On the non-persistence of Hamiltonian identity cycles

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Abstract

We study the leading term of the holonomy map of a perturbed plane polynomial Hamiltonian foliation. The non-vanishing of this term implies the non-persistence of the corresponding Hamiltonian identity cycle. We prove that this does happen for generic perturbations and cycles, as well for cycles which are commutators in Hamiltonian foliations of degree two. Our approach relies on the Chen's theory of iterated path integrals which we briefly resume.

1 Introduction

Let $\mathcal{F}^n$ be a degree n polynomial foliation on the plane $\mathbb{C}^2$. A cycle on a leaf is a non-zero free homotopy class of closed loops on this leaf. Let δ be a closed non-contractible loop on a leaf and denote the holonomy map associated to δ by h_δ. The cycle represented by δ is said to be an *identity cycle* provided that h_δ is the identity map, and *Hamiltonian identity cycle*, provided that $\mathcal{F}^n = \{df = 0\}$. The present paper is motivated by the following:

Conjecture 1. A generic polynomial foliation on $\mathbb{C}^2$ can not have identity cycles.

which follows on its turn from the following more general:

Conjecture 2. [12, p.4], (D.V. Anosov, early 1960s) For a generic polynomial foliation by analytic curves on $\mathbb{C}^k$, all leaves are topological discs except for a countable number of topological cylinders.

Conjecture 1 has its origin in the the famous Petrovskii-Landis paper [18], see the comments of Ilyashenko in [11]. We consider a restricted version of Conjecture 1:

Conjecture 3. Let δ be a Hamiltonian identity cycle. There is a sufficiently small degree n perturbation of $\mathcal{F}^n$ which either destroys δ, or makes it a limit cycle.

For $n > 2$ Conjecture 3 is a Theorem as proved by Ilyashenko and Pyartli [10, Theorem 2]. In the present paper we propose a different approach to Conjecture 3 based on a well known integral formula for the dominant term of the asymptotic expansion of the perturbed holonomy map. Namely, let

$$h_\delta^\varepsilon(t) = t + \varepsilon^k M_k(t) + o(\varepsilon^k)$$

be the asymptotic expansion of the holonomy map h_δ^ε associated to the perturbed foliation

$$df + \varepsilon\omega = 0$$

and to a cycle of the non-perturbed foliation $\{df = 0\}$ (here ω is a polynomial form). Clearly Conjecture 3 would follow from $h_\delta^\varepsilon \neq id$ which on its turn is a consequence of $M_k \neq 0$. The function M_k, the so called Poincaré-Pontryagin-Melnikov function, has an integral representation in terms of iterated path integrals of length at most k, see [3]. Let $(F_t^n)_{n\geq 1}$ be the lower central series of the fundamental group F_t of the fiber $f^{-1}(t)$. The properties of iterated integrals imply that if $\delta \in F_t^k$ then $M_i = 0$ for $i = 1, 2, ..., k-1$. The Poincaré-Pontryagin-Melnikov function M_k is therefore a kind of "linearization" of the holonomy map with respect to the deformation parameter ε. We might expect that, at least for generic ω, the function M_k is non zero. This claim, if true, would be stronger than the claim of Conjecture 3. We shall show, however, that for $k \geq 5$ there exist cycles $\delta \in F_t^k$ such that for every polynomial one-form ω of degree at most n, holds $M_k = 0$ (Example 3).

The present paper is devoted to the study of the Poincaré-Pontryagin-Melnikov function M_k. Our first result is that for generic f, ω and for almost all identity cycles $\delta \in F_t^k/F_t^{k+1}$, the function M_k is not identically zero (Theorem 6). We investigate then in more details the case $n = 2$ (the fibers $f^{-1}(t)$ are elliptic curves). We prove that if $\delta \in F_t^2/F_t^3$ (e.g. $\delta = aba^{-1}b^{-1}$ is a commutator), then $M_2 \neq 0$ for almost all degree n one-forms ω (Theorem 7). In particular the identity cycle δ is destroyed.

The proof of the above results relies in an essential way on the Chen's theory of interated path integral. We formulate and prove the relevant results in §3, which can be read independently. The main results here are Theorem 1 and Theorem 2 (the so called π_1 de Rham theorem). These results are not completely new, see the paper of R. Hain [5]. Our presentation, as well the proofs are different, and we hope simpler, as based on a classical combinatorial theorem of Ree [20]. To the end of this Introduction we resume §3.

Let $\Gamma = \bar{\Gamma} \setminus S$, where $\bar{\Gamma}$ is a compact Riemann surface and $S \subset \bar{\Gamma}$ a non-empty finite set of points. Each holomorphic one-form ω on the *non-compact* Riemann surface Γ defines a linear map $\int \omega \in \mathrm{Hom}_{\mathbb{Z}}(H_1(\Gamma, \mathbb{Z}), \mathbb{C})$ by integration:

$$\int \omega : H_1(\Gamma, \mathbb{Z}) \to \mathbb{C} : \delta \mapsto \int_\delta \omega$$

and he classical de Rham theorem stipulates that $\mathrm{Hom}_{\mathbb{Z}}(H_1(\Gamma, \mathbb{Z}), \mathbb{C})$ is generated by such linear maps. In the context of the present paper $\Gamma = f^{-1}(t)$ is a leaf of the non-perturbed foliation $\{df = 0\}$ and $\int_\delta \omega$ is the first Poincaré-Pontryagin-Melnikov function M_1.

The Chen's de Rham theorem generalizes the de Rham theorem as follows. The fundamental group $F = \pi_1(\Gamma, *)$ is free finitely generated. For $A, B \subset F$ we denote by (A, B) the subgroup of F generated by commutators $(a, b) = aba^{-1}b^{-1}, a \in A, b \in B$. Define by induction the free subgroups $F^k = (F, F^{k-1})$, $F^1 = F$. The Chen's de Rham theorem, e.g. [7], claims that for each k the vector space $\mathrm{Hom}_{\mathbb{Z}}(F^k/F^{k+1}, \mathbb{C})$ is generated by iterated integrals of length k. As $F^1/F^2 = H_1(\Gamma, \mathbb{Z})$ then in the case $k = 1$ we get the usual de

Rham theorem. In the context of the present paper, the Poincaré-Pontryagin-Melnikov function M_k defines an element of $\mathrm{Hom}_{\mathbb{Z}}(F^k/F^{k+1},\mathbb{C})$.

The purpose of §3 is to prove a more precise version of this "π_1 de Rham theorem" by constructing explicitly the space $\mathrm{Hom}_{\mathbb{Z}}(F^k/F^{k+1},\mathbb{C})$ in terms of iterated integrals along Lie elements of length k (see Theorem 1). The construction is purely combinatorial, the only analytic ingredient in the proof being the usual de Rham theorem. Therefore we can formulate the result in algebraic terms as follows:

Let $X=\{x_1,x_2,...,x_m\}$ be a set and $\mathsf{k}\subset\mathsf{K}$ two fields of characteristic zero. In what follows the elements of X will be interpreted either as loops δ_i, or as one-forms ω_j on the Riemann surface Γ. Denote by Ass_X the graded k-algebra of associative but non-commutative polynomials in variables $x_1,x_2,\ldots,x_m$ and by $L_X\subset\mathrm{Ass}_X$ the graded Lie algebra generated by $x_1,\ldots,x_m$. Let also F_X be the free group generated by the elements of X. By an iterated integral we mean any map

$$\begin{array}{rcl}\int: F_X\times\mathrm{Ass}_X & \to & \mathsf{K}\\ (\delta,\omega) & \mapsto & \int_\delta\omega\end{array}$$

which is k-linear in Ass_X and satisfies the four axioms given in the next section. Let Ass_X^k the homogeneous component of degree k of Ass_X. It will follow from the axioms of iterated integral that $\int$ induces a well-defined map :

$$F_X^k/F_X^{k+1}\times\mathrm{Ass}_X^k\to\mathsf{K},\quad k=1,2,\ldots \tag{1}$$

which is $\mathbb{Z}$-linear in the first coordinate and k-linear in the second coordinate. Of course an example of such a map is provided by the usual path integrals of Chen mentioned above. In this context we have $k=K=\mathbb{C}$ and (1) becomes

$$F_{\mathbf{\Delta}}^k/F_{\mathbf{\Delta}}^{k+1}\times\mathrm{Ass}_{\mathbf{\Omega}}^k\to\mathsf{K},\quad k=1,2,\ldots \tag{2}$$

where $\mathbf{\Omega}=\{\omega_1,\omega_2,\ldots,\omega_m\}$ is a set of differential one-forms, and $\mathbf{\Delta}=\{\delta_1,\delta_2,\ldots,\delta_m\}$ a set of loops which generate the first De Rham group $H^1_{DR}(\Gamma)$ and the fundamental group $\pi_1(\Gamma,*)$ respectively.

The π_1 de Rham theorem (Theorem 1) can be formulated as follows: If the vector space of $\mathbb{Z}$-linear maps

$$\begin{array}{rcl}\int\omega: F_X/F_X^2 & \to & \mathsf{K},\ \text{where } \omega\in L_X^1\\ \delta & \mapsto & \int_\delta\omega\end{array}$$

is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(F_X/F_X^2,\mathsf{K})$, then the vector space of $\mathbb{Z}$-linear maps

$$\begin{array}{rcl}\int\omega: F_X^k/F_X^{k+1} & \to & \mathsf{K},\ \text{where } \omega\in L_X^k\\ \delta & \mapsto & \int_\delta\omega\end{array}$$

is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(F_X^k/F_X^{k+1},\mathsf{K})$ for all $k=1,2,\ldots$. We also prove that $\int\omega,\ \omega\in\mathrm{Ass}_X^k$ is identically zero if and only if ω is a linear combination of shuffle elements.

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2 Iterated path integrals

Let k be a field. All modules and algebras are taken over k, unless stated otherwise. We shall use the notations of [21]. For a set $X = \{x_1, x_2, ..., x_m\}$, let Ass_X be the graded free associative algebra on X. Its elements are the non-commutative polynomials in x_i with coefficients in k. Define a Lie bracket in Ass_X by $[x, y] = xy - yx$, and let $L_X \subset \mathrm{Ass}_X$ be the graded free Lie algebra on X. Thus, for instance, $x_1, [x_1, x_2], [[x_1, x_2], x_3]$ belong to L_X but not 1, x_1x_2, $x_1x_2x_3$. Ass_X is the universal algebra of the Lie algebra L_X, see [21] . The graded piece L^1_X is just the k-vector space generated by $x_1, x_2, ..., x_m$, L^2_X is the k-vector space generated by $[x_i, x_j]$ etc. Each element of L_X is called a Lie element. We denote by Ass^k_X (resp. L^k_X) the homogeneous component of degree k of Ass_X (resp. L_X).

Let F_X be the free group on X, its elements are the words in the letters x_i and their formal inverses x_i^{-1}. For $x, y \in F_X$ we define the commutator $(x, y) = xyx^{-1}y^{-1}$. For $A, B \subset F_X$ we denote by (A, B) the subgroup of F_X generated by commutators $(a, b), a \in A, b \in B$. Consider the lower central series F^n_X of F_X, where $F^n_X = (F_X, F^{n-1}_X)$, $F^1_X = F_X$. The associated graded $\mathbb{Z}$-Lie algebra is given by

$$\mathrm{gr}F_X = \sum_{n=1}^{\infty} \mathrm{gr}^n F_X, \quad \mathrm{gr}^n F_X = F^n_X / F^{n+1}_X, \tag{3}$$

$$[xF^{i+1}_X, yF^{j+1}_X] = (x, y)F^{i+j+1}_X.$$

The canonical map $X \to \mathrm{gr}^1 F_X$ which send x_i to x_i induces an isomorphism of Lie algebras

$$\phi : L_X \to (\mathrm{gr}F_X) \otimes_{\mathbb{Z}} \mathsf{k} \tag{4}$$

(e.g.[21] Theorem 6.1, page 24).

Remark 1. *As explained in the Introduction, the elements x_i of the set X are interpreted either as loops δ_i, or as one-forms ω_i. The elements of the associative algebra Ass_X are seen as non-commutative polynomials in one forms ω_i, while the free group F_X is seen as a free group generated by loops δ_i, and hence it is identified to the fundamental group $\pi_1(\Gamma, *)$. The fundamental isomorphism of Lie algebras (4) justifies the identification of ω_i and δ_i.*

For two words $\omega_1 \cdots \omega_r$, $\omega_{r+1} \cdots \omega_{r+s} \in \mathrm{Ass}_X$ define the shuffle product

$$\omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s} \in \mathrm{Ass}_X$$

to be the sum of all words of length $r + s$ that are permutations of $\omega_1 \cdots \omega_r\omega_{r+1} \cdots \omega_{r+s}$ such that both $\omega_1 \cdots \omega_r$ and $\omega_{r+1} \cdots \omega_{r+s}$ appear in their original order, e.g.

$$\omega_1\omega_2 * \omega_3 = \omega_1\omega_2\omega_3 + \omega_1\omega_3\omega_2 + \omega_3\omega_1\omega_2.$$

Let K be a field extension of the field k.

Definition 1. An iterated integral is a map

$$\begin{array}{rcl} \int : F_X \times \mathrm{Ass}_X & \to & \mathsf{K} \\ (\delta, \omega) & \mapsto & \int_\delta \omega \end{array} \tag{5}$$

which is k-linear in the second variable and satisfies the four axioms:

A1 For every non-commutative polynomial $\omega \in \mathrm{Ass}_X$ and $\delta \in F_X$, $\int_1 \omega \in \mathsf{k}$ is the constant term of ω and $\int_\delta 1 = 1$ for all $\delta \in F_X$. We use the convention $\omega_1\omega_2\cdots\omega_r = 1$ for $r = 0$.

A2 For $\alpha, \beta \in F_X$ and $\omega_1, \omega_2, \ldots \omega_r \in \mathrm{Ass}_X^1$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r.$$

A3 For $\alpha \in F_X$ and $\omega_1, \omega_2, \ldots \omega_r \in \mathrm{Ass}_X^1$

$$\int_{\alpha^{-1}} \omega_1\omega_2 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1.$$

A4 For $\alpha \in F_X$ and $\omega_1, \omega_2, \ldots \omega_{r+s} \in \mathrm{Ass}_X^1$ we have

$$\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \int_\alpha \omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s}, \tag{6}$$

where

$$\omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s} = \sum \omega_{k_1}\omega_{k_2} \cdots \omega_{k_{r+s}}$$

is the shuffle product of $\omega_1 \cdots \omega_r$ and $\omega_{r+1} \cdots \omega_{r+s}$.

Let $\{\delta_1, \delta_2, \ldots, \delta_m\}$ be a set which generates F_X freely and $\{\omega_1, \omega_2, \ldots, \omega_m\}$ be a basis of the k-vector space Ass_X^1. **A1**, **A2**, **A3** imply that every iterated integral can be written as a polynomial in

$$\int_{\delta_j} \omega_{i_1}\omega_{i_1} \cdots \omega_{i_r},\ j = 1, 2, \ldots, m,\ i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, m\}. \tag{7}$$

Therefore by **A4** the map (5) defines an iterated integral if and only if the numbers

$$\int_{\delta_j} \omega_{i_1}\omega_{i_2} \cdots \omega_{i_r} = a^j(i_1, \ldots, i_r) \in \mathsf{k}$$

satisfy the "shuffle relations"

$$a^j(i_1, \ldots, i_r)a^j(i_{r+1}, \ldots, i_{r+s}) = \sum a^j(k_1, \ldots, k_{r+s}), \tag{8}$$

where $(k_1, \ldots, k_{r+s})$ runs through all shuffles of $(i_1, \ldots, i_r)$ and $(i_{r+1}, \ldots, i_{r+s})$. The existence of such numbers $a^j(i_1, \ldots, i_r)$ is, however, not obvious.

Example 1. Let $\{\delta_1, \delta_2, \ldots, \delta_m\}$ be a set which generates F_X freely and $\{\omega_1, \omega_2, \ldots, \omega_m\}$ be a basis of the k-vector space Ass_X^1. We set

$$a^j(i_1, \ldots, i_n) = \int_{\delta_j} \omega_{i_1}\omega_{i_2} \cdots \omega_{i_n} = 0 \text{ if at least one of } \omega_{i_s} \text{ is not } \omega_j, \tag{9}$$

$$a^j(j, \ldots, j) = \int_{\delta_j} \omega_j^n = \frac{1}{n!}.$$

The verification that the numbers $a^j(i_1,\dots,i_n)$ define an iterated integral is straightforward. This iterated integral can be interpreted as Chen's iterated integral in the following way: We take the Ceyley diagrams (see [17]) which is the topological space $Y := \cup_{i=0}^{m-1}\mathbb{Z}^i \times \mathbb{R} \times \mathbb{Z}^{m-i-1} \subset \mathbb{R}^m$. Each element δ of the free group F_X is represented by a path $\tilde{\delta}$ in Y with the starting point $0 \in \mathbb{R}^n$ and the end point in some element of $\mathbb{Z}^m$. Each element ω of Ass_X can be interpreted as a differential form $\tilde{\omega}$ substituting dy_i by x_i, where $(y_1, y_2, \dots, y_m)$ is the coordinate system in $\mathbb{R}^m$. Now, $\int_\delta \omega$ is the classical Chen's iterated integral $\int_{\tilde{\delta}} \tilde{\omega}$.

Example 2. As in the Introduction, let Γ be a punctured Riemann surface with free finitely generated fundamental group $F_X = \pi_1(\Gamma, *)$. Let $\{\omega_1, \omega_2, \dots, \omega_m\}$ be a collection of holomorphic one-forms on Γ such that their classes in the de Rham cohomology of Γ form a basis. The Chen's iterated integral

$$a^j(i_1,\dots,i_n) = \int_{\delta_j} \omega_{i_1}\omega_{i_2}\cdots\omega_{i_n},$$

where $\delta_1, \delta_2, \dots, \delta_m$ generates F_X freely, satisfies the shuffle relations (8), see [6, 7], and hence defines an iterated integral in the sense of Definition 1 (in order to follow the terminology used in this text we may identify ω_i with x_i).

3 The π_1 de Rham theorem

Let $\alpha, \beta \in F_X$ and $\omega \in \mathrm{Ass}^1_X = L^1_X$. Then **A3** implies that every iterated integral (1) satisfies

$$\int_{\alpha\beta} \omega = \int_\alpha \omega + \int_\beta \omega, \int_{\alpha\beta\alpha^{-1}} \omega = \int_\alpha \omega, \int_{(\alpha,\beta)} \omega = 0.$$

More generally,

$$\forall \alpha \in F_X^{n+1}, \omega \in \mathrm{Ass}^m_X, \text{ such that } m \le n \text{ holds true } \int_\alpha \omega = 0. \tag{10}$$

Therefore the iterated integral $\int$ induces a map

$$\begin{array}{rcl} \int : \mathrm{gr}^k F_X \times \mathrm{Ass}^k_X & \to & \mathbb{C} \\ (\delta, \omega) & \mapsto & \int_\delta \omega \end{array} \tag{11}$$

which is $\mathbb{Z}$-linear in the first argument and k-linear in the second argument. Suppose that $\omega \in \mathrm{Ass}^k_X$ is a shuffle product of ω_1, ω_2:

$$\omega = \omega_1 * \omega_2, \omega_1 \in \mathrm{Ass}^{k_1}_X, \omega_2 \in \mathrm{Ass}^{k_2}_X, k_1 + k_2 = k$$

and $\alpha \in \mathrm{gr}^k F_X$. Then **A4** and (10) imply

$$\int_\alpha \omega = 0.$$

Thus the vector space S^k_X generated by shuffle products

$$S^k_X = \mathrm{Span}\{\omega_1 * \omega_2 : \omega_1 \in \mathrm{Ass}^{k_1}_X, \omega_2 \in \mathrm{Ass}^{k_2}_X, k_1 + k_2 = k\}$$

is in the kernel of the bilinear map (11). We are ready to formulate the π_1 de Rham theorem:

Theorem 1. *Let $\int$ be an iterated integral in the sense of Definition 1.*

(a) *If the induced bilinear map*

$$\begin{array}{rcl} \int : \mathrm{gr}^k F_X \times L_X^k & \to & \mathsf{K} \\ (\delta, \omega) & \mapsto & \int_\delta \omega \end{array} \tag{12}$$

is non-degenerate for $k = 1$, then then it is non-degenerate for all $k \in \mathbb{N}$.

(b) *Let $\omega \in \mathrm{Ass}_X^k$. The linear map*

$$\begin{array}{rcl} \int \omega : \mathrm{gr}^k F_X & \to & \mathsf{K} \\ \delta & \mapsto & \int_\delta \omega \end{array} \tag{13}$$

is the zero map, if and only if $\omega \in S_X^k$.

In Ass_X we consider the canonical k-bilinear symmetric product given by

$$\langle x_{i_1} x_{i_2} \cdots x_{i_r}, x_{j_1} x_{j_2} \cdots x_{j_s} \rangle = \begin{cases} 1 & i_1 = j_1, \ldots, i_r = j_r, r = s \\ 0 & \text{otherwise} \end{cases}$$

Before proving Theorem 1 we shall need the following auxiliary result:

Theorem 2. *S_X^k and L_X^k are orthogonal vector subspaces of Ass_X^k and for all $k \in \mathbb{N}$*

$$\mathrm{Ass}_X^k = L_X^k \oplus S_X^k.$$

This is a geometric reformulation of the following classical result of Ree [20, Theorem 2.2].

Theorem 3. *A polynomial*

$$\omega = \sum_{n>0} \sum a(i_1, i_2, \ldots, i_n) \omega_{i_1} \omega_{i_2} \cdots \omega_{i_n} \in \mathrm{Ass}_X \tag{14}$$

is a Lie element (i.e. belongs to L_X) if and only if for all $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_s)$ we have:

$$\sum a(k_1, k_2, \ldots, k_{r+s}) = 0 \tag{15}$$

where $(k_1, \ldots, k_{r+s})$ runs through all shuffles of $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_s)$.

Proof of the equivalence of Theorem 2 and Theorem 3. Without loss of generality we can assume that ω in (14) is homogeneous of degree n. The proof follows from the equality:

$$\langle \omega, \omega_{i_1} \omega_{i_2} \cdots \omega_{i_r} * \omega_{j_1} \omega_{j_2} \cdots \omega_{j_s} \rangle = \sum a(k_1, k_2, \ldots, k_{r+s})$$

where $\omega_{i_1} \omega_{i_2} \cdots \omega_{i_r} * \omega_{j_1} \omega_{j_2} \cdots \omega_{j_s}$, $r + s = n$ is a shuffle element and $(k_1, \ldots, k_{r+s})$ runs through all shuffles of $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_s)$. Note that we can formulate Theorem 2 in the following way: The polynomial ω is a Lie element if and only if it is orthogonal to all shuffles $\omega_{i_1} \omega_{i_2} \cdots \omega_{i_r} * \omega_{j_1} \omega_{j_2} \cdots \omega_{j_s}$, $r + s = n$. $\square$

To prove Theorem 1 we note that as the map (12) is non-degenerate for $k=1$, then it can be "diagonalized" as follows: We fix a set of generators $\{\delta_1, \delta_2, \dots, \delta_m\}$ for F_X and find $\Omega = \{\omega_1, \dots, \omega_m\} \subset \mathrm{Ass}^1_X$ such that $\int_{\delta_i} \omega_j = 1$ if $i = j$ and is zero otherwise. Now, Ass_X is freely generated by Ω. Therefore we shall suppose, without loss of generality, that

$$\int_{x_i} x_j = \langle x_i, x_j \rangle = 0 \text{ if } i \neq j \text{ and } 0 \text{ otherwise.} \tag{16}$$

The formula (16) generalizes as follows:

Proposition 1. *We have*

$$\int_\delta \omega = \langle \omega, \phi^{-1}\delta \rangle, \ \forall \omega \in \mathrm{Ass}^k_X, \ \delta \in \mathrm{gr}^k F_X. \tag{17}$$

where ϕ is the isomorphism (4).

Proof. The proof is by induction on k. For $k=1$ it follows from (16). If $\delta = (a, b)$, where $a \in F^p_X, b \in F^q_X$, $p + q = k > 1$, then **A2** implies

$$\begin{aligned}
\int_{(a,b)} x_{j_1} x_{j_2} \cdots x_{j_k} &= \int_a x_{j_1} \cdots x_{j_p} \int_b x_{j_{p+1}} \cdots x_{j_k} \\
&\quad - \int_b x_{j_1} \cdots x_{j_q} \int_a x_{j_{q+1}} \cdots x_{j_k} \\
&= \langle \phi^{-1}(a), x_{j_1} \cdots x_{j_p} \rangle \langle \phi^{-1}(b), x_{j_{p+1}} \cdots x_{j_k} \rangle \\
&\quad - \langle \phi^{-1}(b), x_{j_1} \cdots x_{j_q} \rangle \langle \phi^{-1}(a) x_{j_{q+1}} \cdots x_{j_k} \rangle \\
&= \langle \phi^{-1}(a)\phi^{-1}(b), x_{j_1} \cdots x_{j_p} x_{j_{p+1}} \cdots x_{j_k} \rangle \\
&\quad - \langle \phi^{-1}(b)\phi^{-1}(a), x_{j_1} \cdots x_{j_q} x_{j_{q+1}} \cdots x_{j_k} \rangle \\
&= \langle \phi^{-1}(a)\phi^{-1}(b) - \phi^{-1}(b)\phi^{-1}(a), x_{j_1} x_{j_2} \cdots x_{j_k} \rangle \\
&= \langle \phi^{-1}((a,b)), x_{j_1} x_{j_2} \cdots x_{j_k} \rangle.
\end{aligned}$$

□

Remark 2. For an arbitrary iterated integral map, the same proof implies that for every $\delta \in \mathrm{gr}^k F_X$ such that

$$\phi^{-1}(\delta) = \sum a(i_1, i_2, \dots, i_k) x_{i_1} x_{i_2} \cdots x_{i_k} \in \mathrm{gr}^k L_X$$

holds

$$\int_\delta \omega_{j_1} \omega_{j_2} \cdots \omega_{j_k} = \sum_{i_1, i_2, \dots, i_k} a(i_1, i_2, \dots, i_k) \int_{x_{i_1}} \omega_{j_1} \int_{x_{i_2}} \omega_{j_2} \cdots \int_{x_{i_k}} \omega_{j_k} \tag{18}$$

Proof of Theorem 1. The part (b) follows from Theorem 2 and Proposition 1. Let $\delta \in \mathrm{gr}^k F_X$ and

$$\phi^{-1}(\delta) = \sum a(i_1, i_2, \dots, i_k) x_{i_1} x_{i_2} \cdots x_{i_k} \in \mathrm{gr}^k L_X$$

where in the above sum each non-commutative monomial $x_{i_1} x_{i_2} \cdots x_{i_k}$ is repeated only once. Then $\delta \neq 0$ if and only if

$$\langle \phi^{-1}(\delta), \phi^{-1}(\delta) \rangle = \sum |a(i_1, i_2, \dots, i_k)|^2 \neq 0.$$

Therefore for every $\delta \in \mathrm{gr}^k F_X$ there exists $\omega \in L_X^k$, namely $\omega = \phi^{-1}(\delta)$, such that

$$\int_\delta \omega \neq 0.$$

□

Here we have used strongly the fact that the characteristic of k is zero.

Remark 3. A canonical basis of the k-vector spaces $L_X^k \cong \mathrm{gr}^k F_X$ is given by basic commutators (see for instance [8, 21]). By definition of $\langle \cdot, \cdot \rangle$ if the number of some $x_i, i = 1, 2, \ldots, m$ used in two basic commutators ω_1 and ω_2 are different then $\langle \omega_1, \omega_2 \rangle = 0$. The basic commutators of weight 1 and 2 are dual to each other with respect to the bilinear map $\langle \cdot, \cdot \rangle$. However, this is not the case for weight 3 and the number of generators m bigger than 2. For instance, we have

$$\langle [y, [x, z]], [z, [x, y]] \rangle = 2$$

For $m = 2$ the basic commutators of weight 3 (resp. 4) are orthogonal to each other. In $m = 2$ and $r = 5$ the orthogonality fails. There are two couples in which the number of x is equal to 2 (resp. 3). In fact we have

$$\langle [y, [x, [x, [x, y]]]], [[x, y], [x, [x, y]]] \rangle = -28, \;\; \langle [y, [y, [x, [x, y]]]], [[x, y], [y, [x, y]]] \rangle = -14$$

The computation of basic commutators and scalar products in the free Lie algebra L_X are implemented in the symbolic algebra system Axiom, see [14] and the webpage of the second name author

4 Picard-Fuchs equations and iterated path integrals

Let f be a polynomial of degree d in two variables x, y and suppose, for simplicity, that the highest order homogeneous piece g of f is non-degenerate (has an isolated critical point). Let also $C \subset \mathbb{C}$ be the set of the critical values of f. The cohomology fiber bundle $\cup_{t \in \mathbb{C} \setminus C} H^1(\{f = t\}, \mathbb{C})$ and the corresponding Gauss-Manin connection, for which flat sections are generated by sections with images in $\cup_{t \in \mathbb{C} \setminus C} H^1(\{f = t\}, \mathbb{Z})$, is encoded in the global Brieskorn module $H = \frac{\Omega^1}{df \wedge \Omega^0 + d\Omega^0}$, where Ω^i is the set of polynomial i-forms in $\mathbb{C}^2$ as follows. We note first that H is a $\mathbb{C}[t]$-module ($t.[\omega] = [f\omega]$) generated freely by

(19) $$\omega_i := x^{i_1} y^{i_2} (xdy - ydx), i = (i_1, i_2) \in I$$

where $\{x^{i_1} y^{i_2}, \; i \in I\}$ is a basis of monomials for the $\mathbb{C}$-vector space $\frac{\mathbb{C}[x,y]}{\langle f_x, f_y \rangle}$, see [1, 16]. In particular the rank of H equals the dimension of $H^1(\{f = t\}, \mathbb{Z})$ for generic t. The Gauss-Manin connection of the family of curves $f(x, y) = t, \; t \in \mathbb{C}$ with respect to the parameter t becomes an operator $' : H \to \frac{1}{\Delta} H$ which satisfies the Leibniz rule, where $\Delta = \Delta(t)$ is the discriminant of the polynomial $f(x, y) - t$. In the basis $\omega = (\omega_i)_{i \in I}$ (written in a column) it is of the form

(20) $$\omega' = \frac{1}{\Delta} A \omega,$$

where A is a $m \times m$ matrix with entries in $\mathbb{C}[t]$ and $m = \#I$.

The above construction has a natural generalization based on the π_1 de Rham theorem (Theorem 1), which we discuss briefly. Let

$$F_t = \pi_1(\{f = t\}, *),\ t \in \mathbb{C}\backslash C$$

be the fundamental group of the fiber $\{f = t\}$ (it was denoted F_X in §3). Consider the trivial fiber bundle $\cup_{t\in\mathbb{C}\backslash C}\mathrm{gr}^k F_t \otimes_{\mathbb{Z}} \mathbb{C}$ and its dual $\cup_{t\in\mathbb{C}\backslash C}\check{\mathrm{gr}}^k F_t \otimes_{\mathbb{Z}} \mathbb{C}$. Both of them have a canonical flat connection defined as follows:

Let $X = \{\omega_i\}_{i\in I}$, $\mathsf{k} = \mathbb{C}(t)$ and K the field of locally analytic multi-valued functions on $\mathbb{C}\backslash C$. The Gauss-Manin connection on Ass_X^1 extends canonically to a derivation operator

$$\frac{\partial}{\partial t} =' : \mathrm{Ass}_X \to \frac{1}{\Delta}\mathrm{Ass}_X$$

which respects both the graduation of Ass_X, the direct sum decomposition $\mathrm{Ass}_X = L_X \oplus S_X$, and satisfies the Leibniz rule $(ab)' = a'b + ab',\ a, b \in \mathrm{Ass}_X$. In particular for monomials in Ass_X it is given by:

$$\omega_{i_1}\omega_{i_2}\cdots\omega_{i_r} \to \sum_{j=1}^{r}\omega_{i_1}\omega_{i_2}\cdots\omega_{i_{j-1}}\omega'_{i_j}\omega_{i_{j+1}}\cdots\omega_r.$$

The main property of the map $'$ is the following:

$$\frac{\partial}{\partial t}\int_{\delta(t)}\omega = \int_{\delta(t)}\omega' \tag{21}$$

for all $\omega \in \mathrm{Ass}_X^k$ and continuous family of closed loops $\delta(t) \in F_t^k$. The induced map $' : L_X \to \frac{1}{\Delta}L_X$, where L_X is the k-vector space $L_X \cong \frac{\mathrm{Ass}_X}{S_X}$, is the desired generalization of the usual (algebraic) Gauss-Manin connection.

5 Non-persistence of Hamiltonian identity cycles

Let $\mathcal{F}$ be a holomorphic foliation with singularities on the plane $\mathbb{C}^2$. A cycle on a leaf of $\mathcal{F}$ is a non-zero free homotopy class of closed loops on this leaf. Let δ be a closed non-contractible loop contained in a leaf and denote the holonomy map associated to δ by h_δ. The cycle represented by δ is said to be an *identity cycle* provided that h_δ is the identity map, and *Hamiltonian identity cycle* provided that $\mathcal{F} = \{df = 0\}$ for some polynomial f. It is called a *limit cycle* provided that δ corresponds to an isolated fixed point of h_δ. Let $\mathcal{F}^n$ be a polynomial foliation as above of degree n, and let δ be an identity cycle.

Question: *Is there a sufficiently small degree n perturbation of $\mathcal{F}^n$ which either destroys the cycle δ, or makes it a limit cycle ?*

In the case when $\mathcal{F}^n$ is a plane Hamiltonian foliation the following theorem has been proved by Ilyashenko and Pyartli:

Theorem 4. *([10, Theorem 2]) For any Hamiltonian identity cycle δ in a given leaf of $\mathcal{F}^n = \{df = 0\},\quad \deg(f) = n + 1 > 3$, there exists a perturbation of this foliation in the class of polynomial foliations of degree n which destroys the identity cycle .*

Our approach to the above problem of destroying the Hamiltonian identity cycles is as follows. Let f be a degree $n + 1$ polynomial and let $\omega = Pdx + Qdy$ be a polynomial one-form in $\mathbb{C}^2$ of degree at most n, $\deg P, \deg Q \leq n$. Consider the holomorphic foliation on $\mathbb{C}^2$ defined by

$$df + \epsilon\omega = 0, \epsilon \sim 0. \tag{22}$$

Let t be a regular value of f and $\delta_t \subset f^{-1}(t)$ be a continuous family of non-contractible closed loops. As in the preceding sections we denote by F_t the fundamental group of the fiber $\{f^{-1}(t)\}$. There is an integer $k \geq 1$ such that $\delta(t)$ represents a not zero element in $\mathrm{gr}^k F_t$. We consider the holonomy map h^ϵ_δ along the path $\delta(t)$

$$h^\epsilon_\delta(t) = t + \sum_{i=1}^{\infty} \epsilon^i M_i(t). \tag{23}$$

It is known [3, 15] that $M_1 = \cdots = M_{k-1} = 0$ and M_k is an iterated path integral

$$M_k(t) = \int_{\delta(t)} \omega(\underbrace{\omega(\cdots(\omega(\omega)')'}_{k-1 \text{ times}}\cdots)')' \tag{24}$$

where $\omega' \in \frac{H}{\Delta}$ is defined by (20). It follows in particular that the integral

$$\int_{\delta(t)} \omega'$$

is well defined and does not depend on the particular rational one-form representing the equivalence class of ω'. In a similar way one verifies, using section 2, that the iterated integral in (24) is well defined, and depends only on the equivalence classes of the successive derivatives of ω.

If the function M_k is not identically zero then its zeros correspond to limit cycles of the deformed foliation. In particular the deformation $df + \epsilon\omega$ " destroys" the family of identity cycles $\delta(t)$ in the sense that the holonomy map $h_\epsilon, \epsilon \neq 0$, is not the identity map. This leads to the following:

Question: *Let $\delta(t)$ be a continuous family of closed loops representing a non-trivial element of $\mathrm{gr}^k F_t$. Is there a sufficiently small degree n perturbation of $\{df = 0\}$, such that the corresponding Poincaré-Pontryagin function M_k is not zero ?*

A positive answer to this question implies that the holonomy map (23) is not the identity map. We shall prove that:

(i) The answer to the above question is, in general, negative (Example 3).

(ii) If the loop $\delta(t)$ is generic, then the answer is positive (Theorem 6).

(iii) If the loop is a commutator, $\delta(t) \in \mathrm{gr}^2 F_t$, and the polynomial f is of degree three (case not covered by [10]), then the answer is positive (Theorem 7).

For completeness we consider first the seemingly trivial case $k = 1$ (this is well known to the specialists). We have:

Proposition 2. *Let f be a degree $n+1 > 1$ polynomial, whose degree $n+1$ homogeneous part is non-degenerate (has an isolated critical point). For every fixed non-critical value $t \in \mathbb{C}$ and for every closed loop $\delta \subset f^{-1}(t)$ representing a non-zero homology class in $H_0(f^{-1}(t), \mathbb{Z})$, there is a polynomial one-form $\omega = Pdx + Qdy$ of degree at most n, $\deg P, \deg Q \leq n$, such that*

$$M_1'(t) = \int_\delta \omega' \neq 0.$$

The above proposition implies that $M_1(t)$ is not identically zero and hence the cycle δ is destroyed by the perturbation (22). This also follows (with a *Morse-plus* condition on f) from the following theorem, proved by Ilyashenko:

Theorem 5 ([9])**.** *Let f be a Morse-plus bi-variate polynomial, $\delta(t) \subset H_1(f^{-1}(t), \mathbb{Z})$ a continuous family of vanishing cycles, and ω a polynomial one-form of degree at most n as above. Then $M_1(t) = \int_{\delta(t)} \omega$ is identically zero, if and only if ω is exact.*

The condition that f must be Morse-plus (has n^2 distinct critical values) can not be removed from Ilyashenko's theorem.

Proof of Proposition 2. Consider the one-forms $\omega_i' = \frac{d\omega_i}{df}$ defined by (19). It is well known (and easy to check) that when $|i| = i_1 + i_2 < n-1$, the $n(n-1)/2$ forms ω_i' generate the space of holomorphic differential one-forms on the compact Riemann surface $\Gamma = \overline{f^{-1}(t)}$

$$H^0(\Gamma, \Omega^1) = \mathrm{Span}\{\omega_i' : |i| = i_1 + i_2 < n-1\}$$

while the $n(n+1)/2$ one-forms ω_i', $|i| = i_1 + i_2 \leq n-1$, generate the space of meromorphic one-forms with at most a simple pole at infinity:

$$H^0(\Gamma, \Omega^1(\infty_1 + \infty_2 + \dots \infty_{n+1})) = \mathrm{Span}\{\omega_i' : |i| = i_1 + i_2 \leq n-1\}, \tag{25}$$

where ∞_i are the $n+1$ distinct intersection points of Γ with the line at infinity. This combined to the Hodge decomposition

$$H^1_{DR}(\Gamma) = H^0(\Gamma, \Omega^1) \oplus H^0(\Gamma, \bar{\Omega}^1)$$

and the obvious equalities

$$\dim H^0(\Gamma, \Omega^1(\infty_1 + \infty_2 + \dots \infty_{n+1})) = \dim H^0(\Gamma, \Omega^1) + n,$$

$$\dim H_1(f^{-1}(t)) = \dim H_1(\Gamma) + n$$

implies

$$H^1_{DR}(f^{-1}(t)) = \mathrm{Span}\{\omega_i' : |i| = i_1 + i_2 \leq n-1\} \oplus \mathrm{Span}\{\overline{\omega_i'} : |i| = i_1 + i_2 < n-1\}.$$

Let, for a fixed regular value t, $\delta \subset f^{-1}(t)$ be a closed loop representing a non-zero homology class in $H_0(f^{-1}(t), \mathbb{Z})$. If $\int_\delta \omega_i' = 0, \forall i, |i| = i_1 + i_2 \leq n-1$, then $\int_\delta \overline{\omega_i'} = 0, \forall i, |i| = i_1 + i_2 < n-1$ and hence the homology class of δ is zero. Proposition 2 is proved. $\square$

Let, more generally, $\delta(t) \subset f^{-1}(t)$ be a continuous family of closed loops representing a not zero element in $\mathrm{gr}^k F_t$, $k \geq 1$. Let V_n be the vector space of all pairs (f, ω), where f is a bi-variate polynomial of degree at most $n+1$ and ω is a one-form of degree at most n.

Theorem 6. *There is a dense open subset of V_n with the following property: for every pair (f, ω) of this open subset and for every integer $k \geq 1$ there exists a continuous family of closed loops $\delta(t) \subset f^{-1}(t)$ representing a non-zero homology class in $\mathrm{gr}^k F_t$, such that the corresponding Poincaré-Pontryagin function $M_k(t)$ is not identically zero.*

The following example shows that the result of the above theorem can not be improved.

Example 3. Let f be a polynomial of degree $n+1$, with n^2 distinct critical values, where $n \geq 2$ (such polynomials form a Zarisky open set in the space of degree $n+1$ polynomials). The generic leaf $\{f = t\} \subset \mathbb{C}^2$ is a compact Riemann surface with $n+1$ removed points. Let $\alpha_1(t), \alpha_2(t) \subset \{f = t\}$ be two continuous families of loops which make one turn about two distinct removed points on $\{f = t\}$, and put $X = \{\alpha_1, \alpha_2\}$. The element

$$(((\alpha_1, \alpha_2), \alpha_1), (\alpha_1, \alpha_2))$$

is a basic commutator of degree 5 [21, p.23], and hence represents a non-zero equivalence class in $L_X^5 = F_5/F_6$. Define the continuous family of loops

$$\gamma(t) = (((\alpha_1(t), \alpha_2(t)), \alpha_1(t)), (\alpha_1(t), \alpha_2(t))).$$

We claim that for every polynomial one-form of degree at most n, the Poincaré-Pontrygin function M_5 associated to the deformed foliation (22) and to the family of loops $\gamma(t)$ is identically zero. Indeed, according to (24), the function M_5 is a polynomial in $\int_{\alpha_i(t)} \omega^{(k)}$, $i = 1, 2$, $k = 0, 1, ..., 4$. However, for $k \geq 2$ the one-form $\omega^{(k)}$ has no residues. Therefore (24) takes the form

$$M_k(t) = \int_{\gamma(t)} \omega\omega'\omega'\omega'\omega'$$

and hence

$$\begin{aligned} M_5(t) &= \int_{((\alpha_1(t),\alpha_2(t)),\alpha_1(t))} \omega\omega'\omega' \int_{(\alpha_1(t),\alpha_2(t))} \omega'\omega' \\ &\quad - \int_{(\alpha_1(t),\alpha_2(t))} \omega\omega' \int_{((\alpha_1(t),\alpha_2(t)),\alpha_1(t))} \omega'\omega'\omega' \\ &\equiv 0. \end{aligned}$$

Proof of Theorem 6. Let f^0 be a degree $n+1$ polynomial, non-degenerate at infinity, and ω^0 a degree n one-form. Denote by $\delta(t) \subset \{f^0 = t\}$ a continuous family of closed loops representing a not zero element in $\mathrm{gr}^k F_t$, $k \geq 1$. The function $M_k(t)$, associated to the pair $(f, \omega) \in V_n$ and the family of loops $\delta(t) \subset f^{-1}(t)$ is well defined, at least for (f, ω) sufficiently close to (f^0, ω^0). Therefore the condition that $M_k(t)$ is identically zero defines a closed subset of V_n (possibly equal to V_n). It remains to show that there exists a point $(f^0, \omega^0) \in V_n$ at which the corresponding function $M_k(t)$ is not identically zero. Let f^0 be a homogeneous, degree $n+1$ polynomial, which is non-degenerate at infinity. We shall show that there exists a one-form ω of degree at most n, and a continuous family of closed loops representing a not zero element in $\mathrm{gr}^k F_t$, such that the associated function $M_k(t)$ is not identically zero.

The proof is an application of the π_1 de Rham theorem formulated in §3. Let X be the set of n^2 co-homology classes defined by the one-forms ω_i in $H^1_{DR}(f^{-1}(t))$, see (19) . By making use of the Picard-Fuchs system satisfied by ω_i we can represent the Poincaré-Pontryagin function $M_k(t)$, (24), in the form

$$M_k(t) = \int_{\delta(t)} P_k(\omega), \tag{26}$$

where $P_k(\omega) \in \mathrm{Ass}_X^k$ is a suitable degree k non-commutative polynomial. If the function $M_k(t)$ were identically zero for all family of cycles $\delta(t)$, then the polynomial $P_k(\omega)$ would belong, for every regular value t, to the space S_X^k generated by shuffle products. Let V be the vector space generated by ω_i's. It is easy to check that the set $\{\omega \in V \mid P_k(\omega) \in S_X^k\}$ is algebraic. We are going to prove that it is not the whole V.

Taking into consideration the non-degeneracy of f at infinity, we deduce that the Picard-Fuchs system associated to ω_i is diagonal

$$t\omega_i' = w_i \omega_i, \text{ where } w_i = \text{weight}(\omega_i) := \frac{|i|+2}{n+1}.$$

This allows to deduce an explicit formula for $P_k(\omega)$. Let, for instance, $\omega = \alpha_1\omega_1 + \alpha_2\omega_2$, where weight$(\omega_i) = w_i$. One easily proves by induction that

$$P_k(\omega) = \frac{1}{t^{k-1}} \sum_{i=0}^{k} \alpha_1^i \alpha_2^{k-i} \sum c_{i_1 i_2 \dots i_k} \omega_{i_1}\omega_{i_2}\dots\omega_{i_k} \tag{27}$$

where

$$c_{i_1 i_2 \dots i_k} = w_{i_k}(w_{i_k} + w_{i_{k-1}} - 1)\dots(w_{i_k} + w_{i_{k-1}} + \dots + w_{i_2} - k + 2)$$

and in the second sum of (27), the index $i_1 i_2 \dots i_k$ runs through all the shuffles of

$$\underbrace{(1\dots 1)}_{i \text{ times}} \text{ and } \underbrace{(2\dots 2)}_{k-i \text{ times}}.$$

Note that the first sum in (27) is with respect to all partitions $i + (k-i)$ of k. This fact reflects a further decomposition of S_X^k and L_X^k in direct orthogonal sums corresponding to partitions of k. The polynomial $P_k(\omega)$ belongs to S_X^k if and only if it is orthogonal to L_X^k (Theorem 2). It suffices to show that the coefficient of at least one monomial $\alpha_1^i\alpha_2^{k-i}$ in (27) is not orthogonal to L_X^k. Take for instance the partition $k = 1 + (k-1)$. The coefficient of $\alpha_1\alpha_2^{k-1}$ takes the form

$$\begin{array}{rcl} P_k^1(\omega_1,\omega_2) & = & \omega_1\omega_2^{k-1} w_2(2w_2-1)\dots((k-1)w_2-k+2) \\ & + & \omega_2\omega_1\omega_2^{k-2} w_2(2w_2-1)\dots((k-2)w_2-k+3)((k-2)w_2+w_1-k+2) \\ & & \vdots \\ & + & \omega_2^{k-1}\omega_1 w_1(w_1+w_2-1)\dots((k-2)w_2+w_1-k+2). \end{array}$$

The subspace of L_X^k corresponding to the partition $k = 1 + (k-1)$ is one-dimensional and generated by the Lie polynomial

$$L_k^1(\omega_1,\omega_2) = [[\dots[[\omega_1,\omega_2],\omega_2],\dots,\omega_2],\omega_2].$$

Consider the scalar product

$$C_k(w_1,w_2) = \langle P_k^1(\omega_1,\omega_2), L_k^1(\omega_1,\omega_2)\rangle. \tag{28}$$

The identities

$$\begin{array}{rcl} \langle L_k^1, \omega_{i_1}\omega_{i_2}\dots\omega_{i_k}\rangle & = & \langle [L_{k-1}^1,\omega_2], \omega_{i_1}\omega_{i_2}\dots\omega_{i_k}\rangle \\ & = & \langle L_{k-1}^1, \omega_{i_1}\omega_{i_2}\dots\omega_{i_{k-1}}\rangle\langle\omega_2,\omega_{i_k}\rangle \\ & & -\langle\omega_2,\omega_{i_1}\rangle\langle L_{k-1}^1,\omega_{i_2}\omega_{i_3}\dots\omega_{i_k}\rangle \end{array}$$

applied to the scalar product C_k lead to the following recursive formula

$$C_k(w_1, w_2) = (w_2 - w_1)C_{k-1}(w_1 + w_2 - 1, w_2), k \geq 3, \;\; C_2(w_1, w_2) = w_2 - w_1$$

and hence

$$C_k(w_1, w_2) = (w_2 - w_1)\Pi_{i=1}^{k-2}(i - w_1 - (i-1)w_2).$$

Recall now that $w_i := \frac{|i|+2}{n+1}$, where $|i| + 2 \leq n - 1$. If we choose $w_1 \neq w_2$ and $w_1 < 1$ then the scalar product $C_k(w_1, w_2)$ is not zero. The Theorem is proved. □

Theorem 7. *Let f be a degree three polynomial, whose degree three homogeneous part is non-degenerate (has an isolated critical point). Let $\delta(t) \subset \{f = t\}$ be a continuous family of loops representing a non-zero element in F_X^2/F_X^3, where X is a basis of the first homology group $H_1(\{f = t\}, \mathbb{Z})$. Then there exists a one-form ω of degree at most two, such that*

$$M_2(t) = \int_{\delta(t)} \omega\omega' \not\equiv 0.$$

For a generic f and a particular class of cycles $\delta(t)$ one can say that $M_2 = 0$ if and only if ω is exact. For more details on this see [15]. Before proving the above theorem we need some preparation. For a generic t the affine algebraic curve $\{f = t\}$ is a topological torus with three removed points, and f has four critical points. Let $h_i \in \mathrm{Aut}(H_1(\{f = t_0\}, \mathbb{Z}))$, $i = 1, 2, 3, 4$ be the associated monodromy operators (t_0 is a fixed regular value). Let $\delta_i(t), \alpha_i(t)$, $i = 1, 2$, be continuous family of cycles in $H_1(\{f = t\}, \mathbb{Z})$ which form a basis. We suppose moreover that $\delta_1(t), \delta_2(t)$ are vanishing cycles, while $\alpha_1(t), \alpha_2(t)$ are homologous to zero on the compactified curve $\{f = t\}$. Without loss of generality we have $h_i(\alpha_j) = \alpha_j$, $h_i(\delta_i) = \delta_i$, and

$$h_1(\delta_2) = \delta_2 + \delta_1, h_2(\delta_1) = \delta_1 - \delta_2.$$

Identifying F_X^2/F_X^3 with L_X^2, where

$$X = \{\delta_1, \delta_2, \alpha_1, \alpha_2\}$$

we represent an element $\delta \in F_X^2/F_X^3$ as a linear combination of commutators

$$[\delta_1, \delta_2], [\delta_i, \alpha_j], [\alpha_1, \alpha_2], \;\; i, j = 1, 2.$$

The monodromy operators h_i induce automorphisms of F_X^2/F_X^3 denoted by the same letter.

Proposition 3. *Let $\delta(t)$ be a continuous family of loops representing a non-zero equivalence class in F_X^2/F_X^3. There exists a polynomial P in h_i and with integer coefficients, such that*

$$P(h_1, h_2, h_3, h_4)(\delta) = k[\alpha_1, \alpha_2],$$

where k is a non-zero integer.

Proof. The proof is straightforward. Let δ_i, $i = 1, 2, 3, 4$ be vanishing cycles associated to the critical values of f. The Dynkin diagram of f is of type D_4 and we may suppose that the intersection matrix is

$$(\delta_i \cdot \delta_j) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$\alpha_1=\delta_1-\delta_3$ and $\alpha_2=\delta_1-\delta_4$ and

$$h_i(\delta)=\delta-(\delta\cdot\delta_i)\delta_i$$

(the Picard-Lefschetz formula). Every $\delta\in F_X^2/F_X^3$ is of the form

$$\delta=[\delta_1,a]+[\delta_2,b]+m[\delta_1,\delta_2]+n[\alpha_1,\alpha_2] \tag{29}$$

where $m,n\in\mathbb{Z}$ and a,b are integer linear combinations in α_1,α_2. We have

$$(h_1-id)(\delta) = [\delta_1,b] \tag{30}$$
$$(h_2-id)(\delta) = -[\delta_2,a] \tag{31}$$

which shows that it is enough to prove the Proposition for δ as in (29) with $m=n=0$ (provided that either a or b is non-zero). In the case $m=0$ we have

$$(h_3-id)(\delta) = -[\delta_3,b] \tag{32}$$
$$(h_4-id)(\delta) = -[\delta_4,b] \tag{33}$$

Therefore by (30), (31), (32) there is a polynomial P (a linear combination in h_1,h_3,h_4,id) such that $P(\delta)=[\alpha_1,b]$ or $P(\delta)=[\alpha_2,b]$. If $b\neq 0$ the Proposition is proved. If $b=0$ but $a\neq 0$ then using (31) we replace without loss of generality δ by $[\delta_2,a]$ and conclude as above. It remains the case $a=b=0$ but $m\neq 0$. We have

$$(h_3-id)(\delta) = -[\delta_1,\delta_3] \tag{34}$$
$$(h_4-id)(\delta) = -[\delta_1,\delta_4] \tag{35}$$

and hence $(h_3-h_4)(\delta)=[\delta_1,\delta_4-\delta_3]=[\delta_1,\alpha_1-\alpha_2]$. We may therefore replace δ by $[\delta_1,\alpha_1-\alpha_2]$ and conclude as above. Proposition 3 is proved. □

Proof of Theorem 7. If $M_2(t)$ were identically zero then, by Proposition 3, the function

$$F_\omega(t)=\int_{[\alpha_1(t),\alpha_2(t)]}\omega\omega' = det\begin{pmatrix}\int_{\alpha_1(t)}\omega & \int_{\alpha_2(t)}\omega\\ \int_{\alpha_1(t)}\omega' & \int_{\alpha_2(t)}\omega'\end{pmatrix}$$

would be identically zero too. We shall show that there is a polynomial one-form ω of degree at most two, such that $F_\omega(t)\not\equiv 0$. Indeed, the affine curve $\{f=t\}$ is a topological torus with three removed points $\infty_1,\infty_2,\infty_3$ and the vector space W, (25), of meromorphic one-forms with at most simple poles at ∞_i is three dimensional and generated by

$$\omega_{00}'=\frac{dx}{f_y},\omega_{10}'=2\frac{x\,dx}{f_y},\omega_{01}'=2\frac{y\,dx}{f_y}$$

(see the proof of Proposition 2 and [2, Lemma 3]) where

$$d\omega_{00}=dx\wedge dy, d\omega_{10}=x\,dx\wedge dy, d\omega_{01}=y\,dx\wedge dy$$

and ω' is the Gelfand-Leray residue of $d\omega$. Furthermore, the integrals $\int_{\alpha_i(t)}\omega$ are just the residues of ω at ∞_i. These residues are linear in t when $\omega\in W$. The identity $F_\omega(t)\equiv 0$ (for every fixed $\omega\in W$) and the linearity of $\int_{\alpha_i(t)}\omega$ imply that the direction of the vector

$$(\int_{\alpha_1(t)}\omega,\int_{\alpha_2(t)}\omega)$$

does not depend on $\omega\in W$ neither on t (when it is defined). This shows that a suitable linear combination of $\omega_{10}',\omega_{01}'$ with constant coefficients has no residues and hence is a holomorphic one-form. Therefore the one-forms $\omega_{00}',\omega_{10}',\omega_{01}'$ are linearly dependent, which is a contradiction. Theorem 7 is proved. □

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