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Transfer operators for $\Gamma_0(n)$ and the Hecke operators for the period functions of $\mathrm{PSL}(2, \mathbb{Z})$

By JOACHIM HILGERT

Institut für Mathematik, Universität Paderborn, D-33098 Paderborn, Germany.
e-mail: hilgert@math.uni-paderborn.de

DIETER MAYER

Institut für Theoretische Physik, TU Clausthal, D-38678 Clausthal-Zellerfeld, Germany.
e-mail: dieter.mayer@tu-clausthal.de

AND HOSSEIN MOVASATI

Fachbereich Mathematik, TU Darmstadt, D-64289 Darmstadt, Germany.
e-mail: movasati@mathematik.tu-darmstadt.de

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Abstract

In this paper we report on a surprising relation between the transfer operators for the congruence subgroups $\Gamma_0(nm), n, m \in \mathbb{N}$, and some kind of Hecke operators on the space of vector valued period functions for the groups $\Gamma_0(n)$. We study special eigenfunctions of the transfer operators for the groups $\Gamma_0(nm)$ with eigenvalues ∓ 1 which are also solutions of the Lewis equations for these groups and which are determined by eigenfunctions of the transfer operator for the congruence subgroup $\Gamma_0(n)$. In the language of the Atkin–Lehner theory of old and new forms one should hence call them old eigenfunctions or old solutions of the Lewis equation for $\Gamma_0(n)$. It turns out that certain linear combinations of the components of these old solutions for the group $\Gamma_0(nm)$ determine for any m a solution of the Lewis equation for the group $\Gamma_0(n)$ and hence also an eigenfunction of the transfer operator for this group.

Our construction gives linear operators $\tilde{T}_n$ in the space of vector valued period functions for the group $\Gamma_0(n)$ which are rather similar to the Hecke operators. Indeed, in the case of the group $\Gamma_0(1) = \mathrm{SL}(2, \mathbb{Z})$ these operators are just the well-known Hecke operators on the space of period functions for the modular group, derived previously using the Eichler–Manin–Shimura correspondence between period polynomials and modular forms for this group, and its extension to Maass wave forms by Lewis and Zagier.

1. *Introduction*

This paper has three main ingredients. The first is the transfer operator from statistical mechanics which plays an important role in the ergodic theory of dynamical systems and especially in the theory of dynamical zeta functions (see [**14**, **19**]). Here we are interested in the transfer operators for the geodesic flow on the surfaces $\Gamma\backslash\mathbb{H}$ for Γ any of the congruence subgroups $\Gamma_0(n)$. These operators have been introduced in [**2**, **3**] in the study of the Selberg zeta function for these groups.

The second are certain functions holomorphic in the cut plane, introduced by J. B. Lewis in [**7**] in his study of the Maass wave forms for $\mathrm{PSL}(2,\mathbb{Z})$. They were later named period functions by Zagier (see [**21**]) because of their close relation to the classical period polynomials in the Eichler–Manin–Shimura theory of periods for cusp forms. Period functions for the modular group are solutions of the so called Lewis equation

$$\phi(z) = \phi(z+1) + \lambda z^{-2s}\phi\left(1+\frac{1}{z}\right) \tag{1·1}$$

with $\lambda = \pm 1$, which fulfill certain growth conditions at infinity depending on the weight s. When this weight satisfies $\Re(s) = 1/2$, these solutions are in 1-1 correspondence with the Maass cusp forms (see [**9**]). There is a simple relation between the transfer operator for $\mathrm{PSL}(2,\mathbb{Z})$ and the period functions: they are just the eigenfunctions of this operator with eigenvalue ± 1 (see [**3**]). When s is a negative integer $s = -n$ the space of polynomial solutions of the Lewis equation is in 1-1 correspondence with the space of period polynomials for $\mathrm{PSL}(2,\mathbb{Z})$ (see [**21**]). The Eichler–Shimura–Manin theory of periods however tells us that this space of period polynomials modulo a certain one dimensional space is isomorphic to the direct sum of two copies of the space of cusp forms of weight $2n+2$ in the half plane.

The space of cusp forms is extensively studied in number theory and in particular we have the Hecke algebra acting on it.

A Theorem by Choie and Zagier (see [**4**, section 3, theorem 2]) gives a criterion to find an explicit realization of the corresponding Hecke operators when acting on the space of period polynomials or more generally period functions. Generalizing the description of Hecke operators for Maass wave forms by Manin in [**11**], Choie and Zagier found (see [**4**, theorem 3]) an explicit form for these Hecke operators in the space of period polynomials. Their matrices, however, from which the Hecke operators are constructed via the well-known slash action of the group $\mathrm{Mat}_n(2,\mathbb{Z})$ on smooth functions, have negative entries and hence their action is defined only for entire weights.

The third important ingredient in our paper is a realization of the Hecke operators on period functions for $\mathrm{SL}(2,\mathbb{Z})$ which we first learned from T. Mühlenbruch (to appear in his thesis, see [**18**]). Here one uses matrices in the set

$$S_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a > c \geqslant 0,\ d > b \geqslant 0,\ ad - bc = n \right\}$$

such that the Hecke operators T_n have the form

$$T_n := \sum_{A \in S_n} A \tag{1·2}$$

acting on the period functions via the slash operator, and where the sum is taken in the free abelian group generated by S_n. All the matrices from the set S_n have positive entries and so one can define the Hecke operators on period functions for any weight

$s \in \mathbb{C}$. An analogous realization of the Hecke operators on modular symbols has been given also by L. Merel in [**17**].

The relation between the transfer operator and the period functions was considered originally only for the modular group $\mathrm{SL}(2,\mathbb{Z})$. In this case the Lewis equation is a scalar equation for scalar functions and its solutions can be related explicitly to the period functions of modular and Maass wave forms. Part of this theory has been extended to more general Fuchsian groups. Chang and Mayer began in a series of papers (see [**3**] and its references) to investigate the transfer operator approach to congruence subgroups like $\Gamma_0(n)$, $\Gamma^0(n)$ or $\Gamma(n)$. This lead them to transfer operators acting in Banach spaces of vector valued holomorphic functions. The eigenfunctions of these operators then fulfill general Lewis equations in vector spaces whose dimension is just the index in $\mathrm{SL}(2,\mathbb{Z})$ of the corresponding subgroup.

In this paper we discuss special solutions of these Lewis equations for the congruence subgroups $\Gamma_0(nm)$ for fixed n and $m \in \mathbb{N}$ arbitrary which are determined by the solutions of the Lewis equation for the group $\Gamma_0(n)$. Hence our construction is reminiscent of the theory of Atkin and Lehner of old and new forms (see [**1**]). The exact connection, however, will only be discussed in a forthcoming paper.

To state our main results and to sketch the content of each section we have to fix the notations used throughout the text. For each integer n let $\mathrm{Mat}_n(2,\mathbb{Z})$ (resp. $\mathrm{Mat}_*(2,\mathbb{Z})$) be the set of 2×2-matrices with integer entries and determinant n (resp. nonzero determinant) and $\mathcal{R}_n := \mathbb{Z}[\mathrm{Mat}_n(2,\mathbb{Z})]$ (resp. $\mathcal{R} := \mathbb{Z}[\mathrm{Mat}_*(2,\mathbb{Z})]$) the set of finite linear combinations (with coefficients in $\mathbb{Z}$) of the elements of $\mathrm{Mat}_n(2,\mathbb{Z})$ (resp. $\mathrm{Mat}_*(2,\mathbb{Z})$). Note that $\mathcal{R} = \bigcup_{n\in\mathbb{Z}\setminus\{0\}}\mathcal{R}_n$ and $\mathcal{R}_n \cdot \mathcal{R}_m \subseteq \mathcal{R}_{nm}$. By definition we have

$$\mathrm{GL}(2,\mathbb{Z}) = \mathrm{Mat}_1(2,\mathbb{Z}) \cup \mathrm{Mat}_{-1}(2,\mathbb{Z}).$$

The following four elements of $\mathrm{GL}(2,\mathbb{Z})$ will play a prominent role in this paper:

$$I := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad M := \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad T := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad Q := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

It turns out that instead of the groups $\Gamma_0(n)$ it is more convenient to use their extensions $\overline{\Gamma}_0(n)$ in $\mathrm{GL}(2,\mathbb{Z})$:

$$\overline{\Gamma}_0(n) := \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid c \equiv 0 \mod n\right\} = \Gamma_0(n) \cup \Gamma_0(n)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In Section 4 we recall the definition of the transfer operators for $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$ as used by Chang and Mayer in [**2**, **3**], respectively by Manin and Marcolli in [**10**], discuss briefly their relation and derive the Lewis equation for the eigenfunctions of the operator of Manin and Marcolli. This operator is defined on a space of holomorphic functions with values in the representation space of $\mathrm{GL}(2,\mathbb{Z})$ induced from the trivial representation of $\overline{\Gamma}_0(n)$. In order to describe and solve the corresponding Lewis equation it turns out that the correct indexing of the components of these functions by the set

$$I_n := \overline{\Gamma}_0(n)\backslash\mathrm{GL}(2,\mathbb{Z})$$

is very helpful. The group $\mathrm{GL}(2,\mathbb{Z})$ acts on this coset space on the right in a canonical way. For two positive integers n and m the inclusion $\overline{\Gamma}_0(nm) \subseteq \overline{\Gamma}_0(m)$ induces a canonical map

$$\sigma_{n,m}: I_{nm} \longrightarrow I_m.$$

The detailed structure of I_n will be studied in Section 5 (in particular see Propositions 5·4 and 5·8). The different components of the Lewis equation can then be written for $i \in I_n$ as follows:

$$\phi_i(z) - \phi_{iT^{-1}}(z+1) - \lambda z^{-2s}\phi_{iT^{-1}M}\left(1+\frac{1}{z}\right) = 0, \tag{1·3}$$

where $\lambda = \pm 1$. These equations have to be solved simultaneously with functions ϕ_i holomorphic in the cut plane $\mathbb{C} \setminus (-\infty, 0]$ for all $i \in I_n$. Replacing i by $iT^{-1}MT$ and z by $1/z$, multiplying the resulting equation by λz^{-2s}, and then subtracting it from the original equation we get

$$\phi_i(z) = \lambda z^{-2s}\phi_{iT^{-1}MT}\left(\frac{1}{z}\right),\ i \in I_n. \tag{1·4}$$

We then call ϕ_i and $\phi_{iT^{-1}MT}$ a *symmetric pair*.

Let $\mathcal{I}^\lambda := (I - T - \lambda TM)\mathcal{R}$ be the right ideal generated by $(I - T - \lambda TM)$ in $\mathcal{R}$. Consider then the following system of equations in the right $\mathcal{R}$-module $\mathcal{I}^\lambda \backslash \mathcal{R}$:

$$\psi_i - \psi_{iT^{-1}}T - \lambda\psi_{iT^{-1}M}TM \equiv 0 \mod \mathcal{I}^\lambda, \quad \forall i \in I_n, \tag{1·5}$$

which obviously is closely related to (1·3). Here the ψ_i's are unknown elements in $\mathcal{R}$. The symmetry (1·4) for equations (1·5) reads

$$\psi_i = \lambda\psi_{iT^{-1}MT}M,\ i \in I_n. \tag{1·6}$$

Note the two different matrix actions in these equations: on the one hand matrices act from the right on the index i of ψ_i and on the other hand matrices act from the right on elements ψ_i via the ring multiplication of $\mathcal{R}$. Moreover, in (1·3) we have the familiar slash operation formally defined for $s \in \mathbb{C}$ and $R \in \mathcal{R}$ by $\mathbb{Z}$-linear extension of

$$(\phi \mid_s R)(z) = |\det R|^s (cz+d)^{-2s}\phi(Rz) \tag{1·7}$$

for $R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_*(2, \mathbb{Z})$ with $Rz = (az+b)/(cz+d)$. Now suppose $\psi_i, i \in I_n$, solves (1·5) and ϕ is a solution of the Lewis equation (1·1) for $\mathrm{SL}(2, \mathbb{Z})$. For s an integer the left-hand side of (1·5) can act on ϕ via the usual slash-operator and one obtains a solution $(\phi_i)_{i\in I_n}$ of (1·3) by setting

$$\phi_i := \phi \mid_s \psi_i$$

since $\phi \mid_s \mathcal{I}^\lambda = 0$.

It is well known that $1/z$ is up to a constant factor the only solution of the scalar Lewis equation (1·1) for $\lambda = 1$ and $s = 1$ holomorphic in the complex z-plane cut along $(-\infty, 0]$ (see [**15**]). It follows from a result by Manin and Marcolli (see [**10**, proposition 4·2]) that for these parameter values $(\phi_i)_{i\in I_n}$ with $\phi_i(z) = 1/z$ for all $i \in I_n$ is, up to a trivial scalar factor, also the unique solution of (1·3) holomorphic in the same domain. Hence, if $(\psi_i)_{i\in I_n}$ solves (1·5), then there exists a constant κ such that

$$\frac{1}{z}\bigg|_1 \psi_i = \kappa\frac{1}{z} \quad \forall i \in I_n$$

must hold. Suppose furthermore that $\psi_i = \sum_{A\in P_i} A$, where P_i is some finite subset of $\mathrm{Mat}_n(2,\mathbb{Z})$. Then the above equality reads $\sum_{A\in P_i} 1/(az+b)(cz+d) = \kappa(1/z)$, where $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. The right-hand side of this expression obviously has a pole and a zero only at 0 and ∞. Hence other poles and zeroes of the left-hand side must cancel. This means, however, that the matrices $A \in P_i$ have to be chosen in a very specific way. Explicit calculations for the groups $\Gamma_0(n)$ for small n lead us to an operator $K\colon A \mapsto K(A)$ which attaches to every matrix $A \in P_i$ another matrix KA whose action just cancels the poles and zeros generated by the action of A. In all cases considered only a finite number of matrices A were necessary to get the correct pole and zero structure. We later found that an operator similar to K was indeed used already by Choie and Zagier [**4**] and also by Mühlenbruch (see [**18**, lemma 9]) in their completely different derivation of the Hecke operators. The explicit form of the map K is given as

$$\begin{aligned} K : S_n \setminus Y_n &\longrightarrow S_n \setminus X_n \\ \begin{pmatrix} a & b \\ c & d\end{pmatrix} &\longmapsto T^{\lceil \frac{d}{b}\rceil} Q \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} -c + \lceil \frac{d}{b}\rceil a & -d + \lceil \frac{d}{b}\rceil b \\ a & b\end{pmatrix}, \end{aligned}$$

where

$$X_n := \left\{ \begin{pmatrix} c & a \\ 0 & \frac{n}{c}\end{pmatrix},\ c \mid n,\ 0 \leqslant a < \frac{n}{c} \right\}, \quad Y_n := \left\{ \begin{pmatrix} c & 0 \\ a & \frac{n}{c}\end{pmatrix},\ c \mid n,\ 0 \leqslant a < c \right\}$$

and where for a real r we have denoted by $\lceil r \rceil$ the integer satisfying $\lceil r\rceil - 1 < r \leqslant \lceil r \rceil$. With the usual notation of Gauss brackets we obtain $\lceil r \rceil = -[-r]$. The inverse of K is given by

$$K^{-1} : \begin{pmatrix} a & b \\ c & d\end{pmatrix} \longmapsto M T^{\lceil \frac{a}{c}\rceil} Q M \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} c & d \\ -a + \lceil \frac{a}{c}\rceil c & -b + \lceil \frac{a}{c}\rceil d\end{pmatrix}$$

(see Proposition 6·1). Borrowing terminology from algebraic geometry one might call K a rational automorphism of S_n.

To each index $i \in I_n$ we will attach a matrix (see Definition 5·9)

$$A_i = \begin{pmatrix} c & b \\ 0 & \frac{n}{c}\end{pmatrix},$$

where $c \geqslant 1$, $c \mid n$ and $0 \leqslant b < \frac{n}{c}$ satisfy $\gcd(c, \frac{n}{c}, b) = 1$. The numbers c and b are then uniquely determined by the index i.

Starting now with a matrix $A_i \in X_n$ we apply K repeatedly until we get an element of Y_n where the iteration stops. Since K is injective, two such chains of elements in S_n are either equal or disjoint. For $i \in I_n$ we denote by k_i the number such that $K^j A_i$ is well-defined for $j \leqslant k_i$ and $K^{k_i} A_i \in Y_n$ (see Definition 6·2). Obviously each element in $X_n \cap Y_n$ forms a one-element chain so that $k_i = 0$ for $A_i \in X_n \cap Y_n$. Then we have:

THEOREM 1·1. *The matrices*

$$\psi_i = \sum_{j=0}^{k_i} K^j(A_i), \quad i \in I_m$$

determine a solution of equations (1·5). *Acting by these matrices through the slash operator* *on a solution ϕ of the Lewis equation* (1·1) *for the group* $\mathrm{SL}(2,\mathbb{Z})$ *with weight s gives a solution of equation* (1·3) *for the group* $\overline{\Gamma}_0(m)$ *with the same weight s.*

In the second part of the theorem we used the fact that the slash operator with weight s an arbitrary complex number is indeed well defined. Details will be discussed in Section 3.

Theorem 1·1 shows that any solution ϕ of the scalar Lewis equation (1·1) for $\mathrm{SL}(2,\mathbb{Z})$ determines a solution $(\phi \mid_s \psi_i)_{i\in I_n}$ of the Lewis equation (1·3) for $\overline{\Gamma}_0(m)$. The next step is to generalize this fact in the following way: for any solution $(\phi_i)_{i\in I_n}$ of the Lewis equation (1·3) for $\overline{\Gamma}_0(n)$ we determine a solution $(\phi_i)_{i\in I_{nm}}$ of the Lewis equation (1·3) for $\overline{\Gamma}_0(nm)$, where m is any positive integer. To state our theorem precisely we need the following proposition, which we will prove in Section 6:

PROPOSITION 1·2. *Let $\sigma_{n,m}: I_{nm} \to I_m$ be the canonical map. For any $i \in I_{nm}$ and $0 \leqslant j \leqslant k_{\sigma_{n,m}(i)}$ there exists a unique index $\hat{\imath}_{ij} \in I_n$ such that $A_{\hat{\imath}_{ij}}(K^j A_{\sigma_{n,m}(i)})A_i^{-1}$ is integer-valued and hence in* $\mathrm{SL}(2,\mathbb{Z})$.

The main result of this paper is:

THEOREM 1·3. *Let $\{\psi_{\hat{\imath}}\}_{\hat{\imath}\in I_n}$ (resp. $\{\phi_{\hat{\imath}}\}_{\hat{\imath}\in I_n}$) be any solution of the Lewis equation* (1·5) (*resp.* (1·3)) *for* $\overline{\Gamma}_0(n)$. *Then*

$$\psi_i := \sum_{j=0}^{k_{\sigma_{n,m}(i)}} \psi_{\hat{\imath}_{ij}} K^j A_{\sigma_{n,m}(i)},\ i \in I_{nm} \tag{1·8}$$

$$\textit{resp. } \phi_i := \sum_{j=0}^{k_{\sigma_{n,m}(i)}} \phi_{\hat{\imath}_{ij}} \mid_s K^j A_{\sigma_{n,m}(i)},\ i \in I_{nm}$$

is a solution of the Lewis equation (1·5) (*resp.* (1·3)) *for* $\overline{\Gamma}_0(nm)$. *If $\{\psi_{\hat{\imath}}\}_{\hat{\imath}\in I_n}$ is the special* (*old*) *solution of Theorem* 1·1, *then* (1·8) *coincides* (*modulo* $\mathcal{I}^\lambda$) *with the special* (*old*) *solution of Theorem* 1·1 *for* $\overline{\Gamma}_0(nm)$.

In the case $n=1$ the map $\sigma_{n,m}$ is the identity map and Theorem 1·3 reduces to Theorem 1·1.

Theorem 1·3 shows that any solution $\{\phi_{\hat{\imath}}\}_{\hat{\imath}\in I_n}$ of the system of Lewis equations (1·3) for the group $\overline{\Gamma}_0(n)$ determines a solution $\{\phi_i\}_{i\in I_{nm}}$ of these equations for any of the groups $\overline{\Gamma}_0(nm)$. Furthermore one shows that certain linear combinations of the components of any solution of the Lewis equations (1·3) for the groups $\overline{\Gamma}_0(nm)$ for fixed n and arbitrary $m \in \mathbb{N}$ define a solution of these equation for the group $\overline{\Gamma}_0(n)$. Combining this fact with Theorem 1·3 then allows us to define for any n a family of linear operators $\{\tilde{T}_{n,m}\}_{m\in\mathbb{N}}$ mapping the space of solutions of the Lewis equations for $\overline{\Gamma}_0(n)$ to itself. For these operators we find:

PROPOSITION 1·4. *The linear operators $\tilde{T}_{n,m}$ mapping the space of solutions of the Lewis equations* (1·5) *for the group* $\overline{\Gamma}_0(n)$ *into itself are given by*

$$(\tilde{T}_{n,m}\psi)_{\hat{\imath}} = \sum_{l\in\sigma_{m,n}^{-1}(\hat{\imath})} \sum_{j=0}^{k_{\sigma_{n,m}(l)}} \psi_{\hat{\imath}_{l,j}} K^j A_{\sigma_{n,m}(l)}, \quad \hat{\imath} \in I_n.$$

If $\{\phi_{\hat\imath}\}_{\hat\imath\in I_n}$ is any solution of the Lewis equations (1·3) *for the group $\overline{\Gamma}_0(n)$, then the functions $\{\phi'_{\hat\imath}\}_{\hat\imath\in I_n}$ given by*

$$\phi'_{\hat\imath}(z) = \sum_{l\in\sigma_{m,n}^{-1}(\hat\imath)} \sum_{j=0}^{k_{\sigma_{n,m}(l)}} \phi_{\hat\imath_{l,j}} \mid_s K^j A_{\sigma_{n,m}(l)}(z), \quad \hat\imath \in I_n$$

also determine a solution of these equations for the group $\overline{\Gamma}_0(n)$.

Note that in the above formulas the first sum is taken over $l \in \sigma_{m,n}^{-1}(\hat\imath)$ and not $l \in \sigma_{n,m}^{-1}(\hat\imath)$. In the special case of the group $\overline{\Gamma}_0(1) = \mathrm{GL}(2,\mathbb{Z})$ the operators $\tilde{T}_m := \tilde{T}_{1,m}$ are closely related to the Hecke operators T_m on the period functions for $SL(2,\mathbb{Z})$ in (1·2):

Proposition 1·5. *The operators $\tilde{T}_m$ and the Hecke operators T_m defined in* (1·2) *are related through*

$$T_m = \sum_{d^2|m} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \tilde{T}_{\frac{m}{d^2}}.$$

In particular they coincide if and only if m is a product of distinct primes.

Here we have identified the matrices T_m and $\tilde{T}_{\frac{m}{d^2}}$ with the operators they define via the slash action. The operators $\tilde{T}_m$ have been constructed through special solutions of the Lewis equation for the congruence subgroups $\overline{\Gamma}_0(m)$. There immediately arises the question if this is also the case for the Hecke operators T_m. Indeed, it turns out that the operators $\tilde{T}_{\frac{m}{d^2}}$ appearing in Proposition 1·5 above can also be related to special solutions of the Lewis equation for the group $\overline{\Gamma}_0(m)$: we show in Section 8 that any solution of the Lewis equation for the group $\overline{\Gamma}_0(n)$ determines besides the solution of Theorem 1·3 another trivial solution of the corresponding equation for the group $\overline{\Gamma}_0(nm)$ for arbitrary $m \in \mathbb{N}$. Its components are just identical copies of the former's components (see Proposition 8·3). Taking then for the solution for the group $\overline{\Gamma}_0(n)$ the solution of Theorem 1·1 we get a solution for the group $\overline{\Gamma}_0(nm)$. The sum of its components gives μ-times the operator $\tilde{T}_n$ where μ is the index of $\overline{\Gamma}_0(nm)$ in $\overline{\Gamma}_0(n)$. Thus also the operators $\tilde{T}_{\frac{m}{d^2}}$ can be constructed from special solutions of the Lewis equation and hence from special eigenfunctions of the transfer operator for the group $\overline{\Gamma}_0(m)$. The relation of our operators $\tilde{T}_{n,m}$ for general n with the Hecke operators T_m for the congruence subgroups $\Gamma_0(n)$ will be discussed in another paper. The above results depend in a crucial way on a modified one-sided continued fraction expansion for rational numbers and closely related partitions of $\mathbb{R}$ described in section 2.

The technical results about the slash-operation are provided in section 3 and the transfer operators for $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$ are introduced in section 4. The indexing coset space $\overline{\Gamma}_0(n)\backslash\mathrm{GL}(2,\mathbb{Z})$ is studied in detail in section 5. In section 6 we derive and discuss the operator K and in section 7 we construct our special solutions of the Lewis equations. Finally, in section 8 we show how our results lead to a completely new approach to the Hecke operators on the space of period functions for $\mathrm{SL}(2,\mathbb{Z})$ which basically only uses the transfer operators for the congruence subgroups $\Gamma_0(n)$, respectively $\overline{\Gamma}_0(n)$.

2. *A modified continued fraction expansion*

This section is inspired by the work of Mühlenbruch in [**18**] adapted appropriately to our needs. Mühlenbruch introduces a modified continued fraction expansion for positive rational numbers and attaches to each $x \in \mathbb{Q}^+$ a suitable chain of elements of $\mathcal{R}$ called a partition of x. To explain his construction we begin by collecting some facts which are standard in the theory of continued fractions (see [**5**]). Consider the finite continued fraction expansion of x

$$x = [a_0, a_1, \ldots, a_N] := a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_N}}}$$

and put $p_n/q_n := [a_0, a_1, \ldots, a_n]$ for $0 \leqslant n \leqslant N$ with $\gcd(p_n, q_n) = 1$ and $q_n \geqslant 0$. Then the recursion formulas

$$p_n = a_n p_{n-1} + p_{n-2} \tag{2·1}$$

$$q_n = a_n q_{n-1} + q_{n-2} \tag{2·2}$$

hold. In particular, we have

$$q_0 \leqslant q_1 < \cdots < q_N. \tag{2·3}$$

Moreover, the following equations

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots \leqslant x \leqslant \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1} \tag{2·4}$$

and

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}, \ p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n \tag{2·5}$$

hold. We are going to fill the above sequence (2·4) with more rational numbers. We do that for the left hand side of the sequence, the case we later use. Assume that n is even. The sequence of numbers

$$[a_0, a_1, \ldots, a_{n-1}, t] = \frac{t p_{n-1} + p_{n-2}}{t q_{n-1} + q_{n-2}}, \quad \text{for } t = 0, \ldots, a_n$$

is then strictly increasing from p_{n-2}/q_{n-2} to p_n/q_n. We insert these numbers into the left-hand side of (2·4) and obtain the longer sequence:

$$\cdots < \frac{p_{n-2}}{q_{n-2}} < \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} < \cdots < \frac{(a_n - 1)p_{n-1} + p_{n-2}}{(a_n - 1)q_{n-1} + q_{n-2}} < \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n} < \cdots \tag{2·6}$$

Here we have used the convention $1/(a + 1/0) = 0$. If we denote the rational numbers x_j in this sequence by $x_j = p'_j/q'_j$ with $\gcd(p'_j, q'_j) = 1$, then two consecutive numbers $p'_j/q'_j < p'_{j+1}/q'_{j+1}$ satisfy $q'_j < q'_{j+1}$ and

$$p'_{j+1}q'_j - p'_j q'_{j+1} = 1, \tag{2·7}$$

where the last equality is a consequence of (2·5). Recall (see [**5**]) that for $x \in \mathbb{Q}^+$ there is a unique sequence $a_0, \ldots, a_n \in \mathbb{N}$ such that $a_n > 1$ and $x = [a_0, \ldots, a_n]$: if $x = [b_0, \ldots, b_{m-1}, 1]$ for $b_0, \ldots, b_{m-1} \in \mathbb{N}$, then obviously $x = [b_0, \ldots, b_{m-1} + 1]$, and hence $m = n + 1$ and $a_0 = b_0, \ldots, a_{n-1} = b_{n-1}, a_n = b_n + 1$. This will be used in the following definitions.

Definition 2·1. Given $x \in \mathbb{Q}^+$, the *modified continued fraction expansion* of x is the sequence $x_j,\ j = 0, 1, \ldots$ recursively defined by:

(i) $x_0 := x = [a_0, \ldots, a_N]$ with $a_N > 1$;
(ii) If $x_{j-1} = [b_0, \ldots, b_m]$ with $b_m > 1$, then

$$x_j := \begin{cases} [b_0, b_1, \ldots, b_{m-1}] & \text{if } 2 \nmid m \\ [b_0, b_1, \ldots, b_m - 1] & \text{if } 2 \mid m. \end{cases}$$

If $x_{j-1} = 0$, then $x_j = -\infty$ and the sequence stops.

Note that the length of the modified continued fraction expansion of $x = [a_0, \ldots, a_N]$ with $a_N > 1$ is not greater than $\sum_{i=1,\text{even}}^{N} a_i$.

Proposition 2·2. *Let $x \in \mathbb{Q}^+$ and $x_0, x_1, \ldots, x_{k-1}, x_k$ with $x_0 = x$ and $x_k = -\infty$ be its modified continued fraction expansion. If $x_j = p_j/q_j$ with $\gcd(p_j, q_j) = 1$ and $q_j \geqslant 0$, then we have $p_{j-1}q_j - p_j q_{j-1} = 1$ for $j = 1, \ldots, k$ and $q_0 > q_1 > \cdots > q_{k-1} > q_k = 0$.*

Proof. Suppose that $x_{j-1} = p_{j-1}/q_{j-1} = [b_0, \ldots, b_m]$ with $b_m > 1$. If m is odd we have $x_j = p_j/q_j = [b_0, \ldots, b_{m-1}]$ and the relation $p_{j-1}q_j - p_j q_{j-1} = 1$ follows from (2·5) applied to the continued fraction $[b_0, \ldots, b_m]$, whereas $q_{j-1} > q_j$ is a consequence of the inequalities in (2·3) for $[b_0, \ldots, b_m]$.

In the case where m is even the same calculation leading to (2·7) can be used to derive $p_{j-1}q_j - p_j q_{j-1} = 1$ from the recursion relations for the continued fraction $[b_0, \ldots, b_m]$. Here $q_{j-1} > q_j$ follows also from the recursion relations for the continued fraction $[b_0, \ldots, b_m]$.

Definition 2·3. A sequence $x_0, x_1, \ldots, x_{k-1}, x_k$ of rational numbers is called an *admissible sequence of length* $k + 1$ if the following property holds: if $x_j = p_j/q_j$, where $\gcd(p_j, q_j) = 1$ and $q_j \geqslant 0$, then

$$\det \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} = 1 \quad \forall j = 1, 2, \ldots, k. \tag{2·8}$$

Let x be a positive rational number. A *partition* P of x is an admissible sequence $x_0, x_1, \ldots, x_{k-1}, x_k$ with $x_0 = x$ and $x_k = -\infty$. The number $k + 1$ is called the *length* of the partition. We use the convention $-\infty = -1/0, 0 = 0/1$. A partition P of x is called a *minimal partition* if

$$q_0 > q_1 > \cdots > q_{k-1} > q_k = 0. \tag{2·9}$$

Remark 2·4. From (2·8) it follows that $p_{j-1}q_j > p_j q_{j-1}$ which implies $x_{j-1} > x_j$ for all $j = 1, 2, \ldots, k$. Moreover, (2·8) shows that the equation $p_{j-1}q_j \equiv 1 \mod q_{j-1}$ has a unique solution q_j with $0 \leqslant q_j < q_{j-1}$. Therefore each $x \in \mathbb{Q}^+$ has a unique minimal partition, which we denote by P_x. According to Proposition 2·2 the modified continued fraction expansion of $x \in \mathbb{Q}^+$ satisfies (2·8) and (2·9). Therefore it agrees with the minimal partition P_x. We will show in Proposition 2·6 that there is indeed no partition whose length is less than the length of the minimal partition which justifies the name minimal partition.

Throughout this paper we will use the notations introduced in Definition 2·3.

Remark 2·5. Let $x = x_0, x_1, \ldots, x_{k-1}, x_k$ be a partition of $x \in \mathbb{Q}^+$ and $x_j = p_j/q_j$ with $\gcd(p_j, q_j) = 1$ and $q_j \geqslant 0$.

(i) The equation $p_{k-1}q_k - p_kq_{k-1} = 1$ implies that $-p_k = 1 = q_{k-1}$. If the partition is minimal, Remark 2·4 and the construction of the modified continued fraction expansion of x show, that in addition we have $p_{k-1} = 0$.

(ii) If $q_{j-1} = q_j$ for some $j \in \{1, \ldots, k-1\}$, then (2·8) shows that $q_{j-1} = q_j = 1$, i.e., $x_j = p_j = p_{j-1} - 1 = x_{j-1} - 1$.

For a partition P of x of length $k + 1$ given by $x_0, x_1, \ldots, x_{k-1}, x_k$ and any index $1 \leqslant l \leqslant k-1$, a simple calculation shows that for $x_j = p_j/q_j$ with $\gcd(p_j, q_j) = 1$ for $j = 0, 1, \ldots, k$ the sequence

$$x_0, \ldots, x_{l-1}, \frac{p_{l-1} + p_l}{q_{l-1} + q_l}, x_l, \ldots, x_{k-1}, x_k \tag{2·10}$$

defines a new longer partition $P(l)$ of x. We call it a *Farey extension* of the partition P. One can also introduce the inverse of this construction: if a partition P contains a triple of the type $p_{l-1}/q_{l-1}, (p_{l-1} + p_l)/(q_{l-1} + q_l), p_l/q_l$, then one can delete $(p_{l-1} + p_l)/(q_{l-1} + q_l)$ and obtains in this way a shorter partition $\check{P}(l)$ of x called a *Farey reduction* of P.

Proposition 2·6. *Every partition P of a rational number $x \in \mathbb{Q}^+$ can be obtained from the minimal partition P_x of x by a finite number of Farey extensions $P(l)$. The minimal partition P_x can be derived from any partition P by a finite number of Farey reductions $\check{P}(l)$.*

Proof. Given a partition $x_0, x_1, \ldots, x_{k-1}, x_k$ of x with $x_j = p_j/q_j$ and $\gcd(p_j, q_j) = 1$ it is enough to prove that if the sequence $(q_j)_{j=0,\ldots,k}$ is not decreasing, then there exists a number $l \in \{1, \ldots, k-1\}$ such that

$$\frac{p_l}{q_l} = \frac{p_{l+1} + p_{l-1}}{q_{l+1} + q_{l-1}}.$$

Since $q_k = 0$ there exists for $(q_j)_{j=0,\ldots,k}$ not strictly decreasing an index $l \in \{1, \ldots, k-1\}$ such that

$$q_l > q_{l+1} \quad \text{but} \quad q_l \geqslant q_{l-1}.$$

If $q_l > q_{l-1}$, then the triple x_{l-1}, x_l, x_{l+1} must be of the form $\frac{p_{l-1}}{q_{l-1}}, \frac{e + mp_{l-1}}{f + mq_{l-1}}, \frac{e + (m-1)p_{l-1}}{f + (m-1)q_{l-1}}$, where $m \in \mathbb{N}$ and $\frac{e}{f}$ is the unique rational number such that $p_{l-1}f - q_{l-1}e = 1$ and $0 \leqslant e < q_{l-1}$.

If $q_l = q_{l-1}$, then Remark 2·5 shows that $q_{l-1} = q_l = 1$ and x_{l-1}, x_l, x_{l+1} is of the form $\frac{p_{l-1}}{1}, \frac{p_{l-1} - 1}{1}, \frac{p_{l+1}}{q_{l+1}}$. But then (2·8) shows that $p_{l+1} = (p_{l-1} - 1)q_{l+1} - 1$ so that

$$\frac{p_{l-1} + p_{l+1}}{q_{l-1} + q_{l+1}} = p_{l-1} - 1,$$

which implies the claim also in this case.

Lemma 2·7. *Let $P_x = (x_0, x_1, \ldots, x_k)$ be the minimal partition of $x \in \mathbb{Q}^+$. If $x_j = \frac{p'_j}{q'_j}$ with $\gcd(p'_j, q'_j) = 1$ and $q'_j \geqslant 0$, then we have*

(i) $x < (p'_{j-1} - p'_j)/(q'_{j-1} - q'_j)$ *for* $j = 1, 2, \ldots, k$.

(ii) $\lceil (xq'_{j+1} - p'_{j+1})/(xq'_j - p'_j) \rceil = p'_{j-1}q'_{j+1} - p'_{j+1}q'_{j-1}$ *for* $j = 1, 2, \ldots, k-1$.

Proof. (i) Let $x_0 = [a_0, \ldots, a_N]$ be the continued fraction expansion of x with $a_N > 1$. If p_n and q_n are the corresponding enumerators and denominators defined by (2·1) and (2·2), then Remark 2·4 shows that the sequence $\cdots > \frac{p'_{j-1}}{q'_{j-1}} > \frac{p'_j}{q'_j} > \frac{p'_{j+1}}{q'_{j+1}} > \cdots$ is the same as (2·6) which can also be rewritten as

$$\cdots < \frac{p_{n-2}}{q_{n-2}} = \frac{p_n - a_n p_{n-1}}{q_n - a_n q_{n-1}} < \frac{p_n - (a_n - 1)p_{n-1}}{q_n - (a_n - 1)q_{n-1}} < \cdots < \frac{p_n - p_{n-1}}{q_n - q_{n-1}} < \frac{p_n}{q_n} < \cdots, \tag{2·11}$$

where n is even. For two consecutive elements

$$\frac{p'_j}{q'_j} = \frac{(k-1)p_{n-1} + p_{n-2}}{(k-1)q_{n-1} + q_{n-2}} \quad \text{and} \quad \frac{p'_{j-1}}{q'_{j-1}} = \frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}}$$

in (2·11) we have

$$\frac{p'_{j-1} - p'_j}{q'_{j-1} - q'_j} = \frac{(kp_{n-1} + p_{n-2}) - ((k-1)p_{n-1} + p_{n-2})}{(kq_{n-1} + q_{n-2}) - ((k-1)q_{n-1} + q_{n-2})} = \frac{p_{n-1}}{q_{n-1}}$$

and since n is even (2·4) shows that this is larger than x.

(ii) There are two possible forms for three consecutive elements in the sequence (2·11). The first is

$$\frac{p'_{j-1}}{q'_{j-1}} = \frac{p_n - (k-1)p_{n-1}}{q_n - (k-1)q_{n-1}}, \quad \frac{p'_j}{q'_j} = \frac{p_n - kp_{n-1}}{q_n - kq_{n-1}}, \quad \frac{p'_{j+1}}{q'_{j+1}} = \frac{p_n - (k+1)p_{n-1}}{q_n - (k+1)q_{n-1}}, \tag{2·12}$$

where $k = 1, 2, \ldots, a_n - 1$. Then, using (2·5) and n even, we obtain

$$\begin{aligned} &p'_{j-1}q'_{j+1} - p'_{j+1}q'_{j-1} \\ &\quad = (p_n - (k-1)p_{n-1})(q_n - (k+1)q_{n-1}) - (p_n - (k+1)p_{n-1})(q_n - (k-1)q_{n-1}) \\ &\quad = 2. \end{aligned}$$

On the other hand, using $k \geqslant 1$ and $p_{n-1} - xq_{n-1}, xq_n - p_n > 0$ (again recall that n is even), we calculate

$$\begin{aligned} \left\lceil \frac{(xq'_{j+1} - p'_{j+1})}{(xq'_j - p'_j)} \right\rceil &= \left\lceil \frac{(x(q_n - (k+1)q_{n-1}) - (p_n - (k+1)p_{n-1}))}{(x(q_n - kq_{n-1}) - (p_n - kp_{n-1}))} \right\rceil \\ &= 1 + \left\lceil \frac{(p_{n-1} - xq_{n-1})}{(k(p_{n-1} - xq_{n-1}) + xq_n - p_n)} \right\rceil \\ &= 2. \end{aligned}$$

Thus (ii) is proved for triples of the form (2·12).

The second type of triples appearing in (2·11) is

$$\frac{p'_{j-1}}{q'_{j-1}} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \quad \frac{p'_j}{q'_j} = \frac{p_n}{q_n}, \quad \frac{p'_{j+1}}{q'_{j+1}} = \frac{p_n - p_{n+1}}{q_n - q_{n+1}} \tag{2·13}$$

with even n. This time we have

$$\begin{aligned} p'_{j-1}q'_{j+1} - p'_{j+1}q'_{j-1} &= (p_n + p_{n+1})(q_n - q_{n-1}) - (p_n - p_{n-1})(q_n + q_{n+1}) \\ &= a_{n+1} + 2 \end{aligned}$$

and

$$\left\lceil \frac{(xq'_{j+1} - p'_{j+1})}{(xq'_j - p'_j)} \right\rceil = \left\lceil \frac{(x(q_n - q_{n-1}) - (p_n - p_{n-1}))}{(xq_n - p_n)} \right\rceil = 1 + \left\lceil \frac{(p_{n-1} - xq_{n-1})}{(xq_n - p_n)} \right\rceil.$$

But an easy calculation again using (2·5) shows that $\lceil \frac{p_{n-1} - xq_{n-1}}{xq_n - p_n} \rceil = a_{n+1} + 1$ if and only if $\frac{p_n + p_{n+1}}{q_n + q_{n+1}} \leqslant x < \frac{p_{n+1}}{q_{n+1}}$, which, according to (2·6), is indeed the case.

Definition 2·8. Consider an admissible sequence $P = (x_0, \ldots, x_k)$ of $x_0 = x \in \mathbb{Q}^+$ with $x_j = p_j/q_j$ such that $\gcd(p_j, q_j) = 1$ and $q_j \geqslant 0$. To this partition we attach the following element $m(P)$ of $\mathbb{Z}[\mathcal{R}_1] = \mathbb{Z}[\mathrm{SL}(2, \mathbb{Z})]$

$$m(P) = \begin{pmatrix} q_0 & -p_0 \\ q_1 & -p_1 \end{pmatrix} + \cdots + \begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l & -p_l \end{pmatrix} + \begin{pmatrix} q_l & -p_l \\ q_{l+1} & -p_{l+1} \end{pmatrix} + \cdots + \begin{pmatrix} q_{k-1} & -p_{k-1} \\ q_k & -p_k \end{pmatrix}. \tag{2·14}$$

Given two admissible sequences $P_1 = (x_0, x_1, \ldots, x_k)$ and $P_2 = (y_0, y_1, \ldots, y_l)$ with $x_k = y_0$ we can define the *join*

$$P_1 \vee P_2 = (x_0, x_1, \ldots, x_k, y_1, \ldots, y_l) \tag{2·15}$$

of P_1 and P_2, which is again admissible. Note that in this case we have

$$m(P_1 \vee P_2) = m(P_1) + m(P_2). \tag{2·16}$$

$\mathrm{GL}(2, \mathbb{Z})$ acts on rational numbers from the left in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax + b}{cx + d}.$$

For the next lemma we will need the corresponding right action:

$$x \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} x = \frac{dx - b}{-cx + a}$$

LEMMA 2·9. *Let* $P = (x_0, x_1, \ldots, x_k)$ *be an admissible sequence and* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$ *with*

$$\frac{a}{c} \geqslant x_i, \; i = 0, 1, 2, \ldots, k \tag{2·17}$$

(*which for* $c = 0$ *simply means* $a > 0$). *Then*

$$P \cdot A := \begin{cases} (x_0 A, x_1 A, \ldots, x_{k-1} A, x_k A) & \textit{for } \det A = 1 \\ (x_k A, x_{k-1} A, \ldots, x_1 A, x_0 A) & \textit{for } \det A = -1 \end{cases}$$

defines an admissible sequence with the property

$$m(P)A = \begin{cases} m(P \cdot A) & \textit{for } \det A = 1 \\ Mm(P \cdot A) & \textit{for } \det A = -1, \end{cases}$$

where $(m(P), A) \mapsto m(P)A$ *is the multiplication in* $\mathcal{R}$.

Proof. Condition (2·17) implies that for $x_j = \frac{p_j}{q_j}$ with $\gcd(p_j, q_j) = 1$ and $q_j \geqslant 0$, the number $x_j A = \frac{dp_j - bq_j}{aq_j - cp_j}$ is rational and $\gcd(dp_j - bq_j, aq_j - cp_j) = 1$, since $(r, s)\binom{p}{q} = 1$ implies $((r, s)A)(A^{-1}\binom{p}{q}) = 1$. Moreover, for $\det A = 1$ the matrix

$$\begin{pmatrix} aq_{j-1} - cp_{j-1} & -dp_{j-1} + bq_{j-1} \\ aq_j - cp_j & -dp_j + bq_j \end{pmatrix} = \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2·18}$$

has determinant 1, which implies that $P \cdot A$ is indeed admissible. The equality

$m(P)A = m(P \cdot A)$ is immediate from (2·18). The case $\det A = -1$ can be treated similarly.

3. *The slash operator for complex weight s*

Let $\mathcal{F}$ be the set of functions ϕ holomorphic in the domain $\mathbb{C} \setminus (-\infty, r]$ for some $r = r_\phi$ which we call a *branching point* of ϕ. Note that this does not rule out that ϕ extends to the point r as a holomorphic function. In $\mathcal{F}$ we have the usual addition and multiplication of functions. If $\phi_1, \phi_2 \in \mathcal{F}$, then one can find $r_{\phi_1\phi_2}, r_{\phi_1+\phi_2}$ such that $r_{\phi_1\phi_2}, r_{\phi_1+\phi_2} \leqslant \max\{r_{\phi_1}, r_{\phi_2}\}$. We fix the branch of $\log z$ in $\mathbb{C}\setminus(-\infty, 0]$ which coincides with the ordinary logarithm on $(0,\infty)$ and set $z^s := e^{s \log z}$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $s \in \mathbb{C}$. For each matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{G}$ with

$$\mathcal{G} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_*(2, \mathbb{Z}) \mid (c > 0 \text{ or } (c = 0 \;\&\; a, d > 0)) \right\}$$

one has $(cz + d)^s \in \mathcal{F}$. If $c = 0$ and $\phi \in \mathcal{F}$, then also $\phi(\frac{az+b}{d}) \in \mathcal{F}$. Consider the subset $\mathcal{DS}$ of $\mathcal{F} \times \mathcal{G}$ consisting of those pairs $(\phi, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ such that there exists a branching point r_ϕ for ϕ with

$$a - cr_\phi > 0 \text{ or } (a - cr_\phi = 0 \text{ and } dr_\phi - b < 0) \tag{3·1}$$

If $r_\phi = 0$ this condition reads

$$a > 0 \text{ or } (a = 0 \text{ and } b > 0).$$

Proposition 3·1. *Fix* $s \in \mathbb{C}$. *Then the formula*

$$(\phi \mid_s R)(z) = |\det R|^s (cz + d)^{-2s} \phi\left(\frac{az + b}{cz + d}\right) \tag{3·2}$$

defines a map

$$\mathcal{DS} \longrightarrow \mathcal{F}$$
$$(\phi, R) \longmapsto \phi \mid_s R.$$

If r_ϕ *is a branching point for* ϕ *satisfying* (3·1) *for* $R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{G}$, *then*

$$\max\left\{\frac{dr_\phi - b}{a - cr_\phi}, -\frac{d}{c}\right\}$$

is a branching point for $\phi \mid_s R$, *where we interprete* $-d/c$ *as* $-\infty$ *if* $c = 0$.

Proof. Suppose that $z > -d/c$. Then

$$\frac{az + b}{cz + d} > r_\phi \quad \Longleftrightarrow \quad (a - cr_\phi)z > dr_\phi - b$$

The condition (3·1) implies that $z > \frac{dr_\phi - b}{a - cr_\phi}$. Now the claim is immediate.

Remark 3·2. The *slash-operation* from Proposition 3·1 can be extended by linearity to the subset $\mathcal{DS}_\mathbb{Z}$ of $\mathcal{F} \times \mathbb{Z}[\mathcal{G}]$ consisting of those pairs $(\phi, \sum_{j=1}^m n_j R_m)$ for which all $(\phi, R_j) \in \mathcal{DS}$. In fact, suppose that (3·1) is satisfied for (ϕ, R_j) with branching points $r_{\phi,j}$ for ϕ, then (3·1) is satisfied for all (ϕ, R_j) with branching points $\min_j r_{\phi,j}$.

PROPOSITION 3·3. *Suppose that* $R_1, R_2, R_1R_2 \in \mathcal{G}$ *and* $(\phi, R_1), (\phi\,|_s\, R_1, R_2)$, $(\phi, R_1R_2) \in \mathcal{DS}$. *Then for each* $s \in \mathbb{C}$ *we have*

$$(\phi\;|_s\; R_1)\;|_s\; R_2 = \phi\;|_s\;(R_1R_2).$$

Proof. We argue by analytic continuation. Note first that for $R_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ we have $R_1R_2 = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$ and since $R_1, R_2, R_1R_2 \in \mathcal{G}$ the functions

$$((c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2))^{-2s}$$

and

$$(c_2z + d_2)^{-2s}\left(c_1\left(\tfrac{a_2z + b_2}{c_2z + d_2}\right) + d_1\right)^{-2s} = (c_2z + d_2)^{-2s}\left(\tfrac{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}{c_2z + d_2}\right)^{-2s}$$

are holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and agree on $(0, \infty)$, hence agree everywhere. But then

$$\begin{aligned}
&((\phi\;|_s\; R_1)\;|_s\; R_2)(z)\\
&= |\det R_2|^s (c_2z + d_2)^{-2s} (\phi\;|_s\; R_1)\left(\tfrac{a_2z + b_2}{c_2z + d_2}\right)\\
&= |\det R_2|^s (c_2z + d_2)^{-2s} |\det R_1|^s \left(c_1\left(\tfrac{a_2z + b_2}{c_2z + d_2}\right) + d_1\right)^{-2s} \phi\left(\frac{a_1\left(\frac{a_2z + b_2}{c_2z + d_2}\right) + b_1}{c_1\left(\frac{a_2z + b_2}{c_2z + d_2}\right) + d_1}\right)\\
&= |\det R_1R_2|^s (c_2z + d_2)^{-2s} \left(c_1\left(\tfrac{a_2z + b_2}{c_2z + d_2}\right) + d_1\right)^{-2s} \phi\left(\tfrac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}\right)\\
&= \big(\phi\;|_s\;(R_1R_2)\big)(z)
\end{aligned}$$

proves the proposition.

Remark 3·4. Set

$$\mathcal{G}^+ := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} \mid b, d \geqslant 0 \text{ and } (a > 0 \text{ or } (a = 0 \text{ and } b > 0)) \right\}$$

and

$$\mathcal{F}_0 := \{\phi \in \mathcal{F} \mid 0 \text{ is a branching point of } \phi\}.$$

Then $\mathcal{G}^+$ is a multiplicative subsemigroup of $\mathrm{Mat}_*(2, \mathbb{Z})$ and we have $\mathcal{F}_0 \times \mathcal{G}^+ \subseteq \mathcal{DS}$. Moreover the slash-operation $|_s$ induces a semigroup action $\mathcal{F}_0 \times \mathcal{G}^+ \to \mathcal{F}_0$. In fact, given $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}^+$ and $\phi \in \mathcal{F}_0$, Proposition 3·1 shows that $0 \geqslant \max\{-\frac{b}{a}, -\frac{d}{c}\}$ is a branching point for $\phi\;|_s\; R$. Then Proposition 3·3 implies the identity $(\phi\;|_s\; R_1)\;|_s\; R_2 = \phi\;|_s\;(R_1R_2)$ for all $R_1, R_2 \in \mathcal{G}^+$. Of course we can extend the action to $\mathbb{Z}[\mathcal{G}] \subseteq \mathcal{R}$ by linearity. Note, finally, that I, T, TM, MTM and M are contained in $\mathcal{G}^+$ and for $R \in \mathcal{G}^+, \phi \in \mathcal{F}_0$ the following equality is well defined

$$\phi\;|_s\;(I - T - \lambda TM)R = (\phi\;|_s\;(I - T - \lambda TM))\;|_s\; R.$$

4. *Transfer operators for* $\Gamma_0(n)$ *and* $\overline{\Gamma}_0(n)$

Let W be a μ-dimensional complex vector space and $A, B \in \mathrm{Aut}_{\mathbb{C}}(W)$. We assume the isomorphisms $A^n \in \mathrm{Aut}_{\mathbb{C}}(W)$ to be uniformly bounded in $n \in \mathbb{N}$ w.r.t. one and hence any norm on $\mathrm{Aut}_{\mathbb{C}}(W)$. Consider the Banach space $\mathcal{B}(D)$ of holomorphic functions in the disc $D = \{z \in \mathbb{C} :\; |z - 1| < \frac{3}{2}\}$ which are continuous on $\overline{D}$ with the supnorm. Then the operator $\mathcal{L}_s : \mathcal{B}(D) \otimes W \to \mathcal{B}(D) \otimes W$ with

$$\mathcal{L}_s f(z) = \sum_{m=1}^{\infty} (z+m)^{-2s} A^{m-1} B f\left(\frac{1}{z+m}\right) \tag{4·1}$$

is a nuclear operator for $\Re(2s) > 1$ in this Banach space and $\mathcal{L}_s$ extends to a meromorphic family of nuclear operators in the whole s-plane with possible poles of order one at the points $s = (1-k)/2$ with $k \in \mathbb{N}_0$. The proof follows the same line of argument as in [**2**]. In fact, using the kth Taylor polynomial of f at 0 we have:

$$\mathcal{L}_s f(z) = \mathcal{L}_{s+\frac{k+1}{2}} \tilde{f}(z) + \sum_{l=0}^{k} \zeta_{A,B}(l+2s, z+1) \frac{f^l(0)}{l!}, \tag{4·2}$$

where

$$\tilde{f}(z) := z^{-k-1}\left(f(z) - \sum_{l=0}^{k} \frac{f^l(0)}{l!} z^l \right)$$

and

$$\zeta_{A,B}(a,b) = \sum_{n=0}^{\infty} \frac{A^{n-1}B}{(b+n)^a}$$

is a kind of Hurwitz zeta function. The first term on the right-hand side in expression (4·2) is holomorphic in $\Re(s) > \frac{1-(k+1)}{2}$ and the second term has poles of order one at $\frac{1-l}{2}$, $l = 0, 1, \ldots, k$ (the proof of this last statement is as for the usual Hurwitz zeta function, [**6**], Chapter XIV). This proves our assertion.

By a direct calculation we have

$$\mathcal{L}_s f(z) - (A\mathcal{L}_s) f(z+1) = (z+1)^{-2s} B f\left(\frac{1}{z+1}\right).$$

Therefore any eigenvector f of $\mathcal{L}_s$ with eigenvalue λ satisfies the following three term functional equation:

$$\lambda \left(f(z) - A f(z+1) \right) = (z+1)^{-2s} B f\left(\frac{1}{z+1}\right).$$

It is convenient to make the change of variable $z \mapsto z-1$ and introduce the new function $\Phi(z) = f(z-1)$. For $\lambda \neq 0$ the above equation then takes the form:

$$\Phi(z) - A\Phi(z+1) = \lambda^{-1} z^{-2s} B\Phi\left(1 + \frac{1}{z}\right). \tag{4·3}$$

Since f is defined in the disk D, Φ is defined in the shifted disk $\{z : |z-2| \leqslant \frac{3}{2}\}$. As in [**3**] one shows that any eigenfunction f of the operator $\mathcal{L}_s$ can be extended holomorphically to the entire complex plane $\mathbb{C}$ cut along the line $(-\infty, -1]$. Hence the corresponding function $\Phi(z)$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. In what follows we are interested in solutions of (4·3) in the domain $\mathbb{C} \setminus (-\infty, 0]$ for the eigenvalues $\lambda = \pm 1$. In the scalar case $\mu = 1$ with $A, B = I$ equation (4·3) was introduced by J. Lewis in [**7**]. The derivation of his equation via the transfer operator appeared independently in [**13**]. There one can also find the conditions under which a holomorphic solution of equation (4·3) determines an eigenfunction of the transfer operator with eigenvalue

λ. An interesting property of the solutions of equation (4·3) is described by the following proposition:

PROPOSITION 4·1. *If* $\lambda = \pm 1$ *and* $(BA^{-1})^2 = I$*, then any solution of equation* (4·3) *in* $\mathbb{C} \setminus (-\infty, 0]$ *satisfies*

$$\Phi(z) = \lambda z^{-2s} BA^{-1} \Phi\left(\frac{1}{z}\right). \tag{4·4}$$

Proof. The domain $\mathbb{C} \setminus (-\infty, 0]$ is invariant under $z \mapsto 1/z$. We insert $1/z$ in (4·3), multiply it by $\lambda z^{-2s} BA^{-1}$ and then subtract the result from (4·3). Using the hypotheses we get equality (4·4).

Of special interest for the following is the case $s = 1$: for this let us suppose that A and B are two invertible real matrices with non-negative entries which satisfy

$$A\mathbb{I} = \mathbb{I} = B\mathbb{I}, \tag{4·5}$$

where $\mathbb{I}$ is the μ-dimensional vector with all components equal to 1. This is for instance the case for A and B permutation matrices. Then the vector $\Phi' = \Phi'(z)$ with all entries equal to $1/z$ is obviously a solution of (4·3) with $\lambda = 1$ and $s = 1$. Generalizing the analogous result for the scalar case $\mu = 1$ in section 7·4 of [**15**] one has

PROPOSITION 4·2. Φ' *is up to a constant factor the unique solution of* (4·3) *for* $\lambda = 1$ *and* $s = 1$ *in the Banach space* $\mathcal{B}(D) \otimes W$*. There does not exist any other solution of equation* (4·3) *in this space for the parameter values* $s = 1$ *and* λ *with* $|\lambda| = 1$.

Proof. The proof is a straightforward adaption from [**14**, appendix C], and section 1·2 in [**10**].

Induced representations. Let G be a group and H be a subgroup of finite index $\mu = [G : H]$ of G. For each representation $\chi\colon H \to \mathrm{End}(V)$ we consider the induced representation $\chi_G\colon G \to \mathrm{End}(V_G)$, where

$$V_G := \{f : G \to V \mid f(hg) = \chi(h)f(g) \quad \forall g \in G, h \in H\}$$

and the action of G is given by

$$(\chi_G(g)f)(x) = f(xg) \quad \forall x, g \in G.$$

If $V = \mathbb{C}$ and the initial representation is trivial, the induced representation χ_G is the right regular representation $\rho\colon G \to \mathrm{GL}(\mathbb{C}^{H\backslash G})$. In fact, in this case V_G is the space of complex valued left H-invariant functions on G or, what is the same, complex valued functions on $H\backslash G$, and the action is by right translation in the argument. This also shows that we can view ρ as a homomorphism $G \to \mathrm{GL}(\mathbb{Z}^{H\backslash G})$. Moreover, for each $g \in G$ the operators $\rho(g)^n \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{H\backslash G})$ are uniformly bounded in $n \in \mathbb{N}$.

Remark 4·1. One can identify V_G with V^μ using a set $\{g_1, g_2, \ldots, g_\mu\}$ of representatives for $H\backslash G$, i.e.

$$H\backslash G = \{Hg_1, Hg_2, \ldots, Hg_\mu\}.$$

Then

$$\begin{aligned} V_G &\longrightarrow V^\mu \\ f &\longmapsto (f(g_1), \ldots, f(g_\mu)) \end{aligned}$$

is a linear isomorphism which transports χ_G to the linear G-action on V^μ given by

$$g \cdot (v_1, \ldots, v_\mu) = \left(\chi\left(g_1 g g_{k_1}^{-1}\right) v_{k_1}, \ldots, \chi\left(g_\mu g g_{k_\mu}^{-1}\right) v_{k_\mu}\right),$$

where $k_j \in \{1, \ldots, \mu\}$ is the unique index such that $H g_j g = H g_{k_j}$. To see this one simply calculates

$$(\chi_G(g) f)(g_j) = f(g_j g) = f\left(g_j g g_{k_j}^{-1} g_{k_j}\right) = \chi\left(g_j g g_{k_j}^{-1}\right)\left(f\left(g_{k_j}\right)\right).$$

In the case of the right regular representation the identification $V_G \cong \mathbb{C}^\mu$ yields a matrix realization

$$\rho(g) = \left(\delta\left(g_i g g_j^{-1}\right)\right)_{i,j=1,\ldots,\mu},$$

where $\delta(g) = 1$ if $g \in H$ and $\delta(g) = 0$ otherwise. Note for the following that the matrix $\rho(g)$ is a permutation matrix for all $g \in G$.

In this paper we are primarily interested in the subgroups $\Gamma_0(n) \subseteq \mathrm{PSL}(2, \mathbb{Z})$, respectively their extensions $\overline{\Gamma}_0(n) \subseteq \mathrm{GL}(2, \mathbb{Z})$. The representation χ is in both cases the trivial representation of $\Gamma_0(n)$, respectively $\overline{\Gamma}_0(n)$. The transfer operators for the groups $\Gamma_0(n)$ and $\overline{\Gamma}_0(n)$ have been introduced by Chang and Mayer (see [**2**, **3**]), respectively Manin and Marcolli (see [**10**]). Taking in expression (4·1) for A the matrix $\rho(QT^{\pm 1}Q)$ and for B the matrix $\rho(QT^{\pm 1})$ we get the transfer operators $\mathcal{L}_{s,\pm}$ for $\Gamma_0(n)$ whereas for $A = \rho(T^{-1})$ and $B = \rho(T^{-1}M)$ we have the transfer operator $\mathcal{L}_s$ for the group $\overline{\Gamma}_0(n)$. An easy calculation shows that the operators $\mathcal{L}_{s,+}\mathcal{L}_{s,-}$ and $\mathcal{L}_s^2$ can be conjugated by the matrix $\rho(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$. On the other hand it was shown in [**2**] that the Selberg zeta function $Z_{\Gamma_0(n)}(s)$ for the group $\Gamma_0(n)$ can be expressed in terms of the Fredholm determinant of the operator $\mathcal{L}_{s,+}\mathcal{L}_{s,-}$ as $Z_{\Gamma_0(n)}(s) = \det(1 - \mathcal{L}_{s,+}\mathcal{L}_{s,-})$ and hence also as $Z_{\Gamma_0(n)}(s) = \det(1 - \mathcal{L}_s^2) = \det(1 + \mathcal{L}_s)\det(1 - \mathcal{L}_s)$. This shows that using the operator $\mathcal{L}_s$ the Selberg zeta function for the group $\Gamma_0(n)$ factorizes as in the case of the modular group and hence this transfer operator facilitates also the discussion of the period functions for $\Gamma_0(n)$. In the following we will therefore use this operator. The Lewis equation for $\overline{\Gamma}_0(n)$ derived from the eigenfunction equation for $\mathcal{L}_s$ then has the form

$$\Phi(z) - \rho(T^{-1})\Phi(z+1) - \lambda^{-1} z^{-2s} \rho(T^{-1}M)\Phi\left(1 + \frac{1}{z}\right) = 0. \tag{4·6}$$

For the transfer operators considered above one finds $(BA^{-1})^2 = I$ since $BA^{-1} = \rho(QTQT^{\pm 1}Q)$, respectively $BA^{-1} = \rho(T^{-1}MT)$, and hence the two term equation (4·4) holds. Note that the matrices in both examples are permutation matrices and so also the scalar equations in (4·4) involve only two terms.

5. *The indexing coset space*

In this section we study the fine structure of $I_n = \overline{\Gamma}_0(n)\backslash\mathrm{GL}(2, \mathbb{Z})$ as a right $\mathrm{GL}(2, \mathbb{Z})$-space. To do this we embed $\overline{\Gamma}_0(n)\backslash\mathrm{GL}(2, \mathbb{Z})$ into a natural $\mathrm{GL}(2, \mathbb{Z})$-space with an action by a kind of linear fractional transformations. We start with $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ on which $\mathrm{GL}(2, \mathbb{Z})$ acts from the right via

$$(x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax + cy, bx + dy).$$

We define an equivalence relation $\sim_n$ on $\mathbb{Z}\times\mathbb{Z}$ via

$$(x,y)\sim_n(x',y') \quad :\Longleftrightarrow \quad (\exists k\in\mathbb{Z})\ \gcd(k,n)=1,\ \begin{array}{ll} kx\equiv x' & \mod n \\ ky\equiv y' & \mod n. \end{array}$$

Then the linearity of the action shows that it preserves $\sim_n$ so that the space $[\mathbb{Z}\times\mathbb{Z}]_n := (\mathbb{Z}\times\mathbb{Z})/\sim_n$ of equivalence classes inherits a right $\mathrm{GL}(2,\mathbb{Z})$-action. If, for fixed n, the equivalence class of (x,y) is denoted by $[x:y]_n$, then this action is given by

$$[x:y]_n\begin{pmatrix} a & b \\ c & d\end{pmatrix} = [ax+cy : bx+dy]_n$$

which is of course very reminiscent of linear fractional transformations. Note, however, that even for $n=p$ prime the space $[\mathbb{Z}\times\mathbb{Z}]_p$ is not the projective space $\mathbb{P}^1(\mathbb{Z}_p)$ since we have not excluded the pairs of numbers both divisible by p.

Remark 5·1. The stabilizer of the point $[0:1]_n\in[\mathbb{Z}\times\mathbb{Z}]_n$ is $\overline{\Gamma}_0(n)$ since $(0,1)\sim_n (c,d)$ if and only if $c\equiv 0 \mod n$ and $\gcd(d,n)=1$. Thus the orbit map

$$\begin{aligned} \mathrm{GL}(2,\mathbb{Z}) &\longrightarrow [\mathbb{Z}\times\mathbb{Z}]_n \\ g &\longmapsto [0:1]_n g \end{aligned}$$

factors to the equivariant injection

$$\begin{aligned} \overline{\pi}_n : \overline{\Gamma}_0(n)\backslash \mathrm{GL}(2,\mathbb{Z}) &\longrightarrow [\mathbb{Z}\times\mathbb{Z}]_n \\ \overline{\Gamma}_0(n)\begin{pmatrix} a & b \\ c & d\end{pmatrix} &\longmapsto [c:d]_n. \end{aligned} \tag{5·1}$$

Now we set $I_n := \mathrm{Im}(\overline{\pi}) \subseteq [\mathbb{Z}\times\mathbb{Z}]_n$ and note that I_n is $\mathrm{GL}(2,\mathbb{Z})$-invariant.

PROPOSITION 5·2. $I_n = \{[x:y]_n \mid \gcd(x,y,n)=1\}$.

Proof. "$\supseteq$": if $\gcd(x,y,n)=1$, set $m:=\gcd(x,y)$ and $x':=x/m, y':=y/m$. Then $\gcd(m,n)=\gcd(x',y')=1$ and one can find $a,b\in\mathbb{Z}$ such that $ay'+bx'=1$. Therefore $g:=\left(\begin{smallmatrix} a & -b \\ x' & y'\end{smallmatrix}\right)\in\mathrm{GL}(2,\mathbb{Z})$ and

$$[0:1]_n g = [x':y']_n = [mx':my']_n = [x:y]_n.$$

"$\subseteq$": if $[x:y]_n = [0:1]_n\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = [c:d]_n$, then there exist $k,r,s\in\mathbb{Z}$ such that $\gcd(k,n)=1$ and

$$\begin{aligned} kc - x &= rn \\ kd - y &= sn. \end{aligned}$$

If now $t=\gcd(x,y,n)$, then $t\mid \gcd(c,d)=1$ since $\gcd(t,k)=1$.

LEMMA 5·3. *Given $m,n\in\mathbb{Z}$ and $u,v\in\mathbb{Z}$ such that $c=um+vn=\gcd(m,n)$ one can find $t\in\mathbb{Z}$ such that*

$$\gcd\left(u+\frac{n}{c}t, n\right) = 1.$$

Proof. Let $n=\prod_{j=1}^s p_j^{\alpha_j}$ be the decomposition into prime factors and suppose that they are arranged in such a way that $c=\prod_{j=1}^s p_j^{\beta_j}$ with $\alpha_j=\beta_j$ for $j\leqslant s_1$ and $\alpha_j>\beta_j$ for $j>s_1$. Then $u\frac{m}{c}+v\frac{n}{c}=1$ implies that u cannot contain a prime factor p_j with

$j > s_1$ so that $\gcd(u,n) = \prod_{j=1}^{s_1} p_j^{\gamma_j}$ with $0 \leqslant \gamma_j \leqslant \alpha_j$. We may assume w.l.o.g. that $\gamma_j > 0$ for $j \leqslant s_2$ and $\gamma_j = 0$ for $s_2 < j \leqslant s_1$, i.e.

$$\gcd(u,n) = \prod_{j=1}^{s_2} p_j^{\gamma_j}.$$

Now we pick $t = \prod_{j=s_2+1}^{s_1} p_j$ and comparing which p_j divide respectively u, t, and $\frac{n}{c}$, we see that no p_j divides $u + \frac{n}{c}t$.

Proposition 5·4. *Each element of* I_n *can be written as* $[c:d]_n$, $c \geqslant 1$, $c \mid n$. *Here* c *is determined uniquely, whereas* d *is determined only up to an integer multiple of* $\frac{n}{c}$. *It is possible to choose* $d = kd'$ *with* $d' \geqslant 1, d' \mid n$, $1 \leqslant k < n$ *and* $\gcd(c,d') = 1 = \gcd(k,n)$.

Proof. For $[x:y]_n \in I_n$ set $c = \gcd(x,n)$ and choose $u, v \in \mathbb{Z}$ such that $ux + vn = c$. Using Lemma 5·3 we can find $t \in \mathbb{Z}$ such that $\gcd(u + \frac{n}{c}t, n) = 1$. Set $\nu := u + \frac{n}{c}t$. Then we have

$$\nu x = c + n\left(t\frac{x}{c} - v\right) \equiv c \mod n$$

so that with $d := \nu y$ we obtain $[x:y]_n = [c:d]_n$. Now we set $d' := \gcd(n,d)$ and use Lemma 5·3 in order to find a ν' with $\gcd(\nu',n) = 1$ such that $\nu' d \equiv d' \mod n$. Choosing $k \in \{1,\ldots,n-1\}$ such that $k\nu' \equiv 1 \mod n$ we have $d \equiv kd' \mod n$ and find $[x:y]_n = [c:kd']_n$. This proves the existence part of the proposition since $\gcd(c,d') = \gcd(x,y,n) = 1$.

To prove uniqueness suppose that $[c:d]_n = [c':d']_n$. Then we have $c = lc' + rn, d = ld' + sn$ for some $r, s, l \in \mathbb{Z}$ with $\gcd(l,n) = 1$. If now $c, c' \geqslant 1$, $c, c' \mid n$ the first equality implies that $c = c'$ and $l = -r\frac{n}{c} + 1$. Inserting this l into the second equality we obtain the uniqueness of d up to $\frac{n}{c}\mathbb{Z}$.

Unfortunately the parametrization of the elements of I_n by $[c:d]_n$ with $c \geqslant 1$ and $c \mid n$ is not unique as shown in Proposition 5·4. To achieve an unique parametrization we proceed as follows:

Definition 5·5. For fixed $n \in \mathbb{N}$ and $c \in \{1,\ldots,n\}$ with $c \mid n$ choose $b \in \{0,\ldots,\frac{n}{c} - 1\}$. We call the pair (c,b) *n-admissible* if there exists $k \in \{0,\ldots,c-1\}$ with $\gcd(c, b + k\frac{n}{c}) = 1$. For such a pair we set

$$d_n(c,b) := \min\left\{c + b + k\frac{n}{c} : k \in \{0,\ldots,c-1\}, \gcd\left(c, b + k\frac{n}{c}\right) = 1\right\}. \qquad (5·2)$$

Remark 5·6.

(a) If $\gcd(c,b) = 1$, then $d_n(c,b) = c + b$.
(b) If (c,b) is n-admissible, then $\gcd(c, b, \frac{n}{c}) = 1$.
(c) the pair (c,b) is n-admissible if and only if $\exists k \in \mathbb{Z}$ with $\gcd(c, b + k\frac{n}{c}) = 1$.

We need the following lemma[1]

Lemma 5·7. *Given the numbers a,b,c* $\in \mathbb{Z}$ *we have* $\gcd(a,b,c) = 1$ *if and only if there exists a* $k \in \mathbb{Z}$ *such that* $\gcd(a, b + kc) = 1$.

[1] We thank Ch. Elsholtz for showing us how to prove this lemma.

Proof. If $\gcd(a, b+kc) = 1$ for some $k \in \mathbb{Z}$, then there exist $x, y \in \mathbb{Z}$ such that $ax + (b+kc)y = 1$ and hence $\gcd(a, b, c) = 1$.
Conversely, if $\gcd(a, b, c) = 1$ define

$$t_{ab} := \gcd(a, b), \quad t_{bc} := \gcd(b, c), \quad t_{ac} := \gcd(a, c),$$

respectively

$$t_a := \frac{a}{t_{ab}t_{ac}}, \quad t_b := \frac{b}{t_{ab}t_{bc}}, \quad t_c := \frac{c}{t_{ac}t_{bc}}.$$

Then $\gcd(t_x, t_y) = \gcd(t_{xy}, t_{xz}) = \gcd(t_x, t_{yz}) = 1$ for all $x \neq y \neq z \in \{a, b, c\}$. Obviously

$$a = t_a t_{ab} t_{ac}, \; b = t_b t_{ab} t_{bc}, \; c = t_c t_{ac} t_{bc}.$$

We claim that for $k = \frac{t_a}{\gcd(t_a, t_{a,b})}$ we have $\gcd(a, b+kc) = 1$. In fact we show that for $d = \gcd(t_a, t_{a,b})$ each factor occuring in $a = kdt_{a,b}t_{a,c}$ occurs in precisely one of the summands of $b + kc$:

$$\begin{aligned} 1 &= \gcd(t_{ac}, b) = \gcd(t_{ac}, b+kc), \\ 1 &= \gcd(t_{ab}, kc) = \gcd(t_{ab}, b+kc), \\ 1 &= \gcd(k, b) = \gcd(k, b+kc), \\ 1 &= \gcd(d, kc) = \gcd(d, b+kc). \end{aligned}$$

Now we get a suitable parametrization of the elements in I_n:

PROPOSITION 5·8. *There is a bijection from the set*

$$P_n = \left\{(c, b) : c \geqslant 1, \; c \mid n, \quad b \in \left\{0, \ldots, \frac{n}{c} - 1\right\}, \; (c, b) \; n - admissible\right\}$$

to the set I_n. The map is given by

$$(c, b) \longmapsto [c : d_n(c, b)]_n$$

with $d_n(c, b)$ from Definition 5·5.

Proof. We show first that the above map is surjective. For any $[x : y]_n \in I_n$ by Proposition 5·4 there exist an unique $c \geqslant 1$, $c \mid n$ and a d' with $[x : y]_n = [c : d']_n$. Define $b \in \{0, \ldots, \frac{n}{c} - 1\}$ through $d' \equiv (b+c) \mod \frac{n}{c}$. We claim (c, b) is n-admissible. Indeed, from Proposition 5·2 we see $\gcd(c, d', n) = 1$. Assume $\lambda = \gcd(c, b, \frac{n}{c}) > 1$. But $\lambda \mid \gcd(d', n)$ and hence $\lambda \mid \gcd(c, d', n)$. Hence by Lemma 5·7 there exists $k \in \mathbb{Z}$ with $\gcd(c, b + k\frac{n}{c}) = 1$.
Note that $d' \equiv (b+c) \mod \frac{n}{c}$ and $d_n(c, b) \equiv (b+c) \mod \frac{n}{c}$ imply $d' = d_n(c, b) + l\frac{n}{c}$. Choose $r, s, t \in \mathbb{Z}$ with $rd' - sc - tn = l$. An easy calculation then gives $d_n(c, b) = d' - l\frac{n}{c} = (1 - \frac{n}{c}r)d' + (s + t\frac{n}{c})n$ and trivially $c = (1 - \frac{n}{c}r)c + rn$. We claim $\gcd((1 - \frac{n}{c}r), n) = 1$. Obviously $\gcd((1 - \frac{n}{c}r), \frac{n}{c}) = 1$. Assume then $\gcd((1 - \frac{n}{c}r), n) = m > 1$. Then $\gcd((1 - \frac{n}{c}r), c) = m$. Since $d_n(c, b) = d' - l\frac{n}{c} = (1 - \frac{n}{c}r)d' + (s + t\frac{n}{c})n$ the number m divides also $d_n(c, b)$ and hence $\gcd(d_n(c, b), c) > 1$ in contradiction to the definition of $d_n(c, b)$. Hence $[c : d']_n = [c, d_n(c, b)]_n$.

To show injectivity of the map $(c, b) \mapsto [c : d_n(c, b)]_n$ let us assume (c', b') maps to $[c' : d_n(c', b')]_n$ and $[c : d_n(c, b)]_n = [c' : d_n(c', b')]_n$. Since $c, c' \geqslant 1$ and $c, c' \mid n$, Proposition 5·4 shows $c = c'$. But $b \equiv (d_n(c, b) - c) \mod \frac{n}{c}$ and $b' \equiv (d_n(c', b') - c)$

mod $\frac{n}{c}$. By Proposition 5·4 we know that $d_n(c,b) \equiv d_n(c',b') \mod \frac{n}{c}$. Therefore also $b \equiv b' \mod \frac{n}{c}$ and hence $b = b'$ since both $b, b' \in \{0, \ldots, \frac{n}{c} - 1\}$.

The set P_n can be ordered lexicographically by saying $(c,b) < (c',b')$ if and only if $c < c'$ or $c = c'$ and $b < b'$.

Definition 5·9. Proposition 5·8 allows us to identify each element $i \in I_n$ with a pair $(c,b) \in P_n$. Then we set

$$A_i := \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}, \; B_i := \begin{pmatrix} \frac{n}{c} & 0 \\ b & c \end{pmatrix} \tag{5·3}$$

and define the rational number

$$x_i := \frac{b}{\frac{n}{c}}.$$

Remark 5·10. In particular we now have an one to one correspondences between the sets I_n, P_n and the subsets

$$\tilde{X}_n = \left\{ \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix} \in S_n : \; c \mid n, \; 0 \leqslant b < \frac{n}{c}, \; \gcd\left(c, b, \frac{n}{c}\right) = 1 \right\}$$

$$\tilde{Y}_n = \left\{ \begin{pmatrix} c & 0 \\ b & \frac{n}{c} \end{pmatrix} \in S_n : \; c \mid n, \; 0 \leqslant b < c, \; \gcd\left(c, b, \frac{n}{c}\right) = 1 \right\}$$

of the sets X_n and Y_n from the introduction, where

$$S_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_n(2, \mathbb{Z}) : a > c \geqslant 0, \; d > b \geqslant 0, \right\}.$$

Lemma 5·11. *Given a matrix* $\left(\begin{smallmatrix} c & b \\ 0 & \frac{n}{c} \end{smallmatrix}\right)$ *with relatively prime* $c, b \in \mathbb{Z}$ *such that* $1 \leqslant c$, $c \mid n$ *and* $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, *there is a unique integral matrix of the form* $\left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right) \in X_n$ *such that*

$$\begin{pmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}^{-1}$$

is integer-valued and hence in $\mathrm{SL}(2,\mathbb{Z})$. *Moreover, we have* $\gcd(c', b', \frac{n}{c'}) = 1$, *i.e.* $\left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right) = A_i \in \tilde{X}_n$ *for a uniquely determined* $i \in I_n$.

Proof. Fix an $i \in I_n$ and suppose that $A_i = \left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right)$. Then

$$\begin{aligned} A_i \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}^{-1} &= \frac{1}{n} \begin{pmatrix} c'x + b'z & c'y + b'w \\ \frac{n}{c'}z & \frac{n}{c'}w \end{pmatrix} \begin{pmatrix} \frac{n}{c} & -b \\ 0 & c \end{pmatrix} \\ &= \frac{1}{n} \begin{pmatrix} (c'x + b'z)\frac{n}{c} & (c'y + b'w)c - (c'x + b'z)b \\ \frac{n}{c'}\frac{n}{c}z & \frac{n}{c'}(wc - zb) \end{pmatrix}. \end{aligned}$$

This matrix is integer valued if and only if the following four conditions are satisfied

(1) $c \mid (c'x + b'z)$
(2) $c' \mid \frac{n}{c}z$
(3) $c' \mid (cw - bz)$
(4) $n \mid \big((c'y + b'w)c - (c'x + b'z)b\big) = c'(cy - bx) + b'(cw - bz)$.

Claim. $\gcd(\frac{n}{c}z, cw-bz) = \gcd(n, cw-bz)$.

To prove this claim, pick a prime number p and let α be the maximal exponent such that $p^\alpha \mid \gcd(n, cw-bz)$. Further let $k \leqslant \alpha$ be defined by $p^{\alpha-k} \mid c$ and $p^k \mid \frac{n}{c}$. Then $\gcd(b,c) = 1$ shows

$$p^{\alpha-k} \mid \gcd(c, cw-bz) = \gcd(c, bz) = \gcd(c, z),$$

so that $p^\alpha \mid z\frac{n}{c}$ which in turn yields $p^\alpha \mid \gcd(z\frac{n}{c}, cw-bz)$. Thus we have $\gcd(n, cw-bz) \mid \gcd(z\frac{n}{c}, cw-bz)$.

Conversely, if $p^\alpha \mid \gcd(z\frac{n}{c}, cw-bz)$, then we can choose k such that $p^{\alpha-k} \mid z$ and $p^k \mid \frac{n}{c}$. But then $\gcd(w,z) = 1$ shows

$$p^{\alpha-k} \mid \gcd(z, cw-bz) = \gcd(z, cw) = \gcd(z, c),$$

which in turn yields $p^\alpha \mid n$ and hence the claim.

Now we prove the existence of an $i \in I_n$ of the desired type. We choose $c' := \gcd(\frac{n}{c}z, cw-bz)$ which by definition satisfies (2) and (3). Note that the claim implies $\gcd(\frac{n}{c'}, \frac{cw-bz}{c'}) = 1$. Therefore we can find $b' \in \{0, \dots, \frac{n}{c'} - 1\}$ such that

$$\frac{n}{c'} \;\Big|\; \left((cy-bx) + b'\frac{cw-bz}{c'}\right)$$

which shows that c', b' satisfy (4). But (4) implies that $c \mid (c'x + b'z)b$ so that in view of $\gcd(c,b) = 1$ we have (1). Note that $\left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right) \in X_n$ and we have proven existence.

To prove the uniqueness part, we assume that

$$\begin{pmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}^{-1}, \quad \begin{pmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}^{-1} \in \mathrm{SL}(2,\mathbb{Z})$$

which implies that $\left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right)\left(\begin{smallmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{smallmatrix}\right)^{-1} \in \mathrm{SL}(2,\mathbb{Z})$. But the calculation

$$\begin{pmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{pmatrix}^{-1} = \frac{1}{n} \begin{pmatrix} n\frac{c'}{c''} & -b''c' + c''b' \\ 0 & \frac{nc''}{c'} \end{pmatrix}$$

shows that this is possible only if $\left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right) = \left(\begin{smallmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{smallmatrix}\right)$. Here we use that $|b' - b''| < \frac{n}{c'}$.

It remains to prove that $\gcd(c', b', \frac{n}{c'}) = 1$. Suppose that $p \in \mathbb{N}$ is prime and divides $\gcd(c', b', \frac{n}{c'})$. Note that from $p \mid c'$ and (3) one derives $p \mid (cw-bz)$, whereas $p \mid \frac{n}{c'}$ and (4) together with $p \mid b'$ imply

$$p \;\Big|\; \left(cy - bx + b'\frac{cw-bz}{c'}\right)$$

so that $p \mid (cy-bx)$. Multiplication by x and z respectively, yields

$$p \mid cwx - bzx$$
$$p \mid cyz - bzx$$

and $xw - yz = 1$ shows $p \mid c$. But then we have $p \mid bz$ and $p \mid bx$ so that in view of $\gcd(x,z) = 1$ we obtain $p \mid b$ which contradicts $\gcd(c,b) = 1$ and therefore proves the lemma.

Let n and m be two positive integers and $\sigma := \sigma_{n,m} : I_{nm} \to I_m$ be the canonical map.

LEMMA 5·12. *For all $i \in I_{nm}$ the matrix $A_i A_{\sigma(i)}^{-1}$ is of the form $\left(\begin{smallmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{smallmatrix}\right)$ with $c'', b'' \in \mathbb{Z}$ such that $1 \leqslant c'' \mid n$ and $\gcd(c'', b'', \frac{n}{c''}) = 1$.*

Proof. Let us write

$$i = [c : d]_{nm}, \ \sigma(i) = [c' : d']_m, \ c \mid nm, c' \mid m.$$

Then $[c : d]_m = [c' : d']_m$ and so $m \mid (cd' - c'd)$. Moreover, by Lemma 5·4 (in fact, see the first line of the proof which says how to find c') we have $c' = \gcd(c, m)$ so that $c' \mid c$ and $\frac{c}{c'} \mid n$. Now

$$A_i A_{\sigma(i)}^{-1} = \begin{pmatrix} c & b \\ 0 & \frac{nm}{c} \end{pmatrix} \begin{pmatrix} c' & b' \\ 0 & \frac{m}{c'} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{c}{c'} & \frac{c'b - cb'}{m} \\ 0 & \frac{c'}{c} n \end{pmatrix} =: \begin{pmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{pmatrix}.$$

Using the notation from Proposition 5·8 we see that there exist $k, k' \in Z$ such that $d = b + c + k\frac{nm}{c}$ and $d = b' + c' + k'\frac{m}{c'}$. Therefore

$$c'b - cb' = c'd - cd' - kn\frac{c'}{c}m + k'm\frac{c}{c'}$$

is divisible by m. Thus it only remains to show $\gcd(c'', b'', \frac{n}{c''}) = 1$. Note first that $c = c'c''$ implies $\gcd(c'', m) = 1$. If now $p \in \mathbb{N}$ is prime and divides $\gcd(c'', b'', \frac{n}{c''})$, then p does not divide m, whence $p \mid c'b - cb'$. But then $p \mid c$ implies first $p \mid c'b$, and then $p \mid b$. On the other hand $p \mid \frac{n}{c}c'$ implies $p \mid \frac{n}{c}$ and hence also $p \mid \frac{nm}{c}$. But we have $\left(\begin{smallmatrix} c & b \\ 0 & \frac{nm}{c} \end{smallmatrix}\right) \in \tilde{X}_{nm}$ so that $\gcd(c, b, \frac{nm}{c}) = 1$. This contradiction proves the claim.

Note that in the above lemma we do not have necessarily $0 \leqslant b'' < \frac{n}{c''}$. Thus $\gcd(c'', b'', \frac{n}{c''}) = 1$ does not yield $\left(\begin{smallmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{smallmatrix}\right) \in \tilde{X}_n$!

Definition 5·13. Fix an $l \in I_{nm}$ and a matrix $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. Then, using Lemma 5·12, we can write

$$A_l A_{\sigma(l)}^{-1} = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}$$

with $\gcd(c, b, \frac{n}{c}) = 1$. Then one can find a $k \in \mathbb{Z}$ with $\gcd(c, b + k\frac{n}{c}) = 1$. Set $\tilde{b} := b + k\frac{n}{c}$. Then one can apply Lemma 5·11 to $\left(\begin{smallmatrix} c & \tilde{b} \\ 0 & \frac{n}{c} \end{smallmatrix}\right)$ and finds a unique $\left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right) \in X_n$ such that

$$\begin{pmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} c & \tilde{b} \\ 0 & \frac{n}{c} \end{pmatrix}^{-1}$$

is integral. Moreover, the same lemma also shows that $\left(\begin{smallmatrix} c' & b' \\ 0 & \frac{n}{c'} \end{smallmatrix}\right) \in \tilde{X}_n$ and hence defines an element $i \in I_n$. We say that i is associated with $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ and the index l.

LEMMA 5·14. *For any $j \in I_n$ the following matrices are in* $\mathrm{GL}(2, \mathbb{Z})$:
(i) $A_j T A_{jT}^{-1}$, (*this is of the form T^s with $s \in \mathbb{Z}$*);
(ii) $A_{jM} TMA_{jT}^{-1}$;
(iii) $A_{jT^{-1}MT^{-1}MT} MTMA_j^{-1}$;
(iv) $A_{jT^{-1}MT} MA_j^{-1}$.

Proof. To simplify the notation we define for $r \in \mathbb{Z}$ and $m \in \mathbb{N}$ the number $(r)_m \in \{0, \ldots, m-1\}$ via

$$(r)_m \equiv r \mod m.$$

Let us write the index $j \in I_n$ in the form

$$j = [c : ic']_n, \tag{5·4}$$

with $c, c' \mid n$ and $\gcd(c, c') = 1 = \gcd(i, n)$. It is enough to prove that the above matrices in the lemma are all integer-valued.

(i) Note first that $A_{jT} = \left(\begin{smallmatrix} c & (c'i)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{smallmatrix}\right)$ and $A_j = \left(\begin{smallmatrix} c & (ic'-c)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{smallmatrix}\right)$ and so

$$A_j T A_{jT}^{-1} = T^s$$

where

$$s := \frac{\left((c'i - c)_{\frac{n}{c}} + c - (c'i)_{\frac{n}{c}}\right)}{\frac{n}{c}} \in \mathbb{Z}. \tag{5·5}$$

(ii) We have $jM = [ic' : c] = [c' : \hat{\imath}c]$, where $\hat{\imath}$ is a natural number such that $i\hat{\imath} \equiv 1 \bmod n$.

$$\begin{aligned}
A_{jM} T M A_{jT}^{-1} &= \begin{pmatrix} c' & (\hat{\imath}c - c')_{\frac{n}{c'}} \\ 0 & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & (c'i)_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}^{-1} \\
&= \frac{1}{n} \begin{pmatrix} (c\hat{\imath} - c')_{\frac{n}{c'}} + c' & c' \\ \frac{n}{c'} & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{n}{c} & -(c'i)_{\frac{n}{c}} \\ 0 & c \end{pmatrix} \\
&= \begin{pmatrix} \frac{\left((c\hat{\imath}-c')_{\frac{n}{c'}} + c'\right)}{c} & \frac{-\left((c\hat{\imath}-c')_{\frac{n}{c'}} + c'\right)(c'i)_{\frac{n}{c}} + cc'}{n} \\ \frac{n}{cc'} & -\frac{(c'i)_{\frac{n}{c}}}{c'} \end{pmatrix}.
\end{aligned}$$

Since c and c' are relative prime and divide n we have $cc' \mid n$ so that $c \mid \frac{n}{c'}$ and $c' \mid \frac{n}{c}$. But then

$$c \mid (c' + (\hat{\imath}c - c')_{\frac{n}{c'}} \quad \text{iff} \quad c \mid (c' + \hat{\imath}c - c')$$

and the latter is evident. Similarly $c' \mid (ic')_{\frac{n}{c}}$ reduces to $c' \mid ic'$ which is clear and

$$n \mid \left(- \left(c' + (\hat{\imath}c - c')_{\frac{n}{c'}}\right)(ic')_{\frac{n}{c}} + cc'\right)$$

reduces to $n \mid (-\hat{\imath}cic' + cc')$ which again is evident. Thus the entries of the above matrix are all integral.

(iii) To prove the third item suppose that $j\left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)^{-1}$ has the form (5·4). Then $j = [c : 2c - ic']$ and $jT^{-1}MT^{-1}MT = j\left(\begin{smallmatrix} 2 & 1 \\ -1 & 0 \end{smallmatrix}\right) = [ic' : c] = [c' : \hat{\imath}c]$. We have

$$\begin{aligned}
A_{jT^{-1}MT^{-1}MT} MTMA_j^{-1} &= \begin{pmatrix} c' & (\hat{\imath}c - c')_{\frac{n}{c'}} \\ 0 & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c & (c - ic')_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}^{-1} \\
&= \frac{1}{n} \begin{pmatrix} c' + (\hat{\imath}c - c')_{\frac{n}{c'}} & (\hat{\imath}c - c')_{\frac{n}{c'}} \\ \frac{n}{c'} & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} \frac{n}{c} & -(c - ic')_{\frac{n}{c}} \\ 0 & c \end{pmatrix} \\
&= \begin{pmatrix} \frac{c' + (\hat{\imath}c - c')_{\frac{n}{c'}}}{c} & \frac{-\left(c' + (\hat{\imath}c - c')_{\frac{n}{c'}}\right)(c - ic')_{\frac{n}{c}} + (\hat{\imath}c - c')_{\frac{n}{c'}} c}{n} \\ \frac{n}{cc'} & \frac{-(c - ic')_{\frac{n}{c}} + c}{c'} \end{pmatrix}.
\end{aligned}$$

With the same reasoning as before one can shows that the above matrix is integer valued.

(iv) Let us finally prove the fourth statement. Let jT^{-1} be of the form (5·4). Then $j = [c : ic' + c]$ and $jT^{-1}MT = j\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = [ic' : ic' + c] = [c' : c' + \hat{\imath}c]$. We have

$$A_{jT^{-1}MT}MA_j^{-1} = \begin{pmatrix} c' & (\hat{\imath}c)_{\frac{n}{c'}} \\ 0 & \frac{n}{c'} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & (ic')_{\frac{n}{c}} \\ 0 & \frac{n}{c} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \frac{(\hat{\imath}c)_{\frac{n}{c'}}}{c} & \frac{-(\hat{\imath}c)_{\frac{n}{c'}}(ic')_{\frac{n}{c}} + cc'}{n} \\ \frac{n}{cc'} & -\frac{(ic')_{\frac{n}{c}}}{c'} \end{pmatrix}.$$

With the same reasoning as before one can shows that the above matrix is integer valued.

Lemma 5·15. *With the notations of Lemma 5·14 we have*

$$x_{jM}\big(A_{jM}TMA_{jT}^{-1}\big) = x_j\big(A_jTA_{jT}^{-1}\big), \; (-\infty)\big(A_{jM}TMA_{jT}^{-1}\big) = x_{jT}.$$

The join $P_{x_{jM}}(A_{jM}TMA_{jT}^{-1}) \bigvee P_{x_j}(A_jTA_{jT}^{-1})$ is well-defined and is a partition of x_{jT}.

Proof. We write j in the canonical form (5·4) and use the following notations

$$X =: A_{jM}TMA_{jT}^{-1}, \; z := x_{jM} = \frac{(c\hat{\imath} - c')_{\frac{n}{c'}}}{\frac{n}{c'}}, \; x := x_j = \frac{(c'i - c)_{\frac{n}{c}}}{\frac{n}{c}}, \; y := x_{jT} = \frac{(c'i)_{\frac{n}{c}}}{\frac{n}{c}}.$$

Suppose that P_z is given by $(z = z_0, \ldots, z_m)$. According to Remark 2·4 we have $z = z_0 > z_1 > \ldots > z_m$. Note that for X and P_z condition (2·17) is satisfied. In fact, the number $\frac{a}{c}$ in (2·17) is

$$\frac{\left(\frac{\left((c\hat{\imath}-c')_{\frac{n}{c'}}+c'\right)}{c}\right)}{\left(\frac{n}{cc'}\right)} = z + \frac{{c'}^2}{n} > z \geqslant z_j \quad \forall j = 0, \ldots, m.$$

Now Lemma 2·9 shows that $P_z \cdot X$ exists and since $\det X = -1$ the first element of $P_z \cdot X$ is given by

$$z_mX = -\infty X = \frac{-\frac{(c'i)_{\frac{n}{c}}}{c'}}{-\frac{n}{cc'}} = \frac{(c'i)_{\frac{n}{c}}}{\frac{n}{c}} = y,$$

whereas the last element of $P_z \cdot X$ is given by

$$z_0X = \frac{(c\hat{\imath} - c')_{\frac{n}{c'}}}{\frac{n}{c'}} \begin{pmatrix} \frac{\left((c\hat{\imath} - c')_{\frac{n}{c'}} + c'\right)}{c} & \frac{-\left((c\hat{\imath} - c')_{\frac{n}{c'}} + c'\right)(c'i)_{\frac{n}{c}} + cc'}{n} \\ \frac{n}{cc'} & -\frac{(c'i)_{\frac{n}{c}}}{c'} \end{pmatrix}$$
$$= \frac{(c'i)_{\frac{n}{c}} - c}{\frac{n}{c}}$$
$$= \frac{(c'i - c)_{\frac{n}{c}}}{\frac{n}{c}} - \frac{\left((c'i - c)_{\frac{n}{c}} + c - (c'i)_{\frac{n}{c}}\right)}{\frac{n}{c}}$$
$$= x - s.$$

Note that for $T^s$the number $\frac{a}{c}$ in (2·17) is $\frac{1}{0} = \infty$. Since $\det T^s = 1$, Lemma 2·9 shows that $P_x \cdot T^s$ exists, starts with $xT^s = x - s$ and ends at $-\infty T^s = -\infty$. Thus the join $(P_z \cdot X) \vee (P_x \cdot T^s)$ exists and is a partition of y.

6. *The operator K*

Proposition 6·1. *The formula*

$$K\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T^{\lceil \frac{d}{b} \rceil} Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c + \lceil \frac{d}{b} \rceil a & -d + \lceil \frac{d}{b} \rceil b \\ a & b \end{pmatrix} \tag{6·1}$$

defines a bijection $K: S_n \setminus Y_n \to S_n \setminus X_n$ *with inverse given by the formula*

$$K^{-1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = MT^{\lceil \frac{a'}{c'} \rceil} QM \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} c' & d' \\ -a' + \lceil \frac{a'}{c'} \rceil c' & -b' + \lceil \frac{a'}{c'} \rceil d' \end{pmatrix}. \tag{6·2}$$

Proof. We denote the right-hand side of (6·1) by $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. The condition $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S_n \setminus Y_n$ implies

$$a > c \geqslant 0, \quad d > b > 0, \quad ad - bc = n.$$

From this it is clear that $c' > 0$ so that $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ is not contained in X_n. To show that it is in S_n we note

$$a' = \left\lceil \frac{d}{b} \right\rceil a - c \geqslant a - c \geqslant a = c' > 0,$$

$$0 \leqslant b' = \left\lceil \frac{d}{b} \right\rceil b - d = \left(\left\lceil \frac{d}{b} \right\rceil - \frac{d}{b} \right) b < b = d'$$

and

$$a'd' - b'c' = \left(-c + \left\lceil \frac{d}{b} \right\rceil a \right) b - \left(-d + \left\lceil \frac{d}{b} \right\rceil b \right) a = ad - bc = n.$$

Thus K is well defined. That

$$MT^{\lceil \frac{a'}{c'} \rceil} QMT^{\lceil \frac{d}{b} \rceil} Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

follows from

$$MT^{\lceil r \rceil} QMT^{\lceil s \rceil} Q = \begin{pmatrix} 1 & 0 \\ r - s & 1 \end{pmatrix}$$

and $\lceil \frac{a'}{c'} \rceil = \lceil \frac{d}{b} \rceil$ which in turn is a consequence of $\frac{a'}{c'} = -\frac{c}{a} + \lceil \frac{d}{b} \rceil$ and $-1 < -\frac{c}{a} \leqslant 0$. Similarly we see that

$$MT^{\lceil \frac{d''}{b''} \rceil} QMT^{\lceil \frac{a'}{c'} \rceil} Q \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

where $\left(\begin{smallmatrix} a'' & b'' \\ c'' & d'' \end{smallmatrix}\right)$ denotes the right-hand side of (6·2). All that remains to be seen is that $\left(\begin{smallmatrix} a'' & b'' \\ c'' & d'' \end{smallmatrix}\right) \in S_n \setminus Y_n$ if $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in S_n \setminus X_n$, but that can be checked similarly as the well-definedness of K.

An operator slightly different from the above operator K was used also by Choie and Zagier in [**4**] and by Mühlenbruch in [**18**] in their derivation of the Hecke operators within the Eichler, Manin and Shimura theory of period polynomials. In the following we will use the operator K to attach to any index $i \in I_n$ a sequence of elements in $\mathcal{R}_n$ which are closely related to the minimal partition of the rational number x_i (cf. Definition 5·9).

Definition 6·2. For $i \in I_n$ we denote by k_i the natural number with the property that $K^j(A_i)$ (cf. Definition 5·9) is well-defined for $j \leqslant k_i$ and $K^{k_i}(A_i) \in Y_n$. We call

$$A_i, K(A_i), \ldots, K^{k_i}(A_i)$$

the *chain* associated with $i \in I_n$.

If $A_i \in X_n \bigcap Y_n$, then clearly A_i forms a chain in itself so that $k_i = 0$ in this case.

LEMMA 6·3. *Let* $i = [c:d]_n \in I_n$ *and* $P_{x_i} = (x_0, x_1, \ldots, x_{k-1}, x_k)$ *be the minimal partition of* $x_0 = x_i = \frac{b}{\frac{n}{c}}$ *(cf. Definition* 5·9 *and Remark* 2·4*). Suppose that* $x_j = \frac{p_j}{q_j}$, $\gcd(p_j, q_j) = 1$, *and* $q_j \geqslant 0$. *Then we have* $k_i = k - 1$ *and*

$$K^j(A_i) = \begin{pmatrix} q_{k-1-j} & -p_{k-1-j} \\ q_{k-j} & -p_{k-j} \end{pmatrix} A_i \quad \forall j = 0, \ldots, k-1.$$

Proof. Recall the definition of $b \in \{0, \ldots, \frac{n}{c} - 1\}$ attached to $i = [c:d]_n$ in Definition 5·9. We assume $c \geqslant 1, c \mid n$ and hence

$$A_i = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix}.$$

We claim

$$\begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_i = \begin{pmatrix} cq_{j-1} & bq_{j-1} - \frac{n}{c}p_{j-1} \\ cq_j & bq_j - \frac{n}{c}p_j \end{pmatrix} \in S_n \quad \forall j = 1, \ldots, k. \tag{6·3}$$

In fact, using Lemma 2·7(i) and minimality of the partition we find

$$\left(bq_j - \frac{n}{c}p_j\right) - \left(bq_{j-1} - \frac{n}{c}p_{j-1}\right) = (q_j - q_{j-1})\left(\frac{b}{\frac{n}{c}} - \frac{p_{j-1} - p_j}{q_{j-1} - q_j}\right) > 0,$$

whereas

$$\frac{b}{\frac{n}{c}} = x_0 \geqslant x_{j-1} = \frac{p_{j-1}}{q_{j-1}}$$

implies $bq_{j-1} - \frac{n}{c}p_{j-1} \geqslant 0$ and even

$$bq_j - \frac{n}{c}p_j > 0 \quad \forall j = 1, \ldots, k. \tag{6·4}$$

Since the determinant condition is trivially satisfied we have proved (6·3). But there is more detailed information available: since $\frac{p_0}{q_0} = \frac{b}{\frac{n}{c}}$ and $cq_1 < cq_0$ we have

$$\begin{pmatrix} q_0 & -p_0 \\ q_1 & -p_1 \end{pmatrix}\begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix} = \begin{pmatrix} cq_0 & bq_0 - \frac{n}{c}p_0 \\ cq_1 & bq_1 - \frac{n}{c}p_1 \end{pmatrix} = \begin{pmatrix} cq_0 & 0 \\ cq_1 & bq_1 - \frac{n}{c}p_1 \end{pmatrix} \in Y_n.$$

Moreover, Remark 2·5 shows that $\left(\begin{smallmatrix} q_{k-1} & -p_{k-1} \\ q_k & -p_k \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that

$$\begin{pmatrix} q_{k-1} & -p_{k-1} \\ 0 & p_k \end{pmatrix}\begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix} = \begin{pmatrix} c & b \\ 0 & \frac{n}{c} \end{pmatrix} \in X_n.$$

On the other hand, by (6·4) none of the $\left(\begin{smallmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{smallmatrix}\right)A_i$ in (6·3) with $j = 1, \ldots, k-1$ can be in $X_n \bigcup Y_n$ since $q_j \neq 0$ for these j. Now it suffices to prove

$$K : \begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_i \longmapsto \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_i. \tag{6·5}$$

To prove (6·5) note first that for an arbitrary matrix $A \in S_n \setminus Y_n$ we have

$$K(A)A^{-1} = T^{\lceil \frac{d'}{b'} \rceil} Q,$$

where $A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. For $A := \begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_i$ by Lemma 2·7(ii) we have

$$\lceil \tfrac{d'}{b'} \rceil = \left\lceil \frac{bq_{j+1} - \frac{n}{c} p_{j+1}}{bq_j - \frac{n}{c} p_j} \right\rceil = \lceil \frac{xq_{j+1} - p_{j+1}}{xq_j - p_j} \rceil = p_{j-1}q_{j+1} - p_{j+1}q_{j-1}$$

and using (16) we calculate

$$\begin{aligned}
&\begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_i \left(\begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_i \right)^{-1} \\
&\quad = \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} \begin{pmatrix} -p_{j+1} & p_j \\ -q_{j+1} & q_j \end{pmatrix} \\
&\quad = \begin{pmatrix} p_{j-1}q_{j+1} - p_{j+1}q_{j-1} & -1 \\ 1 & 0 \end{pmatrix} \\
&\quad = T^{(p_{j-1}q_{j+1} - p_{j+1}q_{j-1})} Q \\
&\quad = T^{\lceil \frac{d'}{b'} \rceil} Q
\end{aligned}$$

proving

$$K(A) = \begin{pmatrix} q_{j-1} & -p_{j-1} \\ q_j & -p_j \end{pmatrix} A_i$$

and hence the claim.

Remark 6·4. A construction rather similar to the one in Lemma 6·3 has been used also by L. Merel in [**16**], where he discussed the connection between the ordinary Hecke operators for the group $\Gamma_0(n)$ and continued fractions.

Now we can prove Proposition 1·2.

PROPOSITION 6·5. *Let $\sigma := \sigma_{n,m} : I_{nm} \to I_m$ be the canonical map. For any $i \in I_{nm}$ and $0 \leqslant j \leqslant k_{\sigma(i)}$ there exists an unique index $\hat{\imath}_{i,j} \in I_n$ such that $A_{\hat{\imath}_{i,j}}(K^j(A_{\sigma(i)}))A_i^{-1}$ is integer valued and hence in* $\mathrm{SL}(2,\mathbb{Z})$.

Proof. We apply Lemma 6·3 to $\sigma(i) \in I_m$. Then

$$K^j\left(A_{\sigma(i)}\right) = \begin{pmatrix} q_{k_{\sigma(i)}-1-j} & -p_{k_{\sigma(i)}-1-j} \\ q_{k_{\sigma(i)}-j} & -p_{k_{\sigma(i)}-j} \end{pmatrix} A_{\sigma(i)}$$

so that

$$K^j\left(A_{\sigma(i)}\right)A_i^{-1} = \begin{pmatrix} q_{k_{\sigma(i)}-1-j} & -p_{k_{\sigma(i)}-1-j} \\ q_{k_{\sigma(i)}-j} & -p_{k_{\sigma(i)}-j} \end{pmatrix} \left(A_i A_{\sigma(i)}^{-1}\right)^{-1}.$$

According to Lemma 5·12 we have $A_i A_{\sigma(i)}^{-1} = \begin{pmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{pmatrix}$ with $c'', b'' \in \mathbb{Z}$ such that $1 \leqslant c'' \mid n$ and $\gcd(c'', b'', \frac{n}{c''}) = 1$.

Following the procedure in Definition 5·13 we find an index $\hat{\imath}_{i,j} \in I_n$ such that

$$A_{\hat{\imath}_{i,j}} \begin{pmatrix} q_{k_{\sigma(i)}-1-j} & -p_{k_{\sigma(i)}-1-j} \\ q_{k_{\sigma(i)}-j} & -p_{k_{\sigma(i)}-j} \end{pmatrix} \begin{pmatrix} c'' & b''' \\ 0 & \frac{n}{c''} \end{pmatrix}^{-1} \in \mathrm{SL}(2,\mathbb{Z}),$$

where $b''' = b'' + k\frac{n}{c''}$ is relatively prime to c''. Note that

$$\begin{pmatrix} c'' & b''' \\ 0 & \frac{n}{c''} \end{pmatrix}^{-1} = \left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c'' & b'' \\ 0 & \frac{n}{c''} \end{pmatrix}\right)^{-1}$$
$$= \left(A_i A_{\sigma(i)}^{-1}\right)^{-1}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

which implies

$$A_{\hat{\imath}_{i,j}}\begin{pmatrix} q_{k_{\sigma(i)}-1-j} & -p_{k_{\sigma(i)}-1-j} \\ q_{k_{\sigma(i)}-j} & -p_{k_{\sigma(i)}-j} \end{pmatrix}\begin{pmatrix} c'' & b''' \\ 0 & \frac{n}{c''} \end{pmatrix}^{-1}$$
$$= A_{\hat{\imath}_{i,j}}\begin{pmatrix} q_{k_{\sigma(i)}-1-j} & -p_{k_{\sigma(i)}-1-j} \\ q_{k_{\sigma(i)}-j} & -p_{k_{\sigma(i)}-j} \end{pmatrix}\left(A_i A_{\sigma(i)}^{-1}\right)^{-1}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$
$$= A_{\hat{\imath}_{i,j}} K^j\left(A_{\sigma(i)}\right)A_i^{-1}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}.$$

Thus $\hat{\imath}_{i,j} \in I_n$ also satisfies

$$A_{\hat{\imath}_{i,j}} K^j\left(A_{\sigma(i)}\right)A_i^{-1} \in \mathrm{SL}(2,\mathbb{Z}).$$

This proves the existence statement. Uniqueness is shown as in the proof of Lemma 5·11.

7. *Special solutions for generalized Lewis equations*

From now on we fix an index $i \in I_{nm}$, a solution $(\psi_{\hat{\imath}})_{\hat{\imath}\in I_m}$ of the Lewis equation (1·5) for $\Gamma_0(m)$ and $(\phi_{\hat{\imath}})_{\hat{\imath}\in I_m}$ a solution of the functional Lewis equation (1·3). We assume that 0 is a branching point of each $\phi_{\hat{\imath}}$. Let $\sigma = \sigma_{m,n}\colon I_{nm} \to I_n$ to be the canonical map.

Let $P = (x_0, x_1, \ldots, x_k)$, $x_0 = x_{\sigma(i)}$, $x_k = -\infty$ be an arbitrary partition of $x_{\sigma(i)}$ and $m(P) = \sum_{j=0}^{k-1}\left(\begin{smallmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{smallmatrix}\right)$ be the associated sum (see Definition 2·8). Using Lemmas 5·12 and 5·11 we conclude that for every $0 \leqslant j \leqslant k-1$ there is a unique index $\hat{\imath}_{ij} \in I_m$ associated to $\left(\begin{smallmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{smallmatrix}\right)$ and i, i.e. the matrix

$$X_{ij} := A_{\hat{\imath}_{ij}}\begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix}\left(A_i A_{\sigma(i)}^{-1}\right)^{-1} \tag{7·1}$$

is integer-valued and hence belongs to $\mathrm{SL}(2,\mathbb{Z})$. We define

$$\psi_i(P) := \sum_{j=0}^{k-1} \psi_{\hat{\imath}_{ij}}\begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_{\sigma(i)}$$

$$\phi_i(P) = \sum_{j=0}^{k-1} \phi_{\hat{\imath}_{ij}}|_s\left(\begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_{\sigma(i)}\right).$$

The slash operator in the definition of $\phi_i(P)$ is well-defined and the branching point of $\phi_i(P)$ is zero. One can see these facts using Remark 3·4 and

$$\begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_{\sigma(i)} = \begin{pmatrix} q_j c_{\sigma(i)} & q_j d_{\sigma(i)} - p_j\frac{n}{c_{\sigma(i)}} \\ q_{j+1}c_{\sigma(i)} & q_{j+1}d_{\sigma(i)} - p_{j+1}\frac{n}{c_{\sigma(i)}} \end{pmatrix} \in \mathcal{G}^+, \tag{7·2}$$

where $A_{\sigma(i)} = \begin{pmatrix} c_{\sigma(i)} & \frac{d_{\sigma(i)}}{\frac{n}{c_{\sigma(i)}}} \\ 0 & \end{pmatrix}$. Here we have used $q_j, q_{j+1} \geqslant 0$, $p_j q_{j+1} - p_{j+1} q_j = 1$ and $x_{\sigma(i)} = \frac{d_{\sigma(i)}}{\frac{n}{c_{\sigma(i)}}} = \frac{p_0}{q_0} > \frac{p_j}{q_j}, \frac{p_{j+1}}{q_{j+1}}$ (see Remark 2·4). If $P = P_{x_{\sigma(i)}}$ is the minimal partition of $x_{\sigma(i)}$ we set

$$\psi_i := \psi_i(P_{x_{\sigma(i)}}) = \sum_{j=1}^{k_{\sigma(i)}} \psi_{\hat{\imath}_{ij}} K^j A_{\sigma(i)},\ i \in I_{nm}, \tag{7·3}$$

$$\phi_i := \phi_i(P_{x_{\sigma(i)}}) = \sum_{j=1}^{k_{\sigma(i)}} \phi_{\hat{\imath}_{ij}} \mid_s K^j A_{\sigma(i)},\ i \in I_{nm}. \tag{7·4}$$

LEMMA 7·1. *For any two partitions P_1 and P_2 of $x_{\sigma(i)}$ and any solution $(\psi_{\hat{\imath}})_{\hat{\imath} \in I_m}$ (resp. $(\phi_{\hat{\imath}})_{\hat{\imath} \in I_m}$) of the Lewis equation* (1·5) *(resp.* (1·3)*) for $\overline{\Gamma}_0(m)$ we have*

$$\psi_i(P_1) \equiv \psi_i(P_2) \ \ mod\ \mathcal{I}^\lambda \ \ (resp. \phi_i(P_1) = \phi_i(P_2)).$$

Proof. We are going to prove the statement for $(\phi_{\hat{\imath}})_{\hat{\imath} \in I_m}$. The proof of the state ment for $(\psi_{\hat{\imath}})_{\hat{\imath} \in I_m}$ is the same. We only write $\psi_{\hat{\imath}}$ instead of $\phi_{\hat{\imath}}$ and do not write the symbol $|_s$.

By Proposition 2·6 it is enough to prove the lemma for a partition P and its Farey extension $P(l)$. Define $Q_l = \begin{pmatrix} q_{l-1} & -p_{l-1} \\ q_l & -p_l \end{pmatrix}$. Note that by (7·2) we have $Q_l A_{\sigma(i)} \in \mathcal{G}^+$ (put $l = j+1$). We have

$$\begin{aligned} \phi_i(P(l)) - \phi_i(P) &= \phi_{i_1} \mid_s T\big(Q_l A_{\sigma(i)}\big) + \phi_{i_2} \mid_s MTM\big(Q_l A_{\sigma(i)}\big) - \phi_{i_3} \mid_s \big(Q_l A_{\sigma(i)}\big) \\ &= \phi_{i_1} \mid_s T\big(Q_l A_{\sigma(i)}\big) + \lambda \phi_{i_2 T^{-1} MT} \mid_s TM\big(Q_l A_{\sigma(i)}\big) - \phi_{i_3} \mid_s \big(Q_l A_{\sigma(i)}\big), \end{aligned}$$

where i_1(resp. i_2 and i_3) is associated to the matrix TQ_l(resp. $MTMQ_l$ and Q_l) and the index i (see Definition 5·13). In fact, we use (1·4) to obtain

$$\phi_{i_2} \mid_s MTM(Q_l A_{\sigma(i)}) = (\phi_{i_2} \mid_s M) \mid_s TM\big(Q_l A_{\sigma(i)}\big) = \lambda \phi_{i_2 T^{-1} MT} \mid_s TM\big(Q_l A_{\sigma(i)}\big). \tag{7·5}$$

Since $M, MTMQ_l A_{\sigma(i)}, TMQ_l A_{\sigma(i)} \in \mathcal{G}^+$, by Remark 3·4 the first equality in (7·5) is well-defined. We continue

$$\begin{aligned} \phi_i(P(l)) - \phi_i(P) &= \phi_{i_1} \mid_s T \mid_s \big(Q_l A_{\sigma(i)}\big) + \lambda \phi_{i_2 T^{-1} MT} \mid_s TM \mid_s \big(Q_l A_{\sigma(i)}\big) - \phi_{i_3} \mid_s \big(Q_l A_{\sigma(i)}\big) \\ &= \big(\phi_{i_1} \mid_s T + \lambda \phi_{i_2 T^{-1} MT} \mid_s TM - \phi_{i_3}\big) \mid_s \big(Q_l A_{\sigma(i)}\big). \end{aligned}$$

Since $T, TM, Q_l A_{\sigma(I)} \in \mathcal{G}^+$, the first equality is well-defined.

Now to finish the proof of our lemma it is enough to prove that $i_3 = i_1 T$, $i_2 T^{-1} MT = i_1 M$ or equivalently

$$i_1 = i_3 T^{-1},\ i_2 = i_3 T^{-1} MT^{-1} MT.$$

By definition of i_3 the matrix $A_{i_3} Q_l (A_i A_{\sigma(i)}^{-1})^{-1}$ is in $\mathrm{SL}(2, \mathbb{Z})$. Now if we prove that $A_{i_3 T^{-1}} TQ_l (A_i A_{\sigma(i)}^{-1})^{-1}$ and $A_{i_3 T^{-1} MT^{-1} MT} MTMQ_l (A_i A_{\sigma(i)}^{-1})^{-1}$ are in $\mathrm{SL}(2, Z)$ then by the uniqueness in Lemma 5·11 the above equalities are proved.

$$A_{i_3 T^{-1}} TQ_l \big(A_i A_{\sigma(i)}^{-1}\big)^{-1} = \big(A_{i_3 T^{-1}} TA_{i_3}^{-1}\big)\big(A_{i_3} Q_l \big(A_i A_{\sigma(i)}^{-1}\big)^{-1}\big)$$

$$A_{i_3 T^{-1} MT^{-1} MT} MTMQ_l \big(A_i A_{\sigma(i)}^{-1}\big)^{-1} = \big(A_{i_3 T^{-1} MT^{-1} MT} MTMA_{i_3}^{-1}\big)\big(A_{i_3} Q_l \big(A_i A_{\sigma(i)}^{-1}\big)^{-1}\big).$$

Now by Lemma 5·14(i), (iii) the matrices $A_{i_3 T^{-1}} TA_{i_3}^{-1}$ and $A_{i_3 T^{-1} MT^{-1} MT} MTMA_{i_3}^{-1}$ are integer-valued and our lemma is proved (in the first one put $j = i_3 T^{-1}$).

Proof of Theorem 1·3. We first prove the second statement. Set $\sigma := \sigma_{n,m}$.

$$\psi_i = \sum_{j=0}^{k_{\sigma(i)}} \psi_{\hat{\imath}_{ij}} K^j A_{\sigma(i)} = \sum_{j=0}^{k_{\sigma(i)}} m\big(P_{x_{\hat{\imath}_{ij}}}\big) A_{\hat{\imath}_{ij}} \begin{pmatrix} q_j & -p_j \\ q_{j+1} & -p_{j+1} \end{pmatrix} A_{\sigma(i)}$$
$$= \left(\sum_{j=0}^{k_{\sigma(i)}} m(P_{x_{\hat{\imath}_{ij}}}) X_{ij} \right) A_i = \left(m \left(\bigvee_{j=0}^{k_{\sigma(i)}} P_{x_{\hat{\imath}_{ij}}} X_{ij} \right) \right) A_i$$

where X_{ij} is given in (7·1). We must check that the last equality is well-defined, i.e.

$$x_{\hat{\imath}_{ij}} X_{ij} = (-\infty) X_{i(j-1)}, \ j = 1, 2, \ldots, k_{\sigma(i)}$$

and the inequality (2·17) is true for X_{ij} and $x_{\hat{\imath}_{ij}}$.

Let us write $A_{\hat{\imath}_{ij}} = \left(\begin{smallmatrix} c_{ij} & d_{ij} \\ 0 & \frac{n}{c_{ij}} \end{smallmatrix}\right)$ and $A_i A_{\sigma(i)}^{-1} = \left(\begin{smallmatrix} c & d \\ 0 & \frac{n}{c} \end{smallmatrix}\right)$. Then $x_{\hat{\imath}_{ij}} = \frac{d_{ij}}{\frac{n}{c_{ij}}}$ and

$$X_{ij} = \frac{1}{n} \begin{pmatrix} \frac{n}{c}(c_{ij} q_j + d_{ij} q_{j+1}) & -d(c_{ij} q_j + d_{ij} q_{j+1}) - c(c_{ij} p_j + d_{ij} p_{j+1}) \\ \frac{n}{c} \frac{n}{c_{ij}} q_{j+1} & -\frac{n}{c_{ij}} (q_{j+1} d + p_{j+1} c) \end{pmatrix}.$$

We have

$$\frac{\frac{n}{c}(c_{ij} q_j + d_{ij} q_{j+1})}{\frac{n}{c} \frac{n}{c_{ij}} q_{j+1}} = x_{\hat{\imath}_{ij}} + \frac{q_j}{q_{j+1}} \frac{c_{ij}^2}{n} > x_{\hat{\imath}_{ij}}$$

and so by Remark 2·4 the condition (2·17) is satisfied in our case. Now

$$\begin{aligned} x_{\hat{\imath}_{ij}} X_{ij} &= \frac{d_{ij}}{\frac{n}{c_{ij}}} \begin{pmatrix} \frac{1}{c}(c_{ij} q_j + d_{ij} q_{j+1}) & \frac{-d(c_{ij} q_j + d_{ij} q_{j+1}) - c(c_{ij} p_j + d_{ij} p_{j+1})}{n} \\ \frac{1}{c} \frac{n}{c_{ij}} q_{j+1} & -\frac{1}{c_{ij}} (q_{j+1} d + p_{j+1}) c \end{pmatrix} \\ &= \frac{-\frac{1}{c_{ij}}(q_{j+1} d + p_{j+1}) c \left(\frac{d_{ij}}{\frac{n}{c_{ij}}} \right) + \frac{d(c_{ij} q_j + d_{ij} q_{j+1}) + c(c_{ij} p_j + d_{ij} p_{j+1})}{n}}{-\frac{1}{c} \frac{n}{c_{ij}} q_{j+1} \left(\frac{d_{ij}}{\frac{n}{c_{ij}}} \right) + \frac{1}{c}(c_{ij} q_j + d_{ij} q_{j+1})} \\ &= \frac{d}{\frac{n}{c}} + \frac{p_j}{q_j} \frac{c}{\frac{n}{c}} \\ &= \frac{-\frac{n}{c_{i(j-1)}} (q_j d + p_j c)}{-\frac{n}{c} \frac{n}{c_{i(j-1)}} q_j} \\ &= (-\infty) X_{i(j-1)} \end{aligned}$$

and

$$x_{\hat{\imath}_{i0}} X_{i0} = \frac{d}{\frac{n}{c}} + \frac{p_0}{q_0} \frac{c}{\frac{n}{c}} = \frac{d}{\frac{n}{c}} + \frac{d_{\sigma(i)}}{\frac{m}{c_{\sigma(i)}}} \frac{c}{\frac{n}{c}} = x_i$$

where $A_{\sigma(i)} = \left(\begin{smallmatrix} c_{\sigma(i)} & d_{\sigma(i)} \\ 0 & \frac{m}{c_{\sigma(i)}} \end{smallmatrix}\right)$. In the second equality we have used $x_{\sigma(i)} = \frac{p_0}{q_0} = \frac{d_{\sigma(i)}}{\frac{m}{c_{\sigma(i)}}}$. The last equality is derived from the equality $A_i A_{\sigma(i)}^{-1} = \left(\begin{smallmatrix} c & d \\ 0 & \frac{n}{c} \end{smallmatrix}\right)$. We have also

$$(-\infty) X_{ik_{\sigma(i)}} = \frac{d}{\frac{n}{c}} + \frac{-1}{0} \frac{c}{\frac{n}{c}} = -\infty.$$

Therefore $\bigvee_{j=1}^{k_{\sigma(i)}} P_{x_{\hat{\imath}_{ij}}} X_{ij}$ is a partition of x_i and so by Lemma 7·1 the second statement is proved.

Now let us prove the first part. We prove the statement for $(\psi_{\hat\imath})_{\hat\imath\in I_n}$. The proof of the other is similar. We are going to prove that $\psi_i T + \lambda\psi_{iM}TM = \psi_{iT}$ for all $i \in I_{nm}$.

$$\begin{aligned}\psi_i T + \lambda\psi_{iM}TM &= \sum_{j=0}^{k_{\sigma(i)}} \psi_{\hat\imath_{ij}} K^j A_{\sigma(i)}T + \sum_{j=0}^{k_{\sigma(iM)}} \lambda\psi_{\bar\imath_{ij}} K^j A_{\sigma(iM)}TM \\ &= \Bigg(\sum_{j=0}^{k_{\sigma(i)}} \psi_{\hat\imath_{ij}} Q_j\big(A_{\sigma(i)}TA^{-1}_{\sigma(iT)}\big) \\ &\quad + \sum_{j=0}^{k_{\sigma(iM)}} \psi_{\bar\imath_{ij}T^{-1}MT} MQ'_j\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big)\Bigg) A_{\sigma(iT)},\end{aligned}$$

where $Q_j := \begin{pmatrix} q_{k_{\sigma(i)}-1-j} & -p_{k_{\sigma(i)}-1-j} \\ q_{k_{\sigma(i)}-j} & -p_{k_{\sigma(i)}-j}\end{pmatrix}$ and $Q'_j := \begin{pmatrix} q'_{k_{\sigma(iM)}-1-j} & -p'_{k_{\sigma(iM)}-1-j} \\ q'_{\sigma(iM)-j} & -p'_{\sigma(iM)-j}\end{pmatrix}$. Here $\hat\imath_{ij}$ is associated to Q_j and i, and $\bar\imath_{ij}$ is associated to Q'_j and iM. Now

$$\begin{aligned}&\sum_{j=0}^{k_{\sigma(i)}} Q_j\big(A_{\sigma(i)}TA^{-1}_{\sigma(iT)}\big) + \sum_{j=0}^{k_{\sigma(iM)}} MQ'_j\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big) \\ &\quad = m\big(P_{x_{\sigma(i)}}\big)\big(A_{\sigma(i)}TA^{-1}_{\sigma(iT)}\big) + Mm\big(P_{x_{\sigma(iM)}}\big)\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big) \\ &\quad = Mm\big(P_{x_{\sigma(iM)}}\big)\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big) + m\big(P_{x_{\sigma(i)}}\big)\big(A_{\sigma(i)}TA^{-1}_{\sigma(iT)}\big) \\ &\quad = m\Big(P_{x_{\sigma(iM)}}\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big) \bigvee P_{x_{\sigma(i)}}\big(A_{\sigma(i)}TA^{-1}_{\sigma(iT)}\big)\Big).\end{aligned}$$

The last equality is well-defined, i.e.

$$\big(x_{\sigma(iM)}\big)\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big) = x_{\sigma(i)}\big(A_{\sigma(i)}TA^{-1}_{\sigma(iT)}\big) \tag{7·6}$$

(note that $A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}$ has determinant -1). Also, what is inside $m(.)$, name it P, is a partition of $x_{\sigma(iT)}$, i.e.

$$(-\infty)\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big) = x_{\sigma(iT)}. \tag{7·7}$$

The equalities (7·6) and (7·7) are proved in Lemma 5·15 (Put $j = \sigma(i)$ and replace n with m). We claim that the index $\hat\imath_{ij}$ (resp. $\bar\imath_{ij}T^{-1}MT$) is associated to $Q_j(A_{\sigma(i)}TA^{-1}_{\sigma(iT)})$ (resp. $MQ'_j(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)})$) and iT. If our claim is true then

$$\psi_i T + \lambda\psi_{iM}TM = \psi_i(P)$$

and by Lemma 7·1 the proof is finished. Our claim is equivalent to the fact that the matrices

$$A_{\hat\imath_{ij}} Q_j\big(A_{\sigma(i)}TA^{-1}_{\sigma(iT)}\big)\big(A_{iT}A^{-1}_{\sigma(iT)}\big)^{-1} = \big(A_{\hat\imath_{ij}} Q_j\big(A_i A^{-1}_{\sigma(i)}\big)^{-1}\big)\big(A_i T A^{-1}_{iT}\big)$$

and

$$\begin{aligned}&A_{\bar\imath_{ij}T^{-1}MT} MQ'_j\big(A_{\sigma(iM)}TMA^{-1}_{\sigma(iT)}\big)\big(A_{iT}A^{-1}_{\sigma(iT)}\big)^{-1} \\ &\quad = \big(A_{\bar\imath_{ij}T^{-1}MT} MA^{-1}_{\bar\imath_{ij}}\big)\big(A_{\bar\imath_{ij}} Q'_j\big(A_{iM}A^{-1}_{\sigma(iM)}\big)^{-1}\big)(A_{iM}TMA^{-1}_{iT})\end{aligned}$$

are integer-valued. Thanks to Lemma 5·14 our claim is true.

8. *Hecke operators*

In [**21**, **22**] D. Zagier derived a representation of the Hecke operators on the space of period polynomials for the group $\mathrm{PSL}(2,\mathbb{Z})$ by transferring the action of the classical Hecke operators on the space of cusp forms via the Eichler–Shimura–Manin isomorphism to the space of period polynomials. In his thesis Mühlenbruch ([**18**]) found another representation for these operators in terms of matrices with nonnegative entries which allowed him to extend their action to the space of period functions with arbitrary weight. It turns out that the special solutions of the Lewis equations for the congruence subgroups $\Gamma_0(n)$ we constructed in Theorem 1·1 are closely related to the Hecke operators for $\mathrm{PSL}(2,\mathbb{Z})$ in the form given by Mühlenbruch.

Indeed, since both the maps $T\colon I_n \to I_n$ and $MT\colon I_n \to I_n$ are invertible, any solution Φ of the Lewis equation (1·3) for $\Gamma_0(n)$ given by $\phi_i = \phi_i(z), i \in I_n$, determines a solution $\tilde{\phi} = \tilde{\phi}(z)$ of the Lewis equation (1·1) for the group $\mathrm{PSL}(2,\mathbb{Z})$ with

$$\tilde{\phi}(z) = \sum_{i\in I_n} \phi_i(z). \tag{8·1}$$

Clearly, it can happen that this function vanishes identically. This just signals that the corresponding solution $\phi_i, i \in I_n$ for the group $\Gamma_0(n)$ is not related to any solution ϕ of the group $\mathrm{PSL}(2,\mathbb{Z})$ and hence is either in analogy to the Atkin–Lehner theory of old and new forms a *new solution* of the Lewis equation for $\Gamma_0(n)$ or related to a new solution for one of the groups $\Gamma_0(l)$ with $l \mid n$. The special solution, however, determined in Theorem 1·1 leads to a nontrivial solution $\tilde{\phi}$ which furthermore depends linearly on the solution ϕ of equation (1·1). This shows that the map $\tilde{H}_n\colon \phi \mapsto \tilde{\phi}$ with $\tilde{\phi}$ as defined in equation (8·1) determines a linear operator in the space of period functions of the group $\mathrm{PSL}(2,\mathbb{Z})$. To determine the explicit form of the operator $\tilde{H}_n$ we have to characterize the matrices $K^j(A_i)$ appearing in the definition of the solutions ψ_i in Theorem 1·1 in more detail.

From the definition of the operator K in Proposition 6·1 it is obvious that all matrix elements of $A \in S_n \setminus Y_n$ have greatest common divisor 1 if and only if the matrix elements of the matrix $K(A)$ have this property. Since the entries of the matrix A_i in Definition 5·9 for $i \in I_n$ have greatest common divisor 1 all the matrices appearing in the definition of ψ_i in Theorem 1·1 have this property.

Consider next any matrix $A \in S_n \setminus X_n$ whose entries have greatest common divisor 1. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $K^{-1}A = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ with $c' < c$ and hence there exists $j \in \mathbb{N}$ with $K^{-j}A \in X_n$. But from Proposition 5·8 it follows that any matrix A in X_n whose entries have only 1 as a common divisor appears as A_i for some $i \in I_n$. This shows that any matrix A in the set S_n whose entries have no common divisor besides 1 appears exactly once in one of the components ψ_i in Theorem 1·1.

Denote then by $\tilde{T}_n$ the matrix

$$\tilde{T}_n := \sum_{A\in S_n\,;\gcd(a,b,c,d)=1} A, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then one finds for the operator $\tilde{H}_n$ acting on the space of period functions ϕ for the group $\mathrm{PSL}(2,\mathbb{Z})$

$$\tilde{H}_n\phi = \phi \mid_s \tilde{T}_n.$$

Summarizing we have shown:

THEOREM 8·1. *For any solution $\phi = \phi(z)$ of the Lewis equation* (1·1) *for* $\mathrm{PSL}(2,\mathbb{Z})$ *with arbitrary weight s the function $\tilde{\phi} = \tilde{\phi}(z) = (\tilde{H}_n\phi)(z) = (\phi\ |_s\ \tilde{T}_n)(z)$ is also a solution of equation* (1·1) *with weight s.*

Comparing the operators $\tilde{T}_n$ with the Hecke operators T_n of Mühlenbruch and Zagier in (1·2) we find as a corollary:

COROLLARY 8·2. *The operators $\tilde{T}_n$ and the Hecke operators T_n defined in* (1·2) *are related through*

$$T_n = \sum_{d^2|n} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \tilde{T}_{\frac{n}{d^2}}.$$

The operators coincide if and only if n is a product of distinct primes.

The operators $\tilde{T}_n$ have been constructed from special solutions of the Lewis equation (3) for the group $\Gamma_0(n)$. It turns out that also the Hecke operators T_n can be derived in this way. To achieve this consider any $n_1, n_2 \in \mathbb{N}$ with $n_1 = mn_2$. Denote again for fixed m and n_2 the canonical surjective map $\sigma_{m,n_2} : [\mathbb{Z}\times\mathbb{Z}]_{n_1} \to [\mathbb{Z}\times\mathbb{Z}]_{n_2}$ by σ. This map is equivariant with respect to the right action of the group $\mathrm{GL}(2,\mathbb{Z})$, i.e.

$$\sigma_{m,n_2}([x:y]_{n_1})A = \sigma_{m,n_2}([x:y]_{n_1}A), \quad \forall A \in \mathrm{GL}(2,\mathbb{Z}),\ [x:y]_{n_1} \in [\mathbb{Z}\times\mathbb{Z}]_{n_1}. \quad (8·2)$$

Then we can prove:

PROPOSITION 8·3.

(i) *If the matrices ψ_i, $i \in I_{n_1}$, solve* (1·5) *for $n = n_1 = mn_2$, then $\tilde{\psi}_j := \sum_{i\in\sigma^{-1}(j)} \psi_i$, $j \in I_{n_2}$, solve* (1·5) *for $n = n_2$.*

(ii) *If $\tilde{\psi}_j$, $j \in I_{n_2}$, solve* (1·5) *for $n = n_2$, then $\psi_i := \tilde{\psi}_{\sigma(i)}$, $i \in I_{n_1}$, solve* (1·5) *for $n = n_1 = mn_2$.*

Proof. (8·2) implies that the fibers $\sigma^{-1}(j)$ of σ are invariant under the action of $\mathrm{GL}(2,\mathbb{Z})$, and in particular T^{-1} and $T^{-1}M$.

Proposition 8·3 shows that for any d with $d^2 \mid n$ any solution $\tilde{\Psi} = (\tilde{\psi}_i)_{i\in I_{\frac{n}{d^2}}}$ of equation (1·5) for the group $\Gamma_0(\frac{n}{d^2})$ determines a solution for this equation for the group $\Gamma_0(n)$ whose components coincide with the components for the former group. Indeed, any component shows up μ-times, where μ is the index of $\Gamma_0(n)$ in $\Gamma_0(\frac{n}{d^2})$. Obviously, a similar statement holds for the solutions $\tilde{\Phi} = (\tilde{\varphi}_i)_{i\in I_{\frac{n}{d^2}}}$ of equation (1·3). Taking for $\tilde{\Phi}$ the special solution $\varphi\ |_s\ \psi_i, i \in I_{\frac{n}{d^2}}$, determined in Theorem 1·1 we therefore get

COROLLARY 8·4. *For any solution φ of the Lewis equation* (1·1) *for the group* $\mathrm{PSL}(2,\mathbb{Z})$ *with weight s the functions $\tilde{\varphi}_{j,d} := \varphi\ |_s\ \psi_{\sigma_{d^2,\frac{n}{d^2}}(j)}, j \in I_n$ define a solution Φ_d of the Lewis equation* (1·3) *for the group $\Gamma_0(n)$ with weight s.*

Hence also the function $\tilde{\varphi}_d$ with

$$\tilde{\varphi}_d := \frac{1}{\mu}\sum_{j\in I_n} \tilde{\varphi}_{j,d} = \varphi\ |_s\ \tilde{T}_{\frac{n}{d^2}}$$

defines a solution of the Lewis equation (1·1) for the group $\mathrm{PSL}(2,\mathbb{Z})$. Obviously the matrix inducing this solution $\tilde{\phi}_d$ coincides with the matrix $\tilde{T}_{\frac{n}{d^2}}$. This shows that indeed the Hecke operator T_n on the period functions of $\mathrm{PSL}(2,\mathbb{Z})$ for arbitrary weight s can be derived from special solutions of the Lewis equation for the group $\Gamma_0(n)$ with weight s.

The extension of the above approach to the group $\Gamma_0(n)$ for arbitrary n is straightforward: given any solution $\Phi = (\varphi_i)_{i\in I_n}$ of the Lewis equation (1·3) for the group $\Gamma_0(n)$, Theorem 1·3 together with Proposition 8·3 allow us to construct for any n a family of linear operators $\tilde{T}_{n,m}$, $m = 1, 2, \ldots$ mapping this solution to a new solution of equation (1·3). The explicit form of these operators is given by

$$(\tilde{T}_{n,m}\Phi)_{\hat{\imath}}(z) = \sum_{l\in\sigma_{m,n}^{-1}(\hat{\imath})} \sum_{j=0}^{k_{\sigma_{n,m}(l)}} \varphi_{\hat{\imath}_{l,j}} \,|_s\, K^j A_{\sigma_{n,m}(l)}(z), \quad \hat{\imath} \in I_n.$$

The relation of these operators to the familiar Hecke operators on modular forms for the congruence subgroups $\Gamma_0(n)$ will be discussed in a forthcoming paper.

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