Lie elements, shuffle products and the π_1 de Rham theorem

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Abstract

The purpose of this note is to prove a more precise version of the Chen's π_1 de Rham theorem concerning the fundamental group F of a punctured Riemann surface. We construct explicitly the dual spaces associate to the lower central series of F in terms of iterated integrals along Lie elements. Our construction is purely combinatorial which leads to a more general result in which the iterated integrals are axiomatically defined.

1 Introduction

Let $\Gamma = \bar{\Gamma} \setminus S$ where $\bar{\Gamma}$ is a compact Riemann surface and $S \subset \bar{\Gamma}$ a non-empty finite set of points. Each holomorphic 1-form ω in Γ defines a linear map $\int \omega \in Hom_{\mathbb{Z}}(H_1(\Gamma, \mathbb{Z}), \mathbb{C})$ by integration:

$$\int \omega : H_1(\Gamma, \mathbb{Z}) \to \mathbb{C} : \delta \mapsto \int_\delta \omega$$

and he classical de Rham theorem stipulates that $Hom_{\mathbb{Z}}(H_1(\Gamma, \mathbb{Z}), \mathbb{C})$ is generated by such linear maps. The Chen's de Rham theorem generalizes this fact as follows. The fundamental group $F = \pi_1(\Gamma, *)$ is free finitely generated. For $A, B \subset F$ we denote by (A, B) the the subgroup of F generated by commutators $(a, b) = aba^{-1}b^{-1}, a \in A, b \in B$. Define by induction the free abelian subgroups $F^n = (F, F^{n-1})$, $F^1 = F$. The Chen's de Rham theorem, e.g. [6], claims that for each n the vector space

$$Hom_{\mathbb{Z}}(F^n / F^{n+1}, \mathbb{C})$$

is generated by iterated integrals of length n. As $F^1/F^2 = H_1(\Gamma, \mathbb{Z})$ then in the case $n = 1$ we get the usual de Rham theorem.

The purpose of the paper is to prove a more precise version of this "π_1 de Rham theorem" by constructing explicitly the space $Hom_{\mathbb{Z}}(F^n/F^{n+1}, \mathbb{C})$ in terms of iterated integrals along Lie elements of length n (see Theorem 1). It turns out that our construction is purely combinatorial, the only analytic ingredient in the proof being the usual de Rham theorem. Therefore we can formulate our main result in algebraic terms as follows:

Let $X = \{x_1, x_2, ..., x_m\}$ be a set (which does not represent necessarily loops and one-forms) and $\mathsf{k} \subset \mathsf{K}$ two fields of characteristic zero. Denote by Ass_X the graded k-algebra of associative but non-commutative polynomials in variables $x_1, x_2, \ldots, x_m$ and by $L_X \subset Ass_X$ the graded Lie algebra generated by $x_1, \ldots, x_m$. Let also F_X be a free group generated by the elements of X. By an iterated integral we mean any map

$$\begin{array}{rcl} \int : F_X \times Ass_X & \to & \mathsf{K} \\ (\delta, \omega) & \mapsto & \int_\delta \omega \end{array}$$

which is k-linear in Ass_X and satisfies the four axioms given in the next section. It will follow from these axioms that $\int$ induces a well-defined map:

$$F_X^k / F_X^{k+1} \times Ass_X^k \to \mathsf{K}, \quad k = 1, 2, \ldots \tag{1}$$

which is $\mathbb{Z}$-linear in the first coordinate and k-linear in the second coordinate. Of course an example of such a map is the usual path integrals of Chen mentioned above. Our main result can be described as follows: If the vector space of $\mathbb{Z}$-linear maps

$$\begin{array}{rcl} \int \omega : F_X / F_X^2 & \to & \mathsf{K}, \text{ where } \omega \in L_X^1 \\ \delta & \mapsto & \int_\delta \omega \end{array}$$

is isomorphic to $Hom_{\mathbb{Z}}(F_X/F_X^2, \mathsf{K})$, then the vector space of $\mathbb{Z}$-linear maps

$$\begin{array}{rcl} \int \omega : F_X^k / F_X^{k+1} & \to & \mathsf{K}, \text{ where } \omega \in L_X^k \\ \delta & \mapsto & \int_\delta \omega \end{array}$$

is isomorphic to $Hom_{\mathbb{Z}}(F_X^k/F_X^{k+1}, \mathsf{K})$ for all $k = 1, 2, \ldots$. We also prove that $\int \omega, \ \omega \in Ass_X^k$ is identically zero if and only if ω is a linear combination of shuffle elements.

Example 1. To illustrate the Theorem in the simplest case $k = m = 2$, consider an elliptic curve with one removed point, and call it Γ. Its fundamental group F has two generators δ_1, δ_2, and let ω_1, ω_2 be two holomorphic one-forms which generate $H^1(\Gamma, \mathbb{C})$. The vector space Ass_X^2, where $X = \{\omega_1, \omega_2\}$, is generated by the monomials

$$\omega_1^2, \omega_1\omega_2, \omega_2\omega_1, \omega_2^2$$

and is of dimension four. The subspace L_X^2 is generated by

$$\omega_1\omega_2 - \omega_2\omega_1$$

and the subspace S_X^2 of shuffle products is generated by

$$\omega_1^2, \omega_1\omega_2 + \omega_2\omega_1, \omega_2^2.$$

Clearly we have $Ass_X^2 = L_X^2 \oplus S_X^2$ (Theorem 2). The abelian group F^2/F^3 is generated by $(\delta_1, \delta_2) = \delta_1\delta_2\delta_1^{-1}\delta_2^{-1} \in F_2$ and the dual space $Hom_{\mathbb{Z}}(F^2/F^3, \mathbb{C})$ is generated by the linear form (an iterated integral of length two)

$$\int \omega_1\omega_2 - \omega_2\omega_1 : F^2/F^3 \to \mathbb{C}.$$

Finally, the map

$$\int \omega : F^2/F^3 \to \mathbb{C}$$

where $\omega \in Ass_X^2$, is the zero map, if and only if ω belongs to the space of shuffle products S_X^2.

The paper is organized as follows. In §2 we fix the notations and the definitions used throughout the text. The main result of the present paper is formulated in section 3 and it is proved in §4. §5 is dedicated to some applications and motivations of the methods of the present article.

2 Notations and definitions

Let k be a field. All modules and algebras are taken over k, unless stated otherwise. For a set $X = \{x_1, x_2, ..., x_m\}$, let Ass_X be the graded free associative algebra on X. Its elements are the non-commutative polynomials in x_i with coefficients in k. Define a Lie bracket in Ass_X by $[x, y] = xy - yx$, and let $L_X \subset Ass_X$ be the graded free Lie algebra on X. Thus, for instance, $x_1, [x_1, x_2], [[x_1, x_2], x_3]$ belong to L_X but not 1, x_1x_2, $x_1x_2x_3$. Ass_X is the universal algebra of the Lie algebra L_X, see [14] . The graded piece L_X^1 is just the k-vector space generated by $x_1, x_2, ..., x_m$, L_X^2 is the k-vector space generated by $[x_i, x_j]$ etc. Each element of L_X is called a Lie element. We denote by Ass_X^k (resp. L_X^k) the homogeneous component of degree k of Ass_X (resp. L_X).

Let F_X be the free group on X, its elements are the words in the letters x_i and their formal inverses x_i^{-1}. For $x, y \in F_X$ we define the commutator $(x, y) = xyx^{-1}y^{-1}$. For $A, B \subset F_X$ we denote by (A, B) the the subgroup of F_X generated by commutators $(a, b), a \in A, b \in B$. Consider the lower central series F_X^n of F_X, where $F_X^n = (F_X, F_X^{n-1})$, $F_X^1 = F_X$. The associated graded $\mathbb{Z}$-Lie algebra is given by

$$grF_X = \sum_{n=1}^{\infty} gr^n F_X, \quad gr^n F_X = F_X^n / F_X^{n+1}, \tag{2}$$

$$[xF_X^{i+1}, yF_X^{j+1}] = (x, y)F_X^{i+j+1}.$$

The canonical map $X \to gr^1 F_X$ which send x_i to x_i induces an isomorphism of Lie algebras

$$\phi : L_X \to (grF_X) \otimes_{\mathbb{Z}} \mathsf{k} \tag{3}$$

(e.g.[14] Theorem 6.1). For two words $\omega_1 \cdots \omega_r$, $\omega_{r+1} \cdots \omega_{r+s}$ define the shuffle product $\omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s}$ to be the sum of all words of length $r + s$ that are permutations of $\omega_1 \cdots \omega_r \omega_{r+1} \cdots \omega_{r+s}$ such that both $\omega_1 \cdots \omega_r$ and $\omega_{r+1} \cdots \omega_{r+s}$ appear in their original order, e.g.

$$\omega_1\omega_2 * \omega_3 = \omega_1\omega_2\omega_3 + \omega_1\omega_3\omega_2 + \omega_3\omega_1\omega_2.$$

Let K be a field extension of the field k.

Definition 1. An iterated integral is a map

$$\begin{array}{rcl} \int : F_X \times Ass_X & \to & \mathsf{K} \\ (\delta, \omega) & \mapsto & \int_\delta \omega \end{array} \tag{4}$$

which is k-linear in the second variable and satisfies the four axioms:

A1 For every non-commutative polynomial $\omega \in Ass_X$ and $\delta \in F_X$, $\int_1 \omega \in \mathsf{k}$ is the constant term of ω and $\int_\delta 1 = 1$ for all $\delta \in F_X$. We use the convention $\omega_1\omega_2\cdots\omega_r = 1$ for $r = 0$.

A2 For $\alpha, \beta \in F_X$ and $\omega_1, \omega_2, \ldots \omega_r \in Ass^1_X$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r$$

A3 For $\alpha \in F_X$ and $\omega_1, \omega_2, \ldots \omega_r \in Ass^1_X$

$$\int_{\alpha^{-1}} \omega_1\omega_2 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1.$$

A4 For $\alpha \in F_X$ and $\omega_1, \omega_2, \ldots \omega_{r+s} \in Ass^1_X$ we have

$$\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \int_\alpha \omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s} \tag{5}$$

where

$$\omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s} = \sum \omega_{k_1}\omega_{k_2} \cdots \omega_{k_{r+s}}$$

is the shuffle product of $\omega_1 \cdots \omega_r$ and $\omega_{r+1} \cdots \omega_{r+s}$.

Let $\{\delta_1, \delta_2, \ldots, \delta_m\}$ be a set which generates F_X freely and $\{\omega_1, \omega_2, \ldots, \omega_m\}$ be a basis of the k-vector space Ass^1_X. **A1**, **A2**, **A3** imply that every iterated integral can be written as a polynomial in

$$\int_{\delta_j} \omega_{i_1}\omega_{i_1} \cdots \omega_{i_r},\ j = 1, 2, \ldots, m,\ i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, m\}. \tag{6}$$

Therefore by **A4** the map (4) defines an iterated integral if and only if the numbers

$$\int_{\delta_j} \omega_{i_1}\omega_{i_2} \cdots \omega_{i_r} = a^j(i_1, \ldots, i_r) \in \mathsf{k}$$

satisfy the "shuffle relations"

$$a^j(i_1, \ldots, i_r)a^j(i_{r+1}, \ldots, i_{r+s}) = \sum a^j(k_1, \ldots, k_{r+s}) \tag{7}$$

where $(k_1, \ldots, k_{r+s})$ runs through all shuffles of $(i_1, \ldots, i_r)$ and $(i_{r+1}, \ldots, i_{r+s})$. The existence of such numbers $a^j(i_1, \ldots, i_r)$ is, however, not obvious.

Example 2. Let $\{\delta_1, \delta_2, \ldots, \delta_m\}$ be a set which generates F_X freely and $\{\omega_1, \omega_2, \ldots, \omega_m\}$ be a basis of the k-vector space Ass^1_X. We set

$$a^j(i_1, \ldots, i_n) = \int_{\delta_j} \omega_{i_1}\omega_{i_2} \cdots \omega_{i_n} = 0 \text{ if at least one of } \omega_{i_s} \text{ is not } \omega_j \tag{8}$$

$$a^j(j, \ldots, j) = \int_{\delta_j} \omega_j^n = \frac{1}{n!}.$$

The verification that the numbers $a^j(i_1,\dots,i_n)$ define an iterated integral is straightforward. This iterated integral can be interpreted as Chen's iterated integrals in the following way: We take the Ceyley diagrams (see [11]) which is the topological space $Y := \cup_{i=0}^{m-1}\mathbb{Z}^i \times \mathbb{R} \times \mathbb{Z}^{m-i-1} \subset \mathbb{R}^m$. Each element δ of the free group F_X is represented by a path $\tilde{\delta}$ in Y with the starting point $0 \in \mathbb{R}^n$ and the end point in some element of $\mathbb{Z}^m$. Each element ω of Ass_X can be interpreted as a differential form $\tilde{\omega}$ substituting dy_i by x_i, where $(y_1, y_2, \dots, y_m)$ is the coordinate system in $\mathbb{R}^m$. Now, $\int_\delta \omega$ is the classical Chen's iterated integral $\int_{\tilde{\delta}} \tilde{\omega}$.

Example 3. As in the Introduction, let Γ be a punctured Riemann surface with free finitely generated fundamental group $F_X = \pi_1(\Gamma, *)$. Let $\{\omega_1, \omega_2, \dots, \omega_m\}$ be a collection of holomorphic one forms on Γ such that their classes in the de Rham cohomology of Γ form a basis. The Chen's iterated integral

$$a^j(i_1,\dots,i_n) = \int_{\delta_j} \omega_{i_1}\omega_{i_2}\cdots\omega_{i_n}$$

where $\delta_1, \delta_2, \dots, \delta_m$ generates F_X freely, satisfies the shuffle relations (7), see [5, 6], and hence defines an iterated integral in the sense of Definition 1(in order to follow the terminology used in this text we may identify ω_i with x_i).

3 Statement of the result

Let $\alpha, \beta \in F_X$ and $\omega \in Ass_X^1 = L_X^1$. Then **A3** implies that every iterated integral (1) satisfies

$$\int_{\alpha\beta} \omega = \int_\alpha \omega + \int_\beta \omega, \int_{\alpha\beta\alpha^{-1}} \omega = \int_\alpha \omega, \int_{(\alpha,\beta)} \omega = 0.$$

More generally,

$$\forall \alpha \in F_X^{n+1}, \omega \in Ass_X^m, \text{ such that } m \le n \text{ holds true } \int_\alpha \omega = 0 \tag{9}$$

Therefore the iterated integral $\int$ induces a map

$$\begin{array}{rcl} \int : gr^k F_X \times Ass_X^k & \to & \mathbb{C} \\ (\delta, \omega) & \mapsto & \int_\delta \omega \end{array} \tag{10}$$

which is $\mathbb{Z}$-linear in the first argument and k-linear in the second argument.

Suppose that $\omega \in Ass_X^k$ is a shuffle product of ω_1, ω_2:

$$\omega = \omega_1 * \omega_2, \omega_1 \in Ass_X^{k_1}, \omega_2 \in Ass_X^{k_2}, k_1 + k_2 = k$$

and $\alpha \in gr^k F_X$. Then **A4** and (9) imply

$$\int_\alpha \omega = 0.$$

Thus the vector space S_X^k generated by shuffle products

$$S_X^k = Span\{\omega_1 * \omega_2 : \omega_1 \in Ass_X^{k_1}, \omega_2 \in Ass_X^{k_2}, k_1 + k_2 = k\}.$$

is in the kernel of the bilinear map (10).

The main result of the paper is the following:

Theorem 1. *Let $\int$ be an iterated integral in the sense of Definition 1.*

(a) *If the induced bilinear map*

(11) $$\begin{array}{rcl} \int : gr^k F_X \times L_X^k & \to & \mathsf{K} \\ (\delta, \omega) & \mapsto & \int_\delta \omega \end{array}$$

is non-degenerate for $k = 1$, then then it is non-degenerate for all $k \in \mathbb{N}$.

(b) *Let $\omega \in Ass_X^k$. The linear map*

(12) $$\begin{array}{rcl} \int \omega : gr^k F_X & \to & \mathsf{K} \\ \delta & \mapsto & \int_\delta \omega \end{array}$$

is the zero map, if and only if $\omega \in S_X^k$.

4 Proof of Theorem 1

In Ass_X we consider the canonical k-bilinear symmetric product given by

$$\langle x_{i_1} x_{i_2} \cdots x_{i_r}, x_{j_1} x_{j_2} \cdots x_{j_s} \rangle = \begin{cases} 1 & i_1 = j_1, \ldots, i_r = j_r, r = s \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2. *S_X^k and L_X^k are orthogonal vector subspaces of Ass_X^k and for all $k \in \mathbb{N}$*

$$Ass_X^k = L_X^k \oplus S_X^k.$$

This is a (geometric) reformulation of the following result of Ree [13, Theorem 2.2].

Theorem 3. *A polynomial*

(13) $$\omega = \sum_{n>0} \sum a(i_1, i_2, \ldots, i_n) x_{i_1} x_{i_2} \cdots x_{i_n} \in Ass_X$$

is a Lie element (i.e. belongs to L_X) if and only if for all $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_s)$ we have:

(14) $$\sum a(k_1, k_2, \ldots, k_{r+s}) = 0$$

where $(k_1, \ldots, k_{r+s})$ runs through all shuffles of $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_s)$.

Proof of the equivalence of Theorem 2 and Theorem 3: Without lose of generality we can assume that ω in (13) is homogeneous of degree n. The proof follows from the equality:

$$\langle \omega, x_{i_1} x_{i_2} \cdots x_{i_r} * x_{j_1} x_{j_2} \cdots x_{j_s} \rangle = \sum a(k_1, k_2, \ldots, k_{r+s})$$

where $x_{i_1} x_{i_2} \cdots x_{i_r} * x_{j_1} x_{j_2} \cdots x_{j_s},\ r + s = n$ is a shuffle element and $(k_1, \ldots, k_{r+s})$ runs through all shuffles of $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_s)$. Note that we can formulate Theorem 2 in the following way: The polynomial ω is a Lie element if and only if it is orthogonal to all shuffles $x_{i_1} x_{i_2} \cdots x_{i_r} * x_{j_1} x_{j_2} \cdots x_{j_s},\ r + s = n$.

To prove Theorem 1 we note that as the map (11) is non-degenerate for $k = 1$, then it can be "diagonalized" as follows: We fix a set of generators $\{\delta_1, \delta_2, \ldots, \delta_m\}$ for F_X and find $\Omega = \{\omega_1, \ldots, \omega_m\} \subset Ass_X^1$ such that $\int_{\delta_i} \omega_j = 1$ if $i = j$ and $= 0$ otherwise. Now, Ass_X is freely generated by Ω. Therefore we shall suppose, without loss of generality, that

(15) $$\int_{x_i} x_j = \langle x_i, x_j \rangle = 0 \text{ if } i \neq j \text{ and } 0 \text{ otherwise.}$$

The formula (15) generalizes as follows:

Proposition 1. *We have*

$$\int_\delta \omega = \langle \omega, \phi^{-1}\delta \rangle, \ \forall \omega \in Ass^k_X, \ \delta \in gr^k F_X. \tag{16}$$

where ϕ is the isomorphism (3).

Proof. The proof is by induction on k. For $k = 1$ it follows from (15). If $\delta = (a, b)$, where $a \in F^p_X, b \in F^q_X$, $p + q = k > 1$, then **A2** implies

$$\begin{aligned}
\int_{(a,b)} x_{j_1}x_{j_2}\cdots x_{j_k} &= \int_a x_{j_1}\cdots x_{j_p}\int_b x_{j_{p+1}}\cdots x_{j_k} \\
&\quad - \int_b x_{j_1}\cdots x_{j_q}\int_a x_{j_{q+1}}\cdots x_{j_k} \\
&= \langle \phi^{-1}(a), x_{j_1}\cdots x_{j_p}\rangle\langle\phi^{-1}(b), x_{j_{p+1}}\cdots x_{j_k}\rangle \\
&\quad - \langle\phi^{-1}(b), x_{j_1}\cdots x_{j_q}\rangle\langle\phi^{-1}(a) x_{j_{q+1}}\cdots x_{j_k}\rangle \\
&= \langle\phi^{-1}(a)\phi^{-1}(b), x_{j_1}\cdots x_{j_p}x_{j_{p+1}}\cdots x_{j_k}\rangle \\
&\quad - \langle\phi^{-1}(b)\phi^{-1}(a), x_{j_1}\cdots x_{j_q}x_{j_{q+1}}\cdots x_{j_k}\rangle \\
&= \langle\phi^{-1}(a)\phi^{-1}(b) - \phi^{-1}(b)\phi^{-1}(a), x_{j_1}x_{j_2}\cdots x_{j_k}\rangle \\
&= \langle\phi^{-1}((a,b)), x_{j_1}x_{j_2}\cdots x_{j_k}\rangle.
\end{aligned}$$

□

Remark 1. For an arbitrary iterated integral map, the same proof implies that for every $\delta \in gr^k F_X$ such that

$$\phi^{-1}(\delta) = \sum a(i_1, i_2, \ldots, i_k) x_{i_1}x_{i_2}\cdots x_{i_k} \in gr^k L_X$$

holds

$$\int_\delta \omega_{j_1}\omega_{j_2}\cdots\omega_{j_k} = \sum_{i_1,i_2,\ldots,i_k} a(i_1, i_2, \ldots, i_k)\int_{x_{i_1}}\omega_{j_1}\int_{x_{i_2}}\omega_{j_2}\cdots\int_{x_{i_k}}\omega_{j_k} \tag{17}$$

Proof of Theorem 1: The part (b) follows from Propositions 2 and 1. Let $\delta \in gr^k F_X$ and

$$\phi^{-1}(\delta) = \sum a(i_1, i_2, \ldots, i_k) x_{i_1}x_{i_2}\cdots x_{i_k} \in gr^k L_X$$

where in the above sum each non-commutative monomial $x_{i_1}x_{i_2}\cdots x_{i_k}$ is repeated only once. Then $\delta \neq 0$ if and only if

$$\langle\phi^{-1}(\delta), \phi^{-1}(\delta)\rangle = \sum |a(i_1, i_2, \ldots, i_k)|^2 \neq 0.$$

Therefore for every $\delta \in gr^k F_X$ there exists $\omega \in L^k_X$, namely $\omega = \Phi^{-1}(\delta)$, such that

$$\int_\delta \omega \neq 0.$$

Here we have used strongly the fact that the characteristic of k is zero.

Remark 2. A canonical basis of the k-vector spaces $L_X^k \cong gr^k F_X$ is given by basic commutators (see for instance [7, 14]). By definition of $\langle\cdot,\cdot\rangle$ if the number of some $x_i, i = 1, 2, \ldots, m$ used in two basic commutators ω_1 and ω_2 are different then $\langle\omega_1, \omega_2\rangle = 0$. The basic commutators of weight 1 and 2 are dual to each other with respect to the bilinear map $\langle\cdot,\cdot\rangle$. However, this is not the case for weight 3 and the number of generators m bigger than 2. For instance, we have

$$\langle [y,[x,z]],[z,[x,y]]\rangle = 2$$

For $m = 2$ the basic commutators of weight 3 (resp. 4) are orthogonal to each other. In $m = 2$ and $r = 5$ the orthogonality fails. There are two couples in which the number of x is equal to 2 (resp. 3). In fact we have

$$\langle [y,[x,[x,[x,y]]]],[[x,y],[x,[x,y]]]\rangle = -28,\ \ \langle [y,[y,[x,[x,y]]]],[[x,y],[y,[x,y]]]\rangle = -14$$

The basic commutators are implemented in Axiom in [8]. This and the implementation of $\langle\cdot,\cdot\rangle$ can be found in the second name author's homepage. The above calculations is done using Axiom.

5 Iterated integrals and plane holomorphic foliations

The results of the present paper have applications to the so called infinitesimal 16th Hilbert problem (the problem of finding the number of the limit cycles of a plane vector field close to an integrable one) and it was in fact motivated by it. In this section we discuss briefly this relation.

Let f be a polynomial of degree d in two variables x, y and suppose, for simplicity, that the highest order homogeneous piece g of f is a product of distinct homogeneous lines, i.e $g = \Pi(x - a_i y),\ a_i \neq a_j$. Let also $C \subset \mathbb{C}$ be the set of the critical values of f. The cohomology fiber bundle $\cup_{t\in\mathbb{C}\backslash C} H^1(\{f = t\}, \mathbb{C})$ and the corresponding Gauss-Manin connection whose flat sections are generated by sections with images in $\cup_{t\in\mathbb{C}\backslash C} H^1(\{f = t\}, \mathbb{Z})$ is encoded in the global Brieskorn module $H = \frac{\Omega^1}{df\wedge\Omega^0 + d\Omega^0}$, where Ω^i is the set of polynomial i-forms in $\mathbb{C}^2$ as follows. We note first that H is a $\mathbb{C}[t]$-module ($t.[\omega] = [f\omega]$) generated freely by

$$\omega_i := x^{i_1} y^{i_2}(xdy - ydx),\ i = (i_1, i_2) \in I$$

where $\{x^{i_1} y^{i_2},\ i \in I\}$ is a basis of monomials for the $\mathbb{C}$-vector space $\frac{\mathbb{C}[x,y]}{\langle f_x, f_y\rangle}$. In particular the rank of H equals the dimension of $H^1(\{f = t\}, \mathbb{Z})$ for generic t. The Gauss-Manin connection of the family of curves $f(x, y) = t,\ t \in \mathbb{C}$ with respect to the parameter t becomes an operator $' : H \to \frac{1}{\Delta}H$ which satisfies the Leibniz rule, where $\Delta = \Delta(t)$ is the discriminant of the polynomial $f(x, y) - t$. In the basis $\omega = (\omega_i)_{i\in I}$ (written in a column) it is of the form

$$\omega' = \frac{1}{\Delta} A\omega,$$

where A is a $m \times m$ matrix with entries in $\mathbb{C}[t]$ and $m = \#I$ (for proofs see [1, 10]).

The above construction has a natural generalization based on the π_1 de Rham theorem (Theorem 1), which we describe now. Let $F_t,\ t \in \mathbb{C}\backslash C$ be the fundamental group of the fiber $\{f = t\}$ (it was denoted F_X in section 3). Consider the trivial fiber bundle $\cup_{t\in\mathbb{C}\backslash C} gr^k F_t \otimes_{\mathbb{Z}} \mathbb{C}$ and its $\mathbb{Z}$-dual $\cup_{t\in\mathbb{C}\backslash C} \check{gr}^k F_t \otimes_{\mathbb{Z}} \mathbb{C}$. Both of them have a canonical flat connexion defined as follows.

Let $X = \{\omega_i\}_{i \in I}$, $\mathsf{k} = \mathbb{C}(t)$ and K the field of locally analytic multi-valued functions on $\mathbb{C} \backslash C$. The Gauss-Manin connection on Ass^1_X extends canonically to a derivation operator

$$\frac{\partial}{\partial t} =': Ass_X \to \frac{1}{\Delta} Ass_X$$

which respects both the graduation of Ass_X, the direct sum decomposition $Ass_X = L_X \oplus S_X$, and satisfies the Leibniz rule $(ab)' = a'b + ab',\ a, b \in Ass_X$. In particular for monomials in Ass_X it is given by:

$$\omega_{i_1}\omega_{i_2}\cdots\omega_{i_r} \to \sum_{j=1}^{r} \omega_{i_1}\omega_{i_2}\cdots\omega_{i_{j-1}}\omega'_{i_j}\omega_{i_{j+1}}\cdots\omega_r.$$

The induced map $' : L_X \to \frac{1}{\Delta} L_X$, where L_X is the k-vector space $L_X \cong \frac{Ass_X}{S_X}$, is the desired generalization of the usual (algebraic) Gauss-Manin connexion.

The above construction has applications to the so called infinitesimal 16th Hilbert problem (the problem of finding the number of the limit cycles of a plane vector field close to an integrable one). Namely, let f be a polynomial as above and let ω be a polynomial one-form in $\mathbb{C}^2$. Consider the holomorphic foliation on $\mathbb{C}^2$ defined by

$$df + \epsilon\omega = 0, \epsilon \sim 0.$$

Let b be a regular value of f and $\delta \in F_b$. There is a number k such that δ represents a not zero element in $gr^k F_b$. We consider the holonomy map h_ϵ along δ

$$h_\epsilon(t) = t + \sum_{i=1}^{\infty} \epsilon^i M_i(t)$$

along the path δ. It is known [3, 2, 9] that $M_1 = \cdots = M_{k-1} = 0$ and M_k is an iterated path integral

$$M_k(t) = \int_{\delta_t} \omega(\underbrace{\omega(\cdots(\omega(\omega)')}_{k-1 \text{ times}}{}' \cdots)')' \tag{18}$$

where $\delta_t \in F_t$ is the path over δ in the fiber $\{f = t\}$. If the function M_k is not identically zero then its zeros correspond to limit cycles of the deformed foliation. In particular the deformation $df + \epsilon\omega$ " destroys" the family of cycles δ_t in the sense that the holonomy map h_ϵ is not the identity map ($h_0 = id$). In this relation the following claim can be conjectured

For an arbitrary f of degree d and any family of cycles δ_t in the fibers of f, there is a small perturbation ω of df such that $\deg(\omega) = d - 1$ *and the perturbed holonomy along δ_t is not the identity map.*

We note that the proof of the above conjecture (for $d > 3$) presented in [4] contains a gap (the statement in the 9th line of p. 280 is false). We present the following purely algebraic approach to the above geometric problem in a weaker version. As we explained we consider the deformation $df + \epsilon\omega,\ \omega \in \mathbb{C}\langle X\rangle$, where $\mathbb{C}\langle X\rangle \subset Ass^1_X$ is the $\mathbb{C}$-vector space generated by the elements of X. Now, it is enough to provethe following claim

The set

$$\{\Delta^{k-1}\omega(\underbrace{\omega(\cdots(\omega(\omega)')}_{k-1 \text{ times}}{}' \cdots)')' \mid \omega \in \mathbb{C}\langle X\rangle\}$$

generates the k*-vector space* $\frac{Ass_X}{S_X}$.

If the claim were true this would imply that the k-th order Poincar-Pontryagin-Melnikov function M_k is not identically zero, which on its turn would show that the holonomy map is not the identity map. The claim trivially holds true in the case $k = 1, 2$. We conjecture that it holds true for any k.

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