The moduli of quasi-homogeneous Stein surface singularities

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Abstract

In this article we study good $\mathbb{C}^*$ actions on Stein surfaces and we construct their moduli by means of the resolution data of the dicritical singularity of the action. We also classify $\mathbb{C}^*$ transversal actions around a Riemann surface embedded in a two dimensional manifold.

1 Introduction

A 2-dimensional complex analytic variety V, with a distinguished point $p \in V$, is called a quasi-homogeneous complex surface singularity, if it admits a holomorphic action of the complex multiplicative group $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ such that every non-singular orbit accumulates at $p \in V$ (see for instance [16], page 67 and [5, 14]). Such an action is called a good action. The study of algebraic quasi-homogeneous singularities is a main topic in the theory of singularities. Saito in [13] gave an algebraic description of such singularities in the local context. Orlik and Wagreich ([11], [12], [18]) studied the 2-dimensional affine algebraic varieties embedded in $\mathbb{C}^{n+1}$, with an isolated singularity at the origin, that are invariant by an algebraic action of the form $\sigma_Q(t,(z_0,...,z_n)) = (t^{q_0}z_0,...,t^{q_n}z_n)$ where $Q = (q_0,...,q_n) \in \mathbb{N}^{n+1}$, i.e. all q_i are positive integers. In particular they classified the algebraic surfaces embedded in $\mathbb{C}^3$ endowed with such an action. In this paper we classify the quasi-homogeneous surface singularities which are Stein analytic spaces of dimension two, endowed with an analytic global $\mathbb{C}^*$-action.

Let V be a Stein irreducible complex analytic space of dimension two with normal singularities and $\varphi : \mathbb{C}^* \times V \to V$ a holomorphic action of the group $\mathbb{C}^*$ on V. Denote by $\mathcal{F}_\varphi$ the foliation on V induced by φ. The leaves of this foliation are the one-dimensional orbits of φ, and its singularities are the fixed points of φ. We will assume that there exists a dicritical singularity $p \in V$ for the $\mathbb{C}^*$-action, i.e. for some neighborhood $p \in W \subset V$ there are infinitely many leaves of $\mathcal{F}_\varphi|_W$ accumulating only at p. The closure of any such a local leaf is an invariant local analytic curve called a separatrix of $\mathcal{F}_\varphi$ through p. Thus, a dicritical singularity exhibits infinitely many separatrices. On the other hand, the singularity $p \in V$ of a good action on V is clearly dicritical.

Two pairs (V,φ) and (V',φ') are equivalent if there is a biholomorphic map $f : V \to V'$ such that $f(\varphi(t,x)) = \varphi'(t, f(x))$ for all $t \in \mathbb{C}^*$ and $x \in V$. The set of all pairs (V,φ) up to this equivalent relation is the moduli space which we study in the present paper:

Theorem 1. *The moduli space of pairs (V,φ), where V is a normal Stein analytic space of dimension two and φ is a $\mathbb{C}^*$-action with isolated singularities with at least one dicritical singularity, is the following data*

1. *A Riemann surface σ_0 of genus g and s-points $r_1, r_2, \dots, r_s$ on σ_0 considered up to the automorphism group of σ_0.*

2. *A line bundle L on σ_0 with $c(L) = -k \leq -1$.*

3. For each $i = 1, 2, \ldots, s$ a sequence of integers $-k_j^i$, $j = 1, 2, \ldots, n_i$, $k_j^i \geq 2$, such that

$$\sum_{i=1}^{s} \frac{1}{[k_1^i, k_2^i, \ldots, k_{n_i}^i]} < k,$$

where

$$[k_1^i, k_2^i, ..., k_{n_i}^i] = k_1^i - \cfrac{1}{k_2^i - \cfrac{1}{\ddots}}.$$

Conversely, 1, 2 and 3 imply the existence of a pair (V, φ).

The above data can be read from the minimal resolution of the desingularization at $p \in V$ of the foliation induced by φ. The proof of Theorem 1 will also provide us with the fact that V is indeed an affine variety and φ is an algebraic action of the form σ_Q. Therefore, the GAGA principle is valid for such actions and the algebraic and analytic moduli in Theorem 1 are the same.

The proof of Theorem 1 consists of the following steps. We first analyze in §2 the resolution of the singularity $p \in V$ and obtain Theorem 2 which is an analytic version of a theorem proved in [11]. It turns out that in the divisor of the resolution of $p \in V$ there is only one element, σ_0, of arbitrary genus, on which $\mathbb{C}^*$ acts transversely, i.e. the set of fixed points of the lifted action is σ_0 and there is a 1-dimensional local foliation, transverse to σ_0, invariant by the action. All other elements of the divisor are Riemann spheres and are invariant under the lifted $\mathbb{C}^*$-action. In §3 we show a theorem, which has independent interest, that allows to linearize the lifted $\mathbb{C}^*$-action in a neighborhood of σ_0. The main theorem of this section, Theorem 3, generalizes a theorem on the linearization of foliations transverse to a Riemann surface embedded in a complex surface, published in [2], with the peculiarity that if the foliation is invariant by a $\mathbb{C}^*$-action then no hypothesis is required on the self intersection number of σ_0. In §4 we first introduce the linear model for the resolution of $p \in V$ and then extend the linearization obtained in the previous section to the basin of attraction of $p \in V$. In §5 we prove that the basin of attraction of $p \in V$ is the whole space V. Finally, in §6 we prove our main theorem.

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2 Resolution of singularities

In order to prove Theorem 1 we first describe the resolution of the action φ, at $p \in V$ and then compare it with the resolution of a model good action.

2.1 Holomorphic foliations

We start with the resolution theorem for normal two dimensional singularities (see [8]) and the resolution theorem for holomorphic foliations (see [15], [4]) that combined together assert, first, that there exists a proper holomorphic map $\rho : \tilde{V} \to V$ such that $D{:=}\rho^{-1}(p) = \bigcup_{i=0}^{r} \sigma_i$, is a finite union of compact Riemann surfaces σ_i intersecting at most pairwise at normal crossing points, and then that $\tilde{V}$ is an analytic space of dimension two with no singularities near D. More precisely, the σ_i's are compact Riemann surfaces without singularities such that if $\sigma_i \cap \sigma_j \neq \emptyset$ then σ_i and σ_j have normal crossing and $\sigma_i \cap \sigma_j \cap \sigma_k = \emptyset$ if $i \neq j \neq k \neq i$. Moreover, the intersection matrix $(\sigma_i \cdot \sigma_j)$ is negative definite ([8])and the restriction of ρ to $\tilde{V} \backslash D$ is a biholomorphism onto $V \backslash \{p\}$. By means of this restriction $\mathcal{F}_\varphi$ induces a foliation $\tilde{\mathcal{F}}_\varphi$ on $\tilde{V} \backslash D$ that can be extended to $\tilde{V}$ as a foliation with isolated singularities. Each one of these singularities can be written in local coordinates (x, y) around $0 \in \mathbb{C}^2$ in one of the following forms : (i) simple singularities: $xdy - y(\mu + \cdots)dx = 0$, $\mu \notin \mathbb{Q}_+$, where the points denote higher order terms; (ii) saddle-node singularities: $x^{m+1}dy - (y + ax^m y + \cdots)dx = 0$, $a \in \mathbb{C}$, $m \in \mathbb{N}$. A simple singularity has two invariant manifolds crossing normally, they correspond to the x and y-axes. The saddle-node has an invariant manifold corresponding to the y-axis and, depending on the higher order terms, it may not have another invariant curve (see [10]). The resolution of $\mathcal{F}_\varphi$ can be obtained in such a way that the elements σ_i fall in two categories. Either σ_i is a dicritical component, when $\tilde{\mathcal{F}}_\varphi$ is everywhere transverse to σ_i, or a nondicritical component when σ_i is tangent to $\tilde{\mathcal{F}}_\varphi$. In a similar way, by means of the restriction ρ to $\tilde{V} \backslash D$ the $\mathbb{C}^*$- action φ on $V \backslash \{p\}$ induces a $\mathbb{C}^*$- action $\tilde{\varphi}$ on $\tilde{V} \backslash D$ that can be extended to D as a $\mathbb{C}^*$- action (see [12]). For this it is enough to observe that $D \subset \tilde{V}$ is analytic of codimension one, $\tilde{V}$ is a normal analytic space and $\tilde{\varphi}$ is bounded in a neighborhood of D. We have therefore that the orbits of $\tilde{\varphi}$ are contained in the leaves of the foliation $\tilde{\mathcal{F}}_\varphi$. Moreover, if we denote by $Fix(\tilde{\varphi})$ the set of fixed points of $\tilde{\varphi}$, and by $sing(\tilde{\mathcal{F}}_\varphi)$ the singular set of the foliation $\tilde{\mathcal{F}}_\varphi$, we have that $sing(\tilde{\mathcal{F}}_\varphi) \subset Fix(\tilde{\varphi})$.

The divisor D forms a graph with vertices σ_i and sides the nonempty intersections $\sigma_i \cap \sigma_j$. A star is a contractible connected graph where at most one vertex, called its center, is connected with more than two other vertices. A weighted graph is a graph where at each vertex is associated its genus and its self-intersection number.

2.2 The star weighted graph structure

In this section we describe the resolution of p as a singular point of V and as a singularity of $\mathcal{F}_\varphi$. This description is already in the paper [11].

Theorem 2. *Let V be a normal Stein analytic space of dimension two and φ a $\mathbb{C}^*$-action on V with a dicritical singularity at $p \in V$. Then there is a resolution $\rho : \tilde{V} \to V$ of $\mathcal{F}_\varphi$ at the point $p \in V$ such that*

1. *$\rho^{-1}(p) = \bigcup_{i=0}^r \sigma_i$ is a weighted star graph centered at the Riemann surface σ_0, of genus g, and consisting of Riemann spheres σ_i, $i > 0$;*
2. *σ_0 is the unique dicritical component of $\tilde{\mathcal{F}}_\varphi = \rho^* \mathcal{F}_\varphi$;*
3. *the pull-back action $\tilde{\varphi}$ on $\tilde{V}$ is trivial on σ_0 and nontrivial on each σ_i, $i > 0$, i.e. $Fix(\tilde{\varphi}) \cap \sigma_0 = \sigma_0$, and $Fix(\tilde{\varphi}) \cap \sigma_i$ consists of two points for each $i > 0$;*
4. *The singular points of the foliation $\tilde{\mathcal{F}}_\varphi$ are all simple, $Fix(\tilde{\varphi}) = sing(\tilde{\mathcal{F}}_\varphi) \cup \sigma_0$, and $sing(\tilde{\mathcal{F}}_\varphi) \cap \sigma_0 = \emptyset$.*

In the algebraic context in which V is affine and the $\mathbb{C}^*$-action is algebraic, the above theorem with items *1*, *2* and *3* is a result of Orlik and Wagreich (see [11]). Our proof uses the theory of holomorphic foliations on complex manifolds instead of topological methods. In order to prove Theorem 2 we need the following index theorem.

2.3 The Index theorem

Let σ be a Riemann surface embedded in a two dimensional manifold S ; $\mathcal{F}$ a foliation on S which leaves σ invariant and $q \in \sigma$. There is a neighborhood of q where σ can be expressed by $(f = 0)$ and $\mathcal{F}$ is induced by the holomorphic 1-form ω written as $\omega = hdf + f\eta$. Then we can associate the following index:

$$i_q(\mathcal{F}, \sigma) := -\text{Residue}_q(\frac{\eta}{h})|_\sigma$$

relative to the invariant submanifold σ. In the case of a simple singularity as defined above if σ is locally $(y = 0)$ and $q = 0$, this index is equal to μ (quotient of eigenvalues). In the case of a saddle-node, if σ is equal to $(x = 0)$ and $q = 0$, this index is zero. At a regular point q of $\mathcal{F}$ the index is zero. The index theorem of [4] asserts that the sum of all the indices at the points in σ is equal to the self-intersection number $\sigma \cdot \sigma$:

$$\sum_{q \in \sigma} i_q(\mathcal{F}, \sigma) = \sigma \cdot \sigma.$$

2.4 Proof of Theorem 2

By hypothesis, in the resolution of $p \in V$ there is at least one dicritical component, say σ_0. Then the action $\tilde{\varphi}$ extends to σ_0 as a set of fixed points. We claim that σ_0 is the unique dicritical component. Indeed, at each dicritical component the $\mathbb{C}^*$- action $\tilde{\varphi}$ is trivial. Since V is normal at $p \in V$, $\rho^{-1}(p)$ is connected ([8]), thus if there is another dicritical component, say σ_i, then there would exist $\mathbb{C}^*$- orbits of $\tilde{\varphi}$, with compact analytic closure crossing σ_0 and σ_i transversely contradicting the fact that V is Stein. Thus σ_0 is the only dicritical component, and the action $\tilde{\varphi}$ is trivial on σ_0. The same argument shows that there cannot be cycles of components of D, because this would imply the existence of leaves starting and ending at σ_0. Thus the graph associated to ρ is contractible.

A linear chain at a point $q \in \sigma_0$ is a union of compact Riemann surfaces, elements of the divisor D, say $\sigma_1, ..., \sigma_n$ such that $\sigma_1 \cap \sigma_0 = \{q\}$ and $\sigma_i \cap \sigma_j$ is nonempty if and only if $i = j - 1$ and in this case it is a point, for $j = 2, ..., n$.

Lemma 1. *Suppose that $r_1, r_2, ..., r_s$ are the crossing points at σ_0 of the divisor D. Then the divisor D consists of the union of σ_0 and linear chains of Riemann spheres at each of these crossing points.*

Proof. Consider the divisor D at the point r_1 renamed as p_0. Let σ_1 be such that $p_0 = \sigma_0 \cap \sigma_1$. We claim that the $\mathbb{C}^*$-action $\tilde{\varphi}$ on σ_1 is nontrivial with a fixed point at p_0. Indeed it can be represented in local coordinates (x, y), where $(x = 0) = \sigma_0$, $(y = 0) = \sigma_1$, by the vector field $Y = (Y_1, 0)$ with $Y_1(0, y) = 0$. Consider the restriction of the action $\tilde{\varphi}$ to the subgroup $S^1 \subset \mathbb{C}^*$. Then in the $\mathbb{C}$-plane $(y = y_0)$ the S^1-orbit of a generic point (x, y_0), $x \neq 0$, will turn l times around $(0, y_0)$ and this number, which is different from zero, will be constant as $y_0 \to 0$. Therefore $\tilde{\varphi}$ extends to the x-axis σ_1 as a nontrivial $\mathbb{C}^*$-action. Therefore σ_1 is a Riemann sphere and there is another point $p_1 \in \sigma_1$ which is fixed by $\tilde{\varphi}$. Since p_1 is the unique singularity of $\tilde{\mathcal{F}}_\varphi$ in σ_1 we must have that the index of $\tilde{\mathcal{F}}_\varphi$ with respect to the invariant manifold σ_1 at p_1 is given by ([4])

$$i_{p_1}(\tilde{\mathcal{F}}_\varphi, \sigma_1) = \sigma_1 . \sigma_1 = -k_1, \; k_1 \in \mathbb{N}.$$

Therefore p_1 cannot be a saddle-node, as in this case this index would be zero. This implies that p_1 is simple for $\tilde{\mathcal{F}}_\varphi$. Either the chain ends at σ_1 or there is another component, say σ_2, such that $\{p_1\} = \sigma_1 \cap \sigma_2$. In this last case, p_1 is simple. We claim that the action $\tilde{\varphi}$ on σ_2 is nontrivial. Indeed, let (x, y) be a system of coordinates in a neighborhood $\mathcal{N}$ of $p_1 = (0, 0)$ such that $(x = 0) = \sigma_1 \cap \mathcal{N}$, $(y = 0) = \sigma_2 \cap \mathcal{N}$. By derivation along the parameter of the group, the action φ induces a vector field Y on $\mathcal{N}$. Assuming by contradiction that φ is trivial on σ_2 we would have $Y(x, 0) = 0$ and we can assume, changing coordinates if necessary, that $DY(x, 0) = diag(0, \lambda_x)$, $\lambda_0 \neq 0$. By continuity, $\lambda_x \neq 0$ for x small enough. By the invariant manifold theorem for ordinary differential equations, there is a fibration invariant by Y, transverse to σ_2, whose fibers are the subsets of $\mathcal{N}$ defined as $\tau_x = \{(x, y); \lim_{t\to 0} \varphi(t, (x, y)) = (x, 0)\}$, $\tau_0 = \sigma_1$. Thus σ_2 is a dicritical component of $\tilde{\mathcal{F}}_\varphi$, which is a contradiction. Therefore σ_2 will be a *Riemann sphere* with another fixed point $p_2 \in \sigma_2$ for the action $\tilde{\varphi}$. It is clear that the corresponding index will be given by

$$i_{p_2}(\tilde{\mathcal{F}}_\varphi, \sigma_2) = -k_2 + 1/k_1 \neq 0,\ k_2 = -\sigma_2.\sigma_2 \in \mathbb{N}.$$

More generally, the linear chain will consist of a finite sequence of elements of the divisor $\sigma_0, \sigma_1, ..., \sigma_n$ such that σ_i, for $i \neq 0$, is a Riemann sphere where the action $\tilde{\varphi}$ is nontrivial, and $\sigma_i \cap \sigma_{i+1} = \{p_i\}$ is a simple singularity of $\tilde{\mathcal{F}}_\varphi$ for $i = 1, ..., n-1$. Denote by $-k_i = \sigma_i.\sigma_i, k_i \in \mathbb{N}$. At each point p_i the index of this singularity relative to σ_n is

$$i_{p_j}(\tilde{\mathcal{F}}_\varphi, \sigma_j) = -[k_j, k_{j-1}, ..., k_1],$$

where we have a continued fraction

$$[k_j, k_{j-1}, ..., k_1] = k_j - \cfrac{1}{k_{j-1} - \cfrac{1}{\ddots}}.$$

We claim that the numbers $[k_j, k_{j-1}, ..., k_1]$, $j = 1, ..., n$, are all well defined and different from zero. Indeed, this is a consequence of the fact that the intersection matrix $(\sigma_i \cdot \sigma_i)$ is negative definite ([8]). Let M be a real symmetric $n \times n$ matrix and Q a non-singular real $n \times n$ matrix. Then M is negative definite if and only if $Q^t MQ$ is negative definite. Given the matrix $M = (\sigma_i \cdot \sigma_j)$ we take Q as the matrix with one's in the diagonal, a in the $(1, 2)$ entry, and zeros elsewhere. Then a convenient choice of a will yield a matrix $Q^t MQ$ with $-k_1$ in the $(1, 1)$ entry and zeros in the $(1, 2)$ and $(2, 1)$ entries. Repeating this procedure we obtain that the following diagonal matrix

$$\mathrm{diag}(-k_1, -[k_2, k_1], ..., -[k_n, k_{n-1}, ..., k_1])$$

is negative definite, proving the claim and the lemma.

□

Theorem 2 follows from the above discussion and Lemma 1.

3 Linearization around the dicritical divisor

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. In the previous section we saw that the multiplicative pseudo group $\mathcal{G} = (\mathbb{C}, \mathbb{D}) - \{0\}$ acts on $(\tilde{V}, \sigma_0)$ and the flow of the action φ is transverse to σ_0. The purpose of this section is to show that such an action is biholomorphically conjugated with the canonical $\mathcal{G}$-action on the normal bundle to σ_0 in $\tilde{V}$.

3.1 $\mathcal{G}$-transverse actions to a Riemann surface

Let σ be a Riemann surface embedded in a surface S. We say that ψ is a transverse $\mathcal{G}$-action on (S,σ) if

1. For all $a \in \sigma$ and $t \in \mathcal{G}$ we have $\psi(t,a) = a$.
2. There is a foliation $\mathcal{F}$ on (S,σ), transverse to σ such that each leaf of $\mathcal{F}$ is the closure of $\{\psi(t,a) \mid t \in \mathcal{G}\}$ for some $a \in (S,\sigma) - \sigma$.

A typical example of a $\mathcal{G}$-action is the following: We consider a line bundle L on σ and the embedding $\sigma \hookrightarrow L$ given by the zero section. Now for every $q \in \mathbb{N}$ we have a transverse $\mathcal{G}$-action on (L,σ) given by $(t,a) \mapsto t^q a$. It turns out that up to biholomorphy these are the only transverse $\mathcal{G}$-actions.

Theorem 3 (Linearization theorem)**.** *Let σ be a Riemann surface embedded in a surface S and ψ a transverse $\mathcal{G}$-action on (S,σ). Then ψ is linearizable in the sense that there exist a biholomorphism $h : (S,\sigma) \to (N,\sigma)$, where N is the normal bundle to σ in S, and a natural number q such that $h(\psi(t,a)) = t^q h(a)$ for any $a \in (S,\sigma)$.*

Notice that the linearization of ψ yields also the linearization of the associated foliation. An immediate corollary of the above theorem is that non-linearizable neighborhoods do not admit any transversal $\mathcal{G}$-action. For instance, Arnold's example in which σ is a torus of self-intersection number zero in some complex manifold of dimension two is not linearizable and so it does not admit any transversal $\mathcal{G}$-action (see [1]).

3.2 Local linearization

Let $S = (\mathbb{C}^2, 0)$ and $0 \in \sigma \subset S$ be a smooth curve in S. In a similar way as before we define a $\mathcal{G}$-action on (S,σ) transverse to σ and call it the local transverse $\mathcal{G}$-action.

Lemma 2. *Any local transverse $\mathcal{G}$-action can be written in a local system of coordinates in the form $\psi(t,(x,y)) = (x, t^q y)$.*

Proof. We take a coordinates system (x,y) around $0 \in \mathbb{C}^2$ such that the the foliation $\mathcal{F}_\psi$ is given by $dx = 0$ and σ is given by $y = 0$. In these coordinates the flow ψ_t of the $\mathbb{C}^*$-action is given by:

$$\psi_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0),\ \psi_t(x,y) = (x, p_{t,x}(y)).$$

Since the orbits of ψ tend to σ when t tends to zero, $p_{t,x}$ is a holomorphic function in $t \in (\mathbb{C}, \mathbb{D})$. We have also $p_{t,x}(0) = 0$ because σ is the set of fixed points of ψ. We can write $p_{t,x}(y)$ as a series

$$p_{t,x}(y) = \sum_{i=1} p_i(t,x) y^i.$$

Substituting the above term in $\psi(t_1 t_2, a) = \psi(t_1, \psi(t_2, a))$ we obtain

$$p_1(t_1 t_2, x) = p_1(t_1, x) p_1(t_2, x),\ t_1, t_2 \in \mathcal{G},\ x \in (\mathbb{C}, 0).$$

Since p_1 is holomorphic at $t = 0$, the derivation of the above equality in t_1 implies that $p_1(t,x) = t^q$ for some $q \in \mathbb{N}$. Now, by the Theorem on the linearization of germs of holomorphic mappings, there is a unique $f_{t,x} : (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$ which is tangent to the identity, depends holomorphically on t, x and

$$f_{t,x}^{-1} \circ p_{t,x} \circ f_{t,x}(y) = t^q y.$$

The $\mathbb{C}^*$-action ψ in the coordinates $(\tilde{x}, \tilde{y}) = (x, f_{t,x}(y))$ has the desired form. □

Now consider on S a foliation $\mathcal{F}$ which is transverse to σ (no $\mathcal{G}$-action is considered). Let ω be a 1-form on S such that

$$div(\omega) = \sigma + nL_0,$$

where $n \in \mathbb{Z}$ and L_0 is the leaf of $\mathcal{F}$ through $0 \in S$.

Lemma 3. *Given a local system of coordinates x in σ, there is a unique system of coordinates $(\tilde{x}, \tilde{y})$ in S such that*

1. *The restriction of $\tilde{x}$ to σ is x;*
2. *The 1-form ω in $(\tilde{x}, \tilde{y})$ is of the form $\tilde{x}^n \tilde{y} d\tilde{x}$.*

Proof. For the proof of the existence we take a coordinates system $(\tilde{x}, \tilde{y})$ in a neighborhood of 0 in S such that σ and $\mathcal{F}$ in this coordinate system are given respectively by $\tilde{y} = 0$ and $d\tilde{x} = 0$ and $\tilde{x} \mid_\sigma = x$. We write $\omega = p\tilde{x}^n \tilde{y} d\tilde{x}$, where $p \in \mathcal{O}_S,\ p(0) \neq 0$. By changing the coordinates $(\tilde{x}, \tilde{y}) \to (\tilde{x}, p\tilde{y})$ we obtain the desired coordinate system. The uniqueness follows from the fact that any local biholomorphism $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ which is the identity in $\tilde{y} = 0$ and $f^* \tilde{x}^n \tilde{y} d\tilde{x} = \tilde{x}^n \tilde{y} d\tilde{x}$ is the identity map. □

3.3 Construction of differential forms

Consider a Riemann surface σ embedded in a two dimensional manifold S. We take a meromorphic section s of the normal bundle N of σ in S and set

$$div(s) = \sum n_i p_i,\ n_i \in \mathbb{Z},\ p_i \in \sigma.$$

Lemma 4. *For a transverse $\mathcal{G}$-action ψ on (S, σ), there is a meromorphic function u on (S, σ) such that*

1. $$div(u) = \sigma - \sum n_i p_i,\ n_i \in \mathbb{Z},\ p_i \in \sigma,$$
2. $$u(\psi(t, a)) = t^q u(a),\ a \in (S, \sigma),\ t \in \mathcal{G}.$$

Let $\tilde{v}$ be an arbitrary meromorphic function on σ and v its extension to S along the foliation $\mathcal{F}$. The 1-form

$$\omega = udv$$

has the properties:

1. *ω induces the foliation $\mathcal{F}$;*
2. *The divisor of ω is $\sigma + K$, where K is $\mathcal{F}$-invariant.*
3. *$\psi_t^* \omega = t^q \omega,\ t \in \mathcal{G}$, where $\psi_t(x) = \psi(t, x)$.*

Proof. In a local coordinate system (x_α, y_α) in a neighborhood U_α of a point p_α of σ in S one can write the $\mathcal{G}$-action as follows

$$\psi(t, (x_\alpha, y_\alpha)) = (x_\alpha, t^q y_\alpha),$$

where $\sigma \cap U_\alpha = \{y_\alpha = 0\}$. The meromorphic function $u_\alpha = x_\alpha^{-n} y_\alpha$, where $n = n_i$ if $p = p_i$ for some i and $n = 0$ otherwise, satisfies the conditions *1, 2* in U_α. We define $u_{\alpha\beta} := \frac{u_\alpha}{u_\beta}$. Now $L := \{u_{\alpha\beta}\} \in H^1(S, \pi^{-1}\mathcal{O}_\sigma^*) = H^1(\sigma, \mathcal{O}_\sigma^*)$, where $\pi : S \to \sigma$ is the projection along the fibers. On the other hand, the line bundle associated to σ in S and then restricted to σ is the normal bundle of σ in S and so by definition L restricted to σ is the trivial bundle. This means that there are $a_\alpha \in \pi^{-1}\mathcal{O}_\sigma^*(U_\alpha)$ such that $u_{\alpha\beta} = \frac{a_\alpha}{a_\beta}$. Now, $\frac{u_\alpha}{a_\alpha}$ define a meromorphic function on S with the desired properties. □

Remark 1. In the case in which we have a transverse foliation $\mathcal{F}$ without any transverse ψ action, the linearization of $\mathcal{F}$ requires $\sigma \cdot \sigma < min(2 - 2g, 0)$, where g is the genus of σ (see [2, 3]). In this case, in order to construct u with the first property we used this hypothesis and proved that the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(\sigma)$ is injective. As we saw in the proof of Lemma 4, in the presence of a transverse $\mathcal{G}$-action we do not need any hypothesis on $\sigma \cdot \sigma$.

3.4 Holomorphic equivalence of neighborhoods

Now we consider two embeddings of σ with transverse foliations.

Lemma 5. *Let σ be a Riemann surface embedded in two surfaces S_i, $i = 1, 2$ and let $\mathcal{F}_i$ be a foliation transverse to σ on S_i induced by a 1-form ω_i such that the divisor of ω_i is $\sigma + K_i$, where K_i is $\mathcal{F}_i$-invariant and K_1 and K_2 restricted to σ coincide. Then there is a unique biholomorphism $h : (S_1, \sigma) \to (S_2, \sigma)$ such that $h^*\omega_2 = \omega_1$.*

Proof. Using Lemma 3 we conclude that for a point $a \in \sigma$ there is a unique $h : (S_1, \sigma, a) \to (S_2, \sigma, a)$ such that h restricted to σ is the identity map and $h^*\omega_2 = \omega_1$. The uniqueness implies that these local biholomorphisms coincide in their common domains and so they give us a global biholomorphism $h : (S_1, \sigma) \to (S_2, \sigma)$ with the desired property.

□

3.5 Proof of the linearization theorem

Let us now prove Theorem 3. Take $i = 1, 2$. Let σ be a Riemann surface embedded in two surfaces S_i and let ψ_i be a transverse $\mathcal{G}$-action on (S_i, σ) with the multiplicity q and corresponding foliation $\mathcal{F}_i$. By Lemma 4 we can construct a 1-form ω_i with the properties *1, 2, 3*. By construction of ω_i, if $div(\omega_i) = \sigma + K_i$ then K_i restricted to σ depends only on $\tilde{v}$ and s and so we can take the K_i's so that $K_1 \mid_\sigma = K_2 \mid_\sigma$. Now Lemma 5 implies that there is a unique biholomorphism $h : (S_1, \sigma) \to (S_2, \sigma)$ such that $h^*\omega_2 = \omega_1$. We claim that h conjugates also the ψ_i's. Fix $t \in \mathcal{G}$ and let $\psi_{i,t} : (S_i, \sigma) \to (S_i, \sigma)$ be a biholomorphism defined by

$$\psi_{i,t}(a) := \psi_i(t, a), \; a \in (S_i, \sigma).$$

We have

$$h^*\psi_{2,t}^*\omega_2 = h^*t^q\omega_2 = t^q\omega_1 = \psi_{1,t}^*\omega_1 = \psi_{1,t}^*h^*\omega_2.$$

Since by Lemma 5 the sole $f : (S_2, \sigma) \to (S_2, \sigma)$ such that $f^*\omega_2 = \omega_2$ is the identity map, we conclude that $h^*\psi_{2,t}^* = \psi_{1,t}^*h^*$ and so $h(\psi_1(t, a)) = \psi_2(t, h(a))$.

4 Linearization in the attraction basin

In this section we associate to the foliation $\tilde{\mathcal{F}}_\varphi$ a linear model and prove a linearization result based on the existence of the $\mathcal{G}$-action transverse to σ_0.

4.1 The linear model

We can associate to the pair $(\tilde{\mathcal{F}}_\varphi, \tilde{V})$ a linear model constructed as follows. Let L be the normal bundle of σ_0 in $\tilde{V}$. We denote by L^{-1} the dual of L. We can glue L and L^{-1} together and obtain a compact projective manifold $\bar{L}$ in the following way: Let $\{U_\alpha\}_{\alpha\in I}$ be an open covering of σ_0 and z_α (resp. z'_α) a holomorphic without zero section of L (resp. L^{-1}) on U_α. Then

$$z_\alpha = g_{\alpha\beta} z_\beta,\ z'_\alpha = g^{-1}_{\alpha\beta} z'_\beta,\ L = \{g_{\alpha\beta}\}_{\alpha,\beta\in I} \in H^1(S, \mathcal{O}^*).$$

For a point $a \in L_p, p \in U_\alpha,\ a \neq 0_p$ we define the point $\frac{1}{a} \in L_p^{-1}$ by setting

$$\frac{1}{a} = \frac{z_\alpha(p)}{a} z'_\alpha(p).$$

The map $a \to 1/a$ does not depend on the chart U_α and gives us a biholomorphism between $L - \sigma_0$ and $L^{-1} - \sigma_\infty$, where σ_0 (resp. σ_∞) is the zero section of L (resp. L^{-1}).

For each point $r_i^0 = r_i \in \sigma_0,\ i = 1, 2, \ldots, s$ we denote by r_i^∞ the unique intersection point of σ_∞ and $\bar{L}_{r_i^0}$. By various blow ups starting from r_i^∞ in the chain $\sigma_0, \bar{L}_{r_i^0}, \sigma_\infty$, we can create a chain of divisors

$$\sigma_0, \sigma_1^i, \sigma_2^i, \ldots, \sigma_{n_i}^i, \tilde{\sigma}, \tau_{m_i}^i, \tau_{m_i-1}^i, \ldots, \tau_1^i, \sigma_\infty$$

such that

$$\sigma_j^i \cdot \sigma_j^i = -k_j^i,\ j = 1, 2, \ldots, n_i, \quad \tilde{\sigma} \cdot \tilde{\sigma} = -1, \quad -l_j^i := \tau_j^i \cdot \tau_j^i < -1, j = 1, 2, \ldots, m_i.$$

The chain of self-intersections of the divisors in the blow-up process is given by:

$$(-k, 0, k), (-k, -1, -1, k-1), (-k, -2, -1, -2, k-1), \ldots, (-k, -k_1^i, -1, \underbrace{-2, \cdots, -2}_{k_1^i - 1 \text{ times}}, k-1)$$

$$(-k, -k_1^i, -2, -1, -3, \underbrace{-2, \cdots, -2}_{k_1^i - 2 \text{ times}}, k-1), \cdots, (-k, -k_1^i, -k_2^i, \cdots, -k_{n_i}^i, -1, l_{m_i}^i, \cdots, l_2^i, l_1^i, k-1).$$

Repeating this construction at each point r_i, $i = 1, ..., s$ we obtain a surface X. Let

$$D_\infty = \sigma_\infty + \sum_{i=1}^{s} \sum_{j=1}^{m_i} \tau_j^i,\ D_0 = \sigma_0 + \sum_{i=1}^{s} \sum_{j=1}^{n_i} \sigma_j^i.$$

Now, $\tilde{\mathcal{V}} := X - D_\infty$ is the desired linear model variety. In $\bar{L}$ we have a canonical $\mathbb{C}^*$ action whose orbits are the fibers of L. It gives us a $\mathbb{C}^*$-action $\tilde{\lambda}$ on $\tilde{\mathcal{V}}$. We denote by $\tilde{\mathcal{F}}_\lambda$ the associated foliation on $\tilde{\mathcal{V}}$. The pair $(\tilde{\mathcal{V}}, \tilde{\mathcal{F}}_\lambda)$ will be called the linear approximation of $(\tilde{V}, \tilde{\mathcal{F}}_\varphi)$.

In order to proceed with our discussion we need some definitions: A divisor $Y = \sum_{i=1}^{l} Y_i$ in a two-dimensional surface X is a support of a divisor with positive (resp.

negative) normal bundle if there is a divisor $\tilde{Y} := \sum_{i=1}^{l} a_i Y_i$, where the $a_i,\ i = 1, 2, \ldots, l$ are positive integers, such that $\tilde{Y} \cdot Y_j > 0($ resp. $< 0)$, for $j = 1, \ldots, l$.

We say that the normal bundle of the divisor $\tilde{Y}$ in X is positive (resp. negative). Observe that the normal bundle N of a divisor is positive (resp. negative) if and only if N restricted to each irreducible component of the divisor is positive (resp. negative) (see [7] Proposition 4.3). In fact the above number is the Chern class of $N\,|_{Y_i}$ (see [8] p. 62).

Lemma 6. *The following assertions are equivalent:*

1. *The divisor D_∞ is a support of divisor with positive normal bundle.*
2. *The self-intersection matrix of D_0 is negative definite.*
3. $$\sum_{i=1}^{s} \frac{1}{[k_1^i, k_2^i, \ldots, k_{n_i}^i]} < k.$$
4. $$\sum_{i=1}^{s} \frac{1}{[l_1^i, l_2^i, \ldots, l_{m_i}^i]} > s - k.$$

Proof. 1 $\Rightarrow$ 2. From [7] Theorem 4.2 it follows that one can make a blow down of the divisor D_0 and so the self intersection matrix of D_0 is negative definite.

2 $\Rightarrow$ 3. We remark that the diagonalization of the intersection matrix of D_0 by the procedure given in Lemma 1 leads to

$$\mathrm{diag}(\ldots, -k_{n_i}^i, -[k_{n_i-1}^i, k_{n_i}^i], ..., -[k_1^i, k_2^i, ..., k_{n_i}^i], \ldots, -k + \sum_{i=1}^{s} \frac{1}{[k_1^i, k_2^i, \ldots, k_{n_i}^i]}).$$

Recall that $k_j^i > 1$ for $i = 1, \ldots, s; j = 1, \ldots, n_i$.

3 $\Rightarrow$ 4. Using the index theorem we have

$$\frac{1}{[k_{n_i}^i, k_{n_i-1}^i, \ldots, k_1^i]} + \frac{1}{[l_{m_i}^i, l_{m_i-1}^i, \ldots, l_1^i]} = 1.$$

Notice that the order of the continued fraction is the inverse of the one we need. However we have that: if

$$-k, -k_1^i, -k_2^i, \ldots, -k_{n_i}^i, -1, -l_{m_i}^i, -l_{m_i-1}^i, \ldots, -l_1^i, k-1$$

is obtained by blow-ups as we explained then

$$-k, -k_{n_i}^i, -k_{n_i-1}^i, \ldots, -k_1^i, -1, -l_1^i, -l_2^i, \ldots, -l_{m_i}^i, k-1$$

is also obtained by blow-ups. This can be proved by induction on the number of blow-ups. Notice that to create each branch of the star we have done only one blow-up centered at a point of σ_∞ (the first blow-up) and so after obtaining the desired star the self intersection of σ_0 is $k - s$.

4 $\Rightarrow$ 1. We are looking for natural numbers a and $a_j^i,\ j = 1, 2, \ldots, m_i,\ i = 1, 2, \ldots, s$ such that the normal bundle of $\tilde{Y} = a\sigma_\infty + \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_j^i \tau_j^i$ is ample, i.e $\tilde{Y} \cdot \sigma > 0$ for $\sigma = \sigma_\infty$ and all σ_j^i. These inequalities are translated into:

$$-l_j^i a_j^i + a_{j-1}^i + a_{j+1}^i > 0, a_0^i := n,\ a_{m_i+1}^i := 0,$$

$$a(k-s)+\sum_{i=1}^{s} a^i_{m_i} > 0.$$

We rewrite these inequalities in the following way:

$$\frac{a}{a^i_1} > [l^i_1, \frac{a^i_1}{a^i_2}] > \ldots > [l^i_1, l^i_2, \ldots, l^i_{m_i-1}, \frac{a^i_{m_i-1}}{a^i_{m_i}}] > [l^i_1, l^i_2, \ldots, l^i_{m_i-1}, l^i_{m_i}],$$

$$\sum_{i=1}^{s} \frac{1}{\frac{a}{a^i_1}} > s-k.$$

The existence of positive rational numbers $\frac{a^i_j}{a^i_{j-1}}$ follows from the hypothesis 4. Notice that l^i_j are all greater than 1 and so the $[l^i_1, l^i_2, \ldots, l^i_{m_i-1}, l^i_{m_i}]$'s are positive. □

We denote by $\mathcal{V}$ the variety obtained by the blow down of the divisor D_0 in $\tilde{\mathcal{V}}$. We also denote by λ the $\mathbb{C}^*$-action on $\mathcal{V}$ corresponding to $\tilde{\lambda}$ in $\tilde{\mathcal{V}}$.

Proposition 1. *The variety $\mathcal{V}$ is affine algebraic and the $\mathbb{C}^*$- action λ is given by a good action in some affine coordinates.*

Proof. Since the self intersection matrix of D_0 is negative definite, by Lemma 6 we have that D_∞ is the support of a divisor Y with positive normal bundle. By [7] Theorem 4.2 there exists a birational morphism $f : X \to \tilde{X} \subset \mathbb{P}^\nu$ such that f is an isomorphism in a Zariski open neighborhood of D_∞ and $af(Y)$ for some big positive integer a is a hyperplane section. We have $f = [f_0 : f_1 : \ldots : f_\nu]$, where $f_0, f_1, \ldots, f_\nu$ is a $\mathbb{C}$-basis of $H^0(X, \mathcal{O}_X(aY))$ for $a > 0$ big enough. Here $\mathcal{O}_X(aY)$ is the sheaf of meromorphic functions u on X with $div(u) + aY > 0$. Since $\mathbb{C}^*$ acts on $H^0(X, \mathcal{O}_X(aY))$ we can take f_i's such that $f_i(\lambda(x,t)) = t^{q_i} f_i(x)$ for some $q_i \in \mathbb{N}$. It turns out that f is an isomorphism in $X - D_0$ and the divisor D_0 is mapped to a point of $p \in \mathcal{V}$. □

4.2 Existence of a global linearization

We introduce the attraction basin B_p of p, by the flow φ, as

$$B_p = \{\varphi(t,z); t \in \mathbb{C}^*; z \in U\},$$

where $U \subset V$ is the image of a neighborhood $\tilde{U}$ of σ_0 in $\tilde{V}$ by the resolution map ρ. A theorem of Suzuki in [17] asserts that the foliation $\mathcal{F}_\varphi$ admits a meromorphic first integral. This implies that the singularities of $\tilde{\mathcal{F}}_\varphi$ are linearizable, and together with Theorem 2, that B_p contains an open neighborhood of p. This fact will be proved again during the construction of the conjugacy map between $\mathbb{C}^*$-actions. We aim to construct a conjugacy between φ on B_p and λ on $\mathcal{V}$ establishing the following theorem:

Theorem 4. *The set B_p is an open subset of V and there is a biholomorphism $h : B_p \to \mathcal{V}$ which is a conjugacy between the actions φ and λ, i.e.,*

$$h(\varphi(t,z)) = \lambda(t, h(z)), \text{ for every } (t,z) \in \mathbb{C}^* \times B_p.$$

Proof. It will be enough to show that there is a conjugacy between $\tilde{\varphi}$ on $\tilde{B}_p := \rho^{-1}(B_p)$ and $\tilde{\lambda}$ on $\tilde{\mathcal{V}}$. We start by defining the conjugacy in a neighborhood of σ_0. An immediate consequence of Theorem 3 is that there is a biholomorphic conjugacy $h\colon \tilde{U} \to \tilde{\mathcal{U}}$ between the restrictions of $\tilde{\varphi}$ and $\tilde{\lambda}$, where $\tilde{U}$ is a neighborhood of σ_0 in $\tilde{V}$ and $\tilde{\mathcal{U}}$ is a neighborhood of σ_0 in $\tilde{\mathcal{V}}$. The conjugacy h extends along the flows $\tilde{\varphi}$ and $\tilde{\lambda}$ as follows: For a point $z' \in \tilde{B}_p \backslash D$ there is $t \in \mathbb{C}^*$ such that $z := \varphi(t, z') \in \tilde{U}$. We define $h(z')$ by the equality $h(z') = \tilde{\lambda}(t^{-1}, h(z))$. It remains to extend h to a neighborhood of the invariant manifolds of the fixed points of $\tilde{\varphi}$ in $\bigcup_{i=1}^{r} \sigma_i$. These points are all simple and lie in the linear chains starting at $r_1, ..., r_s$ in σ_0. Fix the linear chain starting at $r_1 {=} p_0$. The linear chain consists of a finite sequence of elements of the divisor $\sigma_0, \sigma_1, ..., \sigma_n$ such that σ_i for $i \neq 0$ is a Riemann sphere where the action $\tilde{\varphi}$ is nontrivial, and $\sigma_i \cap \sigma_{i+1} = \{p_i\}$ is a simple singularity of $\tilde{\varphi}$ for $i = 1, ..., n-1$. Since at each σ_i, $i > 0$, $\tilde{\varphi}$ has two singularities, there is another fixed point of $\tilde{\varphi}$, $p_n \in \sigma_n$. The conjugacy h is already defined on $\sigma_1 \backslash \{p_1\}$. The next lemma will imply that h extends to $\sigma_2 \backslash \{p_2\}$. Proceeding by induction and having already extended h to $\sigma_n \backslash \{p_n\}$ the next lemma will apply again to extend h to the remaining invariant manifold of p_n. The same procedure can be followed on the other linear chains starting at $r_1, ..., r_s$ in σ_0.

Before stating our Lemma we remark that the $\mathbb{C}^*$-action $\tilde{\varphi}$ around each singularity p_i is linearizable i.e. there are natural numbers q_1, q_2 and coordinates (x, y) around p_i such that in these coordinates the $\mathbb{C}^*$-action $\tilde{\varphi}$ is given by

$$\tilde{\varphi}(t, (x, y)) = (t^{q_1} x, t^{-q_2} y), \;\; q_1, q_2 \in \mathbb{N} \tag{4.1}$$

(around p_i the separatrix σ_i is attracting and σ_{i+1} is repelling). In other words, the corresponding vector field is linearizable. This fact follows from generalizations of a theorem due to Cartan, see [9] Theorem 7. This theorem states that any group of holomorphic automorphisms of a bounded domain $\mathcal{D}$ in $\mathbb{C}^n$ with the fixed point $0 \in \mathcal{D}$ can be linearized using a change of coordinates around 0 tangent to the identity. The idea behind the proof of Cartan's Theorem is simple and in our case is as follows: In order to remain in the local context around p_i take the action of $S^1 \subset \mathbb{C}^*$ on an open neighborhood of p_i. The group S^1 acts also in the two dimensional vector space $\mathcal{M}_{p_i}/\mathcal{M}_{p_i}^2$, where $\mathcal{M}_{p_i}$ is the set of germs of holomorphic functions around p_i and so we can take two germs of functions $f_1, f_2 \in \mathcal{M}_{p_i}$ and find two integers $q_1, q_2 \in \mathbb{Z}$ such that $\tilde{\varphi}_t^* f_j = t^{q_j} f_j \bmod \mathcal{M}_{p_i}^2$, $j = 1, 2$ for all $t \in S^1$. We define

$$g_j := \int_0^1 \frac{\tilde{\varphi}^*_{e^{2\pi i\theta}} f_j}{e^{2\pi i \theta q_j}} d\theta, \;\; j = 1, 2$$

Now $(\tilde{x}, \tilde{y}) = (g_1, g_2)$ is the desired change of coordinates. Note that q_1 is positive because by our construction σ_i is attracting for p_i. Note also that q_2 is negative, otherwise all the orbits around p_i are separatrices of p_i. We replace q_2 by $-q_2$ and obtain (4.1) for $t \in S^1$. Since two holomorphic functions which coincide on S^1, coincide also in their common definition domain we get (4.1) for t in a neighborhood of S^1 in $\mathbb{C}^*$.

It only remains to prove the following lemma.

Lemma 7. *Let h be a biholomorphism between neighborhoods of $(\mathbb{C} - \{0\}) \times \{0\}$ in $\mathbb{C}^2$ which is a self-conjugacy of the $\mathbb{C}^*$-action (4.1) . Then h extends to a neighborhood of the origin $0 \in \mathbb{C}^2$ as a biholomorphic map.*

Proof. Let $h = (h_1, h_2)$. We prove that the h_i's extend to $(\mathbb{C}^2, 0) - \{x = 0\}$ and are bounded near the origin and so h extends to 0. A similar argument for h^{-1} implies that h is a biholomorphism. Fix a transversal section $\Sigma = \{x = x_0\}$ near $0 \in \mathbb{C}^2$. For $(x, y) \in (\mathbb{C}^2, 0),\ x \neq 0,\ |x| \leq |x_0|$ we have

$$h_1(x,y) = \frac{x}{x_0} h_1(x_0, (\frac{x}{x_0})^{\frac{q_2}{q_1}} y),\ h_2(x,y) = (\frac{x}{x_0})^{\frac{-q_2}{q_1}} h_2(x_0, (\frac{x}{x_0})^{\frac{q_2}{q_1}} y).$$

These equalities extend h to $(\mathbb{C}^2, 0) - \{x = 0\}$ and show that h_1 and h_2 are bounded. Indeed, h leaves $\{y = 0\}$ invariant and so $h_2 = y\tilde{h}_2$, where $\tilde{h}_2$ is a holomorphic function in a neighborhood of $(\mathbb{C} - \{0\}) \times \{0\}$ in $\mathbb{C}^2$. Therefore, $h_2(x, y) = y\tilde{h}_2(x_0, (\frac{x}{x_0})^{\frac{q_2}{q_1}} y)$ is bounded. Notice also that the extension is one valued because h_1 and h_2 are already defined as one-valued functions in a neighborhood of $(\mathbb{C} - \{0\}) \times \{0\}$ in $\mathbb{C}^2$. □

□

Remark 2. We observe that, as a consequence of Theorem 2 the singularity $p \in V$ is absolutely dicritical in the sense that there is a neighborhood W of p in V such that every leaf of $\mathcal{F}$ intersecting W contains a separatrix of $\mathcal{F}$ through p. In other words, for every leaf L of the restriction $\mathcal{F}\big|_W$ the union $L \cup \{p\}$ is a separatrix of $\mathcal{F}$ through p.

5 Basins of attraction of dicritical singularities

Let φ be a holomorphic action of $\mathbb{C}^*$ on a normal Stein space V of dimension two and let $\mathcal{F}_\varphi$ be the foliation by orbits on V. In [17] Suzuki has shown that: 1. The closure of each leaf of $\mathcal{F}_\varphi$ is an analytic subset of V, 2. There exists a Riemann surface S and a holomorphic surjective map $\pi : V\backslash \text{sing}(\mathcal{F}_\varphi) \to S$ such that the fibers of π are $\mathcal{F}_\varphi$ invariant and the set of points $s \in S$ such that $\pi^{-1}(s)$ is not irreducible has measure zero in S. It follows that the closure of a leaf L is the union of L with some points of $\text{sing}(\mathcal{F}_\varphi)$ as separatrices. Note that for an arbitrary $\mathbb{C}^*$-action on a variety each leaf of $\mathcal{F}_\varphi$ is biholomorphic to either $\mathbb{C}^*$ or torus. Since Stein varieties do not contain compact subvarieties of positive dimension, we conclude that all the leaves are biholomorphic to $\mathbb{C}^*$. The same argument shows that the closure of each leaf L of $\mathcal{F}_\varphi$ contains at most one point in V. The main result of this section is the following.

Theorem 5. *Let φ be a holomorphic action of $\mathbb{C}^*$ on a normal Stein space V of dimension two. If $p \in V$ is a dicritical singularity of $\mathcal{F}_\varphi$ then the attraction basin of p is V. In other words, every orbit of φ on $V \setminus \{p\}$ accumulates on p.*

Proof. Let $\pi : V\backslash \text{sing}(\mathcal{F}_\varphi) \to S$ be as above. The function $\pi \circ \rho$ extends to a function $\tilde{\pi}$ which is defined on σ_0 and its restriction to σ_0 is one to one outside a measure zero subset of S. The Riemann surface S is compact, otherwise there would exist a non-constant holomorphic function on S and hence on σ_0 which is a contradiction. We conclude that $\tilde{\pi}$ is a biholomorphism. The proof of Theorem 5 will be a consequence of the assertions below:

The point p is the only dicritical singularity of φ: If there is another dicritical singularity of φ, namely p', then in the resolution ρ' of p' we find a Riemann surface σ_0' with the property that $\tilde{\mathcal{F}}_\varphi$ is transverse to σ_0'. As before, $\pi \circ \rho' \,|_{\sigma_0'}$ is an isomorphism. This and Suzuki's theorem implies that a generic fiber $\pi^{-1}(s),\ s \in S$ is a separatrix for both p and p'. Since each leaf of $\mathcal{F}_\varphi$ has only one accumulation point in V we get a contradiction.

We have $V = \bar{B}_p$: If this was not the case, then we would have a nonempty open set U in S such that for all $s \in U$, $\pi^{-1}(s)$ has a component in B_p and another in $V - \bar{B}_p$ which is a contradiction.

The set ∂B_p *contains no isolated point*: Each leaf of $\mathcal{F}_\varphi$ in B_p cannot be a separatix of a singularity $p' \in V$ distinct from p. If p' is an isolated point of ∂B_p, its separatrices are in B_p and hence they are separatrices of p which is a contradiction.

∂B_p *does not contain a closed leaf in* V: Suppose that $L_0 \subset \partial B_p$ is a closed leaf of $\mathcal{F}_\varphi$. We know that L_0 is biholomorphic to $\mathbb{C}^*$ and it is an analytic smooth curve in V. Since V is Stein there is a holomorphic function $h \in \mathcal{O}(V)$ such that $\{h = 0\} = L_0$ in V ([6], Theorem 5, p.99). Since L_0 is a real surface diffeomorphic to a cylinder $S^1 \times \mathbb{R}$, we can take a generator $\gamma \colon S^1 \to L_0$ of the homology of L_0 and a holomorphic one-form α on L_0 such that $\int_\gamma \alpha = 1$. Again because V is Stein by Cartan's lemma there is a holomorphic one-form $\tilde{\alpha}$ on V which extends α. Since $\mathcal{F}_\varphi$ has a meromorphic first integral on V, the holonomy of L_0 is finite, say of order n. Let Σ be a small transverse disc to $\mathcal{F}_\varphi$ with $\Sigma \cap L_0 = \{p_0\} \subset \gamma(S^1)$. Then there is a fixed power of γ, say $\gamma_{p_0} := n\gamma$, which has closed lifts $\tilde{\gamma}_z$ to the leaves L_z of $\mathcal{F}_\varphi$ that contain the points $z \in \Sigma$. Thus, for $z \in \Sigma$ close enough to p_0 we have $\big| \int_{\tilde{\gamma}_z} \tilde{\alpha} - \int_{\tilde{\gamma}_{p_0}} \tilde{\alpha} \big| < \frac{1}{2}$, but $\tilde{\gamma}_{p_0} = n\gamma$ and $\int_{\tilde{\gamma}_{p_0}} \tilde{\alpha} = n$. We conclude that $\int_{\tilde{\gamma}_z} \tilde{\alpha} \neq 0$. On the other hand, $L_0 \subset \partial B_p$ and so there are leaves L of $\mathcal{F}_\varphi$ with nonempty intersection with Σ as above and which satisfy $L \subset B_p$. Such a leaf L accumulates on p and therefore $L \cup \{p\}$ is a holomorphic curve biholomorphic to $\mathbb{C}$ and thus with trivial homology, yielding a contradiction.

∂B_p *does not contain a leaf which is not closed in* V: Suppose there is a leaf $L_0 \subset \partial B_p$ such that $\bar{L}_0 = L_0 \cup \{p'\}$, where p' is a nondicritical singularity of φ. Since V is Stein there is a holomorphic function $f \in \mathcal{O}(V)$ such that $\{f = 0\} = \overline{L_0}$ in V ([6], Theorem 5, p.99). Define the meromorphic one-form $\alpha = \frac{df}{f}$ on V, the polar set of α is $\overline{L_0}$. Given a transverse disc $\Sigma \cong \mathbb{D}$ to $\mathcal{F}_\varphi$ with $\Sigma \cap L_0 = \{p_0\}$ we consider a simple loop $\gamma \colon S^1 \to \Sigma$ around $p_0 \in \Sigma$ such that $\int_\gamma \alpha = 2\pi\sqrt{-1}$. We can assume that $\gamma(S^1) \subset B_p$ because $\Sigma \setminus (\Sigma \cap \partial B_p) \subset B_p$. We have a biholomorphic map between B_p and $\mathcal{V}$ which can be contracted to a point. Therefore, $H^1(B_p, \mathbb{R}) = 0$ and it follows that $\int_\gamma \alpha = 0$ yielding a contradiction.

We conclude that ∂B_p is empty.

□

6 Proof of Theorem 1

Let us be given a pair (V, φ) as in Theorem 1. By Theorem 5 the basin of attraction of the dicritical singularity p is the whole space V, i.e $B_p = V$. By Theorem 4 there is a biholomorphic conjugacy $h \colon V \to \mathcal{V}$ between φ and λ. Finally, by Proposition 1 the variety $\mathcal{V}$ is affine and the the action λ of $\mathbb{C}^*$ on $\mathcal{V}$ in some affine coordinates is good. Note that this proof gives an alternative proof of Proposition 1.1.3, page 207 in [11].

We conclude that each data in Theorem 1 gives us a linear model variety $(\mathcal{V}, \lambda)$ and in a similar way we can prove that two linear models are biholomorphic if and only their corresponding data are the same. This finishes the proof.

Remark 3. As a consequence of our Theorem 1 we obtain that if V is a two-dimensional Stein space with a $\mathbb{C}^*$-action having a dicritical singularity at $p \in V$ then $p \in V$ is a quasi-homogeneous surface singularity.

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