zeta_{2d} x_5 = 0, \\ \cdots \\ x_n - \zeta_{2d} x_{n+1} = 0. \end{cases} \qquad \check{\mathbb{P}}^{\frac{n}{2}} : \begin{cases} x_0 - \zeta_{2d} x_1 = 0, \\ \cdots \\ x_{2m} - \zeta_{2d} x_{2m+1} = 0, \\ x_{2m+2} - \zeta_{2d}^3 x_{2m+3} = 0, \\ \cdots \\ x_n - \zeta_{2d}^3 x_{n+1} = 0. \end{cases} \tag{18.1}$$

where $\zeta_{2d} := e^{\frac{2\pi\sqrt{-1}}{2d}}$. These are linear algebraic cycles in the Fermat variety $X_n^d \subset \mathbb{P}^{n+1}$ given by the homogeneous polynomial $x_0^d + x_1^d + \cdots + x_{n+1}^d = 0$, and satisfy $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$. By convention $\mathbb{P}^{-1}$ means the empty set. In general we can take arbitrary linear cycles in the Fermat variety, see (17.6).

Conjecture 18.1 *Let $n \geq 4$ be an even number, $m = \frac{n}{2} - 2$ and let $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ be two linear cycles with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ inside the Fermat variety of degree $d = 3$ or 4 and let Z_∞ be the intersection of a linear $\mathbb{P}^{\frac{n}{2}+1} \subset \mathbb{P}^{n+1}$ with X_n^d. For $(r, \check{r}) = (1, -1)$ the following property holds: there exists a semi-irreducible algebraic cycle Z of dimension $\frac{n}{2}$ in X_n^d such that*

1. *For some $a, b \in \mathbb{Z}$, $a \neq 0$, the algebraic cycle Z is homologous to $a(r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}) + bZ_\infty$.*
2. *The deformation space of the pair (X_n^d, Z), as an analytic variety, contains the intersection of deformation spaces of $(X_n^d, \mathbb{P}^{\frac{n}{2}})$ and $(X_n^d, \check{\mathbb{P}}^{\frac{n}{2}})$ as a proper subset.*

An algebraic cycle $Z = \sum_{i=1}^r n_i Z_i$, $n_i \in \mathbb{Z}$ in a smooth projective variety X is called semi-irreducible if the pair (X, Z) can be deformed into (X_t, Z_t) with Z_t irreducible, for a precise definition see §18.7. If d is a prime number or $d = 4$ or d is relatively prime with $(n+1)!$ then the Hodge conjecture for the Fermat variety X_n^d can be proved using only linear cycles, see [Ran81] and [Shi79a]. Therefore, the existence of the algebraic cycle Z in Conjecture 18.1 is not predicted by the Hodge conjecture for X_n^d. We have derived it assuming the Hodge conjecture for all smooth hypersurfaces of degree d and dimension n and few other conjectures with some computational evidences (Conjectures 18.7, Conjecture 18.9 and Conjecture 18.10). The number a is equal to 1 if the integral Hodge conjecture is true and the term bZ_∞ pops up because the relevant computations are done in primitive (co)homologies. Since the algebraic cycle Z is numerically equivalent to $a(r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}) + bZ_\infty$ this might be used to investigate its (non-)existence, at least for Fermat cubic tenfold.

Let $\mathbb{C}[x]_d = \mathbb{C}[x_0, x_1, \cdots, x_{n+1}]_d$ be the set of homogeneous polynomials of degree d in $n+2$ variables, and let T be the open subset of $\mathbb{C}[x]_d$ parameterizing smooth hypersurfaces X of degree d and $\mathsf{T}_{\underline{1}} \subset \mathsf{T}$ be its subset parameterizing those with a linear $\mathbb{P}^{\frac{n}{2}}$ inside X. We use the notation X_t, $t \in \mathsf{T}$ and denote by $0 \in \mathsf{T}$ the point corresponding to the Fermat variety, and so, $X_0 = X_n^d$. The algebraic variety $\mathsf{T}_{\underline{1}}$ is irreducible, however, as an analytic variety in a neighborhood (usual topology) of $0 \in \mathsf{T}$ it has many irreducible components corresponding to deformations of a linear cycle inside X_n^d. Let us denote by $V_{\mathbb{P}^{\frac{n}{2}}}$ the local branch of $\mathsf{T}_{\underline{1}}$ parameterizing deformations of the pair $\mathbb{P}^{\frac{n}{2}} \subset X_n^d$. In general, for a Hodge cycle in $H_n(X_n^d, \mathbb{Z})$ we define the Hodge locus $V_{\delta_0} \subset (\mathsf{T}, 0)$ which is an analytic scheme and its underlying analytic variety consists of points $t \in (\mathsf{T}, 0)$ such that the monodromy $\delta_t \in H_n(X_t, \mathbb{Z})$ of δ_0 along a path in $(\mathsf{T}, 0)$ is still Hodge. For $[\mathbb{P}^{\frac{n}{2}}] \in H_n(X_n^d, \mathbb{Z})$ we know that $V_{[\mathbb{P}^{\frac{n}{2}}]}$ as analytic scheme is smooth and reduced and moreover $V_{\mathbb{P}^{\frac{n}{2}}} = V_{[\mathbb{P}^{\frac{n}{2}}]}$, see the discussion after Theorem 18.6. This is not true for an arbitrary Hodge cycle. Conjecture 18.1 says that $V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ is a proper subset of the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}] + \check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$, see Figure 18.1. In Conjecture 18.1 the case $m = \frac{n}{2} - 1$ and $r = \check{r} = 1$ is excluded, as the pair $(X_n^d, \mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}})$ can be deformed into a hypersurface containing a complete intersection of type $\underbrace{1, 1, \cdots, 1}_{\frac{n}{2} \text{ times}}, 2$. For small m's, the situation is not also strange.

Theorem 18.1 [i] *Let (n,d,m) be one of the following triples*

$$\begin{aligned}
&(2,d,-1),\ 5\le d\le 14,\\
&(4,4,-1),(4,5,-1),(4,6,-1),(4,5,0),(4,6,0),\\
&(6,3,-1),(6,4,-1),(6,4,0),\\
&(8,3,-1),(8,3,0),\\
&(10,3,-1),(10,3,0),(10,3,1),
\end{aligned}$$

and $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ be linear cycles in (18.1)*. The Hodge locus passing through the Fermat point $0\in\mathsf{T}$ and corresponding to deformations of the Hodge cycle $r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]\in H_n(X_n^d,\mathbb{Z})$ with $\mathbb{P}^{\frac{n}{2}}\cap\check{\mathbb{P}}^{\frac{n}{2}}=\mathbb{P}^m$ and $r,\check{r}\in\mathbb{Z},\ r\neq 0,\ \check{r}\neq 0$ is smooth and reduced. Moreover, its underlying analytic variety is simply the intersection $V_{\mathbb{P}^{\frac{n}{2}}}\cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$.*

The cases $(n,d)=(2,4),(4,3)$ are the only cases such that the $(\frac{n}{2}+1,\frac{n}{2}-1)$ Hodge number of X_n^d is equal to one, and these are out of our discussion as all Hodge loci V_{δ_0} are of codimension one, smooth and reduced. For the discussion of these cases and a baby version of Conjecture 18.1 see §18.7. We expect that Theorem 18.1 for $m=-1$ is always true. In this case $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ do not intersect each other. The restriction on n and d in Theorem 18.1 is due to the fact that our proof is computer assisted, and upon a better computer programing and a better device, it might be improved. The first evidence for Conjecture 18.1 is the fact that for many examples

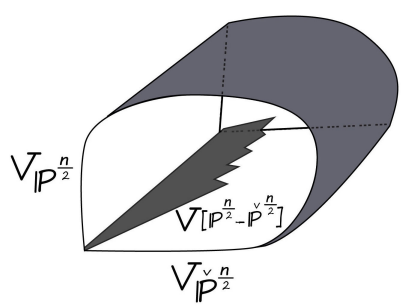

Fig. 18.1 Hodge locus of sum of linear cycles

of n and d, the codimension of the Zariski tangent space of the analytic scheme $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is strictly smaller than the codimension of $V_{\mathbb{P}^{\frac{n}{2}}}\cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ which is smooth. In order to be able to investigate the smoothness and reducedness of this analytic

[i] This theorem for $r=\check{r}=1$ has been announced in [MV19]. Later, R. Villaflor in [Vil20] was able to improve this theorem noting that only $m<\frac{n}{2}-\frac{d}{d-2}$ is needed.

scheme, we have worked out Theorem 18.9 which is just computing a Taylor series. Its importance must not be underestimated. The linear part of such Taylor series encode the whole data of infinitesimal variation of Hodge structures (IVHS) introduced by Griffiths and his coauthors in 1980's, and from this one can derive most of the applications of IVHS, such as global Torelli problem, see [CG80]. In particular, the proof of Theorem 18.1 uses just such linear parts. In a personal communication C. Voisin pointed out the difficulties on higher order approximation of the Noether-Lefschetz locus. This motivated the author to elaborate some of his old ideas in [Mov11] and develop it into Theorem 18.9. The second order approximations in cohomological terms (similar to IVHS), has been formulated in [Mac05], however it is not enough for the investigation of Conjecture 18.1, see Theorem 18.2, and it turns out one has to deal with third and fourth order approximations, see Theorem 18.3. We use Theorem 18.9 to check reducedness and smoothness of components of the Hodge loci. We break the property of being reduced and smooth into N-smooth for all $N \in \mathbb{N}$, see §18.5, and prove the following theorem which is not covered in Theorem 18.1.

Theorem 18.2 *Let (n,d,m) be one of the triples*

$$(6,3,1),(6,3,0),(8,3,1) \tag{18.2}$$

$$(4,4,0),(8,3,2),(8,3,1),(10,3,3),(10,3,2), \tag{18.3}$$

and $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ be linear cycles in (18.1). *For all $r,\check{r} \in \mathbb{Z}$ with $1 \le |r| \le |\check{r}| \le 10$ the analytic scheme $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ is 2-smooth. It is 3-smooth in the cases* (18.2) *and for $(n,d,m,r,\check{r}) = (4,4,0,1,-1)$. It is 4-smooth in the case $(n,d,m,r,\check{r}) = (6,3,1,1,-1)$ and $(n,d,m) = (6,3,0)$.*

Note that the triples in Theorem 18.2 are not covered in Theorem 18.1 and we do not know the corresponding Hodge locus. In order to solve Conjecture 18.1 we will need to identify non-reduced Hodge loci. We prove that:

Theorem 18.3 *Let $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ be linear cycles in* (18.1) *with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$. The analytic scheme $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is either singular at the Fermat point* 0 *or it is non-reduced, in the following cases:*

1. *For all $r,\check{r} \in \mathbb{Z},\ 1 \le |r| \le |\check{r}| \le 10,\ r \ne \check{r}$, $m = \frac{n}{2} - 1$ and (n,d) in the list*

$$(2,d),\ 5 \le d \le 9, \tag{18.4}$$

$$(4,4),(4,5),(6,3),(8,3). \tag{18.5}$$

2. *For all $r,\check{r} \in \mathbb{Z},\ 1 \le |r| \le |\check{r}| \le 10,\ r \ne -\check{r}$ and (n,d,m) in the list*

$$(4,4,0),(6,3,1),(8,3,2).$$

The upper bounds for $|r|$ and $|\check{r}|$ is due to our computational methods, and it would not be difficult to remove this hypothesis. The verification of the case $(n,d,m) = (8,3,2)$ in the second item by a computer takes more than 14 days! Theorem 18.3 in

the case $(n,d,m) = (2,5,0)$ and without the upper bound on $|r|, |\check{r}|$ follows from a theorem of Voisin in [Voi89], see Exercise 2, page 154 [Voi03] and its reproduction in Exercise 16.9. Based on Theorem 18.2 and Theorem 18.3 we may conjecture that for $(n,d,m,r,\check{r}) = (4,4,0,1,-1),(6,3,1,1,-1)$, the analytic scheme $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth and reduced. If this is the case, its underlying analytic variety is bigger than $V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$ (see §18.3), and so, we may try to formulate similar statements as in Conjecture 18.1 in these cases. However, one of the main ingredients of Conjecture 18.1 fails to be true in lower degrees, see Conjecture 18.7 and comments after this. Note that the Hodge conjecture for cubic, quartic and quintic fourfolds and some examples of cubic 8-folds is well-known, see [Lew99] page 92 and the references therein and [Ter90].

18.2 Periods of algebraic cycles

In this section we show how the data of integrals of algebraic differential forms over algebraic cycles can be used in order to prove that algebraic and Hodge cycle deformations of a given algebraic cycle are equivalent.

For a complex projective variety X, an even number n, an element ω of the algebraic de Rham cohomology $\omega \in H^n_{\mathrm{dR}}(X)$ and an irreducible subvariety Z of dimension $\frac{n}{2}$ in X, by a period of Z we simply mean

$$\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_{[Z]}\omega, \tag{18.6}$$

where $[Z] \in H_n(X,\mathbb{Z})$ is the topological class induced by Z. All the homologies with integer coefficients are modulo torsions, and hence they are free $\mathbb{Z}$-modules. We have to use a canonical isomorphism between the algebraic de Rham cohomology and the usual one defined by C^∞-forms in order to say that the integration makes sense, see Grothendieck's article [Gro66]. However, this does not give any clue how to compute such an integral. In general integrals are transcendental numbers, however, in our particular case if X, Z, ω are defined over a subfield k of complex numbers then (18.6) is also in k, see Proposition 1.5 in Deligne's lecture notes in [DMOS82], and so it must be computable. In the C^∞ context many of integrals (18.6) are automatically zero. This is the main content of the celebrated Hodge conjecture, see Chapter 8. Hodge conjecture does not give any information about non-vanishing integrals (18.6). In this section we show that explicit computations of (18.6) lead us to verifications of the following alternative for the Hodge conjecture:

Conjecture 18.2 (Alternative Hodge Conjecture) Let $\{X_t\}_{t\in\mathsf{T}}$ be a family of complex smooth projective varieties of even dimension n, and let Z_0 be a fixed irreducible algebraic cycle of dimension $\frac{n}{2}$ in X_0 for $0 \in \mathsf{T}$. There is an open neighborhood U of 0 in T (in the usual topology) such that for all $t \in U$ if the monodromy $\delta_t \in H_n(X_t,\mathbb{Z})$ of $\delta_0 = [Z_0]$ is a Hodge cycle, then there is an algebraic deformation

$Z_t \subset X_t$ of $Z_0 \subset X_0$ such that $\delta_t = [Z_t]$. In other words, deformations of Z_0 as a Hodge cycle and as an algebraic cycle are the same.

Before explaining the relation of this conjecture with integrals (18.6), we say few words about the importance of Conjecture 18.2. First of all, Conjecture 18.2 might be false in general. P. Deligne in a personal communication (February 4, 2016) pointed out that there are additional obstructions to the hope that algebraic cycles could be constructed by deformation. For instance, the dimension of the intermediate Jacobian coming from the largest sub Hodge structure of $H^{n-1}(X_0,\mathbb{Q}) \cap (H^{\frac{n}{2},\frac{n}{2}-1} \oplus H^{\frac{n}{2},\frac{n}{2}-1})$ might jump down by deformation. The following counterexamples are more concrete.

Take Z_0 to be a sum of two lines with no intersection point in a smooth quartic surface X_0 in $\mathbb{P}^3$ ($K3$ surface), or take Z_0 to be a sum of two linear cycles $\mathbb{P}^2$ in a smooth cubic fourfold which either do not intersect each other, or intersect each other in a point. In both cases the Hodge locus is of codimension one, whereas the codimension of the deformation space of (X_0,Z_0) is ≥ 2, for more details see §18.7. The next examples are taken from a personal communication (November 7, 2018) with P. Deligne.

"Consider a morphism $g : X \to Y$ between smooth projective varieties, and its graph G in $X \times Y$. The cohomology class c of G captures $g^* : H^*(Y) \to H^*(X)$. If we deform X and Y, c will remain Hodge if and only if g^* remains a morphism of Hodge structures, while the cycle G deforms if and only if the morphism g deforms to a morphism between the deformed varieties. Take now for X a hyperelliptic curve of genus ≥ 3, for Y, take $\mathbb{P}^1$, and for g, take the quotient map by the hyperelliptic involution. When we deform X to a curve which is not hyperelliptic, g cannot deform, but g^* is so simple that it has no merit in remaining a morphism of Hodge structures. Variants: take $Y = X$ and for $g : X \to Y$ the hyperelliptic involution. Or take for X a variety whose cohomology is purely Hodge and which admits deformations, for instance a cubic surface, and $Y = X$, g the identity. Another kind of examples: Take a smooth divisor Z on Y, and blow up many points of Y on Z to get X. As cycle C take the pure transform of Z. If we deform X by moving in Y the points to be blown up, the cycle is usually not able to follow. Here it matters what is to be called 'deformation': $C = (C + \text{exceptional divisors}) - (\text{exceptional divisors})$, and both can be deformed."

Despite this abundant number of counterexamples to Conjecture 18.2, we are interested in cases in which it is true, see Theorem 18.5 below. Both Hodge conjecture and Conjecture 18.2 claim that a given Hodge cycle must be algebraic, however, note that Conjecture 18.2 provides a candidate for such an algebraic cycle, whereas the Hodge conjecture doesn't, and so, it must be easier than the Hodge conjecture. Verifications of Conjecture 18.2 support the Hodge conjecture, however, a counterexample to Conjecture 18.2 might not be a counterexample to the Hodge conjecture, because one may have an algebraic cycle homologous to, but different from, the given one in Conjecture 18.2. Note also that in situations where the Hodge conjecture is true, for instance for surfaces, Conjecture 18.2 is still a non-trivial statement:

Theorem 18.4 (Green [Gre88, Gre89], Voisin [Voi88]) *For a smooth hypersurface $X \subset \mathbb{P}^3$ of degree $d \geq 4$ and a line $\mathbb{P}^1 \subset X$, deformations of $\mathbb{P}^1$ as a Hodge cycle and as an algebraic curve are the same.*

Actually, Green and Voisin prove a stronger statement which says that the space of surfaces $X \subset \mathbb{P}^3$ containing a line $\mathbb{P}^1$ is the only component of the Noether-Lefschetz locus of minimum codimension $d-3$. In a similar way some other results of Voisin on Noether-Lefschetz loci, see [Voi90], fit into the framework of Conjecture 18.2. A weaker version of Theorem 18.4 in higher dimensions is generalized in the following way:

Theorem 18.5 ([Mov17b] Theorem 2) *For any smooth hypersurface of degree d and dimension n in a Zariski neighborhood of the Fermat variety with $d \geq 2+\frac{4}{n}$ and a linear projective space $\mathbb{P}^{\frac{n}{2}} \subset X$, deformations of $\mathbb{P}^{\frac{n}{2}}$ as an algebraic cycle and Hodge cycle are the same.*

The relation between integrals (18.6) and Conjecture 18.2 is established through the so-called infinitesimal variation of Hodge structures developed in [CGGH83]. For the Fermat variety this is summarized in Theorem 16.2. Let T be the parameter space of smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$. A hypersurface X_t, $t \in \mathsf{T}$ is given by the projectivization of $f(x_0, x_1, \cdots, x_{n+1}) = 0$, where f is a homogeneous polynomial of degree d. Fix integers $1 \leq d_1, d_2, \ldots, d_{\frac{n}{2}+1} \leq d$ and $\underline{d} := (d_1, d_2, \ldots, d_{\frac{n}{2}+1})$. Let $\mathsf{T}_{\underline{d}} \subset \mathsf{T}$ be the parameter space of hypersurfaces with

$$f = f_1 f_{\frac{n}{2}+2} + \cdots + f_{\frac{n}{2}+1} f_{n+2}, \ \deg(f_i) = d_i, \ \deg(f_{\frac{n}{2}+1+i}) = d - d_i,$$

where f_i's are homogeneous polynomials. The algebraic cycle

$$Z := \mathbb{P}\{f_1 = f_2 = \cdots = f_{\frac{n}{2}+1} = 0\} \subset X \tag{18.7}$$

is called a complete intersection (of type $\underline{d}$) in X. Note that this cycle is a complete intersection in $\mathbb{P}^{n+1}$ and it is not a complete intersection of X with other hypersurfaces. In §17.8 we have computed the codimension of $\mathsf{T}_{\underline{d}}$. Let

$$\omega_i := \mathrm{Resi}\left(\frac{x^i \cdot \sum_{j=0}^{n+1}(-1)^j x_j \, dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1}}{f^k}\right) \tag{18.8}$$

with $k := \frac{n+2+\sum_{e=0}^{n+1} i_e}{d}$, where $\mathrm{Resi} : H^{n+1}_{\mathrm{dR}}(\mathbb{P}^{n+1} - X) \xrightarrow{\sim} H^n_{\mathrm{dR}}(X)_0$ is the residue map and $x^i = x_0^{i_0} \cdots x_{n+1}^{i_{n+1}}$. In §11.7, the restriction of the differential form inside the residue in (18.8) to the affine variety $x_0 = 1$ is denoted by $\frac{\omega_\beta}{f^k}$, where β is obtained by removing i_0 from i. By Griffiths theorem, see Theorem 11.3, we know that $\delta \in H_n(X, \mathbb{Z})_0$ is a Hodge cycle if and only if

$$\int_\delta \omega_i = 0, \ \ \forall i \quad \text{with} \quad \frac{n+2+\sum_{e=0}^{n+1} i_e}{d} \leq \frac{n}{2}.$$

We denote by $\mathrm{Hodge}_n(X,\mathbb{Z})_0$ the $\mathbb{Z}$-modules of n-dimensional primitive Hodge cycles in X. Let us now focus on the Fermat variety X_n^d. We denote by $0 \in \mathsf{T}$ the point corresponding to X_n^d, that is, $X_0 = X_n^d$. The periods of a Hodge cycle $\delta \in \mathrm{Hodge}_n(X_0,\mathbb{Z})_0$ are defined in the following way:

$$\mathsf{p}_i := \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_\delta \omega_i, \tag{18.9}$$

$$\sum_{e=0}^{n+1} i_e = (\frac{n}{2}+1)d-(n+2).$$

Note that the integration of ω_i over the algebraic cycle $Z_\infty := \mathbb{P}^{\frac{n}{2}+1}\cap X$ is automatically zero, and hence, p_i can be also defined as the integration of ω_i over non-primitive algebraic cycles, see §16.4. For natural numbers N, n and d let us define

$$I_N := \left\{(i_0,i_1,\ldots,i_{n+1}) \in \mathbb{Z}^{n+2} \mid 0 \le i_e \le d-2,\ \ i_0+i_1+\cdots+i_{n+1} = N\right\}. \tag{18.10}$$

Assume that n is even and $d \ge 2+\frac{4}{n}$. The periods p_i are indexed by $i \in I_{(\frac{n}{2}+1)d-n-2}$. For any other i which is not in the set $I_{(\frac{n}{2}+1)d-n-2}$, we define p_i to be zero. Let $[\mathsf{p}_{i+j}]$ be a matrix whose rows and columns are indexed by $i \in I_{\frac{n}{2}d-n-2}$ and $j \in I_d$, respectively, and in its (i,j) entry we have p_{i+j}.

Theorem 18.6 [ii] *Let X_n^d be the Fermat variety of degree d and dimension n and let Z be a complete intersection of type $\underline{d}$ inside X_n^d. Let also p_i be the periods of $\delta = [Z]$ defined in* (18.9)*. If*

$$\mathrm{rank}([\mathsf{p}_{i+j}]) = \textit{ the number } (17.29) \tag{18.11}$$

then $\mathsf{T}_{\underline{d}}$ is a component of the Hodge locus. In particular, Conjecture 18.2 is true for smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ containing a complete intersection of type $\underline{d}$, and in a non-empty Zariski open subset of $\mathsf{T}_{\underline{d}}$.

Proof. Theorem 16.2 says that $\mathrm{rank}([\mathsf{p}_{i+j}])$ is the codimension of of the Zariski tangent space of the Hodge locus $V_{[Z]}$ passing through the Fermat point 0 and corresponding to the Hodge cycle $[Z] \in \mathrm{Hodge}_n(X_n^d,\mathbb{Z})$. This together with (18.11) implies that the local branch V_Z of $(\mathsf{T}_{\underline{d}},0)$ corresponding to deformations of (X_n^d,Z) is the same as $V_{[Z]}$. This implies that $V_{[Z]} = V_Z$ and it is reduced and smooth at 0.

We still need to prove that this property is valid for all points in a Zariski open subset of $\mathsf{T}_{\underline{d}}$, and so this is a component of the Hodge locus. The argument is better explained using IVHS which translated into the language of integrals is as follows. It might be useful to recall the proof of Theorem 16.2 as we are going to define the matrix $[\mathsf{p}_{i+j}]$ for an arbitrary smooth hypersurface with an algebraic cycle. We can take global sections $\omega_1,\omega_2,\ldots,\omega_a$ of the n-th de Rham cohomology bundle in some Zariski open neighborhood U of 0 in T such that at each point $t \in U$, they

[ii] The fact that $\mathsf{T}_{\underline{d}}$ is a component of the Hodge locus has been proved in [Dan14] using Macaulay's theorem. One might try to use this in order to prove the equality (18.11).

form a basis of $F^{\frac{n}{2}+1}/F^{\frac{n}{2}+2}$ of $H^n_{\mathrm{dR}}(X_t)$. The linear parts of the integrals $\int_{[Z_t]}\omega_i$ give us a $r\times a$ matrix, $r := \#I_d$, which evaluated at $t=0$ is our $[\mathsf{p}_{i+j}]$. Entries of this matrix are algebraic functions on the affine space which parameterizes $\mathsf{T}_{\underline{d}}$. The rank of this matrix is the constant number (17.29) in $V_{[Z]}\subset \mathsf{T}_{\underline{d}}$. All these imply the same statement in a Zariski open subset of $\mathsf{T}_{\underline{d}}$. $\square$

Our proof of Theorem 18.6 does not imply that the mentioned Zariski open set contains the Fermat point 0. For this stronger statement one has to prove (18.11) for all complete intersection algebraic cycles inside the Fermat variety. This is true for linear algebraic cycles, see Exercise 17.11.

Let $X = X^d_n$ and Z be as in (18.7). It is more desirable to have a comparison of the underlying spaces in (18.11). Recall that we have a canonical inclusion

$$\mathbf{T}_0 V_Z \hookrightarrow \mathbf{T}_0 V_{[Z]}, \tag{18.12}$$

where V_Z is a local irreducible component of $\mathsf{T}_{\underline{d}}$ corresponding to deformations of (X,Z) and $V_{[Z]}$ is the Hodge locus corresponding to $[Z]\in H_n(X,\mathbb{Z})$. This is simply because $V_Z\subset V_{[Z]}$. Here, $\mathbf{T}_0V_Z$ must be understood as the image of a tangent space under the parameterization map of $\mathsf{T}_{\underline{d}}$. We have the following identifications:

$$\begin{aligned}
\mathbf{T}_0 V_Z &\cong \left\{ g_1f_1+g_2f_2+\cdots+g_{n+2}f_{n+2} \;\middle|\; g_i\in \mathsf{k}[x]_{d-a_i}\right\},\\
\mathbf{T}_0 V_{[Z]} &\cong \ker([\mathsf{p}_{i+j}]) = \left\{ v=[v_i,\ i\in I_d]^{\mathsf{tr}} \;\middle|\; [\mathsf{p}_{i+j}]\cdot v = 0\right\},
\end{aligned}$$

where I_d is the set of exponents of all monomials of degree d and $a_i := \deg(f_i)$. The inclusion is given in the following way. For $F\in \mathbf{T}_0V_Z$ we take the coefficients of its monomials and put them in a $1\times \#I_d$ matrix. Note that in general we have

$$\operatorname{rank}([\mathsf{p}_{i+j}]) \le \text{ the number (17.29)}\,. \tag{18.13}$$

Proof (of Theorem 18.5). We need to verify the hypothesis of Theorem 18.6 for $Z=\mathbb{P}^{\frac{n}{2}}\subset X$, that is, for the case $\underline{d}=(1,1,\dots,1)$. This is

$$\operatorname{rank}([\mathsf{p}_{i+j}]) = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2. \tag{18.14}$$

First of all note that the numbers p_i's are not all zero, otherwise the homology class of $\mathbb{P}^{\frac{n}{2}}$ in X would be a multiple of the cycle Z_∞. This cannot happen because the matrix of the intersection form of Z_∞ and $\mathbb{P}^{\frac{n}{2}}$ has non-zero determinant, and hence, by Proposition 8.3 they are linearly independent in homology. Knowing that $\le$ is always true, the proof follows from Problem 21.3 in Chapter 21. Note that for this particular class of algebraic cycles, we have proved the identity (18.14) without computing p_i's. $\square$

Remark 18.1 One of the by-products in the proof of the equality (18.11), for instance for $Z=\mathbb{P}^{\frac{n}{2}}$ as in Theorem 18.5, is the fact that $V_{[Z]}$ is smooth and reduced.

We would like to point out the following conjecture for Fermat varieties.

Conjecture 18.3 (Refined Hodge conjecture for Fermat varieties) *For any Hodge cycle $\delta \in H_n(X_n^d, \mathbb{Z})_0$ such that the Hodge locus V_δ is smooth and reduced there is an algebraic $\frac{n}{2}$-dimensional cycle Z in X and $a \in \mathbb{Z}$, $a \neq 0$ such that $a \cdot \delta = [Z]$, and moreover,* $\mathrm{rank}[\mathsf{p}_{i+j}]$ *is equal to the codimension of the loci of algebraic deformations of Z.*

The Hodge conjecture for all smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$ implies Conjecture 18.3, however, the latter when its hypothesis is satisfied, is strictly stronger than the Hodge conjecture for the Fermat variety X_n^d. We expect that the space of Hodge cycles of the Fermat variety has a basis such that for its element δ, the analytic variety V_δ is smooth and reduced (this is not true for an arbitrary Hodge cycle, see Exercise 16.9). For the Fermat varieties in Theorem 17.1, this follows from the proof Theorem 18.5. We know the Hodge conjecture for X_4^6, for which apart from linear cycles we need the Aoki-Shioda cycles, see [Aok87]. However, the refined Hodge conjecture is still an open problem in this case. One of the reasons for elaborating Theorem 13.3 is to use it and check whether V_δ is smooth and reduced.

We propose two different methods in order to compute integrals (18.6). The first method is purely topological and it is based on the computation of the intersection numbers of algebraic cycles with vanishing cycles. In the case of the Fermat variety, we are able to write down vanishing cycles explicitly, however, they are singular, even though they are homeomorphic to spheres, and many interesting algebraic cycles of the Fermat variety intersect them in their singular points. This makes the computation of intersection numbers harder. We discovered the Hodge cycle $\check{\delta}$ in §15.16 in an effort to push forward this method. The second method is purely algebraic and it is a generalization of Carlson-Griffiths computations in [CG80]. One has to compute the restriction of differential n-forms in X to the top cohomology of Z, and then, one has to compute the so-called trace map. The second method is the main topic of the Ph.D. thesis of R. Villaflor, see [Vil20]. For linear cycles the computation of periods is a direct consequence of Carlson-Griffiths Theorem:

Theorem 18.7 *For $i \in I_{(\frac{n}{2}+1)d-n-2}$ we have*

$$\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_{\mathbb{P}^{\frac{n}{2}}_{a,b}} \omega_i =$$

$$\begin{cases} \frac{\mathrm{sign}(b)}{d^{\frac{n}{2}+1} \cdot \frac{n}{2}!} \zeta_{2d}{}^{\varepsilon} & \text{if} \quad i_{b_{2e-2}} + i_{b_{2e-1}} = d-2, \quad \forall e = 1, \ldots, \frac{n}{2}+1, \\ 0 & \text{otherwise.} \end{cases}$$

where ζ_{2d} is the $2d$-th primitive root of unity and

$$\varepsilon = \sum_{e=0}^{\frac{n}{2}} (i_{b_{2e}} + 1) \cdot (1 + 2a_{2e+1} - 2a_{2e}) \equiv_{2d} \sum_{e=0}^{\frac{n}{2}} (i_{b_{2e}} + 1) - 2 \sum_{e=0}^{n+1} (i_{b_e} + 1) a_e.$$

This is done [MV19] in which we give many applications of this computation in direction of Conjecture 18.2. For the formula of integration over complete intersection

algebraic cycles see also [Vil20]. Theorem 18.5 follows from the verification of the equality (18.14) for periods of linear cycles computed in Theorem 18.7. This verification turns out be an elementary problem which we have formulated in Problem 21.4 in Chapter 21.

The topic of the present section can be reformulated in terms of the so-called infinitesimal variation of Hodge structures (IVHS) developed by Griffiths and his coauthors in the 80's, see [CGGH83]. This is explained in [Mov17b], where the author has tried to keep the classical language of IVHS. The integrals (18.6) are missing in the IVHS formulation and in order to relate them to IVHS, one has to return back to its origin, which is the Gauss-Manin connection of families of varieties and its relation with integrals. This is also fully explained in [Mov17b]. Theorem 18.5 for $n = 2$ is a part of a stronger result which says that there is only one component of the Noether-Lefschetz locus which is of codimension $d-3$ and this is the set of surfaces containing a line. Classifying components of the Hodge locus in each codimension, and in particular reproving the full Green and Voisin's theorem, is of course an important problem, however, the methods developed in this text are suitable for finding more special components crossing the Fermat point. There might be other components with the same dimension of the one we find. There are many works in Noether-Lefschetz locus, for instance, in [Otw03] deals with a generalization of the Noether-Lefschetz locus for curves in three dimensional hypersurfaces, whereas the present text deals with $\frac{n}{2}$ dimensional algebraic cycles in n-dimensional hypersurfaces. The reader is referred to Voisin's expository article [Voi13b] which contains a full exposition and main references on the topic of Hodge and Noether-Lefschetz locus.

In [Gro66] page 103 Grothendieck states a conjecture which is as follows: let $X \to S$ be a smooth morphism of schemes and let S be connected and reduced. A global section α of $H^{2p}_{\mathrm{dR}}(X/S)$ is algebraic at every fiber $s \in S$ if and only if it is a flat section with respect to the Gauss-Manin connection and it is algebraic for one point $s \in S$. Conjecture 18.2, for instance for complete intersections inside hypersurfaces, implies this conjecture in the same context, however the vice versa is not true. The set $\mathsf{T}_{\underline{d}}$ might be a proper subset of a component of the Hodge locus. This would imply that Z is homologous to another algebraic cycle with a bigger deformation space. This cannot happen for the linear case $\underline{d} = (1,1,\cdots,1)$ and the computation of the matrix $[\mathsf{p}_{i+j}]$ and the verification of the equality (18.11) will imply that this never happens. This verification for many examples of n and d has been done in [MV19].

It would be hard to write down a text on algebraic cycles and keep the account of all the available literature on this subject, and in particular the works of S. Bloch. The article [Blo72] is built upon the Grothendieck's conjecture explained above and it considers semi-regular algebraic cycles, that is, the semi-regularity map $\pi : H^1(Z, N_{X/Z}) \to H^{\frac{n}{2}+1}(X, \Omega^{\frac{n}{2}-1})$ is injective. The semi-regularity is a very strong condition. For instance, for curves inside surfaces, [Blo72] only considers the semi-regular curves with $H^1(Z, N_{X/Z}) = 0$. Using Serre duality, one can easily see that this is not satisfied for curves with self intersection less than $2g-2$, where g is the

genus of Z. A simple application of adjunction formula shows that apart from few cases, complete intersection curves inside surfaces do not satisfy this condition.

Years ago when the author started to think about Conjecture 18.2, it seemed to be natural to name it 'Infinitesimal Hodge Conjecture' or 'Variational Hodge Conjecture'. However, a simple internet search for this term gives us few article. The most relevant to the topic of this text is the article [BEK14] and consequently [Blo72]. However, the lack of a strong connection with Conjecture 18.2, was the main reason not to invest on this topic.

18.3 Sum of two linear cycles

Let $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}$ be two linear algebraic cycles in the Fermat variety. We define

$$\mathsf{H}_n^d(m) := \operatorname{rank}\left(\left[\mathsf{p}_{i+j}\left([\mathbb{P}^{\frac{n}{2}}]+[\check{\mathbb{P}}^{\frac{n}{2}}]\right)\right]\right), \text{ where } \mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m. \tag{18.15}$$

Conjecture 18.4 *The number $\mathsf{H}_n^d(m)$ depends only on d,n,m and not on the choice of $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}$.*

We have verified the conjecture for (n,d) in:

$$(2,d),\ \ 5 \le d \le 8,\ (4,4),(6,3),$$

see §20.11 for the computer codes used in this computation. We can use the automorphism group G_n^d of the Fermat variety and we can assume that $\mathbb{P}^{\frac{n}{2}}$ is (17.6) with $a = (0,0,\cdots,0)$ and $b = (0,1,\cdots,n+1)$. In order to avoid Conjecture 18.4 we will fix our choice of linear cycles:

$$\begin{aligned}
\mathbb{P}^{\frac{n}{2}} &= \mathbb{P}^{\frac{n}{2}}_{a,b} \text{ with } a = (0,0,\cdots,0),\ b = (0,1,\cdots,n+1)\\
\check{\mathbb{P}}^{\frac{n}{2}} &= \mathbb{P}^{\frac{n}{2}}_{a,b} \text{ with } a = (\underbrace{0,0,\cdots,0}_{m+1 \text{ times}},1,1,\cdots,1),\ b = (0,1,\cdots,n+1)
\end{aligned}$$

which are those used at the beginning of the present chapter. For examples of $\mathsf{H}_n^d(m)$ see Table 18.1. For a sequence of natural numbers $\underline{a} = (a_1,\ldots,a_s)$ let us define

$$\mathsf{C}_{\underline{a}} = \binom{n+1+d}{n+1} - \sum_{k=1}^{s}(-1)^{k-1} \sum_{a_{i_1}+a_{i_2}+\cdots+a_{i_k}\le d} \binom{n+1+d-a_{i_1}-a_{i_2}-\cdots-a_{i_k}}{n+1}, \tag{18.16}$$

where the second sum runs through all k elements (without order) of a_i, $i = 1,2,\ldots,s$. By our convention, the projective space $\mathbb{P}^{-1}$ means the empty set. By abuse of notation we write

$$a^b := \underbrace{a, a, \cdots, a}_{b \text{ times}}.$$

Hopefully, there will be no confusion with the exponential a^b. Recall our notations in §17.9, Proposition 17.9 and define

$$\mathsf{K}_n^d(m) := \operatorname{codim}(V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}) = 2\mathsf{C}_{1^{\frac{n}{2}+1},(d-1)^{\frac{n}{2}+1}} - \mathsf{C}_{1^{n-m+1},(d-1)^{m+1}}. \tag{18.17}$$

We are now going to analyze the number $\mathsf{H}_n^d(m)$ for $m = \frac{n}{2}, \frac{n}{2}-1, \cdots$. Let us first consider the case $m = \frac{n}{2}$. For the proof of Theorem 18.5 we have verified the first equality in

$$\mathsf{H}_n^d(\frac{n}{2}) = \mathsf{K}_n^d(\frac{n}{2}) = \mathsf{C}_{1^{\frac{n}{2}},(d-1)^{\frac{n}{2}}}.$$

(the second equality follows from Theorem 17.9). One of the by-products of the proof is that $V_{\mathbb{P}^{\frac{n}{2}}}$ as an analytic scheme is smooth and reduced. For $m = \frac{n}{2}-1$, we have

$$\mathsf{H}_n^d(\frac{n}{2}-1) = \mathsf{C}_{1^{\frac{n}{2}},2,(d-1)^{\frac{n}{2}},d-2} \leq \mathsf{K}_n^d(\frac{n}{2}-1) = 2\mathsf{C}_{1^{\frac{n}{2}+1},(d-1)^{\frac{n}{2}+1}} - \mathsf{C}_{1^{\frac{n}{2}+2},(d-1)^{\frac{n}{2}}}.$$

The first equality is conjectural and we can verify it for special cases of n and d by a computer, see [MV19], Section 5. In this case the algebraic cycle $\mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}}$ can be deformed into a complete intersection algebraic cycle of type $(1^{\frac{n}{2}}, 2)$, and so, the inequality is justified. Since the underlying complex variety of the Hodge locus $V_{[\mathbb{P}^{\frac{n}{2}}]+[\check{\mathbb{P}}^{\frac{n}{2}}]}$ contains $V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$, Theorem 16.2 and Theorem 17.9 imply that the inequality

$$\mathsf{H}_n^d(m) \leq \mathsf{K}_n^d(m) \tag{18.18}$$

holds for arbitrary m between -1 and $\frac{n}{2}$. We conjecture that

$$\mathsf{H}_n^d(-1) = 2 \cdot \mathsf{C}_{1^{\frac{n}{2}+1},(d-1)^{\frac{n}{2}+1}}$$

which is the value of $\mathsf{K}_n^d(m)$ (note that $\mathsf{C}_{1^{n+1}} = 0$). This is the same as to say that:

Conjecture 18.5 Let $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}$ be two linear algebraic cycles in the Fermat variety and with no common point. The only deformations of $\mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}}$ as an algebraic or Hodge cycle is again a sum of two linear cycles.

Particular cases of this conjecture has been announced in Theorem 18.1 (those with $m = -1$). It might happen that in (18.18) we have a strict inequality, see for instance Table 18.1.

Conjecture 18.6 *For $n \geq 6$ and $d = 3, 4$ we have*

$$\mathsf{H}_n^d(\frac{n}{2}-2) < \mathsf{K}_n^d(\frac{n}{2}-2). \tag{18.19}$$

Our favorite examples for verifying Conjecture 18.6 are cubic Fermat varieties, that is $d = 3$. For $n \geq 4$ we have the following range:

$$\binom{\frac{n}{2}+1}{3} \leq \operatorname{rank}([\mathsf{p}_{i+j}]) \leq \binom{n+2}{\min\{3, \frac{n}{2}-2\}} \tag{18.20}$$

and in Table 18.1 we have computed $\mathsf{H}_n^3(m)$ for $4 \leq n \leq 10$ and $-1 \leq m \leq \frac{n}{2}$. The following table is the main evidence for Conjecture 18.6. We were also able to

$n\backslash(\frac{n}{2}-m)$	0	1	2	3	4	5	6	7
4	$(1,1)$	$(1,2)$	$(1,2)$	$(1,2)$				
6	$(4,4)$	$(4,7)$	$(6,8)$	$(7,8)$	$(8,8)$			
8	$(10,10)$	$(10,16)$	$(16,19)$	$(19,20)$	$(20,20)$	$(20,20)$		
10	$(20,20)$	$(20,30)$	$(32,36)$	$(38,39)$	$(40,40)$	$(40,40)$	$(40,40)$	
12	$(35,35)$	$(35,50)$	$(55,60)$	$(65,66)$	$(69,69)$	$(70,70)$	$(70,70)$	$(70,70)$

Table 18.1 The numbers $(\mathsf{H}_n^3(m), \mathsf{K}_n^3(m))$.

compute the five-tuples $(n,d,m|\mathsf{H}_n^d(m), \mathsf{K}_n^d(m))$ in the list below:

$$\begin{aligned}
&(4,4,0|11,12),\ (4,4,-1|12,12),\\
&(4,5,0|24,24),\ (4,5,-1|24,24),\\
&(4,6,0|38,38),\ (4,6,-1|38,38),\\
&(6,4,1|36,37),\ (6,4,0|38,38),\ (6,4,-1|38,38).
\end{aligned}$$

We were not able to compute more data such as ? in $(4,7,0|?,54)$. For $n = 2$ and $4 \leq d \leq 14$ we were also able to check Conjecture 18.5. Note that for the quartic Fermat fourfold we have the range $6 \leq \operatorname{rank}([\mathsf{p}_{i+j}]) \leq 21$ and $\mathsf{T}_{1,1,2}$ has codimension 8.

Proof (of Theorem 18.1 for $r = \check{r} = 1$). This is just the outcome of above computations in which $\mathsf{H}_n^d(m) = \mathsf{K}_n^d(m)$. The full proof will be given after Theorem 18.8. For $r = \check{r} = 1$ we have Theorem 18.1 for (n,d,m) in

$$(12,3,-1), (12,3,0), (12,3,1), (12,3,2), \tag{18.21}$$

however, we were not able to verify Theorem 18.8 in these cases. For the computations of $\mathsf{H}_n^d(m)$ and $\mathsf{K}_n^d(m)$ see the codes in §20.10 and §20.11. □

For the convenience of the reader we have also computed the table of Hodge numbers for cubic Fermat varieties. Note that for $d = 3, n = 4$ the Hodge conjecture is well-known, see [Zuc77].

Theorem 18.8 *For all pairs (n,d) in Theorem 18.1 with arbitrary $-1 \leq m \leq \frac{n}{2}$ and all $x \in \mathbb{Q}$ with $x \neq 0$, we have*

n	$\binom{3+n+1}{3}-(n+2)^2$	$\binom{\frac{n}{2}+1}{3}, \binom{n+2}{\min\{3,\frac{n}{2}-2\}}$	Hodge numbers
4	20	1,1	0,1,21,1,0
6	56	4,8	0,0,8,71,8,0,0
8	120	10,45	0,0,0,45,253,45,0,0,0
10	220	20,220	0,0,0,1,220,925,220,1,0,0,0
12	364	35,364	0,0,0,0,14,1001,3432,1001,14,0,0,0,0

Table 18.2 Hodge numbers

$$\operatorname{rank}([\mathsf{p}_{i+j}(\mathbb{P}^{\frac{n}{2}}+x\check{\mathbb{P}}^{\frac{n}{2}})]) = \operatorname{rank}([\mathsf{p}_{i+j}(\mathbb{P}^{\frac{n}{2}}+\check{\mathbb{P}}^{\frac{n}{2}})]) \tag{18.22}$$

and so this number does only depend on (n,d,m) *and not on* x.

Proof. Let $a := \mathsf{H}_n^d(m)$ be the number in the right hand side of (18.22) and let $A(x) := [\mathsf{p}_{i+j}(\mathbb{P}^{\frac{n}{2}}+x\check{\mathbb{P}}^{\frac{n}{2}})]$. Except for a finite number of $x \in \mathbb{Q}$, we have $\operatorname{rank}(A(x)) \geq a$ and in order to prove the equality, it is enough to check it for $a+2$ distinct values of x. This is because if $\operatorname{rank}(A(x)) > a$ then we have a $(a+1)\times(a+1)$ minor of $A(x)$ whose determinant is not zero. This is a polynomial of degree at most $a+1$ in x, and it has $(a+2)$ roots which leads to a contradiction. This argument implies that except for a finite number of values for x we have $\operatorname{rank}(A(x)) = a$. For such exceptional values of x we have $\operatorname{rank}(A(x)) \leq a$, and therefore, in order to get the desired statement for all $x \in \mathbb{Q}$, $x \neq 0$, we find an $a \times a$ minor $B(x)$ of $A(x)$ such that $P(x) := \det(B(x))$ is not identically zero. We only need to check $\operatorname{rank}(A(x)) = a$ for the rational non-zero roots of $P(x) = 0$. For these computations we have used the procedures `GoodMinor` and `ConstantRank` in §20.11. It seems interesting that only for $(n,d,m) = (6,3,1)$, $(6,3,0)$, $(8,3,2)$,$(8,3,1)$,$(6,4,1)$,$(10,3,3)$, $(10,3,2)$ we find a rational root of $P(x)$, and in all these cases it is $x = -1$. This seems to have some relation with Conjecture 18.1 for $(r,\check{r}) = (1,-1)$. □

Proof (of Theorem 18.1). For all the cases in Theorem 18.1

$$\operatorname{rank}\left(\left[r\mathsf{p}_{i+j}(\mathbb{P}^{\frac{n}{2}})+\check{r}\check{\mathsf{p}}_{i+j}(\check{\mathbb{P}}^{\frac{n}{2}})\right]\right) = \operatorname{rank}\left(\left[\mathsf{p}_{i+j}(\mathbb{P}^{\frac{n}{2}})+\check{\mathsf{p}}_{i+j}(\check{\mathbb{P}}^{\frac{n}{2}})\right]\right) = \mathsf{K}_n^d(m),$$

where for the first equality we have used Theorem 18.8. We know that $V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ is the subset of the analytic variety underlying $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ and its codimension is $\mathsf{K}_n^d(m)$. This proves the theorem. □

Remark 18.2 In the proof of Theorem 18.8, one actually observe that $\ker[\mathsf{p}_{i+j}(\mathbb{P}^{\frac{n}{2}}+x\check{\mathbb{P}}^{\frac{n}{2}})]$, $x \neq 0$ (which by Theorem 16.2 is the Zariski tangent space of the Hodge locus $V_{[\mathbb{P}^{\frac{n}{2}}]+x[\check{\mathbb{P}}^{\frac{n}{2}}]}$) is constant. The investigation of this phenomena for other cases might be of some interest.

18.4 Smooth and reduced Hodge loci

Based on the computation in §18.3, we have formulated Conjecture 18.6, and we further claim that:

Conjecture 18.7 *Let $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ be two linear cycles in the Fermat variety X_n^d with $d \geq 2+\frac{4}{n}$ and $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ with $-1 \leq m \leq \frac{n}{2}-1$ and $\mathsf{H}_n^d(m) < \mathsf{K}_n^d(m)$. There is a finite number of coprime non-zero integers $r, \check{r}$ such that the analytic scheme $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth and reduced.*

If $\check{r}=0$ and $r=1$ then we have the Hodge locus $V_{[\mathbb{P}^{\frac{n}{2}}]}$ which is smooth and reduced by Theorem 18.6. Conjecture 18.7 is true in the following case: $m=\frac{n}{2}-1$ and

$$(n,d)=(2,d),\ 4 \leq d \leq 15 \quad (4,d),\ d=3,5,6, \quad (6,d),\ \ d=3,4.$$

In this case, the Hodge locus $V_{[\mathbb{P}^{\frac{n}{2}}]+[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth and reduced at 0 and it parameterizes hypersurfaces with a complete intersection of type $(1^{\frac{n}{2}},2)$, see the comments before Theorem 18.1. The proof can be found in [MV19]. The analytic scheme $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is non-reduced or singular at 0 in the cases covered in Theorem 18.3. Other evidences to Conjecture 18.7 are listed in Theorem 18.2 and Theorem 18.3.

Assuming the Hodge conjecture, the points of the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ parametrizes hypersurfaces with certain algebraic cycles. We do not have any idea how such algebraic cycles look like. In order to verify Conjecture 18.7 without constructing algebraic cycles, we have to analyze the generators $\int_{\delta_t} \omega_i$ of the defining ideal of the Hodge locus in (16.21). These are integrals depending on the parameter $t \in \mathsf{T}$ and their linear part is gathered in the matrix $[\mathsf{p}_{i+j}]$. If Conjecture 18.7 is true in these cases then we have discovered a new Hodge locus, different from $V_{[\mathbb{P}^{\frac{n}{2}}]}$, $V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$ and their intersection. The whole discussion of §18.5 has the goal to provide tools to analyze Conjecture 18.7.

18.5 The creation of a formula

In this section we compute the Taylor series of the integration of differential forms over monodromies of the algebraic cycle $\mathbb{P}^{\frac{n}{2}}_{a,b}$ inside the Fermat variety. Let us consider the hypersurface X_t in the projective space $\mathbb{P}^{n+1}$ given by the homogeneous polynomial:

$$f_t := x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{\alpha} t_\alpha x^\alpha = 0, \tag{18.23}$$

$$t = (t_\alpha)_{\alpha \in I} \in (\mathsf{T}, 0),$$

where α runs through a finite subset I of $\mathbb{N}_0^{n+2}$ with $\sum_{i=0}^{n+1} \alpha_i = d$. In practice, we will take the set $I = I_d$ of all such α with the additional constraint $0 \leq \alpha_i \leq d-2$.

For a rational number r let $[r]$ be the integer part of r, that is $[r] \le r < [r]+1$, and $\{r\} := r-[r]$. Let also $(x)_y := x(x+1)(x+2)\cdots(x+y-1)$, $(x)_0 := 1$ be the Pochhammer symbol. For $\beta \in \mathbb{N}_0^{n+2}$, $\bar\beta \in \mathbb{N}_0^{n+2}$ is defined by the rules:

$$0 \le \bar\beta_i \le d-1, \quad \beta_i \equiv_d \bar\beta_i.$$

Theorem 18.9 *Let $\delta_t \in H_n(X_t,\mathbb{Z})$, $t \in (\mathsf{T},0)$ be the monodromy (parallel transport) of the cycle $\delta_0 := [\mathbb{P}^{\frac{n}{2}}_{\tilde a,b}] \in H_n(X_0,\mathbb{Z})$ along a path which connects 0 to t. For a monomial $x^\beta = x_0^{\beta_0}x_1^{\beta_1}x_2^{\beta_2}\cdots x_{n+1}^{\beta_{n+1}}$ with $k := \sum_{i=0}^{n+1}\frac{\beta_i+1}{d} \in \mathbb{N}$ we have*

$$\frac{C}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_{\delta_t}\mathrm{Resi}\left(\frac{x^\beta\Omega}{f_t^k}\right) = \sum_{a:I\to\mathbb{N}_0}\left(\frac{1}{a!}D_{\beta+a^*}\cdot e^{\pi\sqrt{-1}\cdot E_{\beta+a^*}}\right)\cdot t^a, \quad (18.24)$$

where the sum runs through all #I-tuples $a = (a_\alpha,\ \alpha \in I)$ of non-negative integers such that for $\check\beta := \beta + a^*$ we have

$$\left\{\frac{\check\beta_{b_{2e}}+1}{d}\right\} + \left\{\frac{\check\beta_{b_{2e+1}}+1}{d}\right\} = 1, \quad \forall e = 0,\dots,\frac{n}{2}, \quad (18.25)$$

and

$$\begin{aligned}
C &:= \mathrm{sign}(b)\cdot(-1)^{\frac{n}{2}}\cdot d^{\frac{n}{2}+1}\cdot(k-1)!,\\
t^a &:= \prod_{\alpha\in I} t_\alpha^{a_\alpha}, \qquad |a| := \sum_{\alpha\in I} a_\alpha,\\
a! &:= \prod_{\alpha\in I} a_\alpha!, \qquad a^* := \sum_\alpha a_\alpha\cdot\alpha,\\
D_{\check\beta} &:= \prod_{i=0}^{n+1}\left(\left\{\frac{\check\beta_i+1}{d}\right\}\right)_{\left[\frac{\check\beta_i+1}{d}\right]},\\
E_{\check\beta} &:= \sum_{e=0}^{\frac{n}{2}}\left\{\frac{\check\beta_{b_{2e}}+1}{d}\right\}\cdot(1+2(\tilde a_{2e+1}-\tilde a_{2e}))
\end{aligned}$$

Note that for two types of a the coefficient of t^a in (18.24) is zero. First, when $\beta + a^*$ does not satisfy (18.25). Second, when an entry of $\beta + a^*$ plus one is divisible by d (this is hidden in the definition of $D_{\beta+a^*}$). The coefficients of the Taylor series are in the cyclotomic field $\mathbb{Q}(\zeta_{2d})$.

Proof. This is a direct consequence of Theorem 13.3 and Theorem 18.7, see also Theorem 13.4 for some simplification. □

We are going to explain how to use Theorem 18.9 and give evidences for Conjecture 18.7. Recall the definition of the Hodge locus as an scheme in (16.21). Let

$f_1, f_2, \cdots, f_a \in \mathscr{O}_{\mathsf{T},0}$ be the integrals such that $f_1 = f_2 = \cdots = f_a = 0$ is the underlying analytic variety of the Hodge locus V_{δ_0}. We take $f_1, f_2, \ldots, f_k,\ \ k \leq a$ such that the linear part of $f_1, f_2, \ldots, f_k$ form a basis of the vector space generated by the linear part of all $f_1, f_2, \ldots, f_a$. By Griffiths transversality those of f_i which come from $F^{\frac{n}{2}+2}H^n_{\mathrm{dR}}(X_0)$ have zero linear part and so only $F^{\frac{n}{2}+1}/F^{\frac{n}{2}+2}$ part of the cohomology contribute to the mentioned vector space, see for instance Section 16.5.

The Hodge locus V_{δ_0} is smooth and reduced if and only if the two ideals $\langle f_1, f_2, \ldots, f_k\rangle$ and $\langle f_1, f_2, \ldots, f_a\rangle$ in $\mathscr{O}_{\mathsf{T},0}$ are the same. For this we have to check

$$f \in \langle f_1, f_2, \ldots, f_k, \rangle \ \text{ for } f = f_i,\ \ i = k+1, \ldots, a, \tag{18.26}$$

or equivalently

$$f = \sum_{i=1}^{k} f_i g_i,\ \ g_i \in \mathscr{O}_{\mathsf{T},0}. \tag{18.27}$$

Let $f = \sum_{j=1}^{\infty} f_i,\ \ f_i = \sum_{j=1}^{\infty} f_{i,j}, g_i = \sum_{j=0}^{\infty} g_{i,j}$ be the homogeneous decomposition of f, f_i and g_i, respectively. The identity (18.27) reduces to infinite number of polynomial identities:

$$\begin{aligned} f_1 &= \sum_{i=1}^{k} f_{i,1} g_{i,0}, \\ f_2 &= \sum_{i=1}^{k} f_{i,2} g_{i,0} + \sum_{i=1}^{k} f_{i,1} g_{i,1}, \\ \vdots \quad & \ \vdots \\ f_j &= \sum_{i=1}^{k} f_{i,j} g_{i,0} + \sum_{i=1}^{k} f_{i,j-1} g_{i,1} + \cdots + \sum_{i=1}^{k} f_{i,1} g_{i,j-1}. \end{aligned} \tag{18.28}$$

Definition 18.1 For a Hodge locus V_{δ_0} as in (16.21) and $N \in \mathbb{N}$ we say that it is N-smooth if the first N equations in (18.28) holds for all $f = f_i, i = k+1, k+2, \cdots, a$. In other words (18.27) holds up to monomials of degree $\geq N+1$.

By definition a Hodge locus V_{δ_0} is 1-smooth. Theorem 18.2 and Theorem 18.3, and in particular their computational proof, must be considered our strongest evidence to Conjecture 18.7.

Proof (of Theorem 18.2 and Theorem 18.3). The proof is done using a computer implementation of the Taylor series (18.24). For this we have used the procedures `SmoothReduced` and `TaylorSeries`. For further details see §20.11. In order to be sure that this Taylor series and its computer implementation are mistake-free we have also checked many N-smoothness property which are already proved in Theorem 18.1. In Theorem 18.3 Item 1 we have proved that the corresponding Hodge locus is not 2-smooth except in the following case which we highlight it. Let $\mathbb{P}^1$ and $\check{\mathbb{P}}^1$ be two lines in the Fermat quintic surface intersecting each other in a point. The Hodge locus $V_{r\mathbb{P}^1 + \check{r}\check{\mathbb{P}}^1}$ for all $r, \check{r} \in \mathbb{Z}$ is 2-smooth. Moreover it is not

3-smooth for $0 < |r| < |\check{r}| \leq 10$. In Theorem 18.3 we have proved that the corresponding Hodge locus is not 3-smooth for the cases $(4,4,0)$ and $(8,3,2)$ and it is not 4-smooth in the case $(6,3,1)$. $\square$

The property of being N-smooth for larger N's is out of the capacity of my computer codes, see §18.9 for some comments.

Remark 18.3 Let T_{full} be the open subset of $\mathbb{C}[x]_d$ parameterizing smooth hypersurfaces of degree d and let T be its subset parameterizing hypersurfaces given by (18.23) with $I = I_d$. Throughout the present text we have also used T for T_{full}, and the reason is that any affirmation on Hodge loci for T and T_{full} are equivalent. We have a natural group action $\mathrm{GL}(n+2,\mathbb{C}) \times \mathsf{T}_{\text{full}} \to \mathsf{T}_{\text{full}}$, $(g,f) \mapsto f(g(x))$ which gives us a map $i : \mathrm{GL}(n+2,\mathbb{C}) \times \mathsf{T} \to \mathsf{T}_{\text{full}}$. The derivation of this map at the point $(\text{identity}, \text{Fermat})$ is an isomorphism and so this map is a local biholomrphism (etale) at this point. It can be verified that the pull-back of a Hodge locus in T_{full} by i is the product of $\mathrm{GL}(n+2,\mathbb{C})$ with the Hodge locus in T.

18.6 Uniqueness of components of the Hodge locus

A Hodge cycle $\delta \in H_n(X_n^d, \mathbb{Z})$ is uniquely determined by its periods $\mathsf{p}_i(\delta)$. This data gives the Poincaré dual of δ in cohomology, and hence, the classical Hodge class in the literature. Let Ho_n^d be the $\mathbb{Z}$-module of period vectors p of Hodge cycles. We will also use its projectivization $\mathbb{P}\mathsf{Ho}_n^d$ (two elements p and $\check{\mathsf{p}}$ in the $\mathbb{Z}$-module are the same if there are non-zero integers a and $\check{a}$ such that $a\mathsf{p} = \check{a}\check{\mathsf{p}}$). This $\mathbb{Z}$-module can be described in an elementary linear algebra context without referring to advanced topics, such as homology and algebraic de Rham cohomology, see Chapter 16. Therefore, the conjectures of the present section can be understood by any undergraduate mathematics student! If either d is a prime number or $d = 4$ or d is relatively prime with $(n+1)!$ then we may redefine Ho_d^n the $\mathbb{Z}$-modules generated by $\mathsf{p}^{a,b}$, where

$$\mathsf{p}_i^{a,b} := \begin{cases} \zeta_{2d}^{\sum_{e=0}^{\frac{n}{2}} (i_{b_{2e}}+1)\cdot(1+2a_{2e+1})} & \text{if} \quad i_{b_{2e-2}} + i_{b_{2e-1}} = d-2, \quad \forall e = 1, \ldots, \frac{n}{2}+1, \\ 0 & \text{otherwise.} \end{cases}$$

and a and b are as in (17.6). By Theorem 18.7 and Theorem 17.1 this will be a sub $\mathbb{Z}$-module of the Ho_n^d defined earlier. This will not modify our discussion below. The list of $\mathsf{p}^{a,b}$'s is implemented in the procedure `ListPeriodLinearCycle`, see §20.11. Recall the matrix $[\mathsf{p}_{i+j}]$ in Definition 16.1 and the map $\mathbb{P}\mathsf{Ho}_n^d \to \mathbb{N}, \mathsf{p} \mapsto \mathrm{rank}([\mathsf{p}_{i+j}])$. If $\mathsf{p} \neq 0$ then

$$\binom{\frac{n}{2}+d}{d}-(\frac{n}{2}+1)^2 \leq \mathrm{rank}([\mathsf{p}_{i+j}]) \leq \begin{cases} \binom{\frac{n}{2}d-1}{n+1} & \text{if } d<\frac{2(n+1)}{n-2}, \\ \binom{d+n}{n+1}-(n+2) & \text{if } d=\frac{2(n+1)}{n-2}, \\ \binom{d+n+1}{n+1}-(n+2)^2 & \text{if } d>\frac{2(n+1)}{n-2}. \end{cases} \tag{18.29}$$

Before stating our main conjecture in this section, let us state a simpler one.

Conjecture 18.8 *Let $n \geq 2$ be an even number and $d \geq 3$ an integer with $(n,d) \neq (2,4),(4,3)$. Let also $\mathsf{p} \in \mathsf{Ho}_n^d$ such that*

$$\mathrm{rank}([\mathsf{p}_{i+j}]) = \binom{\frac{n}{2}+d}{d}-(\frac{n}{2}+1)^2. \tag{18.30}$$

Then p, up to multiplication by a rational number, is necessarily of the form $\mathsf{p}^{a,b}$.

One can also formulate a similar conjecture for the next admissible rank. For $n=2$ Voisin's result in [Voi88] tells us that this must be $2d-7 := \mathrm{codim}(\mathsf{T}_{1,2})$. It might happen that in Conjecture 18.8 one must exclude more examples of (n,d). Note that for $(n,d)=(2,4),(4,3)$ both sides of (18.37) are equal to one for all non-zero p.

We need to write down in an elementary language when the linear cycles $\mathbb{P}_i^{\frac{n}{2}}$ and $\check{\mathbb{P}}_j^{\frac{n}{2}}$ underlying two period vectors $\mathsf{p}^i,\ i=(a,b)$ and $\check{\mathsf{p}}^j,\ j=(\check{a},\check{b})$, respectively, have the intersection $\mathbb{P}^m$. This is as follows: A bicycle attached to the permutations b and $\check{b}$ is a sequence $(c_1c_2\ldots c_r)$ with $c_i \in \{0,1,2,\ldots,n+1\}$ and such that if we define $c_{r+1}=c_1$ then for $1 \leq i \leq r$ odd (resp. even) there is an even number k with $0 \leq k \leq n+1$ such that $\{c_i,c_{i+1}\}=\{b_k,b_{k+1}\}$ (resp. $\{c_i,c_{i+1}\}=\{\check{b}_k,\check{b}_{k+1}\}$)) and there is no repetition among c_i's. By definition there is a sequence of even numbers $k_1,k_2,\cdots$ such that

$$\{c_1,c_2\}=\{b_{k_1},b_{k_1+1}\},\ \ \{c_2,c_3\}=\{\check{b}_{k_2},\check{b}_{k_2+1}\},\ \ \{c_3,c_4\}=\{b_{k_3},b_{k_3+1}\},\ldots.$$

Bicycles are defined up to twice shifting c_i's, that is, $(c_1c_2c_3\cdots c_r)=(c_3\cdots c_rc_1c_2)$ etc., and the involution $(c_1c_2c_3\cdots c_{r-1}c_r)=(c_rc_{r-1}\cdots c_3c_2c_1)$. For example, for the permutations

$$b=(0,1,2,3,4,5),\ \check{b}=(1,0,5,3,4,2)$$

we have in total two bicycles $(01),(2354)$. Note that bicycles give us in a natural way a partition of $\{0,1,\ldots,n+1\}$. For such a bicycle we define its conductor to be the sum over k, as before, of the following elements: if $c_i=b_k$ and $c_{i+1}=b_{k+1}$ (resp. $c_i=\check{b}_k$ and $c_{i+1}=\check{b}_{k+1}$) then the element $1+2a_{k+1}$ (resp. $1+2\check{a}_{k+1}$), and if if $c_i=b_{k+1}$ and $c_{i+1}=b_k$ (resp. $c_i=\check{b}_{k+1}$ and $c_{i+1}=\check{b}_k$) then $-1-2a_{k+1}$ (resp. $-1-2\check{a}_{k+1}$). Because of the involution, the conductor is defined up to sign. In our example, the conductor of (01) and (2354) are respectively given by

$$1+2a_1+1+2\check{a}_1,\ \ 1+2a_3-1-2\check{a}_3-1-2a_5+1+2\check{a}_5.$$

A bicycle is called new if $2d$ divides its conductor, and is called old otherwise. Let m_{ij} be the number of new bicycles attached to (i,j) minus one.

Conjecture 18.9 *Let $n \geq 4$ and $d > \frac{2(n+1)}{n-2}$. If for some $\mathsf{p} \in \mathsf{Ho}_n^d$, $\mathsf{p} \neq 0$ we have*

$$\operatorname{rank}[\mathsf{p}_{i+j}] \leq \mathsf{H}_n^d(\frac{n}{2}-2),$$

then p after multiplication with a natural number is in the set

1. $\mathbb{Z}\mathsf{p}^{a,b}$ *and so* $\operatorname{rank}([\mathsf{p}_{i+j}]) = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2$.
2. $\mathbb{Z}\mathsf{p}^{a,b} + \mathbb{Z}\mathsf{p}^{\check{a},\check{b}}$ *with* $m_{a,b,\check{a},\check{b}} = \frac{n}{2}-1$ *and so* $\operatorname{rank}([\mathsf{p}_{i+j}]) = \mathsf{C}_{1^{\frac{n}{2}},2,(d-1)^{\frac{n}{2}},d-2}$.
3. $\mathbb{Z}\mathsf{p}^{a,b} + \mathbb{Z}\mathsf{p}^{\check{a},\check{b}}$ *with* $m_{a,b,\check{a},\check{b}} = \frac{n}{2}-2$ *and so* $\operatorname{rank}[\mathsf{p}_{i+j}] = \mathsf{H}_n^d(\frac{n}{2}-2)$.

A complete analysis of Conjecture 18.9 would require an intensive search for the elements $\mathsf{p} \in \mathsf{Ho}_n^d$ of low $\operatorname{rank}([\mathsf{p}_{i+j}])$. It might be true for $n = 2$ and large d's, and this has to do with the Harris-Voisin conjecture, see [Mov17b], and it is discussed in [Mov20]. Note that the numbers in items 1,2,3 of Conjecture 18.9 for $n = 2$ are respectively $d-3, 2d-7$ and $2d-6$ (for the last one see Conjecture 18.5). We just content ourselves with the following strategy for confirming Conjecture 18.9. Let p^i, $i = 1,2,3$ be three distinct vectors of the form $\mathsf{p}^{a,b}$. We claim that for $d > 3$ we have

$$\operatorname{rank}([\mathsf{p}_{i+j}]) > \mathsf{H}_n^d(\frac{n}{2}-2), \text{ where } \mathsf{p} = \mathsf{p}^1 + \mathsf{p}^2 + \mathsf{p}^3. \tag{18.31}$$

The number $\mathsf{H}_n^d(\frac{n}{2}-2)$ is computed in §18.3 and so we check in total $\binom{N}{3}$ inequalities (18.31), where N is the number of $\mathsf{p}^{a,b}$'s in (17.9). This is too many computations and we have checked (18.31) for samples of p^i's for $(n,d) = (4,6)$. In this way we have also observed that the lower bound for d is necessary as (18.31) is not true for our favorite examples $(n,d) = (4,4),(6,3)$. For $d = 3$, the vector p in (18.31) can be zero. For this computations we have used the procedure `SumThreeLinearCycle`, see §20.11.

The final ingredient of Conjecture 18.1 is the following. In virtue of Theorem 16.2, it compares the Zariski tangent spaces of components of the Hodge locus passing through the Fermat point.

Conjecture 18.10 *Let $n \geq 6$ and $d \geq 3$. There is no inclusion between any two vector spaces of the form*

$$\ker([r\mathsf{p}_{i+j} + \check{r}\check{\mathsf{p}}_{i+j}]), \tag{18.32}$$

where p and $\check{\mathsf{p}}$ ranges in the set of all $\mathsf{p}^{a,b}$ with $m_{a,b,\check{a},\check{b}} = \frac{n}{2}-1, \frac{n}{2}-2$, $r,\check{r} \in \mathbb{Z}$ coprime and $m_{a,b,\check{a},\check{b}} = \frac{n}{2}, r = 1, \check{r} = 0$.

Let δ, $\check{\delta} \in H_n(X_n^d, \mathbb{Q})$ be two Hodge cycles with

$$\ker([\mathsf{p}_{i+j}(\delta)]) \subset \ker([\mathsf{p}_{i+j}(\check{\delta})]), \tag{18.33}$$

that is, the Zariski tangent space of V_δ is contained in the Zariski tangent space of $V_{\check{\delta}}$. The first trivial example to this situation is when $\check{\delta}$ is a rational multiple of $[Z_\infty]$ for which we have $\mathsf{p}(\check{\delta}) = 0$ and $V_{\check{\delta}} = (\mathsf{T}, 0)$. Let us assume that none of $\check{\delta}$ and δ is a rational multiple of $[Z_\infty]$. Next examples for this situation are in Theorem 18.1. In

this theorem the Zariski tangent space of the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]} = V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$, with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ and $r, \check{r} \in \mathbb{Z},\ r \neq 0,\ \check{r} \neq 0$, at the Fermat point does not depend on $r, \check{r}$. For larger m's such as $\frac{n}{2} - 1$, the Zariski tangent spaces of $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ at the Fermat point form a pencil of linear spaces and so there is no inclusion among its members. For $(n,d) = (2,4),(4,3)$, V_δ's are of codimension one, smooth and reduced, and so, any inclusion (18.33) will be an equality and it implies that the period vectors of $\check{\delta}, \check{\delta}$ are the same. This implies that $\delta = a\check{\delta} + b[Z_\infty]$ for some $a, b \in \mathbb{Q}$, and so, $V_\delta = V_{\check{\delta}}$.

We can verify Conjecture 18.10 in the following way. For simplicity we restrict ourselves to the pairs (n,d) in Theorem 18.1 and $r = \check{r} = 1$. Let us take two matrices A and B as inside kernel in (18.32). Let also $A * B$ be the concatenation of A and B by putting the rows of A and B as the rows of $A * B$. Therefore, $A * B$ is a $(2\#I_{\frac{n}{2}d-n-2}) \times (\#I_d)$ matrix. In order to prove that there is no inclusion between $\ker(A)$ and $\ker(B)$ it is enough to prove that

$$\operatorname{rank}(A * B) > \operatorname{rank}(A),\ \operatorname{rank}(B). \tag{18.34}$$

The number of verifications (18.34) is approximately N^4, where N is the number of linear cycles given in (17.9). This is a huge number even for small values of n and d. For this verification we have used the procedure `DistinctHodgeLocus`, see §20.11. Note that the vector space in (18.32) for (n,d,m)'s in Theorem 18.1 is equal to the Zariski tangent space of $V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ at the Fermat point, and hence it does not depend on r and $\check{r}$. This is the main reason why we restrict ourselves to the cases in Conjecture 18.10.

18.7 Semi-irreducible algebraic cycles

Let X be a smooth projective variety and $Z = \sum_{i=1}^r n_i Z_i,\ \ n_i \in \mathbb{Z}$ be an algebraic cycle in X, with Z_i an irreducible subvariety of codimension $\frac{n}{2}$ in X. The following definition is done using analytic deformations and it would not be hard to state it in the algebraic context.

Definition 18.2 We say that $Z = \sum_{i=1}^r n_i Z_i,\ \ n_i \in \mathbb{Z}$ is semi-irreducible if there is a smooth analytic variety $\mathscr{X}$, an irreducible subvariety $\mathscr{Z} \subset \mathscr{X}$ of codimension $\frac{n}{2}$ (possibly singular), a holomorphic map $f : \mathscr{X} \to (\mathbb{C}, 0)$ such that

1. f is smooth and proper over $(\mathbb{C}, 0)$ with X as a fiber over 0. Therefore, all the fibers X_t of f are C^∞ isomorphic to X.
2. The fiber Z_t of $f|_{\mathscr{Z}}$ over $t \neq 0$ is irreducible and $Z_0 = \cup_{i=1}^r Z_i$.
3. The homological cycle $[Z] := \sum_{i=1}^r n_i [Z_i] \in H_n(X, \mathbb{Z})$ is the monodromy of $[Z_t] \in H_n(X_t, \mathbb{Z})$.

It is reasonable to expect that Item 3 is equivalent to a geometric phenomena, purely expressible in terms of degeneration of algebraic varieties. For instance, one might

expect that n_i layers of the algebraic cycle Z_t accumulate on Z_i, and hence semi-irreducibility implies the positivity of n_i's. Moreover, for distinct Z_i and Z_j, the intersection $Z_i \cap Z_j$ is of codimension one in both Z_i and Z_j, because Z_i's are irreducible and of codimension one in $\mathscr{Z}$. In particular, the algebraic cycle $r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m,\ \ m = \frac{n}{2} - 2$ in Conjecture 18.1 is not semi-irreducible.

A smooth hypersurface of degree d and dimension n has the Hodge numbers $h^{n,0} = h^{n-1,1} = \cdots = h^{\frac{n}{2}+2,\frac{n}{2}-2} = 0,\ \ h^{\frac{n}{2}+1,\frac{n}{2}-1} = 1$ if and only if $(n,d) = (2,4),(4,3)$. Recall that Z_∞ is the intersection of a linear $\mathbb{P}^{\frac{n}{2}+1}$ with X_n^d. The following conjecture can be considered as a baby version of Conjecture 18.1.

Conjecture 18.11 *Let $(n,d) = (2,4),(4,3)$ and let Z be an algebraic cycle of dimension $\frac{n}{2}$ and with integer coefficients, in a smooth hypersurface of dimension n and degree d. If $[Z] \in H_n(X_n^d,\mathbb{Q})$ is not a rational multiple of $[Z_\infty]$ then there is a semi-irreducible algebraic cycle $\check{Z}$ of dimension $\frac{n}{2}$ in X_n^d such that $aZ + b\check{Z} + cZ_\infty$ is homologous to zero for some $a,b,c \in \mathbb{Z}$ with $a,b \neq 0$.*

A strategy to prove Conjecture 18.11 is as follows. The algebraic cycle Z induces a homology class $\delta_0 = [Z] \in H_n(X_n^d,\mathbb{Z})$ and the Hodge locus V_{δ_0} is given by the zero locus of a single integral $f(t) := \int_{\delta_t} \omega_0$, where ω_0 is given by (18.8) for $i = (0,0,\cdots,0)$. By our hypothesis on Z, f is not identically zero and since $\delta_0 = [Z]$ it vanishes at $0 \in \mathsf{T}$. We show that V_{δ_0} is smooth and reduced, and for this it is enough to show that the linear part of f is not identically zero. This follows from $\nabla_{\frac{\partial}{\partial t_i}} \omega_0 = \omega_i,\ \ i \in I_d$, $I_d = I_{(\frac{n}{2}+1)d-n-2}$ and the fact that $\omega_0, \omega_i,\ i \in I_d$ form a basis of F^1 of $H^n_{\mathrm{dR}}(X)$. Here, ∇ is the Gauss-Manin connection of the family of hypersurfaces given by (18.23). The Hodge conjecture in both cases is well-known. In the first case it is the Lefschetz $(1,1)$ theorem and in the second case it is a result of Zucker in [Zuc77]. This does not necessarily implies that $\delta_t = [Z_t]$, where $Z_t := \sum_{i=1}^r n_i Z_{i,t},\ Z_{i,t} \subset X_t,\ t \in V_{\delta_0}, \dim(Z_{i,t}) = \frac{n}{2},\ n_i \in \mathbb{Q}$ and for generic t, $Z_{i,t}$ is irreducible and this is the only place where our proof of Conjecture 18.11 fails. If we assume this we proceed as follows. Since $V_{\delta_0} \subset V_{[Z_{i,0}]}$, we conclude that $[Z_{i,0}] = a_i[Z] + b_i[Z_\infty]$ for some $a_i, b_i \in \mathbb{Q}$. By our hypothesis on Z, one of a_i's is not zero let us call it a_1. We get $[Z] = a_1^{-1}[Z_{1,0}] - b_1 a_1^{-1}[Z_\infty]$.

In Conjecture 18.11 let us assume that Z is a sum of linear cycles. It would be useful to see whether the algebraic cycle $\check{Z}$ is a sum of linear cycles. One might start with the sum of two lines in the Fermat surface X_2^4 without any common points (the case $(n,d,m) = (2,4,-1)$).

18.8 How to to deal with Conjecture 18.1?

In this section we sketch a strategy to prove Conjecture 18.1 which follows the same guideline as of the Conjecture 18.11. Let $\delta_0 := r[\mathbb{P}^{\frac{n}{2}}] + \check{r}[\check{\mathbb{P}}^{\frac{n}{2}}] \in H_n(X_n^d,\mathbb{Z})$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m,\ \ m = \frac{n}{2} - 2$ and $\mathsf{H}_n^d(m) < \mathsf{K}_n^d(m)$. Let also $\delta_t \in H_n(X_t,\mathbb{Z}),\ \ t \in (\mathsf{T},0)$ be its monodromy to nearby fibers. Conjecture 18.7 implies that the intersection of

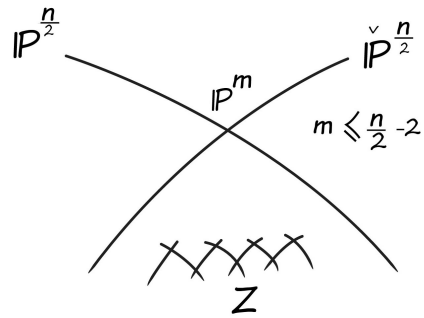

Fig. 18.2 Sum of linear cycles II

$V_{\mathbb{P}^{\frac{n}{2}}}$ and $V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ is a proper subset of the underlying analytic variety of V_{δ_0}. If the Hodge conjecture is true and if we assume that there is an algebraic family of algebraic cycles

$$Z_t := \sum_{k=1}^{r} n_k Z_{k,t},\ Z_{k,t} \subset X_t,\ t \in V_{\delta_0}, \tag{18.35}$$

$$\dim(Z_{k,t}) = \frac{n}{2},\ n_k \in \mathbb{Z},$$

such that $Z_{k,t}$ is irreducible for generic t and Z_t is homologous to a non-zero integral multiple of δ_t, see Figure 18.2. By Conjecture 18.7 we know that V_{δ_0} is smooth and reduced, and so, we have the inclusion of analytic schemes

$$V_{\delta_0} \subset V_{[Z_{k,0}]},\ \ k = 1, 2, \ldots, r \tag{18.36}$$

which implies that

$$\ker[\mathsf{p}_{i+j}(\delta_0)] \subset \ker[\mathsf{p}_{i+j}(Z_{k,0})], \quad \text{and so } \operatorname{rank}([\mathsf{p}_{i+j}(Z_{k,0})]) \leq \operatorname{rank}[\mathsf{p}_{i+j}(\delta_0)]. \tag{18.37}$$

In order to proceed, we consider the cases of Fermat varieties such that linear cycles generates the space of Hodge cycles over rational numbers (these are the cases in Theorem 17.1), or we assume Conjecture 18.9 for Ho_n^d being the lattice of periods of all Hodge cycles and not just linear cycles. We apply Conjecture 18.9 and we conclude that for some linear cycles $\mathbb{P}_k^{\frac{n}{2}}, \check{\mathbb{P}}_k^{\frac{n}{2}}$ in X_n^d with $\mathbb{P}_k^{\frac{n}{2}} \cap \check{\mathbb{P}}_k^{\frac{n}{2}} = \mathbb{P}^{m_k},\ m_k \geq \frac{n}{2} - 2$ and $r_k, \check{r}_k, b_k, c_k \in \mathbb{Z},\ c_k \neq 0$ we have

$$r_k \mathbb{P}_k^{\frac{n}{2}} + \check{r}_k \check{\mathbb{P}}_k^{\frac{n}{2}} + b_k Z_\infty + c_k Z_{k,0} \sim 0, \tag{18.38}$$

where $\sim$ means homologous. The inclusion in (18.37) and (18.38) imply

$$\ker[\mathsf{p}_{i+j}(r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}})] \subset \ker[\mathsf{p}_{i+j}(r_k\mathbb{P}_k^{\frac{n}{2}} + \check{r}_k\check{\mathbb{P}}_k^{\frac{n}{2}})], \tag{18.39}$$

Now, Conjecture 18.10, (18.39) and the fact that r and $\check{r}$ are coprime imply that for some non-zero integer a we have $r_k\mathbb{P}_k^{\frac{n}{2}} + \check{r}_k\check{\mathbb{P}}_k^{\frac{n}{2}} = a(r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}})$, as an equality of algebraic cycles, and hence $\{\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}}\} = \{\mathbb{P}_k^{\frac{n}{2}}, \check{\mathbb{P}}_k^{\frac{n}{2}}\}$. This means that in (18.35) we can assume that Z_t is irreducible for generic t and so we get

$$a(r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}) + bZ_\infty + cZ \sim 0, \quad a,c \neq 0, a,b,c \in \mathbb{Z}, \tag{18.40}$$

where $Z = Z_0$. Taking the intersection of (18.40) with any third linear cycle $\tilde{\mathbb{P}}^{\frac{n}{2}}$ with $\tilde{\mathbb{P}}^{\frac{n}{2}} \cdot \mathbb{P}^{\frac{n}{2}} = \tilde{\mathbb{P}}^{\frac{n}{2}} \cdot \check{\mathbb{P}}^{\frac{n}{2}} = 0$ we get $c \mid b$. Moreover, taking the intersection of (18.40) with any third linear cycle $\tilde{\mathbb{P}}^{\frac{n}{2}}$ with $\tilde{\mathbb{P}}^{\frac{n}{2}} \cdot \mathbb{P}^{\frac{n}{2}} = 1$ and $\tilde{\mathbb{P}}^{\frac{n}{2}} \cdot \check{\mathbb{P}}^{\frac{n}{2}} = 0$ we get $c \mid (ar+b)$. In a similar way, we have $c \mid (a\check{r}+b)$. Since r and $\check{r}$ are coprime we conclude that $c|a$ and, therefore, in (18.40) we can assume that $c = -1$.

One of the most important information about the algebraic cycle $Z \subset X_n^d$ is the data of its intersection numbers with other algebraic cycles of the Fermat variety, and in particular all linear cycles. Recall that $Z_\infty \cdot Z_\infty = d$, for a linear cycle $\mathbb{P}^{\frac{n}{2}} \subset X_n^d$ we have $Z_\infty \cdot \mathbb{P}^{\frac{n}{2}} = 1$, and for two linear cycles $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ we have

$$\mathbb{P}^{\frac{n}{2}} \cdot \check{\mathbb{P}}^{\frac{n}{2}} = \frac{1-(-d+1)^{m+1}}{d} \tag{18.41}$$

This follows from the adjunction formula, see §17.6. Using this we know a and b:

$$b = Z \cdot \tilde{\mathbb{P}}^{\frac{n}{2}}, \text{ where } \tilde{\mathbb{P}}^{\frac{n}{2}} \cdot \mathbb{P}^{\frac{n}{2}} = \tilde{\mathbb{P}}^{\frac{n}{2}} \cdot \check{\mathbb{P}}^{\frac{n}{2}} = 0,$$

and

$$\deg(Z) = a \cdot (r + \check{r}) + b \cdot d.$$

In particular, if $(r, \check{r}) = (1, -1)$ then the degree d of the Fermat variety divides the degree of Z. Another important information about the algebraic cycle Z is a lower bound of the dimension of the Hilbert scheme parameterizing deformations of the pair (X_n^d, Z). One may look for the classification of the components of the Hilbert schemes of projective varieties in order to see whether such a Z exists or not. For instance, we know that if $Z \subset \mathbb{P}^{n+1}$ is an irreducible reduced projective variety of dimension $\frac{n}{2}$ and degree 2 then it is necessarily a complete intersection of type $1^{\frac{n}{2}}, 2$, see [EH87]. One might look for generalizations of this kind of results.

18.9 Final comments

One of the main difficulties in generalizing our main theorems in this chapter for other cases is that the moduli of hypersurfaces of dimension n and degree d is of dimension $\#I_d = \binom{d+n+1}{n+1} - (n+2)^2$ which is two big even for small values of n

and d. One has to prepare similar tables as in Table 18.1 with smaller number of parameters and then start to analyze N-smoothness. For some suggestions see Exercises 15.13, 15.16, 15.17. The author has analyzed statements similar to Theorem 18.2 and Theorem 18.3 for hypersurfaces given by homogeneous polynomials of the form

$$f := A(x_0, x_2, \ldots, x_n) + B(x_1, x_3, \ldots, x_{n+1}). \tag{18.42}$$

The moduli of such hypersurfaces is of dimension $2 \cdot \binom{d+\frac{n}{2}}{\frac{n}{2}} - 2(\frac{n}{2}+1)^2$ and this makes the computations much faster. Here are some sample results mainly in direction of Theorem 18.3. The Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ for $r, \check{r}$ coprime non-zero integers and $|r|, |\check{r}| \leq 10$ is 7-smooth and 4-smooth for $(n,d,m) = (6,3,1)$ and $(4,4,0)$, respectively. Therefore, it seems that we are in situations similar to Theorem 18.1. For $(n,d,m) = (8,3,2), (10,3,3)$ the situation is similar to Theorem 18.2 and Theorem 18.3. Such a Hodge locus is not 3-smooth except for $(r,\check{r}) = (1,\pm 1)$ for which we have even 4-smoothness in the case $(8,3,2)$. The coefficients of the Taylor series in Theorem 18.9 seem to be defined in a reasonable ring, for instance, for $(n,d) = (4,3), (6,3)$ and some sample truncated Taylor series, the ring of coefficients is $\mathbb{Z}[\frac{1}{d}, \zeta_{2d}]$. If so, one may consider them modulo prime ideals, and in this way, study many related conjectures. The tools introduced in this chapter can be used in order to answer the following question which produces an explicit counterexample to a conjecture of J. Harris: determine the integer d (conjecturally less than 10) such that the Noether-Lefschetz locus of surfaces of degree d (resp degree $< d$) has infinite (resp. finite) number of special components crossing the Fermat point. Notice that Voisin's counterexample in [Voi91] is for a very big d. This problem has been studied in the article [Mov20]. For this and its generalization to higher dimensions one needs to classify linear combination of linear cycles in the Fermat variety which are semi-irreducible. The combinatorics of arrangement of linear cycles seems to play some role in this question. The author's favorite examples in this chapter have been cubic varieties, see Manin's book [Man86] for an overview of some results and techniques. Cubic surfaces carry the famous 27-lines which is exactly the number N in (17.9) of linear cycles for the Fermat cubic surface. Hodge conjecture is known for cubic fourfolds (see [Zuc77]), and for a restricted class of cubic 8-folds the Hodge conjecture is also known (see [Ter90]). In general the Hodge conjecture remains open for cubic hypersurfaces of dimension $n \geq 6$. Conjecture 18.1 makes sense starting from cubic tenfolds whose moduli is 220-dimensional. It might be useful to review all the results in this case and to see what one can say more about the algebraic cycle Z in this conjecture. Finally, the knowledge of roots of unity, such as the one discussed in [jL79], might be useful for our main purposes in this book.

Chapter 19
Hodge cycles for cubic hypersurfaces

Every mathematician who is not indifferent to number theory has felt the charm of Fermat's theorem on the sum of two squares of natural numbers. A psychologist of the Jungian school would probably think that such diophantine problems are archetypal to a high degree. The basic idea for the book presented here arose from an attempt to find out what happens in the case of sums of three rational cubes, (Yu. Manin in the introduction of his book [Man86]).

19.1 Abstract

We study an algebraic cycle of the form $Z_0 = r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}},\ r \in \mathbb{N}, \check{r} \in \mathbb{Z},\ \ 1 \leq r, |\check{r}| \leq 10,\ \ \gcd(r,\check{r}) = 1$, inside the cubic Fermat variety X_0 of even dimension $n \geq 4$ and with $\dim\left(\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}}\right) = m$. We take a smooth deformation space S of X_0 such that the triple $(X_0, \mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}})$ becomes rigid. For $m = \frac{n}{2} - 2$ and for many examples of $N \in \mathbb{N}$ and n we show that the N-th order Hodge locus attached to Z_0 is smooth and reduced of positive dimension if and only if $(r,\check{r}) = (1,-1)$. In this case, the underlying algebraic cycles are conjectured to be cubic ruled cycles. For $m = \frac{n}{2} - 3$ the same happens for all choices of coefficients r and $\check{r}$ and we do not know what kind of algebraic cycles might produce such Hodge cycles. The first case gives us a conjectural description of a component of the Hodge locus, and the second case gives us strong computer assisted evidences for the existence of new Hodge cycles for cubic hypersurfaces. Whereas the well-known construction of Hodge cycles due to D. Mumford and A. Weil for CM abelian varieties, and Y. André's motivated cycles can be described in theoretical terms, the full proof of the existence of our Hodge cycle seems to be only possible with more powerful computing machines.

19.2 Introduction

A. Weil starts his article [Wei77] with the following: 'In searching for possible counterexamples to the "Hodge conjecture", one has to look for varieties whose Hodge ring is not generated by its elements of degree 2'. The description of such elements for CM abelian varieties is fairly understood and it is due to D. Mumford and A. Weil himself, see [Mum66a, page 166-167], [Wei77], see also [MZ99] and the references therein for a generalization of this method. Other examples are Y. André's motivated Hodge cycles in [And96]. None of these methods can be applied to hypersurfaces. By Lefschetz hyperplane section theorem, for hypersurfaces of even dimension $n \geq 4$ we have only a one dimensional subspace of $H^{\frac{n}{2},\frac{n}{2}}$ generated by elements of degree 2 (the class of a hyperplane section), and producing interesting Hodge cycles in this case, for which the Hodge conjecture is not known, is extremely difficult, and hence, they might be a better candidate for a counterexample to the Hodge conjecture. Even for Fermat varieties the methods introduced by Z. Ran, T. Shioda in [Ran81], [Shi79a] are not based on explicit description of Hodge cycles, and in Chapter 15 and Chapter 16 we have have worked out a computer implementable description of such cycles. This resulted, for instance, to the verification of integral Hodge conjecture for many examples of the Fermat variety, see [AMV19]. The main goal of Chapter 18 is to describe a computer assisted project in order to classify components of Hodge loci passing through Fermat point, and in this way to discover new Hodge cycles by deforming the class of algebraic cycles. The main difficulty is that the parameter space of hypersurfaces are usually of huge dimensions and it is hard to carry out computations even with the modern computers of today. In November 2018, P. Deligne wrote many comments regarding this chapter, [i] and this gave the author some force to push forward a tiny step toward one of the main goals of the present book. This is namely to find a smaller parameter space for cubic hypersurfaces and describe instances, where a rigid algebraic cycle deforms into a Hodge cycle.

Let $n \geq 4$ be an even number and let us consider the projective space $\mathbb{P}^{n+1}$ with the coordinate system $[x_0 : x_1 : \cdots : x_{n+1}]$ and $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}} \subset \mathbb{P}^{n+1}$ given by:

$$\mathbb{P}^{\frac{n}{2}} : \begin{cases} x_0 - \zeta_6 x_1 = 0, \\ x_2 - \zeta_6 x_3 = 0, \\ x_4 - \zeta_6 x_5 = 0, \\ \cdots \\ x_n - \zeta_6 x_{n+1} = 0, \end{cases} \qquad \check{\mathbb{P}}^{\frac{n}{2}} : \begin{cases} x_0 - \zeta_6 x_1 = 0, \\ \cdots \\ x_{2m} - \zeta_6 x_{2m+1} = 0, \\ x_{2m+2} + x_{2m+3} = 0, \\ \cdots \\ x_n + x_{n+1} = 0, \end{cases} \tag{19.1}$$

where $\zeta_6 := e^{\frac{2\pi\sqrt{-1}}{6}}$. These are linear algebraic cycles in the cubic Fermat variety $X_0 \subset \mathbb{P}^{n+1}$ given by the homogeneous polynomial $x_0^3 + x_1^3 + \cdots + x_{n+1}^3 = 0$, and satisfy $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$. For the main purpose of this chapter we will only consider cubic Fermat varieties and $m = \frac{n}{2} - 2, \frac{n}{2} - 3$. The other cases are fairly discussed in

[i] See Chapter 1 of `w3.impa.br/~hossein/myarticles/Headaches.pdf`.

Chapter 18. We choose a deformation of the Fermat variety

$$X_t : x_0^3 + x_1^3 + \cdots + x_{n+1}^3 - \sum_{\alpha \in I} t_\alpha x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} = 0, \;\; t = (t_\alpha)_{\alpha \in I} \in \mathsf{S} := \mathbb{C}^{\#I}, \quad (19.2)$$

where α runs through a finite subset I of all three elements subsets of $\{0, 1, \ldots, n+1\}$. This deformation is taken in such a way that that the triple $(X_0, \mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}})$ does not deform, see §19.3. For $n = 4, 6, 8, 10$ we have computed such a deformation and the corresponding monomials are listed in Table 19.3 and Table 19.4. For instance, for $n = 4$ we have considered the deformations:

$$X_t : \; x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - (t_1 x_2 + t_2 x_3) x_1 x_5 \text{ the case } m = 0, \quad (19.3)$$

$$X_t : \; x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - (t_1 x_0 + t_2 x_1) x_3 x_5, \text{ the case } m = -1. \quad (19.4)$$

For any algebraic cycle Z_0 of dimension $\frac{n}{2}$ in X_0 let $[Z_0] \in H_n(X_0, \mathbb{Z})$ be the homology class of Z_0 which is a Hodge cycle. For a Hodge cycle $\delta_0 \in H_n(X_0, \mathbb{Z})$, and in particular δ_0 any linear combination of $[\mathbb{P}^{\frac{n}{2}}]$ and $[\check{\mathbb{P}}^{\frac{n}{2}}]$ with $\mathbb{Z}$ coefficients, and $N \in \mathbb{N}$, we can define the Hodge locus V_{δ_0} and N-th order infinitesimal Hodge locus $V_{\delta_0}^N$, see §19.4. The first order Hodge locus $V_{\delta_0}^1$ is the tangent space $\mathbf{T}_0 V_{\delta_0}$ of V_{δ_0} at 0 and its study is mainly done under the name infinitesimal variation of Hodge structures (IVHS) introduced by P. Griffiths and his coauthors in [CGGH83].

Theorem 19.1 *Let $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ be linear cycles in* (19.1) *with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ and $n = 4, 6, 8, 10, 12$. For $m = \frac{n}{2} - 2, \; \frac{n}{2} - 3$ consider the family of hypersurfaces* (19.2) *with monomials coming from Table 19.3 and Table 19.4, respectively.*

1. *($m = \frac{n}{2} - 2$) For all $r, \check{r} \in \mathbb{Z}, \; 1 \le |r|, |\check{r}| \le 10$, the infinitesimal Hodge locus $V^N_{r[\mathbb{P}^{\frac{n}{2}}] + \check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth at 0 for the cases in Table 19.2 with ✓ mark, and further for $r \neq -\check{r}$ it is not smooth at 0 for the cases in Table 19.2 with X mark, and so, for all these cases the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}] + \check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ as analytic scheme is either non-reduced or its underlying analytic variety is singular at the Fermat point 0. Moreover, the infinitesimal Hodge locus $V^N_{[\mathbb{P}^{\frac{n}{2}}] - [\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth for all N's listed in the last row of Table 19.2.*
2. *($m = \frac{n}{2} - 3$) For all $r, \check{r} \in \mathbb{Z}, \; 1 \le |r|, |\check{r}| \le 10$, the infinitesimal Hodge locus $V^N_{r[\mathbb{P}^{\frac{n}{2}}] + \check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth at 0 for all N's listed in Table 19.2. Moreover, their tangent spaces form a pencil with the origin as axis, that is, they have the same dimension and intersect each other only at the origin.*
3. *The codimension of the Zariski tangent space of $V_{r[\mathbb{P}^{\frac{n}{2}}] + \check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ at 0 in both cases as above is listed in the third row of Table 19.2 and Table 19.2, respectively.*
4. *The Fermat variety X_0 together with its algebraic cycles $\mathbb{P}^{\frac{n}{2}}, \; \check{\mathbb{P}}^{\frac{n}{2}}$ with $m = \frac{n}{2} - 2$ (resp. $m = \frac{n}{2} - 3$) is rigid inside the family* (19.2) *with monomials coming from Table 19.3 (resp. 19.4).*

Theorem 19.1 and similar computation in Chapter 18 for the full family of hypersurfaces lead us to speculate many conjectures. The following might be the most evident one.

n	4	6	8	10	12
$\dim(\mathsf{S})$	2	8	19	36	60
$\operatorname{codim}(V^1_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]})$	1	6	16	32	55
$N=2$	✓	✓	✓	✓	✓
$N=3$	✓	✓	X	X	X
$N=4$	✓	X	X	X	X
$Z_0=\mathbb{P}^{\frac{n}{2}}-\check{\mathbb{P}}^{\frac{n}{2}},\quad N=$	∞	14	6	4	3

Table 19.1 Smooth and singular Hodge loci: $m=\frac{n}{2}-2$.

n	4	6	8	10	12
$\dim(\mathsf{S})$	2	8	20	39	66
$\operatorname{codim}(V^1_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]})$	1	7	19	38	65
$Z_0=r\mathbb{P}^{\frac{n}{2}}+\check{r}\check{\mathbb{P}}^{\frac{n}{2}},\ N=$	∞	14	6	4	3

Table 19.2 Smooth Hodge loci: $m=\frac{n}{2}-3$.

Conjecture 19.1 *1. For the full family of cubic hypersurfaces the Hodge locus $V_{[\mathbb{P}^{\frac{n}{2}}]-[\check{\mathbb{P}}^{\frac{n}{2}}]}$ with $\mathbb{P}^{\frac{n}{2}}\cap\check{\mathbb{P}}^{\frac{n}{2}}=\mathbb{P}^{\frac{n}{2}-2}$ is smooth and it is larger than the deformation space of the triple $(X_0,\mathbb{P}^{\frac{n}{2}},\ \check{\mathbb{P}}^{\frac{n}{2}})$.*

2. For the full family of cubic hypersurfaces the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ with $\mathbb{P}^{\frac{n}{2}}\cap\check{\mathbb{P}}^{\frac{n}{2}}=\mathbb{P}^{\frac{n}{2}-3}$ and $|r|,|\check{r}|\in\mathbb{N}$, is smooth and it is larger then the deformation space of the triple $(X_0,\mathbb{P}^{\frac{n}{2}},\ \check{\mathbb{P}}^{\frac{n}{2}})$. The difference of dimensions in this case is 1.

Conjecture 19.1 is true for cubic fourfolds for trivial reasons. In this case $h^{40}=0,\ h^{31}=1$ and so any Hodge locus V_{δ_0}, for $\delta_0\in H_n(X_0,\mathbb{Z})$ a primitive non-zero Hodge cycle, is given by one equation, which turns out that it has non-zero linear part when we consider the full parameter space of cubic hypersurfaces, and hence it is always smooth. Both the integral and rational Hodge conjecture are proved in this case, see [Zuc77] and [Voi13a, Theorem 2.11 and the comments thereafter]. In this case, an effective construction of algebraic cycles for hypersurfaces parameterized by $V_{r[\mathbb{P}^2]+\check{r}[\check{\mathbb{P}}^2]}$ might give some hint to do a similar verification in higher dimensions. For a partial verification of Conjecture 19.1 part 1 see §19.6.

For the proof of Theorem 19.1, the author has written many procedures which are collected in the library `foliation.lib` of SINGULAR, see [GPS01]. In order to check the computations of the present chapter, we first get this library from the author's web page. Then we run the example session of the procedure. For instance, for the procedure `InterTang` used in §19.3 we run

```
LIB foliation.lib;
example InterTang;
```

Modifying, the code in the example session (for instance changing the dimension n or degree d of the hypersurface) we get all the claimed statements.

The present work would not have been possible without the attention of two great mathematicians: thanks go S.-T. Yau for all his effort to create lovely ambients

$n=4$	$x_1x_2x_5,\ x_1x_3x_5$
$n=6$	$x_1x_3x_4,\ x_1x_3x_5,\ x_1x_3x_6,\ x_1x_3x_7,\ x_1x_4x_7,\ x_3x_4x_7,\ x_1x_5x_7,\ x_3x_5x_7$
$n=8$	$x_1x_3x_5\ x_1x_3x_6\ x_1x_5x_6\ x_3x_5x_6\ x_1x_3x_7\ x_1x_5x_7\ x_3x_5x_7\ x_1x_3x_8\ x_1x_5x_8$ $x_3x_5x_8\ x_1x_3x_9\ x_1x_5x_9\ x_3x_5x_9\ x_1x_6x_9\ x_3x_6x_9\ x_5x_6x_9\ x_1x_7x_9\ x_3x_7x_9\ x_5x_7x_9$
$n=10$	$x_1x_3x_5,\ x_1x_3x_7,\ x_1x_5x_7\ x_3x_5x_7\ x_1x_3x_8\ x_1x_5x_8\ x_3x_5x_8\ x_1x_7x_8\ x_3x_7x_8$ $x_5x_7x_8\ x_1x_3x_9\ x_1x_5x_9\ x_3x_5x_9\ x_1x_7x_9\ x_3x_7x_9\ x_5x_7x_9\ x_1x_3x_{10}\ x_1x_5x_{10}$ $x_3x_5x_{10}\ x_1x_7x_{10}\ x_3x_7x_{10}\ x_5x_7x_{10}\ x_1x_3x_{11}\ x_1x_5x_{11}\ x_3x_5x_{11}\ x_1x_7x_{11}\ x_3x_7x_{11}$ $x_5x_7x_{11}\ x_1x_8x_{11}\ x_3x_8x_{11}\ x_5x_8x_{11}\ x_7x_8x_{11}\ x_1x_9x_{11}\ x_3x_9x_{11}\ x_5x_9x_{11}\ x_7x_9x_{11}$
$n=12$	60 monomials

Table 19.3 Monomials of a deformation: $m=\frac{n}{2}-2$.

$n=4$	$x_0x_3x_5,\ x_1x_3x_5$
$n=6$	$x_1x_2x_5,\ x_1x_3x_5,\ x_1x_2x_7,\ x_1x_3x_7,\ x_1x_4x_7,\ x_1x_5x_7,\ x_2x_5x_7,\ x_3x_5x_7$
$n=8$	$x_1x_3x_4,\ x_1x_3x_5,\ x_1x_3x_6,\ x_1x_3x_7,\ x_1x_4x_7,\ x_3x_4x_7,\ x_1x_5x_7,\ x_3x_5x_7,\ x_1x_3x_8$ $x_1x_3x_9,\ x_1x_4x_9,\ x_3x_4x_9,\ x_1x_5x_9,\ x_3x_5x_9,\ x_1x_6x_9,\ x_3x_6x_9,\ x_1x_7x_9,\ x_3x_7x_9$ $x_4x_7x_9,\ x_5x_7x_9$
$n=10$	$x_1x_3x_5,\ x_1x_3x_6,\ x_1x_5x_6,\ x_3x_5x_6,\ x_1x_3x_7,\ x_1x_5x_7,\ x_3x_5x_7,\ x_1x_3x_8,\ x_1x_5x_8$ $x_3x_5x_8,\ x_1x_3x_9,\ x_1x_5x_9,\ x_3x_5x_9,\ x_1x_6x_9,\ x_3x_6x_9,\ x_5x_6x_9,\ x_1x_7x_9,\ x_3x_7x_9$ $x_5x_7x_9,\ x_1x_3x_{10},\ x_1x_5x_{10},\ x_3x_5x_{10},\ x_1x_3x_{11},\ x_1x_5x_{11},\ x_3x_5x_{11},\ x_1x_6x_{11},\ x_3x_6x_{11}$ $x_5x_6x_{11},\ x_1x_7x_{11},\ x_3x_7x_{11},\ x_5x_7x_{11},\ x_1x_8x_{11},\ x_3x_8x_{11},\ x_5x_8x_{11},\ x_1x_9x_{11},\ x_3x_9x_{11}$ $x_5x_9x_{11},\ x_6x_9x_{11},\ x_7x_9x_{11}$
$n=12$	66 monomials

Table 19.4 Monomials of a deformation for $m=\frac{n}{2}-3$.

to do mathematics, from CMSA to TSIMF which I enjoyed both institutes during the preparation of this text, and to P. Deligne for all his enlightening emails and comments to the present book, and in particular this chapter. I would also like to thank D. van Straten for his help in §19.6.

19.3 Smaller deformation space

Let T be the (full) parameter space of smooth degree $d=3$ hypersurfaces of dimension n and X/T be the corresponding family. Let also $X_0,\ 0\in\mathsf{T}$ be a smooth hypersurface given by the zero set of a homogeneous polynomial of the form

$$f=f_1f_{s+1}+f_2f_{s+2}+\cdots+f_sf_{2s},\ \ f_i\in\mathbb{C}[x]_{d_i},\ f_{s+i}\in\mathbb{C}[x]_{d-d_i},\ \ s:=\frac{n}{2}+1. \quad (19.5)$$

We call $Z_0: f_1=f_2=\cdots=f_s=0$ a complete intersection algebraic cycle. Let $V_{Z_0}\subset(\mathsf{T},0)$ be the analytic variety parameterizing deformations of (X_0,Z_0). This corresponds to variation of polynomials f_i as above. This is a branch of the algebraic set $\mathsf{T}_{\underline{d}}\subset\mathsf{T}$ which parameterizes all hypersurfaces given by f of the form 19.5. For $d_1=d_2=\cdots=d_s=1$, $\mathsf{T}_{\underline{1}}$ has $N:=1\cdot3\cdots(n-1)(n+1)d^{\frac{n}{2}+1}$ branches near the Fermat point $0\in\mathsf{T}$, see §17.4. If $f_1,f_2,\cdots,f_{2s}$ have not common zeros in $\mathbb{P}^{n+1}$ then

they form a regular sequence and a Koszul complex argument tells us that V_Z is smooth at 0 and its tangent space at this point is given by the degree d part of the homogeneous ideal $\langle f_1, f_2, \ldots, f_{2s}\rangle$, see Proposition 17.6. In particular, this is the case for two linear cycles

$$\begin{aligned} \mathbb{P}^{\frac{n}{2}} &: f_1 = f_2 = \cdots = f_s = 0, \\ \check{\mathbb{P}}^{\frac{n}{2}} &: \check{f}_1 = \check{f}_2 = \cdots = \check{f}_s = 0, \end{aligned}$$

inside the Fermat variety X_0 and given in (19.1). Moreover, the intersection $V_{\mathbb{P}^{\frac{n}{2}}+\check{\mathbb{P}}^{\frac{n}{2}}} := V_{\mathbb{P}^{\frac{n}{2}}} \cap V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ is smooth at 0 and its tangent space at this point is given by the intersection of tangent spaces of $V_{\mathbb{P}^{\frac{n}{2}}}$ and $V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ at 0 (Proposition 17.9). We conclude that

$$\mathbf{T}_0 V_{\mathbb{P}^{\frac{n}{2}}+\check{\mathbb{P}}^{\frac{n}{2}}} = I_d, \;\; \mathscr{I} := \langle f_1, f_2, \ldots, f_{2s}\rangle \cap \langle \check{f}_1, \check{f}_2, \ldots, \check{f}_{2s}\rangle.$$

We make the identification of T with a Zariski open subset of $\mathbb{C}[x]_d$ such that $f \in \mathsf{T}$ parametrizes the hypersurface given by $x_0^d + x_1^d + \cdots + x_{n+1}^d + f = 0$. In this way, $\mathbf{T_0}\mathsf{T} = \mathbb{C}[x]_d$. We choose a monomial basis x^α, $\alpha \in I$ of degree d piece of the quotient $\mathbb{C}[x]/\mathscr{I}$ and define S to be a linear subspace of $\mathbb{C}[x]$ generated by these monomials. By definition it is perpendicular to $\mathscr{I}_d$, and this is all what we need. The monomials in Table 19.3 and Table 19.4 are obtained in this way. This has been implemented in the procedure `InterTang`.

19.4 Infinitesimal Hodge loci

After P. Griffiths' work [Gri69] we know that for cubic hypersurfaces X_t with t in a Zariski neighborhood of $0 \in \mathsf{T}$, the primitive de Rham cohomology $H^n_{\mathrm{dR}}(X_t)_0$ has a basis of the form

$$\begin{aligned} &\omega_\beta := \mathrm{Resi}\left(\frac{x_{\beta_1} x_{\beta_2} \cdots x_{\beta_{3k-n-2}} \Omega}{f^k}\right), \text{ where} \\ &\Omega = \sum_{i=0}^{n+1} (-1)^i x_i dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_i \wedge \cdots \wedge dx_{n+1}, \\ &k \in \mathbb{N}, \;\; \frac{n+2}{3} \le k \le \frac{2(n+2)}{3}, \;\; \beta := \{\beta_1, \beta_2, \ldots, \beta_{3k-n-2}\} \subset \{0, 1, 2, \ldots, n+1\}, \end{aligned}$$

where $\mathrm{Resi} : H^{n+1}_{\mathrm{dR}}(\mathbb{P}^{n+1} - X_t) \to H^n_{\mathrm{dR}}(X_t)_0$ is the residue map, β_i's are distinct and the subindex 0 refers to primitive cohomology. Moreover, this basis is compatible with the Hodge filtration, that is, $F^{n+1-a} H^n_{\mathrm{dR}}(X_t)_0$ is generated by ω_β with $k \le a$. For the definition of a Hodge locus we need $a = \frac{n}{2}$. For a Hodge cycle $\delta_0 \in H_n(X_0, \mathbb{Z})$, the Hodge locus $V_{\delta_0} \subset (\mathsf{T}, 0)$ is an analytic scheme given by the ideal

$$\left\langle \int_{\delta_t} \omega_\beta \;\middle|\; \forall \beta \subset \{0,1,2,\ldots,n+1\} \text{ with }, \ \#\beta = 3k-n-2, \ k \leq \frac{n}{2} \right\rangle \subset \mathscr{O}_{\mathsf{T},0}. \tag{19.6}$$

Let $\mathscr{M}_{\mathsf{T},0}$ be the maximal ideal of $\mathscr{O}_{\mathsf{T},0}$, that is, the set of germs of holomorphic functions in $(\mathsf{T},0)$ vanishing at 0. The N-th order infinitesimal scheme $V^N_{\delta_0}$ is the induced scheme by (19.6) in the infinitesimal scheme $\mathsf{T}^N := \mathrm{Spec}(\mathscr{O}_{\mathsf{T},0}/\mathscr{M}^{N+1}_{\mathsf{T},0})$. We denote by $\mathsf{X}^N/\mathsf{T}^N$ the N-th order infinitesimal deformation of X_0 induced by X/T. Let $\mathrm{cl}(Z_0) \in H^n_{\mathrm{dR}}(X_0)$ be the class of an algebraic cycle Z_0 of codimension $\frac{n}{2}$ in X_0. Let us consider the Gauss-Manin connection

$$\nabla : H^n_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathscr{O}_{\mathsf{T}}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}).$$

It induces a connection in $H^n_{\mathrm{dR}}(\mathsf{X}^N/\mathsf{T}^N)$ which we call it again the Gauss-Manin connection. There is a unique section s of $H^n_{\mathrm{dR}}(\mathsf{X}^N/\mathsf{T}^N)$ such that $\nabla(\mathsf{s}) = 0$ and $\mathsf{s}_0 = \mathrm{cl}(Z_0)$. This is called the horizontal extension of $\mathrm{cl}(Z_0)$ or a flat section of the cohomology bundle. An equivalent definition for $V^N_{[Z_0]}$ is as follows.

Definition 19.1 The Hodge locus $V^N_{[Z_0]}$ is a subscheme of T^N given by the conditions

$$\nabla(\mathsf{s}) = 0, \tag{19.7}$$

$$\mathsf{s} \in F^{\frac{n}{2}} H^n_{\mathrm{dR}}(\mathsf{X}^N/\mathsf{T}^N), \tag{19.8}$$

$$\mathsf{s}_0 = \mathrm{cl}(Z_0). \tag{19.9}$$

19.5 Proof of Theorem 19.1

The main ingredient of the proof is the closed formula for the Taylor expansion of the holomorphic functions $\int_{\delta_t} \omega_\beta$ introduced in §18.5. The proof of Item 1 and Item 2 are the same as the proof of Theorem 18.2, Theorem 18.3. For Item 3 note that these numbers are the dimension of the $\mathbb{C}$-vector space generated by linear parts of $\int_{\delta_t} \omega_\beta$ in (19.6). Item 4 is just the consequence of our construction of S in §19.3. Note that in a neighborhood of 0 in S, S intersects $V_{\mathbb{P}^{\frac{n}{2}}+\check{\mathbb{P}}^{\frac{n}{2}}}$ only at 0. The following computer code has been used for the proof of statements in Theorem 19.1 Item 1 and Item 2 involving arbitrary coefficients. A simple modification of it can be used for the case $(r,\check{r}) = (1,-1)$.

```
LIB "foliation.lib";
int n=10; int d=3; int m=(n div 2)-2; int tru=4; int zb=10;
intvec zaribl=l,-zb; intvec zarib2=zb, zb;
intvec mlist=d;  for (int i=l;i<=n; i=i+l){mlist=mlist,d;}
ring r=(0,z), (x(l..n+l)),dp;
poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(l));
cp=subst(cp, x(l),z); minpoly =number(cp);
list lcycles=SumTwoLinearCycle(n,d,m,l); lcycles;
list ll=InterTang(n,d,lcycles);
SmoothReduced(mlist,tru, lcycles, zaribl, zarib2,lcycles);
```

The tangent spaces of $V_{[\mathbb{P}^{\frac{n}{2}}]} = V_{\mathbb{P}^{\frac{n}{2}}}$ and $V_{[\check{\mathbb{P}}^{\frac{n}{2}}]} = V_{\check{\mathbb{P}}^{\frac{n}{2}}}$ are of the form $\ker(A)$ and $\ker(\check{A})$, where A and $\check{A}$ are two $\dim(\mathsf{S}) \times h^{\frac{n}{2}+1,\frac{n}{2}-1}$ matrices with entries in $\mathbb{Q}(\zeta_6)$ which can be constructed from the periods of the underlying algebraic cycles, see

§16.5. The tangent space of $V_{[\mathbb{P}^{\frac{n}{2}}]+x[\check{\mathbb{P}}^{\frac{n}{2}}]}$, $x \in \mathbb{Q}$ at 0 is given by $\ker(A + x\check{A})$. By our construction of S, $\ker(A) \cap \ker(\check{A}) = \{0\}$ which is the tangent space of the deformation space of $(X_0, \mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}})$. For two distinct $x_1, x_2 \in \mathbb{Q}$, we have $\ker(A + x_1\check{A}) \cap \ker(A + x_2\check{A}) = \ker(A) \cap \ker(\check{A}) = \{0\}$ and the last affirmation in Item 2 follows. Note that in all the cases considered in Table 19.2, $\ker(A + x_1\check{A})$ is a one dimensional space which is a byproduct of our computations above.

19.6 Finding algebraic cycles

After the first draft of this chapter was written, there were many email exchanges in January 2019 with P. Deligne in order to verify the Hodge conjecture for the Hodge cycles in Theorem 19.1 part 1. Most of the content of this section is the result of this joint effort. We were not able to give similar descriptions for Theorem 19.1 part 2. Let

$$\mathbb{P}^{\frac{n}{2}}_{a_1,a_2} : \begin{cases} x_0 - \zeta_6 x_1 = 0, \\ \cdots \\ x_{n-4} - \zeta_6 x_{n-3} = 0, \\ x_{n-2} - \zeta_6^{2a_1+1} x_{n-3} = 0, \\ x_n - \zeta_6^{2a_2+1} x_{n+1} = 0. \end{cases} \quad \text{where } 0 \le a_1, a_2 \le 2. \tag{19.10}$$

Using this notation we have $\mathbb{P}^{\frac{n}{2}}_{0,0} = \mathbb{P}^{\frac{n}{2}}, \mathbb{P}^{\frac{n}{2}}_{1,1} = \check{\mathbb{P}}^{\frac{n}{2}}$. It turns out that

$$\mathbb{P}^{\frac{n}{2}}_{0,0} - \mathbb{P}^{\frac{n}{2}}_{1,1} = \mathbb{P}^{\frac{n}{2}}_{0,0} + \mathbb{P}^{\frac{n}{2}}_{0,1} + \mathbb{P}^{\frac{n}{2}}_{2,1} \tag{19.11}$$

which is written modulo $\mathbb{P}^{\frac{n}{2}+1}$ slices of the Fermat X_n^3, and hence, both sides of (19.11) induce the same element in the primitive cohomology and have the same Hodge locus. Let C_0 be the algebraic cycle in the right hand side of (19.11). We write the three cycles in C_0 as

$$\begin{aligned} \mathbb{P}^{\frac{n}{2}}_{0,0} &: g_1 = g_2 = \cdots = g_{\frac{n}{2}-1} = f_{11} = f_{21} = 0, \\ \mathbb{P}^{\frac{n}{2}}_{0,1} &: g_1 = g_2 = \cdots = g_{\frac{n}{2}-1} = f_{11} = f_{32} = 0, \\ \mathbb{P}^{\frac{n}{2}}_{2,1} &: g_1 = g_2 = \cdots = g_{\frac{n}{2}-1} = f_{22} = f_{32} = 0, \end{aligned}$$

where g_i and f_{ij}'s are homogeneous linear polynomials. Now, we can easily see that this algebraic cycle deforms into:

$$C : g_1 = g_2 = \cdots = g_{\frac{n}{2}-1} = 0, \quad \operatorname{rank} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix} \le 1, \tag{19.12}$$

where g_i's and f_{ij}'s are deformed homogeneous linear polynomials, and for simplicity, we have not introduced new notation for deformed polynomials. The cycle C deforms to C_0 by setting f_{12}, f_{31} equal to zero. In a personal communication Duco van Straten pointed out the determinantal structure of C and the fact that for $\frac{n}{2} = 2$, C is the cubic ruled surface/Hirzebruch surface F_1. It is isomorphic to $\mathbb{P}^2$ blown up in a single point, embedded by the linear system of quadrics through the point. For this reason, we call C a cubic ruled cycle of dimension $\frac{n}{2}$.

The ideal given by the determinantal variety in (19.12) is radical, and hence, if a smooth hypersurface of degree d and dimension n given by the homogeneous polynomial f contains C then

$$f = g_1 *_1 + g_2 *_2 + \cdots + g_{\frac{n}{2}-1} *_{\frac{n}{2}-1} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}, \tag{19.13}$$

where $*_i$'s are homogeneous polynomials of degree $d - \deg(g_i)$ and the first two columns consist of linear homogeneous polynomials and the last column of degree $d-2$ homogeneous polynomials. Let $\check{\mathsf{T}} \subset \mathsf{T}$ be the space of smooth hypersurfaces of the form (19.13) and $V_{C_0} \subset \check{\mathsf{T}}$ be the branch of $\check{\mathsf{T}}$ corresponding to deformations of (X_0, C_0). It is now natural to conjecture that $V_{[C_0]} = V_{C_0}$. For this it is enough to prove that the codimension of $\check{\mathsf{T}}$ in T is the same as the codimension of $\mathbf{T_0} V_{[C_0]}$ in $\mathbf{T_0}\mathsf{T}$ which is computed in Table 19.2. The author was not able to find a closed formula for the codimension of $\check{\mathsf{T}}$. Instead, it is possible to take random points in $\check{\mathsf{T}}$ and compute the tangent space of $\check{\mathsf{T}}$ at such points for all examples of n in Table 19.2. It turned out that one gets the desired codimension. For this computer assisted verification, we have used `CodRuledCubic`.

It might be possible to prove that $\check{\mathsf{T}}$ is a component of the Hodge locus, that is, for generic X_t, $t \in \check{\mathsf{T}}$ with $C \subset X$ as above, the homology class of C cannot be deformed into a Hodge cycle in $X_t, t \in \mathsf{T} \backslash \check{\mathsf{T}}$. For sum of two linear cycles such statements are proved in [MV19] using computer calculations and in [Vil20] using long theoretical methods.

It is early to claim that the conjectural Hodge cycles introduced in Theorem 19.1 part 2 are counterexamples to the Hodge conjecture, as this needs more time and effort of other people who might try to verify the Hodge conjecture for such cycles. However, it seems to the author that they are better candidates for this than the well-known ones. Components of low codimension of the Noether-Lefschetz loci (Hodge loci for $n = 2$) parameterize surfaces with rather simple algebraic cycles such as lines or quadrics, see for instance [Gre89, Voi88]. The minimal codimension of components of the Hodge loci for cubic hypersurfaces in a Zariski neighborhood of the Fermat point $0 \in \mathsf{T}$ is $\binom{\frac{n}{2}+3}{3} - (\frac{n}{2}+1)^2 = \binom{\frac{n}{2}+1}{3}$ which is obtained by $\mathsf{T}_{\underline{1}}$ introduced in §19.3, see [Mov17b, Theorem 2]. The next admissible codimension seems to be of $\check{\mathsf{T}}$ introduced in this section. Note that for cubic hypersurfaces all the components $\mathsf{T}_{\underline{d}}$ are the same as $\mathsf{T}_{\underline{1}}$. After this, we have conjecturally an infinite number of components $V_{r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}}$, $m = \frac{n}{2} - 3$ of the same codimension. For the convenience of the reader we have also computed Table 19.7 which contains the

dimension of the full moduli, Hodge numbers and the range of codimensions of Hodge loci for cubic hypersurfaces. Finally, note that it follows from [DMOS82, Theorem 2.12, Principle B] that the Hodge cycles of the present text are absolute.

19.7 Learning from cubic fourfolds

Despite the fact that the integral Hodge conjecture for cubic fourfolds is well-known, effective construction of algebraic cycles in this case might help us in a better understanding of the Hodge cycles in Theorem 19.1. For cubic fourfolds, Hodge loci is a union of codimension one irreducible subvarieties $\mathscr{C}_D,\ D \equiv_6 0,2, D \geq 8$ of T, see [Has00]. Here, D is the discriminant of the saturated lattice generated by $[Z]$ and $[Z_\infty]$ in $H_4(X,\mathbb{Z})$ (in [Has00] notation $[Z_\infty] = h^2$), where Z is an algebraic cycle $Z \subset X,\ X \in \mathscr{C}_D$ whose homology class together $[Z_\infty]$ form a rank two lattice. The loci of cubic fourfolds containing a plane (resp. cubic ruled surface) is $\mathsf{T}_{1,1,1} = \mathscr{C}_8$ (resp. $\mathscr{C}_{12}$). A cubic ruled surface is also called cubic scroll. Let $Z = r\mathbb{P}^2 + \check{r}\check{\mathbb{P}}^2$ be as in Theorem 19.1 with $\mathbb{P}^2 \cap \check{\mathbb{P}}^2 = \mathbb{P}^m,\ \ m = -1,0,1,\ r,\check{r}$ are two non-zero coprime integers and $r > 0$. Using adjunction formula, see for instance §17.6, we can easily see that the discriminant of the lattice generated by Z and Z_∞ is

$$D := \begin{cases} 8(r^2+\check{r}^2) - 2r\cdot\check{r}\ m = -1, & \text{two disjoint planes,} \\ 8(r^2+\check{r}^2) + 4r\cdot\check{r}\ m = 0, & \text{two planes intersecting in a point,} \\ 8(r^2+\check{r}^2) - 8r\cdot\check{r}\ m = 1, & \text{two planes intersecting in a line.} \end{cases}$$

The case $m = -1$ is our main interest and it is always $\equiv_6 0,2$, however, the corresponding lattice might not be saturated. The first values of D above are $14,18,36$ which are obtained by $(r,\check{r}) = (1,1),(1,-1),(2,1)$, respectively. Below, we will describe generalizations of $\mathscr{C}_{14}$ and $\mathscr{C}_{20}$ for cubic n-folds. They seems to be far from any possible algebraic cycle for the verification of Theorem 19.1 part 2.

The loci $\mathscr{C}_{14}$ parametrizes cubic fourfolds with a quartic scroll. This is the image of

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5,\ (x,y) \mapsto [f_1g_1 : f_1g_2 : f_1g_3 : f_2g_1 : f_2g_2 : f_2g_3],$$

where f_i's (resp. g_i's) form a basis of $\mathbb{C}[x,y]_1$ (resp. $\mathbb{C}[x,y]_2$). Putting $f_1 = x,\ f_2 = y,\ g_1 = x^2,\ g_2 = xy,\ g_3 = y^2$, the ideal of a quartic scroll is given by

$$C:\ f_{21}f_{22} - f_{11}f_{32} = 0,\ \ f_{21}^2 - f_{11}f_{31} = 0,\ f_{22}^2 - f_{12}f_{32} = 0 \quad \text{rank}\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix} \leq 1, \tag{19.14}$$

where f_{ij}'s are linear equations. In a similar way the image of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ by degree 2 monomials is given by

$$\begin{aligned} C:\ & f_{11}f_{21} - f_{32}^2 = f_{11}f_{31} - f_{22}^2 = f_{21}f_{31} - f_{12}^2 = 0, \\ & f_{12}f_{22} - f_{31}f_{32} = f_{12}f_{32} - f_{21}f_{22} = f_{22}f_{32} - f_{11}f_{12} = 0. \end{aligned} \tag{19.15}$$

The loci of cubic fourfolds with a Veronese surface as above is $\mathscr{C}_{20}$.

It is now natural to generalize the quartic scroll (19.14) and Veronese (19.15) by adding $\frac{n}{2}-2$ linear equations into the defining equation of C and obtain $\frac{n}{2}$ dimensional subvarieties of $\mathbb{P}^{n+1}$. We call them a quartic scroll and Veronese cycle, respectively; carrying in mind that they are $\frac{n}{2}$-dimensional. Let T_{QS} (resp. T_{V}) be the space of smooth degree d hypersurfaces in $\mathbb{P}^{n+1}$ containing a quadric scroll (resp. Veronese) cycle. The ideal of both cycles are radical and written in the standard basis (Groebner basis) and so the defining equation f of a hypersurface containing such a cycle can be written as $f=\sum_{i=1}^{s} f_i g_i$, where f_i's are the polynomials in (19.14) (resp. (19.15)) and g_i's are other homogeneous polynomials such that $\deg(f_i g_i)=d$. We have computed the codimension of the image of the derivation of the maps parameterizing T_{QS} and T_{V} at many random points for cubic hypersurfaces. These are conjecturally the codimensions of T_{QS} and T_{V} and they are listed in Table 19.7. It would not be hard to prove that these are actual codimensions and in the following we are going to assume this. We have used the procedures `CodQuarticScroll` and `CodVeronese` for this codimension computations. In this table L, CS, QS, V means linear cycle $\mathbb{P}^{\frac{n}{2}}$, cubic scroll/ruled, quadric scroll and Veronese, respectively, and the numbers below them are the codimension of the corresponding sub loci of T. The column under M is the codimension of the mysterious components of the Hodge loci that we get in Theorem 19.1, part 2.

Remark 19.1 One loci in Table 19.7 is definitely not a component of the Hodge loci. This is namely, the loci of cubic six folds containing a Veronese cycle. Its codimension is 10, whereas the upper bound for components of the Hodge loci in this case is 8. Assuming the Hodge conjecture in this case, it means that the Veronese cycle is homologous to another algebraic cycle in primitive homology such that the new cycle has bigger deformation space. The same reasoning implies that the loci of cubic six folds containing a quartic scroll is a component of the Hodge loci as its codimension is the maximal one $h^{51}=8$.

Remark 19.2 A natural generalization of $\mathscr{C}_D$ for arbitrary cubic n-folds seems to have increasing codimension with respect to D. If this is the case, even constructing algebraic cycles in the cubic fourfold case might not help to understand the Hodge loci of Theorem 19.1 part 2 whose codimension seems to be below $\operatorname{codim}(\mathsf{T}_{\mathrm{QS}})$.

Remark 19.3 For quartic scroll in (19.14), by setting $f_{12}=f_{31}=0$, degenerates into a sum of two planes and a line $C_0:\mathbb{P}^2_1+\mathbb{P}^2_2+\mathbb{P}^1_3$:

$$\begin{aligned}
&\mathbb{P}^2_1 : f_{21}=f_{22}^2=f_{11}=0,\\
&\mathbb{P}^2_2 : f_{21}^2=f_{22}=f_{32}=0,\\
&\mathbb{P}^1_3 : f_{21}^2=f_{22}^2=f_{21}f_{22}=f_{11}=f_{32}=0.
\end{aligned}$$

Note that $\mathbb{P}^2_i$'s are non-reduced, the intersection $\mathbb{P}^2_1\cap\mathbb{P}^2_2$ is reduced and it is equal to the underlying reduced line of $\mathbb{P}^1_3$. It is not clear whether the Fermat fourfold contains a quartic scroll or its degeneration C_0. Note that if C_0 is inside a cubic

fourfold X, using topological arguments it defines a class $[C_0] \in H^n_{\mathrm{dR}}(X)$, however, it is not clear how to define it algebraically.

$\dim(X_0)$	$\dim(\mathrm{T})$	range of codimensions	L	CS	M	QS	V	Hodge numbers
n	$\binom{n+2}{3}$	$(\binom{\frac{n}{2}+1}{3}, \binom{n+2}{\min\{3, \frac{n}{2}-2\}})$						$h^{n,0}, h^{n-1,1}, \dots, h^{1,n-1}, h^{0,n}$
4	20	$1,1$	1	1	$\boxed{1}$	1	1	$0,1,21,1,0$
6	56	$4,8$	4	6	$\boxed{7}$	8	$\boxed{10}$	$0,0,8,71,8,0,0$
8	120	$10,45$	10	16	$\boxed{19}$	23	25	$0,0,0,45,253,45,0,0,0$
10	220	$20,220$	20	32	$\boxed{38}$	45	47	$0,0,0,1,220,925,220,1,0,0,0$
12	364	$35,364$	35	55	$\boxed{65}$	75	77	$0,0,0,0,14,1001,3432,1001,14,0,0,0,0$

Table 19.5 Codimensions of the components of the Hodge/special loci for cubic hypersurfaces.

We started to prepare Table 19.2 and Table 19.2 with a computer with processor `Intel Core i7-7700`, 16 GB Memory plus 16 GB swap memory and the operating system `Ubuntu 16.04`. It turned out that for many cases such as $(n,m,N) = (12,3,3)$ in Table 19.2 we get the 'Memory Full' error. Therefore, we had to increase the swap memory up to 170 GB. Despite the low speed of the swap which slowed down the computation, the computer was able to use the data and give us the desired output. The computation for this example took more than 21 days. We only know that at least 18 GB of the swap were used. Other time consuming computations are the cases $(n,m,N) = (10,2,4), (12,4,3)$.

19.8 Quartic hypersurfaces

Computations in §18.3 show that we have a similar situation as in Theorem 19.1 for quartic hypersurfaces. Recently, R. Villaflor in [MV19] has proved that the Zariski tangent space of $V_{r\mathbb{P}^{\frac{n}{2}}+\check{r}\check{\mathbb{P}}^{\frac{n}{2}}}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$, $m < \frac{n}{2}$ for hypersurfaces of degree d and dimension n is strictly bigger than the dimension of the deformation space of the triple $(X_0, \mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}})$ if and only if $m \geq \frac{n}{2} - \frac{d}{d-2}$. Knowing that the case $m = \frac{n}{2} - 1$ is expected, see the comments before Theorem 18.1, we are reduced to the cases $d = 3$ with $m = \frac{n}{2} - 3$ or $\frac{n}{2} - 2$ and $d = 4$ with $m = \frac{n}{2} - 2$. The first case is the main topic of the present text. In the second case (quartic hypersurfaces) the author was only able to do computations for quartic six folds. In this case, we use the recipe in §19.3 and consider a deformation of the Fermat variety with 37 monomials, and prove that the infinitesimal Hodge locus $V^3_{r[\mathbb{P}^3]+\check{r}[\check{\mathbb{P}}^3]}$ with $\mathbb{P}^3 \cap \check{\mathbb{P}}^3 = \mathbb{P}^1$ for all $r, \check{r} \in \mathbb{Z}$, $1 \leq |r|, |\check{r}| \leq 10$, is smooth at 0. Also in this case, the tangent spaces $\mathbf{T_0} V_{r[\mathbb{P}^3]+\check{r}[\check{\mathbb{P}}^3]}$ form a pencil of one dimensional spaces with the origin as axis.

Chapter 20
Online supplemental items

In theory there is no difference between theory and practice. In practice there is, (a quote attributed to Yogi Berra or Jan L. A. van de Snepscheut).

20.1 Introduction

In algebra the difference between theory and practice can be simply explained by the following: given a polynomial and an ideal in many variables how can we check that the polynomial belongs to the ideal? In theory, there is no problem at all to think about, whereas in practice we have to develop the beautiful theory of Gröbner bases in order to answer this question. Many arguments and proofs of the present text rely on some computer computations. For this purpose, we have used Singular, [GPS01], a computer programming language for polynomial computations. We have also written the library `foliation.lib` in Singular which contains the implementation of most of the algorithms developed in the present book. In this chapter, we explain how to use this library. The author is not at all a good programmer and so the implementation of many algorithms throughout the book might not be efficient. For instance, in a visit to Kaiserslautern, where Singular is developed, the author was told that considering parameters of a tame polynomial as parameters of the base ring of Singular will highly slow down the computations, and so, it would be necessary to treat such parameters as variables. Since the library `foliation.lib` works perfectly for many simple examples, a better implementation of its content is postponed for some time in the future.

20.2 How to start?

One has to run Singular in the same directory, where `foliation.lib` lies. Then in Singular's command line one has to type:

```
LIB "foliation.lib";
```

In order to get an example and help of a command, for instance `PeriodMatrix`, one has to type respectively:

```
example PeriodMatrix;
help PeriodMatrix;
```

In this chapter I will only sketch few procedures related to the topic of the present text. For more information the reader might consult the help and example of each procedure.

20.3 Tame polynomials

In order to deal with tame polynomials, we have to define a ring with parameters and variables. In order to explain the related commands let us focus on the most well-studied tame polynomial $f = y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3$.

```
ring r=(0,t(1..3)), (x,y),wp(2,3);
poly f=y^2-4*(x-t(1))^3+t(2)*(x-t(1))+t(3);
```

`okbase`: This procedure computes the set of monomials x^β, $\beta \in I$ introduced in §10.6. For this we use the procedure `lasthomo` in order to compute the last homogeneous piece g of f and `jacob` to compute the Jacobian ideal of g:

```
  poly g=lasthomo(f);
  okbase(std(jacob(g)));
//->_[1]=x
//->_[2]=1
```

That is a basis of V_g is given by $x, 1$. For computer implementations we need an order for these monomials . In the following this order is going to be the order in the output of `okbase`. In particular, when we write an element of H in terms of the basis in Theorem 10.1 and Corollary 10.1 then the coefficients are stored in a matrix with the same ordering for its columns or rows.

`discriminant`: This computes the discriminant of a tame polynomial as it is explained in §10.9:

```
  ring r=(0,t(1..3)), (x,y),wp(2,3);
  poly f=y^2-4*(x-t(1))^3+t(2)*(x-t(1))+t(3);
  discriminant(f);
//-> (-1/27*t(2)^3+t(3)^2)
```

In many situation we need to compute the discriminant $\Delta(s)$ of the tame polynomial $f - s$ for an additional parameter s and then the differential n-form η_f in (12.5). For this we use the procedure `etaof`. The discriminant $\Delta(s)$ might be huge which makes the computation of η_f impossible. However, a factor of $\Delta(s)$ might be sufficient for the equality (12.5).

`infoof`: Attached to a tame polynomial f we have many data such as $x^\beta, \beta \in I$, discriminant, η_f etc., and we may want to compute all these once for all, and use in other procedures. This can be done by `infoof`.

`linear`: Let us consider the element $x^n dx \wedge dy$ in the Brieskorn module H'' of f and write down it in terms of the basis $xdx \wedge dy, dx \wedge dy$ given in Theorem 10.1. For this we use the procedure `linear`. For instance for $n = 0,1,2,3$ we use:

```
  for (int i=0;i<=3 ;i=i+1)
      { print(linear(f, x^i)[1]);}
//-> 0,1
//-> 1,0
//-> (2*t(1)),(-t(1)^2+1/12*t(2))
//-> (3*t(1)^2+3/20*t(2)),(-2*t(1)^3+1/10*t(1)*t(2)+1/10*t(3))
```

In this way we can check the equalities in Exercise 2.8. In a similar way the procedure `linearp` is used for the Brieskorn module H'.

`gaussmanin`: This procedure computes the Gauss-Manin connection of tame polynomials. For instance for our main example, let us compute the Gauss-Manin connection of $xdx \wedge dy$ with respect to parameters t_1, t_2, t_3:

```
list l=t(1),t(2),t(3);
gaussmanin(f,l,x);
//-> [1]:
//->    -27/(t(2)^3-27*t(3)^2)
//-> [2]:
//->    _[1,1]=0
//->    _[1,2]=(-1/27*t(2)^3+t(3)^2)
//-> [3]:
//->    _[1,1]=(1/6*t(1)*t(3)-1/108*t(2)^2)
//->    _[1,2]=(-1/6*t(1)^2*t(3)+1/54*t(1)*t(2)^2-1/72*t(2)*t(3))
//-> [4]:
//->    _[1,1]=(-1/9*t(1)*t(2)+1/6*t(3))
//->    _[1,2]=(1/9*t(1)^2*t(2)-1/3*t(1)*t(3)+1/108*t(2)^2)
```

The first entry of the output is one over the discriminant of f. If we multiply it with the other entries of the output we get the data of the Gauss-Manin connection. For instance, the whole data in the output means that:

$$\nabla_{\frac{\partial}{\partial t_3}}(xdx \wedge dy) =$$

$$\frac{-27}{t_2^3 - 27t_3^2}\left(\left(-\frac{1}{9}t_1t_2 + \frac{1}{6}t_3\right)xdx \wedge dy + \left(\frac{1}{9}t_1^2t_2 - \frac{1}{3}t_1t_3 + \frac{1}{108}t_2^2\right)dx \wedge dy\right).$$

In this way we can prove all the ingredient equalities in Exercise 12.2. The Gauss-Manin connection matrix of the tame polynomial f in the basis of Theorem 10.1 is computed by the procedure `gaussmaninmatrix`. Another useful procedure is `gaussmaninvf` which computes the Gauss-Manin connection of an element of the Brieskorn module H in direction of a vector field.

`PFeq`: This procedure computes the Picard-Fuchs equation of tame polynomials over a parameter ring. Let us take the historical examples of Picard-Fuchs equations of the Legendre and Weierstrass family of elliptic curves in Exercise 12.1.

```
  ring r=(0,t), (x,y),wp(2,3);
```

```
  poly L=y^2-x*(x-1)*(x-t);
  PFeq(L,1,t);
//-> _[1,1]=1
//-> _[1,2]=(8t-4)
//-> _[1,3]=(4t2-4t)
  poly W=y^2-x^3+3*t*x-2*t;
  PFeq(W,-2,t);
//-> _[1,1]=(27t+4)
//-> _[1,2]=(288t2-144t)
//-> _[1,3]=(144t3-144t2)
```

For some examples the procedure `PFequ` works faster.

`sysdif`: Let us be given a linear differential equation $Y' = A(z)Y$, where $A(z)$ is a $\mu \times \mu$ matrix with entries in $\mathbb{C}(z)$. For instance, this can be the Gauss-Manin connection matrix of a tame polynomial depending on the parameter z. For a 1-matrix with entries in $\mathbb{C}(z)$ we can compute the linear differential equation satisfied by $B \cdot Y$. This is implemented in `sysdif`. In the example below we have computed the Gauss equation from the Gaussian system:

```
  ring r=(0,z,a,b,c),x,dp;
  matrix A0[2][2]=0,-b,0,1-c; matrix A1[2][2]=0,0,a,c-a-b-1;
  matrix A[2][2]=(1/z)*A0+(1/(z-1))*A1; print(A);
//-> 0,        (-b)/(z),
//-> (a)/(z-1),(-za-zb+c-1)/(z2-z)
  matrix B[1][2]=1,0; sysdif(A,B,z);
//-> _[1,1]=(ab)
//-> _[1,2]=(za+zb+z-c)
//-> _[1,3]=(z2-z)
```

`dbeta`: This procedure is the implementation of the algorithm in Proposition 11.8. Its first output is the list of numbers d_β in this proposition and Theorem 11.5.

```
  ring r=(0,t), (x,y),dp;
  poly f=2*(x3+y3)-3*(x2+y2)-t;
  dbeta(f,2);
//->[1]:
//->   1,1,2,4
//->[2]:
//->   _[1,1]=(-3/2t3-9/2t2-3t)
//->   _[1,2]=(6t+6)
//->   _[1,3]=-3
//->   _[1,4]=-3
```

The second output is a matrix such that the multiplication of its entries is $\tilde{\Delta}$ in the proof of Proposition 11.8.

`LinearRelations`: This procedure takes a tame polynomial f with zero discriminant and returns a list of polynomials P such that Pdx is zero in the Gauss-Manin system of f. These are derived from the equality (12.17) in §12.9.

```
  ring r=(0,a,b,d),(x,y,w),wp(8,9,6);
  poly f=y^2*w-4*x^3+3*a*x*w^2+b*w^3-(1/2)*(d*w^2+w^4);
  LinearRelations(f);
//->[1]:.....
```

Up to multiplications by rational numbers, the output is reproduced in Proposition 12.8.

20.4 Generalized Fermat variety

Recall that Chapter 15 and Chapter 16 are dedicated to the tame polynomial

$$x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}} - 1$$

and its zero set L which is called the generalized Fermat variety. A smooth compactification of L is denoted by X. Most of the procedures explained below have the integer vector $m_1, m_2, \ldots, m_{n+1}$ as input. Inside most of the procedures we have used a ring with the following commands. For some of them one has to run these commands before using them.

```
LIB "foliation.lib";
intvec mlist=2,4,5; //--substitue this with any sequence of
                    //--positive integers m_1,m_2,...,m_{n+1}
int n=size(mlist)-1;   int d=lcm(mlist);
list wlist;         //--weight of the variables
for (int i=1; i<=size(mlist); i=i+1)
    { wlist=insert(wlist, (d div mlist[i]), size(wlist));}
ring r=(0,z), (x(1..n+1)),wp(wlist[1..n+1]);
poly cp=cyclotomic(d);   int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z);
minpoly =number(cp); //--z is the d-th root of unity---
```

It defines a ring with $n+1$ variables x_i of weight $\nu_i := \frac{d}{m_i}$ and the base field $\mathbb{Q}(\zeta_d)$, where d is the lowest common multiple of m_i's. The last four lines are used in the procedures which deal with the periods of the generalized Fermat variety.

`MixedHodgeFermat`: Its output is a list of lists. The third list is simply the basis of monomials (15.11). This list for us represents both the basis $\delta_\beta,\ \delta \in I$ of $H_n(L,\mathbb{Z})$ and $\omega_\beta, \beta \in I$ of $H^n_{\mathrm{dR}}(L)$. The first two lists are the monomial ingredients of the pieces of the mixed Hodge structure of $H^n_{\mathrm{dR}}(L)$. The first list is a list of $n+1$ list; its k-th list contains the monomials of (15.17). The second list is a list of n lists; its k-th list contains the monomials (15.16).

```
  ring r=0, (x,y,z),wp(10,5,4);
  list li=MixedHodgeFermat(intvec(2,4,5));
  list h20=li[1][1]; list h11=li[1][2]; list h02=li[1][3];
  h20[1..size(h20)]; h11[1..size(h11)]; h02[1..size(h02)];
//-> 1
//-> yz3 z3 y2z2 yz2 z2 y2z yz z y2 y
//-> y2z3
```

`HodgeNumber`: The Hodge numbers (15.19) and (15.18) are simply the cardinality of the first and second list of the procedure `MixedHodgeFermat`. Note that these are the Hodge numbers of the primitive cohomologies $H^n_{\mathrm{dR}}(X)_0$ and $H^n_{\mathrm{dR}}(Y)_0$.

```
  HodgeNumber(intvec(5,5,5));
//->[1]:
//->   [1]:
//->      4
//->   [2]:
//->      44
//->   [3]:
```

```
//->      4
//->[2]:
//->   [1]:
//->      6
//->   [2]:
//->      6
```

`PeriodMatrix`: The period matrix in §15.9 is implemented in this procedure. It gives the period matrix of a list of elements of the de Rham cohomology of the generalized Fermat variety (the first list of input) over a list of vanishing cycles (the second list of the input). The third input is the d-th root of unity of the base ring.

```
  ring r=(0,z), (x,y),dp;
  minpoly=z^2+z+1;
  poly f=x3+y3;
  ideal I=std(jacob(f)); I=kbase(I);
  list Il=I[1..size(I)]; Il[1..size(Il)];
//->xy y x 1
  print(PeriodMatrix(Il,Il, z));
//->-3,    3,      3,      -3,
//->(-3z), (-3z-3),(3z),   (3z+3),
//->(-3z), (3z),   (-3z-3),(3z+3),
//->(3z+3),3,      3,      (-3z)
```

`DimHodgeCycles`: This procedure gives us the dimension of the primitive part of the vector space of Hodge cycles of the generalized Fermat variety. These are Hodge cycles with support in L. Its input is the integer vector $m_1, m_2, \ldots, m_{n+1}$. It has a second optional input. If it is 0 then the procedure gives the integer valued matrix $\tilde{\mathbf{P}}^{\frac{n}{2}+1}$ introduced in §15.10. Therefore, any perpendicular vector to its columns correspond to an affine Hodge cycle. For the optional input one can also give a list $\tilde{I}$ of exponents $\beta \in I$. In this case, a similar matrix $\tilde{\mathbf{P}}^{\frac{n}{2}+1}$ is given using $\omega_\beta, \beta \in \tilde{I}$ instead of $\omega_\beta,\ \beta \in I,\ A_\beta < \frac{n}{2},\ A_\beta \notin \mathbb{N}$. Using this procedure we have produced Table 15.7. In my computer it took some hours to compute

$$\dim_{\mathbb{Q}} \Big(\mathrm{Hodge}_4(X_4^6, \mathbb{Q}) \Big) = 1752 \tag{20.1}$$

Note that for the classical Fermat variety the dimension $\dim_{\mathbb{Q}} \big(\mathrm{Hodge}_4(X_4^6, \mathbb{Q}) \big) = \dim_{\mathbb{Q}} \big(\mathrm{Hodge}_4(X_4^6, \mathbb{Q})_0 \big) + 1$.

```
  ring r=0,x,dp;
  DimHodgeCycles(intvec(6,6,6,6,6));
//->1751
  ring r=0,x,dp;
  print(DimHodgeCycles(intvec(2,4,5),0));
//->0, 2, -2,0, 0, 0, 2, 2,
//->0, 2, 2, 0, 0, 0, -2,2,
//->0, -2,2, 0, 0, 0, -2,-2,
//->-2,0, 4, 2, -2,0, 2, -2,
//->2, 0, -4,2, 2, 0, -2,-2,
//->2, 0, -4,-2,2, 0, -2,2,
//->2, 2, -2,-4,4, 2, -2,-2,
//->-2,2, 2, -4,-4,2, 2, -2,
//->-2,-2,2, 4, -4,-2,2, 2,
```

```
//->2, -2,0, 2, -2,-4,0, 2,
//->-2,-2,0, 2, 2, -4,0, 2,
//->-2,2, 0, -2,2, 4, 0, -2
```

For the computation of the rank of the elliptic curve $y^2+x^3+z^d=1$ over $\mathbb{C}(z)$ introduced in §15.13, we have used the following code:

```
ring r=0,x,dp;
for (int i=3; i<=80; i=i+1)
{i, DimHodgeCycles(intvec(2,3,i));}
```

`IntersectionMatrix`: This procedure gives us the intersection form $\langle\cdot,\cdot\rangle$ in the basis δ_β defined in §15.7. Its input is any set of monomials x^β representing the topological cycle δ_β.

```
  ring r=0, (x,y,z),dp;
  poly f=x3+y3+z3;
  ideal I=std(jacob(f)); I=kbase(I);
  list Il=I[1..size(I)]; Il[1..size(Il)];
//-> xyz yz xz z xy y x 1
    print(IntersectionMatrix(Il));
//-> -2,1, 1, -1,1, -1,-1,1,
//-> 1, -2,0, 1, 0, 1, 0, -1,
//-> 1, 0, -2,1, 0, 0, 1, -1,
//-> -1,1, 1, -2,0, 0, 0, 1,
//-> 1, 0, 0, 0, -2,1, 1, -1,
//-> -1,1, 0, 0, 1, -2,0, 1,
//-> -1,0, 1, 0, 1, 0, -2,1,
//-> 1, -1,-1,1, -1,1, 1, -2
```

20.5 Hodge cycles and periods

`BasisHodgeCycles`: The output of this matrix is the matrix `X` introduced in §15.14. The rows of this matrix form a basis of the space of affine Hodge cycles of the generalized Fermat variety over rational numbers and written in the basis of vanishing cycles $\delta_\beta, \beta\in I$. It includes the cycles at infinity. The second input is optional, if it is the d-th root of unity then the output is a list of two matrices `X` and `H` introduced in §15.14. The first matrix is as before. The rows of the second matrix are the periods of Hodge cycles using the differential forms ω_β, $\frac{n}{2}<A_\beta<\frac{n}{2}+1$ (a basis of $F^{\frac{n}{2}}/F^{\frac{n}{2}+1}$ piece of $H^n_{\mathrm{dR}}(X)_0$). In this case, the first $n+1$ variable of the base ring are reserved for the equation of the Fermat variety. Their weights must be $\nu_i:=\frac{d}{m_i}$. If the optional input is a d-th root of unity and a list $\tilde{I}$ of integer vectors $\beta\in I$, then the procedure uses $\omega_\beta, \beta\in\tilde{I}$ instead of ω_β, $\beta\in I$, $A_\beta<\frac{n}{2}$, $A_\beta\notin\mathbb{N}$.

```
  ring r=0,x,dp;
  print(BasisHodgeCycles(intvec(2,4,5)));
//->0,0,0,0,0,0,1,0,1,0,0,0,
//->1,0,1,0,0,0,0,0,0,0,0,0,
//->0,0,0,1,0,1,0,0,0,0,0,0,
//->0,0,0,0,0,0,0,0,0,1,0,1
```

```
  ring rr=(0,z),x(1..3),wp(10,5,4);
  minpoly =z8-z6+z4-z2+1; //20-th root of unity
  list A=BasisHodgeCycles(intvec(2,4,5),z);
  print(A[2], "%s");
//->(8*z^6-8*z^2),0,0,(8*z^4+8*z^2),0,0,(-16*z^6+8*z^4-8*z^2+8),0,0,
//->(8*z^6-16*z^4+8*z^2-8),
//->(-8*z^6+16*z^4-8*z^2+8),0,0,(16*z^6-8*z^4+8*z^2-8),0,0,
//->(-8*z^4-8*z^2), 0,0,(-8*z^6+8*z^2),
//->(8*z^6-8*z^4+16*z^2-8),0,0,(-8*z^6-8*z^4),0,0,(8*z^6+8*z^4),0,0,
//->(-8*z^6+8*z^4-16*z^2+8),
//->(-8*z^6-8),0,0,(-8*z^2-8),0,0,(8*z^6-8*z^4+8*z^2-16),0,0,(8*z^4-8)
```

As it is noticed in Remark 15.1 we can write Hodge cycles in terms of vanishing cycles by adding more vanishing conditions. For instance, the K3 surface $x^4+y^4+z^4=1$ has the Picard rank 20. We can write $19=20-1$ primitive Hodge cycles in terms of $27=3^3$ vanishing cycles δ_β. We first run the code in §20.4 with `mlist=4,4,4;` and then

```
  list ll=MixedHodgeFermat(mlist); list J=ll[1][1]+ll[2][1];
  for (i=2; i<=(n div 2); i=i+1){J=J+ll[1][i]+ll[2][i];}
  list Jexp;  for (i=1; i<=size(J); i=i+1)
            {Jexp=insert(Jexp, leadexp(J[i]),  size(Jexp));}
  list A=BasisHodgeCycles(mlist,z, Jexp);
  rank(DimHodgeCycles(mlist, Jexp));  print(A[1]);
```

```
//->8
//->1, 0, 1, 0, 0,0,1,   0,   1,0,0,0,0,0,0,0,0,0,   0,   0,0,0,0,0,0,0,
//->0, 0, 0, -1,0,0,-1/2,-1/2,0,1,0,0,0,0,0,0,0,0,1/2, 1/2, 0,0,0,0,0,0,0,
//->0, 0, -1,1, 0,0,1/2, 1/2, 0,0,1,0,0,0,0,0,0,0,-1/2,1/2, 0,0,0,0,0,0,0,
//->0, 1, 1, -1,0,0,-1/2,-1/2,0,0,0,1,0,0,0,0,0,0,-1/2,-1/2,0,0,0,0,0,0,0,
//->0, 1, 0, 0, 0,0,-1/2,1/2, 0,0,0,0,1,0,0,0,0,0,-1/2,1/2, 0,0,0,0,0,0,0,
//->-1,0, 0, 0, 0,0,-1/2,-1/2,0,0,0,0,0,1,0,0,0,0,-1/2,-1/2,0,0,0,0,0,0,0,
//->1, -1,1, 0, 0,0,1/2, -1/2,0,0,0,0,0,0,1,0,0,0,1/2, -1/2,0,0,0,0,0,0,0,
//->0, -1,0, 1, 0,0,1/2, -1/2,0,0,0,0,0,0,0,1,0,0,-1/2,-1/2,0,0,0,0,0,0,0,
//->1, 0, 1, -1,0,0,1/2, -1/2,0,0,0,0,0,0,0,0,1,0,1/2, -1/2,0,0,0,0,0,0,0,
//->0, 0, -1,1, 0,0,1/2, 3/2, 0,0,0,0,0,0,0,0,0,1,1/2, 1/2, 0,0,0,0,0,0,0,
//->0, 1, 1, -1,0,1,-1,  -1,  0,0,0,0,0,0,0,0,0,0,0,   0,   0,0,0,0,0,0,0,
//->0, 0, -1,1, 1,0,0,   1,   0,0,0,0,0,0,0,0,0,0,0,   0,   0,0,0,0,0,0,0,
//->1, 0, 1, 0, 0,0,0,   0,   0,0,0,0,0,0,0,0,0,0,1,   0,   1,0,0,0,0,0,0,
//->0, -1,0, 1, 0,0,0,   0,   0,0,0,0,0,0,0,0,0,0,0,   -1,  0,1,0,0,0,0,0,
//->1, 0, 1, -1,0,0,0,   -1,  0,0,0,0,0,0,0,0,0,1,   0,   0,0,1,0,0,0,0,
//->0, 0, -1,1, 0,0,1,   1,   0,0,0,0,0,0,0,0,0,0,0,   1,   0,0,0,1,0,0,0,
//->1, 0, 0, 0, 0,0,1,   0,   0,0,0,0,0,0,0,0,0,1,   0,   0,0,0,0,1,0,0,
//->0, 1, 0, 0, 0,0,0,   1,   0,0,0,0,0,0,0,0,0,0,   1,   0,0,0,0,0,1,0,
//->-2,0, -1,0, 0,0,-1,  0,   0,0,0,0,0,0,0,0,0,-1,  0,   0,0,0,0,0,0,1
```

The vanishing cycles in this case are ordered according to:

$$x^2y^2z^2, xy^2z^2, y^2z^2, x^2yz^2, xyz^2, yz^2, x^2z^2, xz^2, z^2, x^2y^2z, xy^2z, y^2z, x^2yz,$$

$$xyz, yz, x^2z, xz, z, x^2y^2, xy^2, y^2, x^2y, xy, y, x^2, x, 1.$$

The number 8 computed above is the codimension of the space of Hodge cycles in the 27-dimensional space $H_2(L,\mathbb{Q})$.

`Matrixpij`: This procedure computes the matrix $[\mathsf{p}_{i+j}]$ introduced in §16.4.

```
  ring r=0,(x,y,z), wp(20,15,12);
  matrix P[1][20]; P[1,1..20]=1..20;
  print(Matrixpij(intvec(3,4,5),P));
//->0,0,0,1,2,3,4,5, 6, 7, 8, 9, 10,11,12,13,14,15,
//->3,4,5,6,7,8,9,10,12,13,14,15,0, 16,17,18,19,20
```

In this example the monomials corresponding to I_d and $I^{1,1}$ are respectively:

$$yz^3,\ xz^3,\ z^3,\ y^2z^2,\ xyz^2,\ yz^2,\ xz^2,\ z^2,\ y^2z,\ xyz,\ yz,\ xz,\ z,\ xy^2,\ y^2,\ xy,\ y,\ x,$$

and $z, 1$. For further applications of this procedure see §20.8.

20.6 De Rham cohomology of Fermat varieties

`TranCoho`: This procedure returns $I_{\mathrm{tra}}^{\frac{n}{2},\frac{n}{2}}$ in §16.3.

```
  ring r1=0,x,dp;
  intvec V=5,5,5;
  list li1=TranCoho(V); li1[1..size(li1)];
//->1,1,3 2,2,0 1,3,1 2,0,2 3,1,1 0,2,2 1,1,1 2,2,2
  ring r2=0,(x,y,z),dp;
  list li2=TranCoho(V,0); li2[1..size(li2)];
//->xyz3 x2y2 xy3z x2z2 x3yz y2z2 xyz x2y2z2
```

The B-factor of all these monomials are conjectured to be transcendental numbers. For instance, the B-factor of the first monomial is $B(\frac{2}{5},\frac{2}{5},\frac{4}{5})$. By Exercise 16.5 we know that the dimension of the primitive Hodge cycles of the Fermat variety is equal to the cardinality of $I_{\mathrm{alg}}^{\frac{n}{2},\frac{n}{2}}$. Therefore, we can use this procedure and we can compute $\mathrm{dimHodge}_n(X,\mathbb{Q})_0$.

```
   ring r=0,x,dp; intvec V;
   for (int d=2; d<=20; d=d+1)
      {
      V=d,d,d;
      HodgeNumber(V)[1][2]-size(TranCoho(V));
      }
//->1,6,19,36,85,90,175,216,361,270,643,
//->396,805,834,871,720,1657,918,1987
```

These numbers plus one are the Picard numbers in Table 15.12. In a similar way we can compute the dimension of primitive Hodge cycles of the Fermat fourfold.

```
   ring r=0,x,dp; intvec V;
   for (int d=2; d<=8; d=d+1)
      {
      V=d,d,d,d,d;
      HodgeNumber(V)[1][3]-size(TranCoho(V));
      }
//->1, 20, 141, 400, 1751, 1860, 5881
```

20.7 Hodge cycles supported in linear cycles

`LinearCoho`: This procedure returns $I_{\mathbb{P}^{\frac{n}{2}}}^{\frac{n}{2},\frac{n}{2}}$ defined in (16.22).

```
  ring r1=0,x,dp;  intvec V=2,4,5;
  list li1=LinearCoho(V); li1[1..size(li1)];
//->0,1,3 0,1,2 0,1,1 0,1,0
  ring r2=0,(x,y,z),dp;
  list li2=LinearCoho(V,0); list l1=li2[1];
  l1[1..size(l1)];
//->yz3 yz2 yz y
```

If you want to have the list of differential forms in $I^{\frac{n}{2},\frac{n}{2}} \backslash I^{\frac{n}{2},\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}}$ use the procedure `RemoveList`. We compute the dimension of $\mathrm{Hodge}_n(X,\mathbb{Q})_{\mathbb{P}^{\frac{n}{2}}}$ for Fermat surfaces:

```
ring r=0,x,dp;
for (int d=2; d<=20; d=d+1){size(LinearCoho(intvec(d,d,d)));}
//->1,6,19,36,61,90,127,168,217,270,331,396,469
//->546,631,720,817,918,1027
```

This together with the computations in §20.6 proves Theorem 17.1 for small degrees. These numbers are collected in Table 15.12. For Fermat fourfolds of degree $d = 2,3,\ldots,10$ the dimension of $\mathrm{Hodge}_n(X,\mathbb{Q})_{\mathbb{P}^{\frac{n}{2}}}$ is given by

```
//-> 1,20,141,400,1001,1860,3301,5120,7761
```

In particular, for sextic Fermat fourfold we prove Proposition 16.5.

`PeriodsLinearCycle`: This procedure gives us the periods of linear cycles described in Theorem 18.7. Below we have verified the equality (18.14) for a linear cycle in the Fermat quintic fourfold.

```
  int d=5; intvec aa=0,0,0,0,0,0;  intvec pp=0,1,2,3,4,5;
  intvec mlist=d,d,d,d,d; int n=size(mlist)-1;
  ring r=(0,z), (x(1..n+1)),dp;
  poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
  cp=subst(cp, x(1),z); minpoly =number(cp);
  matrix Per=PeriodsLinearCycle(mlist, aa, pp,z);
  matrix A=Matrixpij(mlist, Per);
  rank(A), binomial(n div 2+d,d)-(n div 2+1)^2;
//->  12 12
```

The procedure `CodComIntZar` is designed to investigate the matrix $[\mathsf{p}_{i+j}]$ for a sum of linear cycles in the Fermat variety which is a complete intersection. It has been used to prove the main theorem in [MV19]. In a similar way `SumTwoLinearCycle` is designed to investigate the matrix $[\mathsf{p}_{i+j}]$ for a sum of two linear cycles.

20.8 General Hodge cycles

In this section we explain the computer code which is used to verify Conjecture 16.2 in particular cases such as Proposition 16.6 and Proposition 16.7. Upon the availability of a better computer and a better computer code, conjecture 16.2 can be proved for more cases. Below is the computer code for proving that the Fermat surface given by $x^6+y^6+z^6=1$ in some affine chart, has at least one general Hodge cycle. For

other case, replace `mlist=6,6,6;` with any sequence of $m_1, m_2, \ldots, m_{n+1}$ with $2\sum_{i=0}^{n+1} \frac{1}{m_i} \leq n$. Here are some remarks, apart from those in the code, which might help its understanding. 1. We compute Hodge cycles in $\mathrm{Hodge}_n(X,\mathbb{Z})_{\mathbb{P}^{\frac{n}{2}}}$ defined in (16.23). 1. For this class of Hodge cycles we know the values (16.24) of B_β. 3. Because of this we need the $2d$-th primitive root of unity. 4. The formula (16.25) is used. 5. We choose the Hodge cycle corresponding to the first row of `H` defined in (16.3).

```
LIB "foliation.lib";
ring r=0,x,dp; //Any ring
intvec mlist=6,6,6; int n=size(mlist)-1; int d=lcm(mlist);
list wlist;    //weight of the variables
for (int i=1; i<=size(mlist); i=i+1)
  { wlist=insert(wlist, (d div mlist[i]), size(wlist));}
ring r=(0,z), (x(1..n+1)),wp(wlist[1..n+1]);
poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z);
minpoly =number(cp); //z is the 2d-th primitive root of unity
list ll=MixedHodgeFermat(mlist);  list J=ll[1][1];
int hFn=size(ll[1][1]); int nh=n div 2;
for (i=2; i<=nh; i=i+1){hFn=hFn+size(ll[1][i]); J=J+ll[1][i];}
list Hn2n2=ll[1][nh+1]; list Pn2=LinearCoho(mlist,0);
J=J+RemoveList(ll[1][nh+1], Pn2[1]);
list Jli; for (i=1; i<=size(J); i=i+1)
              {Jli=insert(Jli, leadexp(J[i]),size(Jli));}
matrix BHC=BasisHodgeCycles(mlist, z^2,Jli)[2];
int mhn=ncols(BHC);  matrix P[1][mhn]=BHC[1,1..mhn];
           //We insert the algebraic numbers due to B_beta's.
int s; int N; int j;
for (i=1; i<=size(Hn2n2); i=i+1)
    {  for (j=1; j<=size(Pn2[1]); j=j+1)
            {  if (Hn2n2[i]==Pn2[1][j])
                  {
                     for (s=2; s<=nh+1; s=s+1)
                         { N=int( d*Pn2[2][j][s]);
                           P[1,i]=P[1,i]/(z^N+z^(d-N));
                         }
                  }
          }
     }
matrix Mpij=Matrixpij(mlist,P);
rank(Mpij), nrows(Mpij), ncols(Mpij);

//-> 10 10 68
```

20.9 The Hodge cycle $\check{\delta}$

In this section we explain how we handle the Hodge cycle $\check{\delta}$ defined in §15.16. In particular, we have proved Proposition 15.13. This is done by analyzing the matrix `myhodge` in the code below. We have used the formula of intersection matrix in Definition 15.3 and Proposition 15.12.

```
LIB "foliation.lib";
intvec mlist=4,4,4,4,4; int n=size(mlist)-1; int d=lcm(mlist);
list wlist;  //weights of variables
for (int i=1; i<=size(mlist); i=i+1)
    { wlist=insert(wlist, (d div mlist[i]), size(wlist));}
ring r=(0,z), (x(1..n+1)),wp(wlist[1..n+1]);
poly cp=cyclotomic(d);  cp=subst(cp, x(1),par(1));
minpoly =number(cp);
list komak=MixedHodgeFermat(mlist);
list mhf=komak[1]; list I=komak[3];
    //mhf serves as the elements of the cohomology
    //I serves as the elements of the homology
 list deX; for (i=1; i<=n+1; i=i+1)
              { deX=deX+mhf[i];}
 int cmu=size(deX);int mu=size(I); list I1=I[1..cmu];
 matrix intermat=IntersectionMatrix(I1);
 matrix permat=PeriodMatrix(deX,I1, par(1));
 matrix myh[1][mu];for (i=1; i<=mu; i=i+1)
                    {
                    if  (var(1)*(I[i]/var(1))==I[i]){myh[1,i]=0;}
                    else   {myh[1,i]=I[i];}
```

```
                    }
for (i=1; i<=n+1; i=i+1){myh=subst(myh, var(i),-1);}
matrix myhodge=submat(myh, 1,1..cmu)*inverse(transpose(intermat))*permat;
          for (i=1; i<=cmu; i=i+1){ myhodge[i]*deX[i];}
int hn1; for (i=1; i<=n div 2; i=i+1){ hn1=hn1+size(mhf[i]);}
int hn2=size(mhf[1+(n div 2)]);
matrix final=submat(myhodge, 1, hn1+1..hn1+hn2);
matrix Mpij=Matrixpij(mlist,final); rank(Mpij);
```

20.10 Codimension of the components of the Hodge locus

`CodModInt`: The procedure computes the number (17.29). This is the codimension of the locus of hypersurfaces of dimension n and degree d containing a complete intersection of type $\underline{d}$.

```
  CodComInt(4,6,intvec(3,3,3));
//->141
```

Using this procedure we have prepared Table 17.3. For Table 17.4 we may use the following:

```
  int n=2; int d=10; int d1; int d2; int hd=(d div 2);
  for (d2=1 ; d2<=hd; d2=d2+1)
      {for (d1=1 ; d1<=d2; d1=d1+1)
           {d1,d2, CodComInt(2,d,intvec(d1,d2));}
      }
```

The procedure `Codim` is a slight modification of `CodModInt` and it computes the number $\mathsf{C}_{\underline{a}}$ in §17.9.

20.11 Hodge locus and linear cycles

In this section we explain the computer codes used in Chapter 18.

`SumTwoLinearCycle`: This procedure returns the number $\mathsf{H}_n^d(m)$ defined in (18.15). For instance, for the computation of this number in the case $(n,d,m) = (10,3,3)$ we have used the following:

```
  int n=10; int d=3; int m=3; ring r=(0,z), (x(1..n+1)),dp;
  poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
  cp=subst(cp, x(1),z); minpoly =number(cp);
  SumTwoLinearCycle(n,d,m);
//->32
```

`TaylorSeries`: This procedure is the computer implementation of the Taylor series in Theorem 13.3, Theorem 13.4 and Theorem 18.9. In this procedure, we have considered an arbitrary deformation space.

`HodgeLocusIdeal`: This gives us the ideal of the Hodge locus corresponding to a Hodge cycle which is identified with its periods.

`SmoothReduced`: This has been used in the proof of Theorem 18.2 and Theorem 18.3. For example let us consider the case $(n,d,m) = (6,3,1)$. We would like

to know all pairs $(r,\check{r}) \in \mathbb{Z}^2$ with $1 \le r \le 10$ and $-10 \le \check{r} \le 10$ such that the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ is 4-smooth. The output of the following code is a list of two lists. The first list contains all such pairs.

```
int n=6; int d=3; int m=1; int tru=4; int zb=10;
intvec zarib1=1,-zb; intvec zarib2=zb, zb;
intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
ring r=(0,z), (x(1..n+1)),dp;
poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z); minpoly =number(cp);
list lcycles=SumTwoLinearCycle(n,d,m,1); lcycles;
SmoothReduced(mlist,tru, lcycles, zarib1, zarib2);
```

`DistinctHodgeLocus`: This procedure verifies that there is no inclusion between the Zariski tangent space of two Hodge loci corresponding to two distinct sum of two linear cycles. It is designed to investigate Conjecture 18.10.

`aIndex, bIndex`: These procedures generate the (a,b) index of linear cycles $\mathbb{P}^{\frac{n}{2}}_{a,b}$ defined in (17.6). For instance, in the case $(n,d) = (2,3)$ we have 27 linear cycles and these are indexed by the output of the following:

```
  int n=2; int d=3; list aI=aIndex(intvec(0,0), intvec(d-1,d-1));
  list bI=bIndex(n+2); int i; int j;
  for(i=1;i<=size(bI);i=i+1)
     {for(j=1;j<=n+2;j=j+1){bI[i][j]=bI[i][j]-1;}}
  aI[1..size(aI)]; bI;
//->0,0 0,1 0,2 1,0 1,1 1,2 2,0 2,1 2,2
//->[1]:
//->   0,1,2,3
//->[2]:
//->   0,2,1,3
//->[3]:
//->   0,3,1,2
```

`ListPeriodLinearCycle`: This procedure's output is a list of two lists. The first is the list of (a,b) indices of linear cycles of the Fermat variety of dimension n and degree d. The second is the list of period vectors of such linear cycles. In case of interest it also gives only the first list (without computing the second list).

`mTwoLinearCycle`: This procedure computes the dimension of the intersection of two linear cycles $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$, that is the number m, in $\mathbb{P}^{\frac{n}{2}} \cap \mathbb{P}^{\frac{n}{2}} = \mathbb{P}^m$.

```
int n=4; int d=4;
list P1= intvec(0,0,0,0,0,0), intvec(0,1,2,3,4,5);
list P2= intvec(0,0,0,0,0,1), intvec(0,1,2,3,4,5);
mTwoLinearCycle(n,d,P1,P2);
```

For an arbitrary number of cycles use `mLinearCycle`.

`ndm`: This procedure verifies that the rank of the matrix $[\mathsf{p}_{i+j}]$ for sum of two linear cycles in the Fermat variety of dimension n and degree d depends only on n, d and the dimension m of the intersection of the linear cycles. This has been used for the verification of Conjecture 18.4 for particular cases of n and d.

`GoodMinor, ConstantRank`: The first procedure for a given matrix A returns a square minor of A of size $\operatorname{rank}(A)$ and with non-zero determinant. The second

procedure checks whether the matrix $[\mathsf{p}_{i+j}]$ for sum of two linear cycles with arbitrary non-zero coefficients and with intersection $\mathbb{P}^m$ has constant rank. This has been used in the proof of Theorem 18.8.

`SumThreeLinearCycle`: This procedure computes the codimensions of the Zariski tangent space of the Hodge locus corresponding to deformations of a sum of three linear cycles inside the Fermat variety. It has been used in order to get some evidences in favor of conjecture 18.9.

20.12 Join Hodge cycles

In this section we explain the code which finds a basis of the $\mathbb{Q}$-vector space (16.33). For this the new procedure `ListMon` has been written.

```
  int de=4; intvec  mlist=de,de;
  int n=size(mlist)-1; int d=lcm(mlist);
  list wlist;    //--weight of the variables
  for (int i=1; i<=size(mlist); i=i+1)
      { wlist=insert(wlist, (d div mlist[i]), size(wlist));}
  ring r=(0,z), (x(1..n+1)),wp(wlist[1..n+1]);
  poly cp=cyclotomic(d);  cp=subst(cp, x(1),z);
  minpoly =number(cp); //--z is the d-th root of unity---
  number ub=number(n+1)/2-1/number(d); list J=ListMon(mlist, ub )[1];
  list Jexp;  for (i=1; i<=size(J); i=i+1)
                {Jexp=insert(Jexp, leadexp(J[i]),  size(Jexp));}
  list A=BasisHodgeCycles(mlist,z, Jexp); print(A[1]); print(A[2]);
  list ll=MixedHodgeFermat(mlist); list  befmid=ll[1][(n div 2)+1];
  befmid[1..size(befmid)];
//->1, 0, 1,0,0,0,0,0,0,
//->0, -1,0,1,0,0,0,0,0,
//->1, 0, 0,0,1,0,0,0,0,
//->0, 1, 0,0,0,1,0,0,0,
//->1, 0, 0,0,0,0,1,0,0,
//->0, 1, 0,0,0,0,0,1,0,
//->-1,0, 0,0,0,0,0,0,1

//->0,        (4*z-4), 0,
//->(-4*z),   (4*z),   0,
//->-4,       -4,      0,
//->(4*z),    4,       0,
//->(4*z-4), 0,        0,
//->(4*z+4), 0,        0,
//->(-4*z+4),(-4*z+4),0

//->x(2) x(1) 1
```

The output A is a list of two matrices $A[1]$ and $A[2]$. The rows of the matrix $A[1]$ form a basis of (16.33) over rational numbers and written in the basis of vanishing cycles δ_β. It includes the cycles at inifinity. The rows of the second matrix are the periods of such cycles (without the Beta factor) written in the canonical basis ω_β, $\frac{m-1}{2} < A_\beta < \frac{m+1}{2}$ of $H^{\frac{m+1}{2},\frac{m-1}{2}}$. For instance, for $(m,d) = (1,4)$ these are rep-

resented by $x_2, x_1, 1$. Using this code we can experimentally observe that for $m = 1$ only for $d = 3,4,6$ the matrix $A[2]$ is not identially zero. In a similar way, after a modification of this code we observe that for $m = 0$ only for $d = 3,4$ the matrix $A[2]$ is not identially zero.

20.13 Exercises

20.1. Let us consider the following family of hyperelliptic curves of genus 2:

$$L: \quad y^2 - x(x^2-1)^2 + z = 0$$

The Picard-Fuchs equation of the differential form $\frac{(-5x^2+1)dx}{y}$ is given by

$$-4096\theta^2(\theta-1)^2 + 5z^2(10\theta+11)(10\theta+3)(10\theta+7)(10\theta-1)$$

where $\theta = z\frac{\partial}{\partial z}$.

20.2. The moduli space of cubic hypersurface in $\mathbb{P}^{n+1}$ invariant by the permutation of variables is of dimension one and in a neighborhood of the Fermat variety one can choose the chart $z \in \mathbb{C}$ with the following family:

$$x_0^3 + x_1^3 + \cdots + x_{n+1}^3 + z(\sum_{0 \le i < j < k \le n+1} x_i x_j x_k) = 0$$

For $n = 1, 2, \ldots,$ compute the Picard-Fuchs equation of the differential form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}$ given in the affine chart $x_0 = 1$.

20.3. Compute the Gauss-Manin connection of the following family of cubic surfaces

$$x^3 + y^3 + z^3 - t_1 x - t_2 y - t_3 - t_4 yz - t_5 xz - t_6 yz. \tag{20.2}$$

In the following basis of the Brieskorn module:

$$p dx \wedge dy \wedge dz, \quad p = yz,\ xz,\ z,\ xy,\ y,\ x.$$

20.4. * Using the computer code in 20.9, prove Proposition 15.13 for small values of degree d and dimension n. Can you prove it in general?

20.5. * If we compare the ξ-invariant of the Hodge cycle $\check{\delta}$ in Table 16.1 with Table 17.3, entry $(1,3,3)$, and Table 17.4, entries $(1,2),(1,3),(1,4),(1,5)$, we may conjecture that $\check{\delta}$ in homologous to the primitive part of a complete intersection of type $(1, \frac{d}{2}, \cdots, \frac{d}{2})$. Gather further evidences for this conjecture, and in the best case prove it.

20.6. Use the procedure `PeriodMatrix` and try to formulate a conjectural closed formula for the determinant (21.8) in Problem 21.8, Chapter 21.

Chapter 21
Some mathematical olympiad problems

If there is a problem you can't solve, then there is an easier problem you can solve: find it, (G. polya in his book 'How to solve it').

21.1 Introduction

One of the main goals of the present text has been to make Hodge theory and Hodge conjecture accessible for a broader public of mathematicians. In our way we have tried to state our problems and theorems as elementary as possible, using only the indispensable notations and terminologies. Some of these problems can be understood by high school mathematics or at most by knowing basics of (linear) algebra, like groups, rings, rank of matrices etc. In the present chapter, we collect some of these problems and refer to the main body of the book for their origin. This chapter is specially for students who like challenging problems, and in particular those which stem from some advanced mathematics. Hopefully, this might motivate them to study advanced topics in mathematics. Having the quotation above in mind, for any problem in this chapter the corresponding easier problems are special cases of the original one.

21.2 Some problems on the configuration of lines

Problem 21.1 Let us consider a set of d lines in the plane in general position, that is, no two of them are parallel and no three are concurrent. Let also A be the free abelian group of formal linear combinations with integer coefficients of $\frac{d(d-1)}{2}$ points of line intersection. For each line l we consider the alternating sum of the intersection points of l with the other lines (ordered as one meets them going along l, and so it is defined up to sign). The complement of the lines in the plane has $\frac{(d-1)(d-2)}{2}$ bounded polygons and to each of them we consider the sum of the vertices. Show

that the subgroup of A generated by these elements attached to lines and bounded polygons has the full rank $\frac{d(d-1)}{2}$ and the quotient is annihilated by d.

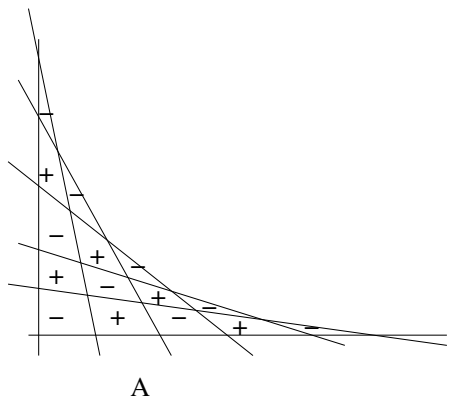

Fig. 21.1 An arrangement of lines

We consider a larger abelian group G generated by the elements of A as above and B, where B is a free abelian group of formal linear combinations with integer coefficients of $\frac{(d-1)(d-2)}{2}$ bounded polygons of the complement of lines. Therefore, in G we have two types of elements attached to a bounded polygon: the sum of vertices as above and an element in B. Let us divide the set of bounded polygons as above in two positive and negative sets so that a positive (resp. negative) polygon has only common edge with negative (resp. positive) polygons, see Figure 21.1. In G we define an antisymmetric bilinear map $\langle\cdot,\cdot\rangle$ as follows: $\langle\delta,\delta'\rangle = 1$ if $\delta \in B$ is a bounded polygon and $\delta' \in A$ is its vertex or if δ is a positive bounded polygon and δ' is a negative adjacent bounded polygon. We have $\langle\delta,\delta'\rangle = -1$ if $\langle\delta',\delta\rangle = 1$, and in all other cases $\langle\delta,\delta'\rangle = 0$.

Problem 21.2 Prove that the group

$$C := \left\{\delta \in G \middle| \langle\delta,\delta'\rangle = 0,\ \ \forall\delta' \in G\right\} \tag{21.1}$$

is generated by the elements of A associated to the lines.

If we multiply the linear equations of the lines then we get a tame polynomial f of degree d in the sense of Chapter 10. The $\mathbb{Z}$-module G is interpreted as the first homology group of a regular fiber of f and $\langle\cdot,\cdot\rangle$ is the intersection form in homology. The Picard-Lefschetz theory of f gives rise to these problems. They originally appear in the article [Mov04] in which the author has also given a non-elementary solution of the problems using Picard-Lefschetz theory, and an elementary solution for the particular arrangement of lines in Figure 21.1. An exposition together with the origin of the problems, is also presented in the preprint [Mov01]. An elementary

solution together with the structure of the torsion quotient group in Problem 21.1 is given by O. A. Camarena in his homepage, see [AC16].

21.3 A very big matrix: how to compute its rank?

Problem 21.3 For natural numbers N, n and d let us define the set

$$I_N := \left\{ (i_0, i_1, \ldots, i_{n+1}) \in \mathbb{Z}^{n+2} \middle| 0 \le i_e \le d-2, \;\; i_0 + i_1 + \cdots + i_{n+1} = N \right\}. \quad (21.2)$$

Assume that n is even and $d \ge 2 + \frac{4}{n}$. We will only need the three sets

$$I_d,\; I_{(\frac{n}{2}+1)d-n-2},\; I_{\frac{n}{2}d-n-2}.$$

Consider a collection of numbers

$$\mathsf{p} := (\mathsf{p}_i, \;\; i \in I_{(\frac{n}{2}+1)d-n-2}).$$

For any other i which is not in the set $I_{(\frac{n}{2}+1)d-n-2}$, p_i by definition is zero. Let $[\mathsf{p}_{i+j}]$ be the matrix whose rows and columns are indexed by $i \in I_{\frac{n}{2}d-n-2}$ and $j \in I_d$, respectively, and in its (i,j) entry we have p_{i+j}. Prove that if the numbers p_i are not simultaneously zero then

$$\binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2 \le \operatorname{rank}([\mathsf{p}_{i+j}]). \quad (21.3)$$

A solution of Problem 21.3 uses the following problem:

Problem 21.4 For any $k \in I_{(\frac{n}{2}+1)d-(n+2)}$ prove that

$$\#\left\{ (i,j) \in I_{\frac{n}{2}d-(n+2)} \times I_d \mid k = i+j \right\} \quad \ge \quad \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2 \quad (21.4)$$

We can even give numbers p_i's such that the lower bound in (21.3) is attained.

Problem 21.5 For $i \in I_{(\frac{n}{2}+1)d-n-2}$, odd numbers $a_0, a_2, \ldots, a_n$ and a permutation σ of $\{0,1,2,\ldots,n+1\}$, let

$$\mathsf{p}_i = \begin{cases} \zeta_{2d}^{a_0 i_{\sigma_0} + a_2 i_{\sigma_2} + \ldots + a_n i_{\sigma_n}} & \text{if} \quad i_{\sigma_{2l-2}} + i_{\sigma_{2l-1}} = d-2, \quad \forall l = 1, \ldots, \frac{n}{2}+1, \\ 0 & \text{otherwise.} \end{cases} \quad (21.5)$$

Then

$$\operatorname{rank}[\mathsf{p}_{i+j}] = \binom{\frac{n}{2}+d}{d} - \left(\frac{n}{2}+1\right)^2. \quad (21.6)$$

Prove or disprove: Let $\mathsf{p} := \left(\mathsf{p}_i,\ \ i \in I_{(\frac{n}{2}+1)d-n-2}\right)$ be an element in the $\mathbb{Z}$-module generated by those in (21.5) for all choices of a's and σ's. If we have (21.6) then p, up to multiplication by a constant, is necessarily of the form (21.5). This is not true for an arbitrary p, for instance take all p_i's equal to zero except one of them.

The origin of the matrix $[\mathsf{p}_{i+j}]$ and Problem 21.3, 21.5 comes from the notion of infinitesimal variation of Hodge structures (IVHS) developed by P. Griffiths and his school, see [CGGH83]. It is computed for IVHS of the Fermat variety in [Mov17b]. An elementary derivation of this matrix using only the machinery of integrals is done using Theorem 13.9 and Theorem 16.2. After computing the periods of linear cycles, see [MV19], one gets Problem 21.5.

Problem 21.6 For $n = 2,\ d \geq 3$ show that there are no complex numbers $\mathsf{p}_i, i \in I_{(\frac{n}{2}+1)d-n-2}$ such that

$$d-3 < \operatorname{rank}[\mathsf{p}_{i+j}] < 2d-7.$$

Prove or disprove: if $\operatorname{rank}[\mathsf{p}_{i+j}] = 2d-7$ then the vector p is a sum of two vectors as in (21.5).

This problem is inspired by Voisin's result in [Voi89]. Here is a reformulation of Problem 21.3 for the case $d = 3$.

Problem 21.7 Let $n \geq 4$ be an even number and let $I_k,\ k = 3, \frac{n}{2}-2, \frac{n}{2}+1$ be the set of subsets of $\{0,1,2,\ldots,n+1\}$ of cardinality k. Consider a collection of numbers

$$\mathsf{p} := (\mathsf{p}_i,\ \ i \in I_{\frac{n}{2}+1}).$$

For any other i which is not in the set $I_{\frac{n}{2}+1}$, p_i by definition is zero. Let $[\mathsf{p}_{i\cup j}]$ be the matrix whose rows and columns are indexed by $i \in I_{\frac{n}{2}-2}$ and $j \in I_3$, respectively, and in its (i,j) entry we have $\mathsf{p}_{i\cup j}$. Therefore, if $i \cap j$ is not empty then by definition $\mathsf{p}_{i\cup j} = 0$. Prove that if the numbers p_i are not simultaneously zero then

$$\binom{\frac{n}{2}+1}{3} \leq \operatorname{rank}([\mathsf{p}_{i\cup j}]) \leq \binom{n+2}{\min\{3, \frac{n}{2}-2\}}. \tag{21.7}$$

21.4 Determinant of a matrix

Problem 21.8 Let $m_1, m_2, \ldots, m_{n+1} \geq 2$ be positive integers and let I be the set of $(n+1)$-tuples $\beta = (\beta_1, \beta_2, \cdots, \beta_{n+1})$ with $\beta_i \in \mathbb{Z}$ and $0 \leq \beta_i \leq m_i - 2$. Let also $\zeta_{m_i} = e^{\frac{2\pi\sqrt{-1}}{m_i}}$ be the m_i-th primitive root of unity. Show that

$$\det\left[\prod_{i=1}^{n+1}\left(\zeta_{m_i}^{(\beta_i+1)(\beta_i'+1)} - \zeta_{m_i}^{\beta_i(\beta_i'+1)}\right)\right]_{\beta,\beta'\in I} \neq 0. \tag{21.8}$$

The cardinality of I is $\mu := (m_1-1)(m_2-2)\cdots(m_{n+1}-1)$ and we have a $\mu\times\mu$ matrix in (21.8) whose (β,β')-entry for $\beta,\beta\in I$ is given by the product in (21.8). The matrix in (21.8) is the period matrix of the affine Fermat variety, see Proposition 15.1, and Problem 21.8 follows from the duality between singular homology and de Rham cohomology. One may use the procedure `PeriodMatrix` discussed in Chapter 20 in order to conjecture the value of the determinant (21.8) in a closed formula. Using a Galois action one can easily see that the square of this number is in the integer ring of $\mathbb{Q}(\zeta_d)$, where d is the lowest common multiple of m_i's. For $n=1$ and small values of $d := m_1 = m_2$ the author with the help of S. Reiter was able to find that the absolute value of this number must be $d^{(d-1)^2+(d-1)}$. For $n=2$ and $d := m_1 = m_2 = m_3 = 3,4,5,6$ this value is respectively $3^{18}, 4^{44}\zeta_4, 5^{120}, 6^{225}$.

Problem 21.9 We define two matrices A and B whose rows and columns are indexed by $\beta,\beta'\in I$, respectively:

$$A := \left[\prod_{i=1}^{n+1}\left(\zeta_{m_i}^{(\beta_i+1)(\beta_i'+1)} - \zeta_{m_i}^{\beta_i(\beta_i'+1)}\right)\right]_{\beta,\beta'\in I,\ \sum_{i=1}^{n+1}\frac{\beta_i'+1}{m_i}\notin\mathbb{Z}} \tag{21.9}$$

The matrix B is quadratic and is defined by the rules:

1. $B_{\beta,\beta'} = (-1)^n B_{\beta',\beta},\ \ \beta,\beta'\in I.$
2. $B_{\beta,\beta} = (-1)^{\frac{n(n-1)}{2}}(1+(-1)^n),\ \beta\in I.$
3. We have
$$B_{\beta,\beta'} = (-1)^{\frac{n(n+1)}{2}}(-1)^{\sum_{k=1}^{n+1}\beta_k'-\beta_k}$$
for those $\beta,\beta'\in I$ such that for all $k=1,2,\ldots,n+1$ we have $\beta_k\le\beta_k'\le\beta_k+1$ and $\beta\neq\beta'$.
4. In the remaining cases, except those arising from the previous ones by a permutation, we have $B_{\beta,\beta'}=0$.

Show that a $1\times\mu$ matrix C, where $\mu := \#I$, with integer entries satisfies $C\cdot A=0$ if and only if $C\cdot B=0$.

The matrix A is a submatrix of the period matrix of the affine Fermat variety, see Proposition (15.1). The matrix B is the intersection matrix of vanishing cycles, see §7.10. Problem 21.9 is the same as Proposition 15.5 and it follows from Proposition (15.1) and Proposition 5.10.

21.5 Some modular arithmetic

Problem 21.10 Let $m_1,m_2,\cdots,m_{n+1}\ge 2$ be positive integers, n an even number and

$$I := \{1,2,\cdots,m_1-1\}\times\{1,2,\cdots,m_2-1\}\times\cdots\times\{1,2,\cdots,m_{n+1}-1\}.$$

Let also d be the lowest common multiple of $m_1, m_2, \cdots, m_{n+1}$ and $\left(\frac{\mathbb{Z}}{d\mathbb{Z}}\right)^\times$ be the group of invertible elements of $\frac{\mathbb{Z}}{d\mathbb{Z}}$. For $a := (a_1, a_2, \cdots, a_{n+1}) \in I$ and $p \in \left(\frac{\mathbb{Z}}{d\mathbb{Z}}\right)^\times$ we define

$$A_{pa} := \sum_{i=1}^{n+1} \left\langle \frac{pa_i}{m_i} \right\rangle.$$

Here, for a rational number r by $\langle r \rangle$ we denote the unique rational number with $r - \langle r \rangle \in \mathbb{Z}$ and $0 \leq \langle r \rangle < 1$. Prove that

1. If $a \in I$ has this property that for all $p \in \left(\frac{\mathbb{Z}}{d\mathbb{Z}}\right)^\times$, the integer part $[A_{pa}]$ of A_{pa} is constant, then this constant is $\frac{n}{2}$.
2. Let $a \in I$ such that $\frac{n}{2} < A_a < \frac{n}{2} + 1$ and a does not satisfy the condition of item 1. Then there is a $b \in I$ with $A_b < \frac{n}{2}$ and $p \in \left(\frac{\mathbb{Z}}{d\mathbb{Z}}\right)^\times$ such that $a_i = [p \cdot b_i]$ for all $i = 1, 2, \ldots, n+1$, where for an integer r and $i = 1, 2, \ldots, n+1$ by $[r]$ we denote the unique positive integer with $0 \leq [r] < m_i$.
3. Let $\Gamma : \mathbb{Q} \to \mathbb{C}$ be any function such that for all $z \in \mathbb{Q}$

$$\Gamma(z+1) = z \cdot \Gamma(z), \tag{21.10}$$

$$\frac{1}{2\pi i}\Gamma(z)\Gamma(1-z) = \frac{1}{e^{\pi i z} - e^{-\pi i z}}, \tag{21.11}$$

$$\frac{1}{(2\pi i)^{\frac{m-1}{2}}} \frac{1}{\Gamma(m \cdot z)} \prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}) = m^{\frac{1}{2} - m \cdot z} i^{-\frac{m-1}{2}}. \tag{21.12}$$

Let also a be as in Problem 21.10, item 1. The following number is algebraic

$$\frac{1}{(2\pi i)^{\frac{n}{2}}} \frac{\Gamma(\frac{a_1}{m_1})\Gamma(\frac{a_2}{m_2}) \cdots \Gamma(\frac{a_{n+1}}{m_{n+1}})}{\Gamma(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_{n+1}}{m_{n+1}})} \tag{21.13}$$

and the algebraicity of the number (21.13) follows from (21.10),(21.11) and (21.12). For instance, using Mathematica and in the case of classical Γ-function one can check that:

$$\frac{\Gamma(\frac{1}{12})\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})}{\Gamma(\frac{4}{3})} = 3 \cdot 2^{\frac{1}{4}} \cdot 3^{\frac{3}{8}} \cdot \sqrt{1+\sqrt{3}} \cdot \pi, \tag{21.14}$$

$$\frac{\Gamma(\frac{1}{12})\Gamma(\frac{1}{3})\Gamma(\frac{3}{4})}{\Gamma(\frac{7}{6})} = 2 \cdot 2^{\frac{7}{12}} 3^{\frac{7}{8}} \cdot \sqrt{1+\sqrt{3}} \cdot \pi.$$

Problem 21.10 is the final step in the proof of Theorem (16.1). For a reformulation and a not-so-elementary proof of items 1 and 3 see the appendix of N. Koblitz and A. Ogus in P. Deligne's article [Del79]. For a computer implementation of the algorithm which returns the list of a's with the property in item 2 see §20.6. For instance for $n = 2$ and $m_1 = m_2 = m_3 = 5$ the set of such $a = (\beta_1 + 1, \beta_2 + 1, \beta_3 + 1)$ is given by β's:

$$(1,1,3), (2,2,0), (1,3,1), (2,0,2), (3,1,1), (0,2,2), (1,1,1), (2,2,2).$$

Problem 21.11 Let $z = \{z_0, z_1, z_2, \cdots, z_{n+1}\}$ be a set of $n+2$ rational numbers $z_i \in \mathbb{Q}$, possibly with repetition. Let also

$$A_z := \sum_{i=0}^{n+1} \langle z_i \rangle.$$

Here, for a rational number r by $\langle r \rangle$ we denote the unique rational number with $r - \langle r \rangle \in \mathbb{Z}$ and $0 \leq \langle r \rangle < 1$. For such a z, we define d to be the lowest common multiple of the denominators of z_i's. We are interested to classify the set M_n of all z's with the property:

$$A_{pz} = \frac{n}{2} + 1, \ \ \forall p \in (\frac{\mathbb{Z}}{d\mathbb{Z}})^{\times}, \tag{21.15}$$

where $\left(\frac{\mathbb{Z}}{d\mathbb{Z}}\right)^{\times}$ is the set of invertible elements of $\frac{\mathbb{Z}}{d\mathbb{Z}}$.

1. Show that the following elements satisfy (21.15).
   a. $\{z_1, 1-z_1, z_2, 1-z_2, \ldots, z_{\frac{n}{2}}, 1-z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, 1-z_{\frac{n}{2}+1}\}$ for n an even number.
   b. $\{z_1, z_1 + \frac{1}{n+1}, z_1 + \frac{2}{n+1}, \ldots, z_1 + \frac{n}{n+1}, -(n+1)z\}$ for arbitrary n.
2. Let us consider the decomposition $n+2 = (a+2)+(b+2)$ and $z \in M_a,\ w \in M_b$. Then $z * w := z \cup w \in M_n$.
3. For $n = 2$ and $2 \leq d \leq 11$ show that any z with the property (21.15) is obtained by the $*$ operation in (2) from the elements of the form (1a) and (1b). For $d = 12$ and $n = 2$, the followings cannot be obtained in this way:

$$z = \{\frac{1}{12}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}, \ \ \{\frac{1}{12}, \frac{1}{3}, \frac{3}{4}, \frac{5}{6}\}.$$

4. Compute the numbers $\Gamma(z) := (2\pi i)^{-\frac{n}{2}-1} \prod_{i=0}^{n+1} \Gamma(z_i)$ for z in (1a) and (1b).
5. Show that for arbitrary $z \in M_n$, $\Gamma(z)$ is an algebraic number. Can you give an algorithm to compute this algebraic number.
6. Show that a z with the property (21.15) is obtained by $*$ operation from (1a) or (1b) if and only if $d = p^s$ or $2p^s$ for some prime number p.

The main content of Problem 21.11 is taken from [Aok15] and the appendix of [Del79]. In Problem 21.11 n is not necessarily even and affine version of this Problem can be formulated by neglecting z_0. In this way we see its relation with Problem 21.10. The elements in item 1a include the set of z such that $\prod_{i=0}^{n+1}(x - e^{2\pi\sqrt{-1}z_i})$ is a product of cyclotomic polynomials. The algebraic number in (21.14) (without π factor) has been used in [Aok15] in order to find new curves in the Fermat variety of degree 12.

21.6 Last step in the proof of Theorem 17.1

Problem 21.12 Let $n,d \geq 2$ be positive integers and assume that n is even. Let also S_{n+2} be the permutation group in $n+2$ elements $0,1,\dots,n+1$. We need the subset $\hat{\mathsf{S}}_{n+2} \subset \mathsf{S}_{n+2}$ consisting of permutations b such that $b_0 = 0$ and for i an even number b_i is the smallest number in $\{0,1,\dots,n+1\}\backslash\{b_0,b_1,b_2,\dots,b_{i-1}\}$. The cardinality of $\hat{\mathsf{S}}_{n+2}$ is $(n+1)\cdot(n-1)\cdots 3\cdot 1$. We are going to define a matrix A whose rows and columns are indexed by:

$$I := \hat{\mathsf{S}}_{n+2} \times \{0,1,2,\dots,d-1\}^{\{1,3,5,\dots,n+1\}}.$$

An element of the above set is denoted by $i := ((b_0,b_1,\cdots,b_{n+1}),(a_1,a_3,\dots,a_{n+1}))$, and another element j with the same notation but with $\check{}$ on top of its ingredients, $\check{b}_0$ etc. In the following note that the order of elements inside $(\cdots)$ does matter, whereas the order of elements inside $\{\cdots\}$ does not matter. A bicycle attached to the permutations b and $\check{b}$ is a sequence $(c_1c_2\dots c_r)$ with $c_i \in \{0,1,2,\dots,n+1\}$ and such that if we define $c_{r+1} = c_1$ then for $1 \leq i \leq r$ odd (resp. even) there is an even number k with $0 \leq k \leq n+1$ such that $\{c_i,c_{i+1}\} = \{b_k,b_{k+1}\}$ (resp. $\{c_i,c_{i+1}\} = \{\check{b}_k,\check{b}_{k+1}\}$)) and there is no repetition among c_i's. By definition there is a sequence of even numbers $k_1,k_2,\cdots$ such that

$$\{c_1,c_2\} = \{b_{k_1},b_{k_1+1}\},\ \ \{c_2,c_3\} = \{\check{b}_{k_2},\check{b}_{k_2+1}\},\ \ \{c_3,c_4\} = \{b_{k_3},b_{k_3+1}\},\dots.$$

Bicycles are defined up to twice shifting c_i's, that is, $(c_1c_2c_3\cdots c_r) = (c_3\cdots c_rc_1c_2)$ etc., and the involution $(c_1c_2c_3\cdots c_{r-1}c_r) = (c_rc_{r-1}\cdots c_3c_2c_1)$. For example, for the permutations

$$b = (0,1,2,3,4,5),\ \check{b} = (1,0,5,3,4,2)$$

we have in total two bicycles $(01),(2354)$. Note that bicycles give us in a natural way a partition of $\{0,1,\dots,n+1\}$. For such a bicycle we define its conductor to be the sum over k, as before, of the following elements: if $c_i = b_k$ and $c_{i+1} = b_{k+1}$ (resp. $c_i = \check{b}_k$ and $c_{i+1} = \check{b}_{k+1}$) then the element $1+2a_{k+1}$ (resp. $1+2\check{a}_{k+1}$), and if if $c_i = b_{k+1}$ and $c_{i+1} = b_k$ (resp. $c_i = \check{b}_{k+1}$ and $c_{i+1} = \check{b}_k$) then $-1-2a_{k+1}$ (resp. $-1-2\check{a}_{k+1}$). Because of the involution, the conductor is defined up to sign. In our example, the conductor of (01) and (2354) are respectively given by

$$1+2a_1+1+2\check{a}_1,\ \ 1+2a_3-1-2\check{a}_3-1-2a_5+1+2\check{a}_5.$$

A bicycle is called new if $2d$ divides its conductor, and is called old otherwise. Let a_{ij} be the number of new bicycles attached to (i,j). For example, for $i=j$ we have $\frac{n}{2}+1$ new bicycles $(b_0b_1),(b_2b_3),\cdots,(b_nb_{n+1})$ and no old bicycle. The (i,j)-entry of A is

$$A_{ij} := (-d+1)^{a_{ij}}$$

Let $s(n,d)$ be the maximum number with the property that there is $J \subset I$ such that the cardinality of J is $s(n,d)$ and the corresponding $J \times J$ sub matrix A_J of A has non-zero determinant. Show that

1.

$$s(n,d) \leq \#\left\{ a \in \mathbb{Z}^{n+2} \middle| 0 \leq a_e \leq d-2,\ \sum_{e=0}^{n+1} a_e = (\frac{n}{2}+1)(d-2) \right\} \tag{21.16}$$

2. For $d = 3,4,6$, (21.16) is an equality and these are the only cases with this property.

Problem 21.13 Let $2 \leq d \in \mathbb{N}$ and let I be the set of $(n+1)$-tuples $\beta = (\beta_1, \beta_2, \cdots, \beta_{n+1})$ with $\beta_i \in \mathbb{Z}$ and $0 \leq \beta_i \leq d-2$. Let also $\zeta_d = e^{\frac{2\pi\sqrt{-1}}{d}}$ be the d-th primitive root of unity. We consider the matrix

$$A := \left[\prod_{i=1}^{n+1} \left(\zeta_d^{(\beta_i+1)(\beta_i'+1)} - \zeta_d^{\beta_i(\beta_i'+1)} \right) \right] \tag{21.17}$$

whose rows and columns are indexed by

$$\beta, \beta' \in I,\ \sum_{i=1}^{n+1} \beta_i' \leq d\frac{n}{2} - (n+2),$$

respectively. A primitive Hodge cycle C is a $1 \times (d-1)^{n+1}$ matrix with rational entries such that $C \cdot A = 0$. Show that if d is a prime number or $d = 4$ or d is relatively prime with $(n+1)!$ then the number $s(n,d)$ in Problem 21.12 is the dimension of the $\mathbb{Q}$-vector space of Hodge cycles.

The number $s(n,d)$ is the dimension of the space of primitive Hodge cycles for the Fermat variety, that is, $s = \dim_{\mathbb{Q}} \mathrm{Hodge}_n(X_n^d, \mathbb{Q})_0$. This is discussed in the end of §17.7. The space of Hodge cycles $\mathrm{Hodge}_n(X_n^d, \mathbb{Q})_0$ can be embedded into $H_0^{\frac{n}{2},\frac{n}{2}}$ of the Fermat variety and the right hand side of (21.16) is just the primitive Hodge number $\dim_{\mathbb{C}} H_0^{\frac{n}{2},\frac{n}{2}}$. Problem 21.13 is the last missing step in the proof of Theorem 17.1. A very challenging problem is to compute the minimum of $|\det(A_J)|$ in Problem 21.12. This number encodes the failure of integral Hodge conjecture for the Fermat variety of degree d and dimension n. This is discussed in the end of §8.9.

References

AC16. Omar Antolín Camarena. Linear algebra around a configuration of lines in the plane. http://www.math.ubc.ca/~oantolin/notes/genposlines.html, 2016.

ACG11. Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II. With a contribution by Joseph Daniel Harris.* Berlin: Springer, 2011.

AGZV88. V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Monodromy and asymptotics of integrals Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1988.

AH62. M. F. Atiyah and F. Hirzebruch. Analytic cycles on complex manifolds. *Topology*, 1:25–45, 1962.

AMSY16. M. Alim, H. Movasati, E. Scheidegger, and S.-T. Yau. Gauss-Manin connection in disguise: Calabi-Yau threefolds. *Comm. Math. Phys.*, 334(3):889–914, 2016.

AMV19. E. Aljovin, H. Movasati, and R. Villaflor Loyola. Integral Hodge conjecture for Fermat varieties. *Jou. of Symb. Comp.*, 95:177–184, 2019.

And96. Yves André. Pour une théorie inconditionnelle des motifs. *Publ. Math., Inst. Hautes Étud. Sci.*, 83:5–49, 1996.

And04. Yves André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2004.

Aok83. Noboru Aoki. On some arithmetic problems related to the Hodge cycles on the Fermat varieties. *Math. Ann.*, 266:23–54, (Erratum: Math. Ann. 267 (1984), no. 4, 572), 1983.

Aok87. Noboru Aoki. Some new algebraic cycles on Fermat varieties. *J. Math. Soc. Japan*, 39(3):385–396, 1987.

Aok13. Noboru Aoki. On sign ambiguities of multiplicative relations between Gauss sums. *Comment. Math. Univ. St. Pauli*, 62(2):143–171, 2013.

Aok15. Noboru Aoki. Algebraic cycles on Fermat varieties. *Talk available at* http://gauss.ms.u-tokyo.ac.jp/conf/pdf-file/aoki.pdf , 2015.

Ara12. Donu Arapura. *Algebraic geometry over the complex numbers.* Berlin: Springer, 2012.

AS83. Noboru Aoki and Tetsuji Shioda. Generators of the Néron-Severi group of a Fermat surface. Arithmetic and geometry, Pap. dedic. I. R. Shafarevich, Vol. I: Arithmetic, Prog. Math. 35, 1-12 (1983)., 1983.

AS10. Noboru Aoki and Tetsuji Shioda. Correction and supplement to the paper "Generators of the Néron-Severi group of a Fermat surface" [mr0717587]. *Comment. Math. Univ. St. Pauli*, 59(1):65–75, 2010.

BC16. Jean-Benoît Bost and François Charles. Some remarks concerning the Grothendieck period conjecture. *J. Reine Angew. Math.*, 714:175–208, 2016.

BCP97. Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

Bea14. Arnaud Beauville. Quelques surfaces dont le nombre de Picard est maximal. *J. Éc. Polytech., Math.*, 1:101–116, 2014.

BEK14. Spencer Bloch, Hélène Esnault, and Moritz Kerz. Deformation of algebraic cycle classes in characteristic zero. *Algebr. Geom.*, 1(3):290–310, 2014.

BH61. A. Borel and A. Haefliger. La classe d'homologie fondamentale d'un espace analytique. *Bull. Soc. Math. France*, 89:461–513, 1961.

BHPV04. Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces. 2nd enlarged ed.* Berlin: Springer, 2nd enlarged ed. edition, 2004.

Blo72. Spencer Bloch. Semi-regularity and deRham cohomology. *Invent. Math.*, 17:51–66, 1972.

BM77. E. Bombieri and D. Mumford. Enriques' classification of surfaces in char. p. II. Complex Anal. Algebr. Geom., Collect. Pap. dedic. K. Kodaira, 23-42 (1977)., 1977.

BM00. Gilles Bailly-Maitre. Connexion de Gauss-Manin polynômes hyper-elliptiques. *Bull. Sci. Math.*, 124(8):621–657, 2000.

Bon03. Philippe Bonnet. Relative cohomology of polynomial mappings. *Manuscripta Math.*, 110(4):413–432, 2003.

Bre93. Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

Bri70. Egbert Brieskorn. Die Monodromie der isolierten Singularitäten von Hyperflächen. *Manuscripta Math.*, 2:103–161, 1970.

Bro88. S. A. Broughton. Milnor numbers and the topology of polynomial hypersurfaces. *Invent. Math.*, 92(2):217–241, 1988.

BT82. Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

BT14. Christopher Brav and Hugh Thomas. Thin monodromy in Sp(4). *Compos. Math.*, 150(3):333–343, 2014.

Bv95. Victor V. Batyrev and Duco van Straten. Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties. *Commun. Math. Phys.*, 168(3):493–533, 1995.

CD11. Adrian Clingher and Charles F. Doran. Note on a geometric isogeny of K3 surfaces. *Int. Math. Res. Not. IMRN*, (16):3657–3687, 2011.

CD12. Adrian Clingher and Charles F. Doran. Lattice polarized K3 surfaces and Siegel modular forms. *Adv. Math.*, 231(1):172–212, 2012.

CDK95. Eduardo H. Cattani, Pierre Deligne, and Aroldo G. Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.

CdlOGP91. Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.

CEZGuT14. Eduardo Cattani, Fouad El Zein, Phillip A. Griffiths, and Lê D ung Tráng, editors. *Hodge theory*, volume 49 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2014.

CG80. James A. Carlson and Phillip A. Griffiths. Infinitesimal variations of Hodge structure and the global Torelli problem. Journees de geometrie algebrique, Angers/France 1979, 51-76 (1980)., 1980.

CGGH83. James Carlson, Mark Green, Phillip Griffiths, and Joe Harris. Infinitesimal variations of Hodge structure. I, II,III. *Compositio Math.*, 50(2-3):109–205, 1983.

Cha14. Ching-Li Chai. Riemann's theta function. *Preprint*, 2014.

Ché91. Denis Chéniot. Topologie du complémentaire d'un ensemble algébrique projectif. *Enseign. Math. (2)*, 37(3-4):293–402, 1991.

Ché96. D. Chéniot. Vanishing cycles in a pencil of hyperplane sections of a non-singular quasi-projective variety. *Proc. London Math. Soc. (3)*, 72(3):515–544, 1996.

CHM88. Ciro Ciliberto, Joe Harris, and Rick Miranda. General components of the Noether-Lefschetz locus and their density in the space of all surfaces. *Math. Ann.*, 282(4):667–680, 1988.

CK99. David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

CMSP03. James Carlson, Stefan Müller-Stach, and Chris Peters. *Period mappings and period domains*, volume 85 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.

CMT00. Jasbir Chahal, Matthijs Meijer, and Jaap Top. Sections on certain $j = 0$ elliptic surfaces. *Comment. Math. Univ. St. Pauli*, 49(1):79–89, 2000.

Col93. Pierre Colmez. Périodes des variétés abéliennes à multiplication complexe. *Ann. Math. (2)*, 138(3):625–683, 1993.

Cox90. David A. Cox. The Noether-Lefschetz locus of regular elliptic surfaces with section and $p_g \geq 2$. *Amer. J. Math.*, 112(2):289–329, 1990.

Dan14. A. Dan. Noether-Lefschetz locus and a special case of the variational Hodge conjecture. *ArXiv e-prints*, April 2014.

Deg15. Alex Degtyarev. Lines generate the Picard groups of certain Fermat surfaces. *J. Number Theory*, 147:454–477, 2015.

Del68. P. Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Inst. Hautes Études Sci. Publ. Math.*, (35):259–278, 1968.

Del71a. Pierre Deligne. Théorie de Hodge. I. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 425–430. Gauthier-Villars, Paris, 1971.

Del71b. Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, -(40):5–57, 1971.

Del73. Pierre Deligne. Cohomologie des intersections complètes. Sém. Géom. algébrique Bois-Marie 1967–1969, SGA 7 II, Lect. Notes Math. 340, Exp. No. XI, 39-61 (1973)., 1973.

Del74. Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, -(44):5–77, 1974.

Del79. P. Deligne. Valeurs de fonctions L et périodes d'intégrales. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 313–346. Amer. Math. Soc., Providence, R.I., 1979. With an appendix by N. Koblitz and A. Ogus.

Del94. P. Deligne. À quoi servent les motifs? In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 143–161. Amer. Math. Soc., Providence, RI, 1994.

Del97. P. Deligne. Local behavior of Hodge structures at infinity. In *Mirror symmetry, II*, volume 1 of *AMS/IP Stud. Adv. Math.*, pages 683–699. Amer. Math. Soc., Providence, RI, 1997.

Del06. P. Deligne. The Hodge conjecture. In *The millennium prize problems*, pages 45–53. Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute, 2006.

Del81. P. Deligne. Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes. *Mém. Soc. Math. France (N.S.)*, (2):23–33, 1980/81. Abelian functions and transcendental numbers (Colloq., École Polytech., Palaiseau, 1979) (French).

Dim87. Alexandru Dimca. *Topics on real and complex singularities*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1987. An introduction.

Dim92. Alexandru Dimca. *Singularities and topology of hypersurfaces*. Universitext. Springer-Verlag, New York, 1992.

DMOS82. Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982. Philosophical Studies Series in Philosophy, 20.

DMWH16. Ch. Doran, H. Movasati, U. Whitcher, and A. Harder. Humbert surfaces and the moduli of lattice polarized k3 surfaces. *Proceedings of Symposia in Pure Mathematics, String-Math conference proceeding 2014*, 93, 2016.

DN01. Alexandru Dimca and András Némethi. On the monodromy of complex polynomials. *Duke Math. J.*, 108(2):199–209, 2001.

Dol82. Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

Duq20. Jorge Duque. Periods of Hodge cycles and special values of the hypergeometric function. *Preprint*, 2020.

EH87. David Eisenbud and Joe Harris. On varieties of minimal degree (a centennial account). In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 3–13. Amer. Math. Soc., Providence, RI, 1987.

Ehr47. Charles Ehresmann. Sur les espaces fibrés différentiables. *C. R. Acad. Sci. Paris*, 224:1611–1612, 1947.

Eis95. David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

Eis05. David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

ES52. Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology*. Princeton University Press, Princeton, New Jersey, 1952.

EZ91. Fouad El Zein. *Introduction à la théorie de Hodge mixte*. Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1991.

Fal88. Gerd Faltings. p-adic Hodge theory. *J. Amer. Math. Soc.*, 1(1):255–299, 1988.

Fre17. Javier Fresán. Periods of Hodge structures and special values of the gamma function. *Invent. Math.*, 208(1):247–282, 2017.

Gav01. L. Gavrilov. The infinitesimal 16th Hilbert problem in the quadratic case. *Invent. Math.*, 143(3):449–497, 2001.

GAZ10. Duco van Straten Gert Almkvist, Christian van Enckevort and Wadim Zudilin. Tables of Calabi-Yau equations. *Preprint, arXiv:math/0507430*, 2010.

GH85. Phillip Griffiths and Joe Harris. On the Noether-Lefschetz theorem and some remarks on codimension-two cycles. *Math. Ann.*, 271(1):31–51, 1985.

GH94. Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

GM07. L. Gavrilov and H. Movasati. The infinitesimal 16th Hilbert problem in dimension zero. *Bull. Sci. Math.*, 131(3):242–257, 2007.

GMV94. M. Green, J. Murre, and C. Voisin. *Algebraic cycles and Hodge theory*, volume 1594 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994. Lectures given at the Second C.I.M.E. Session held in Torino, June 21–29, 1993, Edited by A. Albano and F. Bardelli.

GPS01. G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 2.0. A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2001. `http://www.singular.uni-kl.de`.

Gre88. Mark L. Green. A new proof of the explicit Noether-Lefschetz theorem. *J. Differential Geom.*, 27(1):155–159, 1988.

Gre89. Mark L. Green. Components of maximal dimension in the Noether-Lefschetz locus. *J. Differential Geom.*, 29(2):295–302, 1989.

Gre98. Gert-Martin Greuel. Some aspects of Brieskorn's mathematical work. In *Singularities (Oberwolfach, 1996)*, volume 162 of *Progr. Math.*, pages xv–xxii. Birkhäuser, Basel, 1998.

Gri66. Phillip A. Griffiths. The residue calculus and some transcendental results in algebraic geometry. I, II. *Proc. Nat. Acad. Sci. U.S.A.*, 55:1303–1309; 1392–1395, 1966.

Gri68a. Phillip A. Griffiths. Periods of integrals on algebraic manifolds. I. Construction and properties of the modular varieties. *Amer. J. Math.*, 90:568–626, 1968.

Gri68b. Phillip A. Griffiths. Periods of integrals on algebraic manifolds. II. Local study of the period mapping. *Amer. J. Math.*, 90:805–865, 1968.

Gri69. Phillip A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math. (2) 90 (1969), 460-495; ibid. (2)*, 90:496–541, 1969.

Gri70. Phillip A. Griffiths. Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems. *Bull. Amer. Math. Soc.*, 76:228–296, 1970.

Gri79. Phillip A. Griffiths. A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles. *Amer. J. Math.*, 101(1):94–131, 1979.

Gri03. Phillip A. Griffiths. *Selected works of Phillip A. Griffiths with commentary. Part 2*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2003. Algebraic geometry, Edited by Enrico Arbarello and Joe Harris.

Gro66. Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

Gro69. A. Grothendieck. Standard conjectures on algebraic cycles. In *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, pages 193–199. Oxford Univ. Press, London, 1969.

Gro78. Benedict H. Gross. On the periods of abelian integrals and a formula of Chowla and Selberg. *Invent. Math.*, 45(2):193–211, 1978. With an appendix by David E. Rohrlich.

Gro79. Benedict H. Gross. On an identity of Chowla and Selberg. *J. Number Theory*, 11(3 S. Chowla Anniversary Issue):344–348, 1979.

GSW92. Phillip Griffiths, Donald Spencer, and George Whitehead. Solomon lefschetz 1884-1972. *National Academy of Sciences*, 1992.

HA55. W. V. D. Hodge and M. F. Atiyah. Integrals of the second kind on an algebraic variety. *Ann. of Math. (2)*, 62:56–91, 1955.

Hai87. R. Hain. The geometry of the mixed Hodge structure on the fundamental group. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 247–282. Amer. Math. Soc., Providence, RI, 1987.

Har77. Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

Has00. Brendan Hassett. Special cubic fourfolds. *Compos. Math.*, 120(1):1–23, 2000.

Hei12. Bas Heijne. The maximal rank of elliptic Delsarte surfaces. *Math. Comput.*, 81(278):1111–1130, 2012.

Her02. Claus Hertling. *Frobenius manifolds and moduli spaces for singularities*, volume 151 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002.

Hir76. Morris W. Hirsch. *Differential topology*. Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 33.

Hir95. Friedrich Hirzebruch. *Topological methods in algebraic geometry*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition.

HLTY18. Shinobu Hosono, Bong H. Lian, Hiromichi Takagi, and Shing-Tung Yau. K3 surfaces from configurations of six lines in $\mathbb{P}^2$ and mirror symmetry I. *arXiv e-prints*, page arXiv:1810.00606, Oct 2018.

HLY96. Shinobu Hosono, Bong H. Lian, and Shing-Tung Yau. GKZ-generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces. *Commun. Math. Phys.*, 182:535–578, 1996.

HLZ16. An Huang, Bong H. Lian, and Xinwen Zhu. Period integrals and the Riemann-Hilbert correspondence. *J. Differ. Geom.*, 104(2):325–369, 2016.

Hod41. W.V.D. Hodge. The theory and applications of harmonic integrals. . London: Cambridge University Press, 1941.

Hod57. W.V.D. Hodge. Professor Lefschetz's contributions to algebraic geometry: An appreciation. *Princeton Math. Ser.*, 12:3–23, 1957.

Hod86. William Hodge. Solomon Lefschetz. Algebraic geometry, Proc. Lefschetz Centen. Conf., Mexico City/Mex. 1984, Pt. I, Contemp. Math. 58, 27-46 (1986)., 1986.

Hum93. G. Humbert. Théorie générale des surfaces hyperelliptiques. *Journ. de Math. (4)*, 9:29–170, 1893.

jL79. H. W. jun. Lenstra. Vanishing sums of roots of unity. Proc. bicenten. Congr. Wiskd. Genoot., Part II, Amsterdam 1978, Math. Cent. Tracts 101, 249-268 (1979)., 1979.

KK98. Vik. S. Kulikov and P. F. Kurchanov. Complex algebraic varieties: periods of integrals and Hodge structures. In *Algebraic geometry, III*, volume 36 of *Encyclopaedia Math. Sci.*, pages 1–217, 263–270. Springer, Berlin, 1998.

Kle94. Steven L. Kleiman. The standard conjectures. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 3–20. Amer. Math. Soc., Providence, RI, 1994.

KO68. Nicholas M. Katz and Tadao Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.*, 8:199–213, 1968.

Kob78. Neal Koblitz. Gamma function identities and elliptic differentials on Fermat curves. *Duke Math. J.*, 45(1):87–99, 1978.

Kol92. János Kollár. Trento examples. In *Classification of irregular varieties (Trento, 1990)*, volume 1515 of *Lecture Notes in Math.*, pages 134–139. Springer, Berlin, 1992.

KS53. K. Kodaira and D. C. Spencer. Divisor class groups on algebraic varieties. *Proc. Nat. Acad. Sci. U. S. A.*, 39:872–877, 1953.

Kul98. Valentine S. Kulikov. *Mixed Hodge structures and singularities*, volume 132 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998.

KZ01. Maxim Kontsevich and Don Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

Lai16. Pierre Lairez. Computing periods of rational integrals. *Mathematics of computation*, 85:1719–1752, 2016.

Lam81. Klaus Lamotke. The topology of complex projective varieties after S. Lefschetz. *Topology*, 20(1):15–51, 1981.

Lan73. Alan Landman. On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities. *Trans. Amer. Math. Soc.*, 181:89–126, 1973.

Lan78. Serge Lang. Relations de distributions et exemples classiques. In *Séminaire Delange-Pisot-Poitou, 19e année: 1977/78, Théorie des nombres, Fasc. 2*, pages Exp. No. 40, 6. Secrétariat Math., Paris, 1978.

Lef24a. S. Lefschetz. *L'analysis situs et la géométrie algébrique*. Gauthier-Villars, Paris, 1924.

Lef24b. Solomon Lefschetz. *L'analysis situs et la géométrie algébrique*. Gauthier-Villars, Paris, 1924.

Lef68. Solomon Lefschetz. A page of mathematical autobiography. *Bull. Amer. Math. Soc.*, 74:854–879, 1968.

Lew99. James D. Lewis. *A survey of the Hodge conjecture*, volume 10 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, second edition, 1999. Appendix B by B. Brent Gordon.

LS18. Pierre Lairez and Emre Can Sertöz. A numerical transcendental method in algebraic geometry. *arXiv e-prints*, page arXiv:1811.10634, Nov 2018.

LSY13. Bong H. Lian, Ruifang Song, and Shing-Tung Yau. Periodic integrals and tautological systems. *J. Eur. Math. Soc. (JEMS)*, 15(4):1457–1483, 2013.

Lui97. S. H. Lui. An interview with Vladimir Arnold. *Notices Amer. Math. Soc.*, 44(4):432–438, 1997.

LY96a. Bong H. Lian and Shing-Tung Yau. Arithmetic properties of mirror map and quantum coupling. *Comm. Math. Phys.*, 176(1):163–191, 1996.

LY96b. Bong H. Lian and Shing-Tung Yau. Mirror maps, modular relations and hypergeometric series I, II. *In Proceedings of the Conference on S-duality in Triste ICTP 1995, Nucl. Phys.*, pages 248–262, 1996.

LY13. Bong H. Lian and Shing-Tung Yau. Period integrals of CY and general type complete intersections. *Invent. Math.*, 191(1):35–89, 2013.

Mac05. Catriona Maclean. A second-order invariant of the Noether-Lefschetz locus and two applications. *Asian J. Math.*, 9(3):373–399, 2005.

Man64. Yu.I. Manin. Algebraic curves over fields with differentiation. *Transl., Ser. 2, Am. Math. Soc.*, 37:59–78, 1964.

Man86. Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

Mas91. William S. Massey. *A Basic Course in Algebraic Topology*. Harcourt, Brace & World, Inc., New York, 1991.

Mat73. John N. Mather. Stratifications and mappings. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 195–232. Academic Press, New York, 1973.

Mil. James Milne. *Jacobian varieties*. www.jmilne.org.

Mil62. J. Milnor. On axiomatic homology theory. *Pacific J. Math.*, 12:337–341, 1962.

Mil68. John Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J., 1968.

Moo93. B.J.J. Moonen. The Picard number of certain algebraic surfaces. *J. Pure Appl. Algebra*, 85(3):317–330, 1993.

Mor93. David R. Morrison. Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians. *J. Amer. Math. Soc.*, 6(1):223–247, 1993.

Mov00. H. Movasati. On the topology of foliations with a first integral. *Bol. Soc. Brasil. Mat. (N.S.)*, 31(3):305–336, 2000.

Mov01. H. Movasati. Picard-Lefschetz theorie der anordnungen. `http://w3.impa.br/~hossein/myarticles/plta.pdf`, 2001.

Mov04. H. Movasati. Center conditions: rigidity of logarithmic differential equations. *J. Differential Equations*, 197(1):197–217, 2004.

Mov06. H. Movasati. Relative cohomology with respect to a Lefschetz pencil. *J. Reine Angw. Math.*, 594, 2006.

Mov07a. H. Movasati. Calculation of mixed Hodge structures, Gauss-Manin connections and Picard-Fuchs equations. In *Real and complex singularities*, Trends Math., pages 247–262. Birkhäuser, Basel, 2007.

Mov07b. H. Movasati. Mixed Hodge structure of affine hypersurfaces. *Ann. Inst. Fourier (Grenoble)*, 57(3):775–801, 2007.

Mov08. H. Movasati. Moduli of polarized Hodge structures. *Bull. Braz. Math. Soc. (N.S.)*, 39(1):81–107, 2008.

Mov11. H. Movasati. *Multiple Integrals and Modular Differential Equations*. 28th Brazilian Mathematics Colloquium. Instituto de Matemática Pura e Aplicada, IMPA, 2011.

Mov12. H. Movasati. Quasi-modular forms attached to elliptic curves, I. *Ann. Math. Blaise Pascal*, 19(2):307–377, 2012.

Mov17a. H. Movasati. Gauss-Manin connection in disguise: Calabi-Yau modular forms. *Surveys of Modern Mathematics, Int. Press, Boston.*, 2017.

Mov17b. H. Movasati. Gauss-Manin connection in disguise: Noether-Lefschetz and Hodge loci. *Asian Journal of Mathematics*, 21(3):463–482, 2017.

Mov20. H. Movasati. Special components of noether-lefschetz loci. *To appear in Rend. Circ. Mat. Palermo (2)*, 2020.

MR04. Vincent Maillot and Damian Roessler. On the periods of motives with complex multiplication and a conjecture of Gross-Deligne. *Ann. Math. (2)*, 160(2):727–754, 2004.

MR06. H. Movasati and S. Reiter. Hypergeometric series and Hodge cycles of four dimensional cubic hypersurfaces. *Int. Jou. of Number Theory.*, 2(6), 2006.

Mum66a. D. Mumford. Families of abelian varieties. Proc. Sympos. Pure Math. 9, 347-351 (1966)., 1966.

Mum66b. D. Mumford. On the equations defining Abelian varieties. I-III. *Invent. Math.*, 1:287–354, 1966.

MV19. H. Movasati and R. Villaflor. Periods of linear algebraic cycles. *To appear in Pure and Applied Mathematics Quarterly*, 2019.

MZ99. B.J.J. Moonen and Yu. J. Zarhin. Weil classes and Rosati involutions on complex abelian varieties. *Contem. Math.*, 224:229–235, 1999.

Nag51. Trygve Nagell. Introduction to number theory. Stockholm: Almqvist & Wiksell. New York: John Wiley & Sons, Inc. 309 pp. (1951)., 1951.

Nar68. Raghavan Narasimhan. Compact analytical varieties. *Enseignement Math. (2)*, 14:75–98, 1968.

NS12. András Némethi and Ágnes Szilárd. *Milnor fiber boundary of a non-isolated surface singularity*, volume 2037 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

Oes88. Joseph Oesterlé. Nouvelles approches du "théorème" de Fermat. *Astérisque*, (161-162):Exp. No. 694, 4, 165–186 (1989), 1988. Séminaire Bourbaki, Vol. 1987/88.

OR16. John J O'Connor and Edmund F Robertson. *MacTutor History of Mathematics archive*. http://www-groups.dcs.st-and.ac.uk/∼history/, 2016.

Otw03. Ania Otwinowska. Composantes de petite codimension du lieu de Noether-Lefschetz: un argument asymptotique en faveur de la conjecture de Hodge pour les hypersurfaces. *J. Algebraic Geom.*, 12(2):307–320, 2003.

Pha65. Frédéric Pham. Formules de Picard-Lefschetz généralisées et ramification des intégrales. *Bull. Soc. Math. France*, 93:333–367, 1965.

Pha79. Frédéric Pham. Singularités des systèmes différentiels de Gauss-Manin. Avec des contributions de Lo Kam Chan, Philippe Maisonobe et Jean-Etienne Rombaldi. Progress in Mathematics. 2. Boston-Basel-Stuttgart: Birkhäuser. V, 339 p. SFr. 34.00 (1979)., 1979.

Poh68. Henry Pohlmann. Algebraic cycles on abelian varieties of complex multiplication type. *Ann. of Math. (2)*, 88:161–180, 1968.

Poi87. H. Poincaré. Sur les résidus des intégrales doubles. *Acta Math.*, 9(1):321–380, 1887.

Poi95. H. Poincaré. Analysis situs. *Journal de l'École Polytechnique*, 2(1):1–123, 1895.

Poi02. H. Poincaré. Sur certaines surfaces algébriques. IIIième complément à l'analysis situs. *Bull. Soc. Math. Fr.*, 30:49–70, 1902.

PS06. É. Picard and G. Simart. Théorie des fonctions algébriques de deux variables indépendantes. Vol. I, II. Paris: Gauthier-Villars. IV + 206 S. gr. 8° (1900)., 1897, 1906.

PS08. Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.

Ran81. Ziv Ran. Cycles on Fermat hypersurfaces. *Compositio Math.*, 42(1):121–142, 1980/81.

Rem95. Reinhold Remmert. Complex analysis in "Sturm und Drang". *Math. Intelligencer*, 17(2):4–11, 1995.

Rie53. Bernhard Riemann. *Collected Works of Bernhard Riemann (Gesammelte Mathematische Werke) Editor: Heinrich Weber*. Dover, 1953.

RS14. Martin Raussen and Christian Skau. Interview with Pierre Deligne. *Notices Amer. Math. Soc.*, 61(2):177–185, 2014.

S$^+$20. W. A. Stein et al. *Sage Mathematics Software*. The Sage Development Team, 2020. `http://www.sagemath.org`.

Sab99. Claude Sabbah. Hypergeometric period for a tame polynomial. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(7):603–608, 1999.

Sab07. Claude Sabbah. *Isomonodromic deformations and Frobenius manifolds*. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, french edition, 2007. An introduction.

Sab08. Claude Sabbah. Fourier-Laplace transform of a variation of polarized complex Hodge structure. *J. Reine Angew. Math.*, 621:123–158, 2008.

Sai76. Kyoji Saito. On a generalization of de-Rham lemma. *Ann. Inst. Fourier (Grenoble)*, 26(2):vii, 165–170, 1976.

Sai82. Morihiko Saito. Gauss-Manin system and mixed Hodge structure. *Proc. Japan Acad. Ser. A Math. Sci.*, 58(1):29–32, 1982.

Sai89. Morihiko Saito. Introduction to mixed Hodge modules. *Astérisque*, (179-180):10, 145–162, 1989. Actes du Colloque de Théorie de Hodge (Luminy, 1987).

Sai01. Kyoji Saito. Primitive automorphic forms. In *Mathematics unlimited—2001 and beyond*, pages 1003–1018. Springer, Berlin, 2001.

Sas74. T. Sasai. Monodromy representations of homology of certain elliptic surfaces. *J. Math. Soc. Japan*, 26:296–305, 1974.

Sch88. Norbert Schappacher. *Periods of Hecke characters*, volume 1301 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

Sch05. Mathias Schulze. Good bases for tame polynomials. *J. Symbolic comput.*, 39:103–126, 2005.

Sch15. Matthias Schütt. Picard numbers of quintic surfaces. *Proc. Lond. Math. Soc. (3)*, 110(2):428–476, 2015.

Sei08. Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

Ser56. Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

Ser19. Emre Can Sertöz. Computing periods of hypersurfaces. *Math. Comput.*, 88(320):2987–3022, 2019.

Sha16. S. Shahshahani. *An Introductory Course on Differentiable Manifolds [Hendeseye manifold in persian]*. Dover Modern Math Originals. Aurora, 2016.

Shi79a. Tetsuji Shioda. The Hodge conjecture for Fermat varieties. *Math. Ann.*, 245(2):175–184, 1979.

Shi79b. Tetsuji Shioda. The Hodge conjecture and the Tate conjecture for Fermat varieties. *Proc. Japan Acad., Ser. A*, 55:111–114, 1979.

Shi81a. Tetsuji Shioda. On the Picard number of a complex projective variety. *Ann. Sci. École Norm. Sup. (4)*, 14(3):303–321, 1981.

Shi81b. Tetsuji Shioda. On the Picard number of a Fermat surface. *J. Fac. Sci., Univ. Tokyo, Sect. I A*, 28:725–734, 1981.

Shi86. Tetsuji Shioda. An explicit algorithm for computing the Picard number of certain algebraic surfaces. *Am. J. Math.*, 108:415–432, 1986.

Shi91. Tetsuji Shioda. Theory of Mordell-Weil lattices. In *Proceedings of the international congress of mathematicians (ICM), August 21-29, 1990, Kyoto, Japan. Volume I*, pages 473–489. Tokyo etc.: Springer-Verlag, 1991.

Shi92. Tetsuji Shioda. Some remarks on elliptic curves over function fields. In *Journées arithmétiques. Exposés présentés aux dix-septièmes congrès à Genève, Suisse, 9-13 septembre 1991*, pages 99–114. Paris: Société Mathématique de France, 1992.

Sie74. Dirk Siersma. *Classification and deformation of singularities*. University of Amsterdam, Amsterdam, 1974. With Dutch and English summaries, Doctoral dissertation, University of Amsterdam.

Sim08. Carlos Simpson. Algebraic cycles from a computational point of view. *Theor. Comput. Sci.*, 392(1-3):128–140, 2008.

SS85. J. Scherk and J. H. M. Steenbrink. On the mixed Hodge structure on the cohomology of the Milnor fibre. *Math. Ann.*, 271(4):641–665, 1985.

SSvL10. Matthias Schütt, Tetsuji Shioda, and Ronald van Luijk. Lines on Fermat surfaces. *J. Number Theory*, 130(9):1939–1963, 2010.

Ste51. Norman Steenrod. *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

Ste57. Norman E. Steenrod. The work and influencc of Professor S. Lefschetz in algebraic topology. *Princeton Math. Ser.*, 12:24–43, 1957.

Ste77a. J. H. M. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

Ste77b. Joseph Steenbrink. Intersection form for quasi-homogeneous singularities. *Compositio Math.*, 34(2):211–223, 1977.

SV05. C. Soulé and C. Voisin. Torsion cohomology classes and algebraic cycles on complex projective manifolds. *Adv. Math.*, 198(1):107–127, 2005.

SXZ13. Mao Sheng, Jinxing Xu, and Kang Zuo. Maximal families of Calabi-Yau manifolds with minimal length Yukawa coupling. *Commun. Math. Stat.*, 1(1):73–92, 2013.

SZ85. Joseph Steenbrink and Steven Zucker. Variation of mixed Hodge structure. I. *Invent. Math.*, 80(3):489–542, 1985.

SZ10. Mao Sheng and Kang Zuo. Polarized variation of Hodge structures of Calabi-Yau type and characteristic subvarieties over bounded symmetric domains. *Math. Ann.*, 348(1):211–236, 2010.

Ta07. Mihai Tibu ar. *Polynomials and vanishing cycles*, volume 170 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.

Tat65. John T. Tate. Algebraic cycles and poles of zeta functions. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 93–110. Harper & Row, New York, 1965.

Tat94. John Tate. Conjectures on algebraic cycles in l-adic cohomology. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 71–83. Amer. Math. Soc., Providence, RI, 1994.

Ter90. Tomohide Terasoma. Hodge conjecture for cubic 8-folds. *Math. Ann.*, 288(1):9–19, 1990.

Tho15. Jack A. Thorne. E_6 and the arithmetic of a family of non-hyperelliptic curves of genus 3. *Forum Math. Pi*, 3:e1, 41, 2015.

Tot97. Burt Totaro. Torsion algebraic cycles and complex cobordism. *J. Amer. Math. Soc.*, 10(2):467–493, 1997.

Tot13. Burt Totaro. On the integral Hodge and Tate conjectures over a number field. *Forum Math. Sigma*, 1:e4, 13, 2013.

Ulm14. D. Ulmer. Rational curves on elliptic surfaces. *ArXiv e-prints*, July 2014.

Usu08. Hisashi Usui. On the Mordell-Weil lattice of the elliptic curve $y^2 = x^3 + t^m + 1$. IV. *Comment. Math. Univ. St. Pauli*, 57(1):23–63, 2008.

Ver76. Jean-Louis Verdier. Stratifications de Whitney et théorème de Bertini-Sard. *Invent. Math.*, 36:295–312, 1976.

Vil20. Roberto Villaflor. Periods of complete intersection algebraic cycles. *arXiv:1812.03964*, 2020.

Voi88. Claire Voisin. Une précision concernant le théorème de Noether. *Math. Ann.*, 280(4):605–611, 1988.

Voi89. Claire Voisin. Composantes de petite codimension du lieu de Noether-Lefschetz. *Comment. Math. Helv.*, 64(4):515–526, 1989.

Voi90. Claire Voisin. Sur le lieu de Noether-Lefschetz en degrés 6 et 7. *Compositio Math.*, 75(1):47–68, 1990.

Voi91. Claire Voisin. Contrexemple à une conjecture de J. Harris. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(10):685–687, 1991.

Voi99. Claire Voisin. *Mirror symmetry*, volume 1 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI, 1999. Translated from the 1996 French original by Roger Cooke.

Voi02a. Claire Voisin. A counterexample to the Hodge conjecture extended to Kähler varieties. *Int. Math. Res. Not.*, (20):1057–1075, 2002.

Voi02b. Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.

Voi03. Claire Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. Translated from the French by Leila Schneps.

Voi07. Claire Voisin. Hodge loci and absolute Hodge classes. *Compos. Math.*, 143(4):945–958, 2007.

Voi13a. Claire Voisin. Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal. *J. Algebraic Geom.*, 22(1):141–174, 2013.

Voi13b. Claire Voisin. Hodge loci. In *Handbook of moduli. Vol. III*, volume 26 of *Adv. Lect. Math. (ALM)*, pages 507–546. Int. Press, Somerville, MA, 2013.

Wal06. Michel Waldschmidt. Transcendence of periods: the state of the art. *Pure Appl. Math. Q.*, 2(2):435–463, 2006.

Wal16. Michel Waldschmidt. Irrationality and transcendence of values of special functions. *Proceedings of the XVth Annual Conference of Society for Special Functions and their Applications*, 2016.

Wei49. André Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc.*, 55:497–508, 1949.

Wei77. Abelian varieties and the hodge ring. *André Weil: Collected papers III*, pages 421–429, 1977.

Wei99. Charles A. Weibel. History of homological algebra. In *History of topology*, pages 797–836. Amsterdam: Elsevier, 1999.

Wol83. J. Wolfart. Eine arithmetische Eigenschaft automorpher Formen zu gewissen nicht-arithmetischen Gruppen. *Math. Ann.*, 262(1):1–21, 1983.

WW85. Jürgen Wolfart and Gisbert Wüstholz. Der Überlagerungsradius gewisser algebraischer Kurven und die Werte der Betafunktion an rationalen Stellen. *Math. Ann.*, 273(1):1–15, 1985.

Zuc77. Steven Zucker. The Hodge conjecture for cubic fourfolds. *Compositio Math.*, 34(2):199–209, 1977.

Zuc84. Steven Zucker. Degeneration of Hodge bundles (after Steenbrink). In *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982)*, volume 106 of *Ann. of Math. Stud.*, pages 121–141. Princeton Univ. Press, Princeton, NJ, 1984.

Zuc87. Steven Zucker. Degeneration of mixed Hodge structures. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 283–293. Amer. Math. Soc., Providence, RI, 1987.

Mathematicians

Here, is a list of mathematicians whose works have considerable impact on the content of the present book, and not necessarily the Hodge theory in general. The main source of information for this has been many mathematical works and also the web page [OR16] created by John J. O'Connor and Edmund F. Robertson. We only mention the contributions relevant to the content of this book. This may not be the major contribution of the corresponding mathematician.

Adrien-Marie Legendre (1752-1833) His major contribution is on elliptic integrals. Legendre relation between elliptic integrals is a first concrete manifestation of an algebraic cycle.

Johann Carl Friedrich Gauss (1777-1855) His name in the 'Gauss-Manin connection' represents the works of many mathematicians before 1900. Gauss hypergeometric function with rational parameter is an abelian integral (up to some constant gamma factors) and the corresponding Gauss hypergeometric equation is an example of a Picard-Fuchs equation.

Johann Friedrich Pfaff (1785-1825) In 1815 he published his work on Pfaffian forms which was developed into Élie Cartan's calculus of differential forms.

Augustin Louis Cauchy (1789-1857) He is the founder of the theory of integrals in the complex domain $\mathbb{C}$.

Niels Henrik Abel (1802-1829) He as a beginner was frustrated with two big mathematicians of his time. These are namely Cauchy and Gauss. Despite his short life, his name in Abelian integrals will live forever.

Carl Gustav Jacob Jacobi (1804-1851) "It gives me great satisfaction to see two young mathematicians such as you [Jacobi] and [Abel] cultivate with such success a branch of analysis [elliptic functions and integrals] which for such a long time has been my favorite topic of study but which had not been received in my own country as well as it deserves. By your works you place yourselves in the ranks of

the best analysts of our era", (A quotation from A.-M. Legendre, see [OR16]).

Georg Friedrich Bernhard Riemann (1826-1866) His article 'Über die Hypothesen welche der Geometrie zu Grunde liegen' announced in 1853 and published in 1868, two years after his death, despite being of philosophical nature rather than mathematical, is one of the foundational works in higher dimensional topology and geometry. In two dimensions his work on 'Riemann surfaces' paved the road for a transfer from abelian integrals to multiple integrals.

Leo August Pochhammer (1841-1920) He contributed to finding solutions of differential equations, and in particular generalized hypergeometric equation, as integrals and convergent series. In this book we have used the notions 'Pochhammer symbol' and 'Pochhammer cycle'.

Max Noether (1844-1921) "He was of course never content without algebraic or arithmetic proof but had sometimes to be satisfied with an incomplete proof.", (F. S. Macaulay, see [OR16]). The rigorous proof of Noether-Lefschetz theorem is one of the main applications of the tools developed in Hodge theory.

Georges Simart (1846-1921) In [PS06] he is introduced as 'capitaine de frégate, répétiteur a l'École Polytechnique'. In the preface of the same book, which is only signed by Picard, one reads "Mon ami, M. Simart, qui m'a déjà rendu de grands services dans la publication de mon Traité d'Analyse, ayant bien voulu me promettre son concours, a levé mes hésitations. J'ai traité cet hiver dans mon cours de la Théorie des surfaces algébriques, et nous avons, M. Simart et moi, rassemblé ces Leçons dans le Tome premier, que nous publions aujourd'hui". The two volume book [PS06] may not existed without Simart's assistance, and that is why, we have included his name in this list of great mathematicians.

Jules Henri Poincaré (1854-1912) "He can be said to have been the originator of algebraic topology and, in 1901, he claimed that his researches in many different areas such as differential equations and multiple integrals had all led him to topology. For 40 years after Poincaré published the first of his six papers on algebraic topology in 1894, essentially all of the ideas and techniques in the subject were based on his work", (see [OR16]).

Giuseppe Veronese (1854-1917) He is mainly famous for his works in projective geometry. The Veronese embedding is now a classical tool in algebraic geometry.

Thomas Joannes Stieltjes (1856-1894) Poincaré in [Poi02] mentions an unpublished work of Stieltjes on rational double integrals. This has been one of the motivations for Poincaré to study residues of higher dimensional integrals in [Poi87].

Charles Émile Picard (1856-1941) His book [PS06] is one of the main inspirations for the author of the present book.

Marie Georges Humbert (1859-1921) His work on hyperelliptic surfaces [Hum93] is without doubt the beginning of the study of double integrals.

Élie Cartan (1869-1951) In 1899 he published his work on the Pfaff problem and officially introduced differential forms as objects independent from partial differential equations.

Emmy Amalie Noether (1882-1935) She was one of the founders of abstract algebra. This paved the road for algebraic geometry to be more algebraic rather than geometric.

Solomon Lefschetz (1884-1972) "He had the misfortune to lose both his hands in a laboratory accident in November 1907 when they were burnt off in a transformer explosion." (see [OR16]). After this he decided to be a mathematician.

William Vallance Douglas Hodge's (1903-1975) Even though Hodge theory was founded after his book [Hod41], the techniques introduced there to deal with harmonic integrals have been somewhat isolated from the whole theory. This might be because of their non-algebraic and mainly functional analysis and Riemannian geometry nature.

Georges de Rham (1903-1990) He is mainly responsible for de Rham theorem, proved by him in 1931, which says that the de Rham cohomology is dual to singular homology with real coefficients.

Charles Ehresmann (1905-1979) Even though Picard-Lefschetz theory is built upon Ehresmann's fibration theorem, most of the theory were developed before its rigorous proof.

Jean Leray (1906-1998) "While at the camp Leray and some of his fellow captives organized a *université en captivité* and Leray became its rector. Not wishing the Germans to know that he was an expert in hydrodynamics, since he feared that if they found out he would be forced to undertake war work for them, Leray claimed to be a topologist. He worked only on topological problems for the years he was held captive in the camp" quotation from [OR16].

Norman E. Steenrod (1910-1971) He together with Eilenberg axiomatized the homology and cohomology theory.

Samuel Eilenberg (1913-1998) "Thanks to Sammy's insight and his enthusiasm, this text [the book *Foundations of algebraic topology* written together with N. Steenrod] drastically changed the teaching of topology" (see the quotation of S. Mac Lane in [OR16]).

Werner Gysin (1915-??) He published only one paper in mathematics, however, his name apprears in this book many times.

René Thom (1923-2002) In his autobiography he writes "Relations with my colleague Grothendieck were less agreeable for me." (see [OR16]).

Michael Francis Atiyah (1929-) His joint work with W. Hodge paved the road for definition of algebraic de Rham cohomology by Alexander Grothendieck.

Alexander Grothendieck (1928-2014) His definition of algebraic de Rham cohomology in [Gro66] was the culmination of more than a century of

mathematics, from the works on integrals to the introduction of sheaves and cohomologies in Algebraic Geometry.

John Willard Milnor (1931-) His works on the the topology of singularities has a great impact on our study of tame polynomials.

Klaus Lamotke (1936-) In the author's opinion his article [Lam81] is one of the most beautiful articles on the topology of algebraic varieties.

Vladimir Igorevich Arnold (1937-2010) His contribution to singularity theory and the school he created on this subject are essential for understanding many concepts related to tame polynomials. Whereas Grothendieck's school tried to replace Picard and Lefschetz's works in a more abstract context, useful for arithmetic purposes, his school continued to explore those works in a more and less original style.

Yuri Ivanovitch Manin (1937-) "Two main things I was always interested in were number theory on the one hand and physics on the other. So I think in both domains I always tried to use the intuition developed in both domains." (Y. Manin in *The Berlin Intelligencer*, 1998, p. 16-19). The concept of Gauss-Manin connection is just one of those with applications ranging from numer theory to physics.

Phillip Augustus Griffiths (1938-) His works on the algebraic de Rham cohomology of hypersurfaces and its Hodge filtration is one of the fundamental works in Hodge theory.

Pierre René Deligne (1944-) "Which of those [results] that you worked on after the proof of the Weil conjecture are you particularly fond of? Deligne: I like my construction of a so-called mixed Hodge structure on the cohomology of complex algebraic varieties," (P. Deligne in [RS14], page 182). Picard would be happy to see Deligne's mixed Hodge theory in his mathematical language.

Tetsuji Shioda It is possible to do advanced mathematics based on an example and Shioda's work on Fermat varieties is an evidence for this statement.

Shing-Tung Yau (1949-) His solution of Calabi conjecture resulted in the introduction of Calabi-Yau manifolds and connected Hodge theory to other branches of mathematics ranging from symplectic geometry to automorphic forms. It also brought it to its origin which is the study of multiple integrals.

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