December 5, 2008

Dear Movasati,

The first sentence of your introduction is ambiguous: on a variety of dimension $n$, here $n = 1$, canonical bundle often means $\Omega^n$. I hope you mean not $\Omega^1$ of the moduli space, but the bundle $\omega$ on the moduli stack of elliptic curves whose fiber at $E$ is $\mathrm{Lie}(E)^\vee$. I guess "tensor product" is a misprint for "tensor power".

Why do you say "there is no way to generalize the first interpretation" when you essentially do just that? As you tell, an elliptic curve $E$ defines a torsor $P$ under the Borel subgroup $B$ of SL(2): $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$: the space of $(\omega_1, \omega_2)$ you consider. Any linear representation of $B$ defines a vector bundle on the moduli stack $M$ of elliptic curves, and you show that (forgetting the condition at $\infty$) quasi-modular forms can be identified with the sections of a vector bundle so constructed (on a covering of $M$ corresponding to $\Gamma$). Bundles attached to representations of $B$ should be thought as vector bundles deduced from the vector bundle $\mathcal{H}$ of de Rham cohomology (fiber $H^1_{\mathrm{DR}}(E)$ at $E$), with the multilinear algebra structures it carries: the cup product, with value in $\mathcal{O}$, and the Hodge filtration, $F^1 \subset \mathcal{H}$.

The §4 definition of quasi-modular forms tries to hide that they can be viewed as sections of the bundle of $n$-jets of sections of $\omega^{\otimes(m-2n)}$: the object with a nice transformation law is $(f_0, \ldots, f_n)$, and corresponds point by point to such an $n$-jet. Near $z_0$, one has a local coordinate $z$, as well as a trivialization of $\omega$, and to an $n$-jet, viewed as an $n$-jet of function: $\sum_0^n \varphi_i (z - z_0)^i$, one attaches $\varphi_n$(the <u>last</u> coefficient).

For an elliptic curve $E$, it amounts to the same to deform $E$, or to move the line $F^1$ in $H^1_{\mathrm{DR}}(E)$, and if one identifies the moduli space near $E$ with the space of $F$ near ($F^1$ of $E$), the fiber of $\omega$ at $F$ is the line $F$ itself. In algebraic geometry, this remains true for formal deformations, in char.0, and expresses the fact that the Gauss Manin connection induces $F \xrightarrow{\sim} (\mathcal{H}/F) \otimes \Omega^1$.

If $E$ is $\mathbb{C}/\langle 1, z\rangle$, in $H^1(E,\mathbb{Z})\otimes\mathbb{C}$ and for a natural basis of $H^1(E,\mathbb{Z})$, $F$ is spanned by $(1,z)$ and has a supplement $G$ spanned by $(0,1)$. Near $E$, one moves $F$ to the line spanned by $(1,z)+\lambda(0,1)$. This is the trivialization of $\omega$ near $E$ used, and the local coordinate used, to speak of the last coefficient of an $n$-jet.

What you do is to observe that if one chooses any supplement $G$ to $F$ at $E$, it again makes sense to take the last coefficient of an $n$-jet of section of $\omega^{\otimes k}$. It is an element of $\omega^{\otimes(k+2n)}$ depending on $G$. You tell that one should use all $G$, rather than at $z$ one adapted to $z$. The space of all $G$ is an affine line ($\mathbb{P}(H^1_{\mathrm{DR}}(E))$ minus the point "$F$"), and at any point of the moduli stack, functions of $G$ of degree $\leq n$ correspond one to one to $n$-jets at that point.

Best,

P. Deligne

P.S. Concordance with your notations: the pairs $(\omega_1,\omega_2)$ you consider are determined by $\omega_2$ alone, subject to $\omega_2\notin F^1$. Instead of considering $f(G)$ with value in $\omega^{\otimes m}$, you use the $\omega_1 =$ "$1/\omega_2$" attached to $\omega_2$ and the scalar function $f(\omega_2) := f(\text{line}\,\langle\omega_2\rangle)/\omega_1^{\otimes m}$. The condition becomes an homogeneity condition and degree $\leq\cdots$ on the affine lines $\omega_2+F^1$. Your vector field is of course the Gauss Manin connection in disguise: the choice of $\omega_2$ at a point $p$ of the moduli stack defines a tangent vector at $p$ (as $\omega^{\otimes 2}\simeq\Omega^1$), and the Gauss-Manin connection lifts it to a vector at $\omega_2$ on the bundle $H^1_{\mathrm{DR}}$. So described, the bundle with fiber $H^1_{\mathrm{DR}}-F^1$ on the moduli stack, and the vector field, make sense in any characteristic _ better, over $\mathbb{Z}$. Only the use of the Weierstrass form creates trouble for $p=2,3$. The jet interpretation is another matter: trouble can be expected for $n$-jets when $n\geq p$.