Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.31, No. 3, 305–336

On the topology of foliations with a first integral

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Abstract. The main objective of this article is to study the topology of the fibers of a generic rational function of the type $\frac{F^p}{G^q}$ in the projective space of dimension two. We will prove that the action of the monodromy group on a single Lefschetz vanishing cycle δ generates the first homology group of a generic fiber of $\frac{F^p}{G^q}$. In particular, we will prove that for any two Lefschetz vanishing cycles δ_0 and δ_1 in a regular compact fiber of $\frac{F^p}{G^q}$, there exists a mondromy h such that $h(\delta_0) = \pm\delta_1$.

Keywords: Lefschetz pencil and vanishing cycle, homology groups, fiber bundle.

Mathematical subject classification: 14D99 - 57R30.

0 Introduction

Let F and G be two homogeneous polynomials in $\mathbb{C}^{n+1}$. The following function is well-defined

$$f = \frac{F^p}{G^q} : \mathbb{C}P(n)\backslash\mathcal{R} \to \overline{\mathbb{C}},$$

$$f(x) = \frac{F(x)^p}{G(x)^q},\ x = [x_0; x_1; \cdots ; x_n]$$

where $\mathcal{R} = \{F = 0\} \cap \{G = 0\}$, $\frac{deg(F)}{deg(G)} = \frac{q}{p}$ and p and q are relatively prime numbers. We can view the fibration of f as a codimension one foliation in $\mathbb{C}P(n)$ given by the 1-form

$$\omega = pGdF - qFdG$$

Received 5 October 2000.
*Partially supported by CNPq-Brazil.

Let $\mathcal{P}_a$ denote the set of homogeneous polynomials of degree a in $\mathbb{C}^n$.

Proposition 0.1. *There exists an open dense subset U of $\mathcal{P}_a \times \mathcal{P}_b$ such that for any $(F, G) \in U$ we have:*

1. *$\{F = 0\}$ and $\{G = 0\}$ are smooth varieties in $\mathbb{C}P(n)$ and intersect each other transversally;*
2. *The restriction of f to $\mathbb{C}P(n)\backslash(\{F = 0\} \cup \{G = 0\})$ has nondegenerate critical points, namely $p_1, p_2, \ldots, p_r$, with distinct images in $\overline{\mathbb{C}}$, namely $c_1, c_2, \ldots, c_r$ respectively.*

Throughout the text the elements of U will be called generic elements. We will prove this proposition in Appendix A. Put

$$C = \{c_1, c_2, \ldots, c_r\}$$

From now on, we will work with the function f which has the generic properties as in Proposition 0.1. The foliation $\mathcal{F}$ associated to f has the following singular set

$$Sing(\mathcal{F}) = \{p_1, p_2, \ldots, p_r, \mathcal{R}\}$$

The value 0 (resp. ∞) is a critical value of f if and only if $p > 1$ (resp. $q > 1$). Let A be a subset of $\{0, \infty\}$ which consists of only critical values. For example if $p = 1, q = 1$ then A is empty. The set of critical values of $\frac{F^p}{G^q}$ is $C \cup A$.

Proposition 0.2. *f is a C^∞ fiber bundle map over $\overline{\mathbb{C}}\backslash(C \cup A)$.*

We will prove this proposition in Section 3.

The above proposition enables us to use the arguments of Picard-Lefschetz Theory to study the topology of the fibers of f. But, for example, the critical fiber $\{F = 0\}$, when $p > 1$, is not considered in that theory (as far as I know). To overcome with this obstacle, we will construct a ramification map $\tau : \mathbb{C}\tilde{P}(n) \to \mathbb{C}P(n)$ for the multivalued function $f^{\frac{1}{pq}}$. The pull-back function $\tilde{f}^{\frac{1}{pq}} = f^{\frac{1}{pq}} \circ \tau$ of $f^{\frac{1}{pq}}$ is univalued and has no more the critical values 0 and ∞. Next, we will embed the complex manifold $\mathbb{C}\tilde{P}(n)$ in some $\mathbb{C}P(N)$ in such a way that the pull-back foliation $\tilde{\mathcal{F}}$ is obtained by the intersection of the hyperplanes of a generic Lefschetz pencil with $\mathbb{C}\tilde{P}(n)$.

The study of the topology of an algebraic variety by intersecting it with hyperplanes of a pencil has been started systematicly by Lefschetz in his famous article [12]. We will use the arguments of this area of mathematics, specially the articles [11],[3], to understand the topology of the leaves of $\mathcal{F}$. Note that the leaves of $\mathcal{F}$ do not contain the points of the set $\{F = 0\} \cap \{G = 0\}$.

In the first section we will construct such ramification map τ, and in the second section we will review Picard-Lefshetz Theory. In the third section we will apply our results to the foliation $\mathcal{F}$.

I want to use this opportunity to express my thanks to my advisor Alcides Lins Neto. Discussions with him during this work have provided valuable information and new ideas. Also, thanks go to Cesar Camacho and Paulo Sad for their interest and support. I also thank Eduardo Esteves for his comments on algebraic geometry. The author is also grateful to the exceptional scientific atmosphere in IMPA that made this work possible.

1 Ramification Maps

In the first part of this section we will introduce ramification maps with a normal crossing divisor. In the second part we will use a method which gives us some examples of ramification maps. This method will be enough for our purpose. The following isomorphism will be used frequently during this section:

Leray (or Thom-Gysin) Isomorphism: If a closed submanifold N has pure real codimension c in M, then there is an isomorphism

$$\tau : H_{k-c}(N) \tilde{\to} H_k(M, M \backslash N)$$

holding for any k, with the convention that $H_s(N) = 0$ for $s < 0$. Roughly speaking, given $x \in H_{k-c}(N)$, its image by this isomorphism is obtained by thickening a cycle representing x, each point of it growing into a closed c-disk transverse to N in M (see [3] p. 537).

Let N be a connected codimension one submanifold of the complex manifold M. Write the long exact sequence of the pair $(M, M \backslash N)$ as follows:

$$\cdots \to H_2(M, M\backslash N) \xrightarrow{\sigma} H_1(M\backslash N) \xrightarrow{i} H_1(M) \to \cdots \quad (1)$$

where σ is the boundary operator and i is induced by inclusion. Since N has real codimension two in M, $H_2(M, M\backslash N)$ ($\simeq H_0(N) \simeq \mathbb{Z}$) is generated by the disk Δ transverse to N at a point $y \in N$. By the above long exact sequence it follows that if a closed cycle x in $M\backslash N$ is homologous to zero in M then it is homologous to a multiple of $\sigma(\Delta) = \delta$ in $M\backslash D$. The cycle δ is called a simple loop around the point $y \in N$ in $M\backslash N$.

1.1 Normal Crossing Divisors

The following well-known fact will be used frequently:

Proposition 1.1.1. *Let $\tau : \tilde{M} \to M$ be a finite covering map of degree p. Then the following statements are true:*

1. *$\tau_* : \pi_1(\tilde{M}) \to \pi_1(M)$ is one to one, where $\pi_1(M)$ denotes the fundamental group of M;*

2. *If $\pi_1(M)$ is abelian then $\pi_1(\tilde{M})$ is also abelian and $\pi_1(M)^p \subset \tau_*(\pi_1(\tilde{M}))$, where $\pi_1(M)^p = \{\gamma^p \mid \gamma \in \pi_1(M)\}$.*

Proof: The proof of the first statement can be found in [19]. If $\pi_1(M)$ is abelian then

$$\tau_*(aba^{-1}b^{-1}) = \tau_*(a)\tau_*(b)\tau_*(a)^{-1}\tau_*(b)^{-1} = 1$$

where $a, b \in \pi_1(\tilde{M})$. The map τ_* is one to one and so $aba^{-1}b^{-1} = 1$ which implies that $ab = ba$.

For any closed path $a \in \pi_1(M, x)$, its inverse image by τ is a union of closed paths $a_1, a_2, \ldots, a_k$ in $\tilde{M}$. Choose a point y in $\tilde{M}$ and points x_i in the path a_i, for $i = 1, \ldots, k$ such that $\tau(x_i) = \tau(y) = x$ and put $b_i = A_i^{-1}a_iA_i$ and $b = b_1b_2 \cdots b_k$, where A_i is a path in $\tilde{M}$ which connects y to x_i. The image of A_i by τ is a closed path in M and $\pi_1(M)$ is abelian, therefore

$$\tau_*(b) = \Pi\tau_*(A_i)^{-1}\tau_*(a_i)\tau_*(A_i) = \Pi\tau_*a_i = a^p$$

The last equality is true because τ has degree p and the paths a_i's are the inverse image of a. □

In what follows, if the considered group is abelian, we use the additive notations of groups; for example instead of a^p we write pa.

Definition 1.1.1. Let M be a complex manifold of dimension n. By a reduced normal crossing divisor we mean a union of finitely many connected closed submanifolds, namely $D_1, D_2, \ldots, D_s$ of M and of codimension one, which intersect each other transversally i.e., for any point $a \in M$ there is a local coordinate $(x, y) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$, $k \le s$ around a such that in this coordinate $a = (0, 0)$ and for any $j = 1, \ldots, k$, the component D_{i_j} is given by $x_j = 0$, where $\{i_1, \ldots, i_k\} = \{i \mid a \in D_i\}$. We say this coordinate normalizing coordinate of D at a. We will denote a reduced normal crossing divisor by

$$D = \sum_1^s D_i$$

When D has only one component i.e., $s = 1$, then D is called simple. The following fact is a direct consequence of the definition.

Proposition 1.1.2. *Let D be a reduced normal crossing divisor in a complex manifold M. For any subset I of $\{1, 2, \dots, s\}$, the set*

$$M_I = \cap_{i \in I} D_i$$

is a complex manifold of codimension $\# I$ in M and

$$D_I = D \cap M_I = \sum_{i \notin I} D_i \cap M_I$$

is a reduced normal crossing divisor in M_I.

In what follows for a given function $\tau : \tilde{M} \to M$, and for any subset x of M (resp. a meromorphic function x on M) we denote by $\tilde{x}$ the set $\tau^{-1}(x)$ (resp. the meromorphic function $x \circ \tau$ on $\tilde{M}$).

Definition 1.1.2. Let M and $\tilde{M}$ be two compact complex manifolds of the same dimension, $D = \sum_1^s D_i$ be a reduced normal crossing divisor in M and $p_1, p_2, \dots, p_s$ be positive integer numbers greater than one. The holomorphic map $\tau : \tilde{M} \to M$ is called a ramification map with divisor D and ramification index p_i at D_i if

1. $\tau^{-1}(D) = \tilde{D}$ is a reduced normal crossing divisor;

2. For any point $\tilde{a} \in \tilde{M}$ and a normalizing coordinate $(x, y) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$ around $a = \tau(\tilde{a})$, there is a normalizing coordinate $(\tilde{x}, \tilde{y}) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$ around $\tilde{a}$ such that in these coordinates $a = (0, 0)$, $\tilde{a} = (0, 0)$ and τ is given by:

$$(\tilde{x}, \tilde{y}) \to (x, y) = ((\tilde{x}_1^{p_{i_1}}, \tilde{x}_2^{p_{i_2}}, \dots, \tilde{x}_k^{p_{i_k}}), \tilde{y})$$

From the definition, we can see that

1. The critical points and values of τ are $\tilde{D}$ and D, respectively;

2. $\tau \mid_{\tilde{M} \setminus \tilde{D}}$ is a finite covering map of some degree p;

3. For any subset I of $\{1, 2, \dots, s\}$, if M_I is not empty then $\Pi_{i \in I} p_i$ divides p.

We have also the subramification maps of τ, which is stated in bellow.

Proposition 1.1.3. *Keeping the notations in Definition 1.1.2 and Proposition 1.1.2, for any ramification map $\tau : \tilde{M} \to M$ and I a subset of $\{1, 2, \ldots, s\}$, the restriction of τ to $\tilde{M}_I$, namely τ_I, is a ramification map with divisor D_I and ramification multiplicity p_i at $D_i \cap M_I$, where $i \notin I$. Moreover, If τ is of degree p then τ_I is of degree $\frac{p}{\prod_{i\in I} p_i}$.*

Proposition 1.1.4. *Let $\tau : \tilde{M} \to M$ be a ramification map of degree p and with reduced normal crossing divisor D and let $\pi_1(M\backslash D)$ be abelian. Then the following statement are true:*

1. *$\pi_1(\tilde{M})$ is abelian;*
2. *$p\pi_1(M) \subset \tau_*(\pi_1(\tilde{M}))$;*
3. *If D is simple then $\tau_* : \pi_1(\tilde{M}) \to \pi_1(M)$ is one to one;*
4. *If $D^* = \sum_{i=1}^{s^*} D_i^*$ is a reduced divisor in M such that $D^* + D$ is a normal crossing divisor, then $\tilde{D}^* + \tilde{D}$ is also a normal crossing divisor.*

Proof: The set D is a finite union of some submanifolds of M with real codimension greater than two, therefore every path in $\pi_1(M, x)$, where $x \in M\backslash D$, is homotopic to some path in $\pi_1(M\backslash D, x)$. Now the first and second statements are the direct consequences of Proposition 1.1.1, Definition 1.1.2 and the mentioned fact.

Let Σ be a small disk transverse to D at $y \in D$ and $x \in \Sigma \cap M\backslash D$. Let also $a \in \pi_1(\tilde{M}, \tilde{x})$, where $\tau(\tilde{x}) = x$, and $\tau_*(a)$ be homotopic to zero in M and a do not intersect $\tilde{D}$.

Considering the long exact sequence (1) and the fact that $i(\tau_*(a)) = 0$, we conclude that $\tau_*(a)$ is homotopic to $k\delta$ in $M\backslash D$, where k is an integer number and δ is a simple loop in Σ around y. By Proposition 1.1.1, this means that a is homotopic to a closed path around $\tilde{y}$ in $\tau^{-1}(\Sigma)$, where $\tau(\tilde{y}) = y$, which means that a is homotopic to the point $\tilde{y}$ in $\tilde{M}$, and this proves the third statement.

Let $a \in (\cap_{i\in I} D_i)\cap(\cap_{i\in I^*} D_i^*)$, where $I \subset \{1, 2, \ldots, s\}$ and $I^* \subset \{1, 2, \ldots, s^*\}$. Choose a normalizing coordinate $(x, x^*, y) \in \mathbb{C}^r \times \mathbb{C}^{r^*} \times \mathbb{C}^{n-r-r^*}$ around a such that the components of D (resp. D^*) through p are represented by the coordinate x (resp. x^*), where $r = \#I$ and $r^* = \#I^*$. Now the fourth statement is a direct consequence of Definition 1.1.2. $\square$

1.2 Construction of Ramification Maps

For any abelian group G and a positive integer number p define $G_p = G/pG$. We have the following properties with respect to G_p:

Proposition 1.2.1. *The following statements are true:*

1. *Every morphism $f : G \to G'$ of abelian groups induces a natural morphism $f_p : G_p \to G'_p$;*

2. *$(G_p)_q = G_{(p,q)}$, where (p, q) denotes the greatest common divisor of p and q;*

3. *If $f : G \to G'$ is surjective then f_p is also surjective. If f is one to one then f_p may not be one to one and so we cannot rewrite exact sequences of abelian groups by this change of groups and maps;*

4. *Let $f : G \to G'$ be a morphism of abelian groups and p,q be two positive integer numbers. If f is one to one, $pG' \subset f(G)$ and $(p, q) = 1$ then f_q is an isomorphism between G_q and G'_q;*

Proof: We only prove the fourth statement, since others are trivial. There exist integer numbers x, y such that $px + qy = 1$.
For any $a \in G'$ we have

$$a - q(ay) = p(ax) \in f(G)$$

and so f_q is surjective.
If $f_q(a) = 0$ then $f(a) = qb$ for some $b \in G'$. We have

$$b = p(bx) + f(ay) = f(s), \ s \in G$$

which implies that $f(a - qs) = 0$. The morphism f is one to one and so we have $a = qs$, which means that f_q is one to one. □

The following statement gives us an example of ramification map with simple divisor.

Proposition 1.2.2. *Let M be a complex manifold with $\pi_1(M) = 0$ and D be a simple divisor whose complement in M has abelian fundamental group. Let also p' be a positive integer number and*

$$\pi_1(M \backslash D)_{p'} = \mathbb{Z}_p$$

Then there exists a degree p ramification map with divisor D and ramification multiplicity p at D.

Proof: For any $e \in D$ define $\pi_1(M\backslash D, e) = \{0_e\}$ and

$$\tilde{M} = \cup_{e \in M} \pi_1(M\backslash D, e)_{p'}$$

$\tilde{M}$ has the structure of a complex manifold. For any $[a_e] \in \pi_1(M\backslash D, e)$ we must define a base open set and a chart map around $[a_e]$. Consider two cases:

1. $e \in M\backslash D$

Let V_e be a simply connected open neighbourhood of e in $M\backslash D$. The following function is well-defined:

$$\eta : V_e \to \tilde{M}, \quad \eta(y) = [A_{ey} a_e A_{ey}^{-1}]$$

where A_{ey} is a path which connects e to y in V_e. The image of η is a base open set around a_e and η is a chart map.

2. $e \in D$

Let $(V_e, (x, y))$, $(x, y) \in (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$, be a coordinate around e such that in this coordinate $e = (0, 0)$ and D is given by $y = 0$. By Leray isomorphism, for any $e' = (x', y') \in V_e$ the group $\pi_1(M\backslash D, e')_{p'} \simeq \mathbb{Z}_p$ is generated by a simple loop around $(x', 0)$ in

$$\Sigma_{(x',0)} = \{(x, y) \in (\mathbb{C}^{n-1} \times \mathbb{C}, 0) \mid x = x'\}$$

In particular, we have $p \mid p'$. This gives us the following construction of a chart map around $[a_e]$:

For any $y_0 \in (\mathbb{C}, 0)$, let $j(y_0)$ be a point in $(\mathbb{C}, 0)$ such that

$$y_0^p = j(y_0)^p \;\&\; 0 \leq arg(j(y_0)) < \frac{2\pi}{p}$$

and let δ_{y_0} be the path which connects y_0 to $j(y_0)$ in $\{y \in (\mathbb{C}, 0) \mid y^p = y_0^p\}$ in the clock direction.

The image of δ_{y_0} by the map $i(y) = y^p$, $(\delta_{y_0})^p$, is a closed path with initial and end point y_0^p and so the following function is well-defined

$$\eta : (\mathbb{C}^{n-1} \times \mathbb{C}, 0) \to \tilde{M}, \quad \eta(x, y) = \{x\} \times (\delta_y)^p$$

The image of η is a base open set around the point e and η is a chart map. The reader can verify easily that $\tilde{M}$ with these base open sets and chart maps is a complex manifiold and the natural function $\tau : \tilde{M} \to M$ is the desired ramification map. $\square$

Theorem 1.2.1. *Let M be a complex manifold with $\pi_1(M) = 0$ and $D = \sum_{i=1}^{s} D_i$ be a reduced normal crossing divisor such that the complement of each D_i in M has abelian fundamental group. Let also $p'_1, p'_2, \dots p'_s$ be positive integer numbers which are prime to each other. Put $p_i = \#\pi_1(M\backslash D_i)_{p'_i}$. Then there is a degree $p_1 p_2 \cdots p_s$ ramification map with divisor D and ramification multiplicity p_i at D_i, $i = 1, 2, \dots, s$.*

Proof: The proof is by induction on s. For $s = 1$ it is Proposition 1.2.2.

Suppose that the theorem is true for $s-1$. Let $\tau : \tilde{M} \to M$ be a degree p_1 ramification map with simple divisor D_1 and multiplicity p_1 at D_1. We check the assumptions of the theorem for the divisor $\tilde{D}_2 + \ldots + \tilde{D}_s$ in the manifold $\tilde{M}$, to apply the hypothesis of the induction.

By the third part of Proposition 1.1.4 $\tau_* : \pi_1(\tilde{M}) \to \pi_1(M)$ is one to one and by hypothesis $\pi_1(M) = 0$, therefore $\pi_1(\tilde{M}) = 0$.

Applying Proposition 1.1.4 to the ramification map $\tau\mid_{\tilde{M}\backslash\tilde{D}_i}$, we see that $\pi_1(\tilde{M}\backslash\tilde{D}_i)$ is abelian; also 4 of Proposition 1.1.4 implies that $\tilde{D}$ is a normal crossing divisor.

The morphism

$$\tau_* : \pi_1(\tilde{M}\backslash\tilde{D}_i) \to \pi_1(M\backslash D_i)$$

is one to one and by 2 of Proposition 1.1.4

$$p_1\pi_1(M\backslash D_i) \subset \tau_*(\pi_1(\tilde{M}\backslash\tilde{D}_i)$$

But $g.c.d.(p_1, p'_i) = 1$ and so by 4 of Proposition 1.2.1 we have

$$\pi_1(\tilde{M}\backslash\tilde{D}_i)_{p'_i} \simeq \pi_1(M\backslash D_i)_{p'_i}$$

Now we can apply the hypothesis of the induction to $\tilde{M}$ and $D' = \sum_2^s D_i$. There exists a degree $p_2 \cdots p_s$ ramification map $\tau' : \tilde{M}' \to \tilde{M}$ with divisor D' and multiplicity p_i at D_i, $i = 2, \dots, s$. The reader can check that the map $\tau \circ \tau'$ is the desired ramification map. $\square$

1.3 Multivalued Functions

To study multivalued functions, we will need to study a certain class of ramification maps. First, we give the precise definition of multivalued functions.

Definition 1.3.1. Let $\tau : \tilde{M} \to M$ be a degree p holomorphic map between two complex manifold $\tilde{M}$ and M i.e., τ is a finite covering map of degree p out of its critical points. Every meromorphic function g on $\tilde{M}$ is called a p-valued meromorphic function on M. Roughly speaking, the image of a point $x \in M$ under g is the set $g(\tau^{-1}(x))$. The map τ is called the ramification map of g and the set of critical values of the map τ is called the ramification divisor of the multivalued function g.

Given a complex manifold M, a meromorphic function f on it and an integer number N. Can we construct a ramification map of the multivalued function $f^{\frac{1}{N}}$ according to the above definition? Here we will answer to this question for some limited classes of meromorphic functions.

Let f be a meromorphic function on the manifold M and $\tau : \tilde{M} \to M$ be a ramification map with reduced normal crossing divisor $D = \sum_{i=1}^{s} D_i$ and multiplicity p_i at D_i, $i = 1, 2, \ldots, s$. Let also $div(f) = \sum m_j V_j$. Then

$$div(\tilde{f}) = \sum a_j m_j V_j$$

where $\tilde{f} = f \circ \tau$, $a_j = p_i$ if $V_j = D_i$ for some i and $a_j = 1$ otherwise.

Proposition 1.3.1. *Keeping the notations used above, suppose that $H^1(\tilde{M}, \mathbb{Z}_N) = 0$ and $N \mid div(\tilde{f})$ i.e., N divides the multiplicities of the components of $div(\tilde{f})$, where N is a positive integer number. Then $\tilde{f}^{\frac{1}{N}}$ is a well-defined meromorphic function on $\tilde{M}$. Therefore, we can view $f^{\frac{1}{N}}$ as a multivalued function on M with the ramification map τ.*

Proof: For any point $x \in \tilde{M}$ there is a neighbourhood V_i of x such that in this neighbourhood $\tilde{f} = g_i^N$, where g_i is a meromorphic function on V_i. Let $c_{ij} = \frac{g_i}{g_j}$, then $c_{ij}^N = 1$. Since $H^1(\tilde{M}, \mathbb{Z}_N) = 0$, there exist complex numbers c_i's such that $c_{ij} = \frac{c_i}{c_j}$. Now $g \mid_{V_i} = \frac{g}{c_i}$'s define the global meromorphic function which is the desired candidate for $\tilde{f}^{\frac{1}{N}}$. □

2 Picard-Lefschetz Theory

In 1924 S. Lefschetz published his famous article [12] on the topology of algebraic varieties. In his article, in order to study the topology of an algebraic variety, he considered a pencil of hyperplanes in general position with respect to that variety. Many of the Lefschetz intuitive arguments are made precise by appearance a critical fiber bundle map. In the first part of this section we introduce the basic concepts of Picard-Lefschez Theory and in the second part we introduce the Lefschetz pencil and state our two basic theorems 2.2.1, 2.3.2. This section is mainly based on the articles [11],[3]. Homologies are considered in an arbitrary field of characteristic zero except it mentioned explicitly.

2.1 Critical Fiber Bundle Maps

The following theorem gives us a huge number of fiber bundle maps.

Theorem 1. *(Ehresmann's Fibration Theorem [7]). Let $f : Y \to B$ be a proper submersion between the manifolds Y and B. Then f fibers Y locally trivially i.e., for every point $b \in B$ there is a neighbourhood U of b and a C^∞-diffeomorphism $\phi : U \times f^{-1}(b) \to f^{-1}(U)$ such that $f \circ \phi = \pi_1 =$ the first projection. Moreover if $N \subset Y$ is a closed submanifold such that $f\mid_N$ is still a submersion then f fibers Y locally trivially over N i.e., the diffeomorphism ϕ above can be chosen to carry $U \times (f^{-1}(b) \cap N)$ onto $f^{-1}(U) \cap N$.*

The map ϕ is called the fiber bundle trivialization map. Ehresmann's theorem can be rewrite for manifolds with boundary and also for stratified analytic sets. In the last case the result is known as the Thom-Mather theorem.
In the above theorem let f not be submersion, and let C' be the union of critical values of f and critical values of $f\mid_N$, and C be the closure of C' in B. By a critical point of the map f we mean the point in which f is not submersion. Now we can apply the theorem to the function

$$f : Y\backslash f^{-1}(C) \to B\backslash C = B'$$

For any set $K \subset B$, we use the following notations

$$Y_K = f^{-1}(K),\ Y'_K = Y_K \cap N,\ L_K = Y_K\backslash Y'_K$$

and for any point $c \in B$, by Y_c we mean the set $Y_{\{c\}}$. By $f : (Y, N) \to B$ we mean the mentioned map and we call it the critical fiber bundle map.

Definition 2.1.1. Let $A \subset R \subset S$ be topological spaces. R is called a strong deformation retract of S over A if there is a continuous map $r : [0, 1] \times S \to S$ such that

1. $r(0, .) = id$;
2. $r(1, x) \in R \,\&\, r(1, y) = y\ \forall x, y \in S,\ y \in R$;
3. $r(t, x) = x\ \forall t \in [0, 1],\ x \in A$.

Here r is called the contraction map. In a similar way we can do this definition for the pairs of spaces $(R_1, R_2) \subset (S_1, S_2)$, where $R_2 \subset R_1$ and $S_2 \subset S_1$.

We use the following important theorem to define generalized vanishing cycle and also to find relations between the homology groups of $Y\backslash N$ and the generic fiber L_c of f.

Theorem 2.1.1. *Let $f : Y \to B$ and C' as befor, $A \subset R \subset S \subset B$ and $S \cap C$ be a subset of the interior of A in S, then every retraction from S to R over A can be lifted to a retraction from L_S to L_R over L_A.*

Proof: According to Ehresmann's fibration theorem $f : L_{S\backslash C} \to S\backslash C$ is a C^∞ locally trivial fibre bundle. The homotopy covering theorem, see 14,11.3[19], implies that the contraction of $S\backslash C$ to $R\backslash C$ over $A\backslash C$ can be lifted so that $L_{R\backslash C}$ becomes a strong deformation retract of $L_{S\backslash C}$ over $L_{A\backslash C}$. Since $C \cap S$ is a subset of the interior of A in S, the singular fibers can be filled in such a way that L_R is a deformation retract of L_S over L_A. □

Monodromy: Let λ be a path in $B' = B\backslash C$ with the initial and end points b_0 and b_1. In the sequel by λ we will mean both the path $\lambda : [0, 1] \to B$ and the image of λ; the meaning being clear from the text.

Proposition 2.1.1. *There is an isotopy*

$$H : L_{b_0} \times [0, 1] \to L_\lambda$$

such that for all $x \in L_{b_0},\ t \in [0, 1]$

$$H(x, 0) = x,\ \ H(x, t) \in L_{\lambda(t)} \tag{2}$$

For every $t \in [0, 1]$ the map $h_t = H(., t)$ is a homeomorphism between L_{b_0} and $L_{\lambda(t)}$. The different choices of H and paths homotopic to λ would give the class of homotopic maps $\{h_\lambda : L_{b_0} \to L_{b_1}\}$ where $h_\lambda = H(., 1)$.

Proof: The interval $[0, 1]$ is compact and the local trivializations of L_λ can be fitted together along γ to yield an isotopy H. □

The class $\{h_\lambda : L_{b_0} \to L_{b_1}\}$ defines the maps

$$h_\lambda : \pi_*(L_{b_0}) \to \pi_*(L_{b_1})$$
$$h_\lambda : H_*(L_{b_0}) \to H_*(L_{b_1})$$

In what follows we will consider the homology class of cycles, but many of the arguments can be rewritten for their homotopy class.

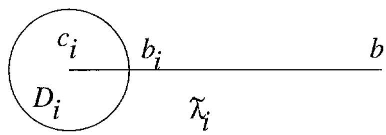

Figure 1:

Definition 2.1.2. For any regular value b of f, we can define

$$\begin{aligned} h : \pi_1(B', b) \times H_*(L_b) &\to H_*(L_b) \\ h(\lambda, .) &= h_\lambda(.) \end{aligned}$$

$\pi_1(B', b)$ is called the monodromy group and its action h on $H_*(L_b)$ is called the action of monodromy on the homology groups of L_b. Following the article [3], we give the generalized definition of vanishing cycles.

Definition 2.1.3. *Let K be a subset of B and b be a point in $K \backslash C$. Any relative k-cycle of L_K modulo L_b is called a k-thimble above (K, b) and its boundary in L_b is called a vanishing $(k-1)$-cycle above K.*

Let us consider the case that we will need. Let Y be a complex compact manifold, N be a submanifold of Y of codimension one, $B = \overline{\mathbb{C}}$ and f be a holomorphic function. The set of critical values of f, C, is a finite set.
Let $c_i \in C$ (which is an isolated point of C in $\overline{\mathbb{C}}$), D_i be an small disk around c_i and $\tilde{\lambda}_i$ be a path in B' which connects $b \in B'$ to $b_i \in \partial D_i$. Put λ_i the path $\tilde{\lambda}_i$ plus the path which connects b_i to c_i in D_i (see Figure 1). Define the set K in the three ways as follows:

$$K^s = \begin{cases} \lambda_i & s = 1 \\ \lambda_i \cup D_i & s = 2 \\ \tilde{\lambda}_i \cup \partial D_i & s = 3 \end{cases} \tag{3}$$

In each case we can define the vanishing cycle in L_b above K^s. K^1 and K^3 are subsets of K^2 and so the vanishing cycle above K^1 and K^3 is also vanishing above K^2. In the first case we have the intuitional concept of vanishing cycle. If c_i is a critical point of $f\mid_N$ we can see that the vanishing cycle above K^2 may not be vanishing above K^1.
The third case gives us the vanishing cycles obtained by a monodromy around c_i. In this case we have the Wang isomorphism

$$v : H_{k-1}(L_b) \tilde{\to} H_k(L_K, L_b)$$

Roughly speaking, The image of the cycle α by v is the footprint of α, taking the monodromy around c_i. Let γ_i be the closed path which parametrize K_3 i.e., γ_i starts from b, goes along $\tilde{\lambda}_i$ until b_i, turn around c_i on ∂D_i and finally comes back to b along $\tilde{\lambda}_i$. Let also $h_{\gamma_i} : H_k(L_{b_i}) \to H_k(L_{b_i})$ be the monodromy around the critical value c_i. It is easy to check that

$$\sigma \circ v = h_{\lambda_i} - I$$

where σ is the boundary operator, therefore the cycle α is a vanishing cycle above K^3 if and only if it is in the image of $h_{\lambda_i} - I$. For more information about the generalized vanishing cycle the reader is referred to [3].

Lefschetz Vanishing Cycle: Let f have a nondegenerate critical point p_i in $Y \backslash N$ and p_i be the unique critical point of $f : (Y, N) \to \overline{\mathbb{C}}$ within Y_{c_i}, where $c_i = f(p_i)$.

Proposition 2.1.2. *In the above situation, the following statements are true:*

1. *For all $k \neq n$ we have $H_k(L_{\lambda_i}, L_b) = 0$. This means that there is no $(k-1)$-vanishing cycle along λ_i for $k \neq n$;*

2. *$H_n(L_{\lambda_i}, L_b)$ is infinite cyclic generated by a hemispherical homology class $[\Delta_i]$ which is called the Lefschetz thimble and its boundary is called the Lefschetz vanishing cycle;*

3. *Let λ_i' be another path which connects b to c_i in B' and is homotopic to λ_i in B' (with fixed initial and end point), then we have the same, up to homotopy and change of sign, Lefschetz vanishing cycle in L_b.*

For the proof of above Proposition see 5.4.1 of [11] . By a hemispherical homology class, we mean the image of a generator of infinite cyclic group $H_n(\mathbb{B}^n, \mathbb{S}^{n-1})$ under the homeomorphism induced by a continuous mapping of the closed n-ball $\mathbb{B}^n$ into L_{λ_i} which sends its boundary, the $(n-1)$-sphere $\mathbb{S}^{n-1}$, to L_b. Let B be a small ball around p_i such that in B we can write f in the Morse form

$$f = c_i + x_1^2 + x_2^2 + \cdots + x_n^2$$

For b such that $b - c_i$ is positive real, the Lefschetz vanishing cycle in the fiber L_b is given by:

$$\delta_i = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum x_j^2 = b - c_i\}$$

which is the boundary of the thimble

$$\Delta_i = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum x_j^2 \leq b - c_i\}$$

In the above situation the monodromy h_i around the critical value c_i is given by the Picard-Lefschetz formula

$$h(\delta) = \delta + (-1)^{\frac{n(n+1)}{2}} < \delta, \delta_i > \delta_i, \quad \delta \in H_{n-1}(L_b)$$

where $< ., . >$ denotes the intersection number of two cycles in L_b.

Remark: In the above example vanishing above K^1 and K^2 are the same. Also by the Picard-Lefschetz formula the reader can verify that three types of the definition of a vanishing cycle coincide. In what follows by vanishing along the path λ_i we will mean vanishing above K^2.

2.2 Vanishing Cycles as Generators

Now let $\{c_1, c_2, \ldots, c_s\}$ be a subset of the set C of critical values of f, and $b \in \overline{\mathbb{C}} \backslash C$. Consider a system of s paths $\lambda_1, \ldots, \lambda_s$ starting from b and ending at $c_1, c_2, \ldots, c_s$, respectively, and such that:

1. each path λ_i has no self intersection points ;
2. two distinct path λ_i and λ_j meet only at their common origin $\lambda_i(0) = \lambda_j(0) = b$ (see Figure 2).

This system of paths is called a distinguished system of paths. The set of vanishing cycles along the paths λ_i, $i = 1, \ldots, s$ is called a distinguished set of vanishing cycles related to the critical points $c_1, c_2, \ldots, c_s$.

Theorem 2.2.1. *Suppose that $H_{k-1}(L_{\overline{\mathbb{C}} \backslash \{a\}}) = 0$ for some positive integer number k and $a \in \overline{\mathbb{C}}$, which may be a critical value. Then a distinguished set of vanishing $(k-1)$-cycles related to the critical points in the set $C \backslash \{a\} = \{c_1, c_2, \ldots, c_r\}$ generates $H_{k-1}(L_b)$.*

Proof: We use the arguments of the article [11] Section 5. Note that in our case the fiber is $L_b = Y_b \backslash N$ and not Y_b.

We consider our system of distinguished paths inside a large disk D_+ so that $a \in \overline{\mathbb{C}} \backslash \overline{D}_+$, the point b is in the boundary of D_+ and all critical values c_i's in $C \backslash \{a\}$ are interior points of D_+. Small disks D_i with centers c_i $i = 1, \cdots, r$ are chosen so that they are mutually disjoint and contained in D_+. Put

$$K_i = \lambda_i \cup D_i, \;\; K = \cup_{i=1}^r K_i$$

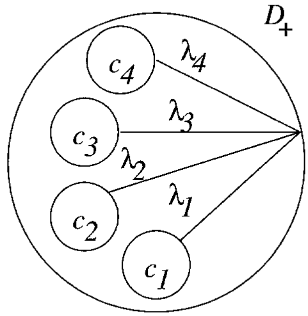

Figure 2:

The pair (K, b) is a strong deformation retract of (D_+, b) and so by Theorem 2.1.1 (L_K, L_b) is a strong deformation retract of (L_{D_+}, L_b). The set $\tilde{\lambda} = \cup\tilde{\lambda}_i$ can be retract whithin itself to the point b and so (L_K, L_b) and $(L_K, L_{\tilde{\lambda}})$ have the same homotopy type. By the excision theorem (see [14]) we conclude that

$$H_k(L_{D_+}, L_b) \simeq \sum_{i=1}^{r} H_k(L_{K_i}, L_b) \simeq \sum_{i=1}^{r} H_k(L_{D_i}, L_{b_i})$$

Write the long exact sequence of the pair (L_{D_+}, L_b):

$$\ldots \to H_k(L_{D_+}) \to H_k(L_{D_+}, L_b) \xrightarrow{\sigma} H_{k-1}(L_b) \to H_{k-1}(L_{D_+}) \to \ldots \quad (4)$$

Knowing this long exact sequence, it is enough to prove that $H_{k-1}(L_{D_+}) = 0$. A contraction from $\overline{\mathbb{C}}\backslash\{a\}$ to D_+ can be lifted to the contraction of $L_{\overline{\mathbb{C}}\backslash\{a\}}$ to L_{D_+} which means that L_{D_+} and $L_{\overline{\mathbb{C}}\backslash\{a\}}$ have the same homotopy type and so by the hypothesis $H_{k-1}(L_{D_+}) = 0$. □

2.3 Lefschetz Pencil

In this section we repeat some notations and propositions of [11] Section 2. All the proofs can be found there.

The hyperplanes of $\mathbb{C}P(N)$ are points of the dual projective space $\mathbb{C}\check{P}(N)$. We use the following notation:

$$H_y \subset \mathbb{C}P(N)\ ,\ \ y \in \mathbb{C}\check{P}(N)$$

Let X be a closed irreducible subvariety of $\mathbb{C}P(N)$ and let $X_e \subset X$ be the nonempty open subset of its regular points. Define

$$V_X' = \{(x, y) \in \mathbb{C}P(N) \times \mathbb{C}\check{P}(N) \mid x \in X_e \ \&\ H_y \text{ is tangent to } X \text{ at } x\}$$

This is a quasiprojective subset of $\mathbb{C}P(N) \times \mathbb{C}\check{P}(N)$, because the set

$$\tilde{V} = \{(x, y) \in \mathbb{C}P(N) \times \mathbb{C}\check{P}(N) \mid \\ x \text{ is a singular point of } X \text{ or } H_y \text{ is tangent to} X \text{ at } x\}$$

is closed in $\mathbb{C}P(N) \times \mathbb{C}\check{P}(N)$ and V_X' is a zariski open in $\tilde{V}$. The closure V_X of V_X' is called the tangent hyperplane bundle of X. Consider the second projection

$$\pi_2 : V_X \to \mathbb{C}\check{P}(N)\ ,\ (x, y) \to y$$

its image $\check{X}$ is a closed irreducible subvariety of $\mathbb{C}\check{P}(N)$ of dimension at most $n-1$ which is called the dual variety of X. If X is a smooth variety then

$$\check{X} = \{y \in \mathbb{C}\check{P}(N) \mid H_y \text{ is tangent to } X \text{ at some point } \}$$

In general $\check{X}$ has singularities even if X does not. If $dim(\check{X}) = N-1$ the degree of $\check{X}$ is well-defined and if $dim(\check{X}) < N-1$ we define $deg(\check{X}) = 0$.

Proposition 2.3.1. *(Duality Theorem [11] 2.2) The tangent hyperplane bundles of X and $\check{X}$ coincide*

$$V_X = V_{\check{X}} \textit{ and hence } \check{\check{X}} = X$$

A pencil in $\mathbb{C}P(N)$ consists of all hyperplanes which contain a fixed $(N-2)$-dimensional projective space A, which is called the axis of the pencil. We denote a pencil by $\{H_t\}_{t\in G}$ or G itself, where G is a projective line in $\mathbb{C}\check{P}(N)$.

The pencil $\{H_t\}_{t\in G}$ is in general position with respect to X if G is in general position with respect to $\check{X}$. From now on, fix a pencil $\{H_t\}_{t\in G}$ in general position with respect to X.

Proposition 2.3.2. *([11], 1.6.1) The axis A intersects X transversally.*

For the pencil $\{H_t\}_{t\in G}$ put

$$X_t = X \cap H_t,\ L_t = X_t \backslash A,\ C = G \cap \check{X} = \{c_1, c_2, \ldots, c_r\},\ X' = A \cap X$$

We will sometimes parametrize G by $\overline{\mathbb{C}}$ and denote the pencil by $\{H_t\}_{t\in\overline{\mathbb{C}}}$. In order to have a map whose level surfaces are the X_t's, we need to do a "blow up" along the variety X'. Let

$$Y = \{(x,t) \in X \times \overline{\mathbb{C}} \mid x \in H_t\}$$

There are two projections

$$X \overset{p}{\leftarrow} Y \overset{f}{\rightarrow} \overline{\mathbb{C}}$$

Put $Y' = p^{-1}(X') = X' \times \overline{\mathbb{C}}$ then

Proposition 2.3.3. *([11] 1.6.2, 1.6.3, 1.6.4) If X is a smooth variety then*

1. *The modification Y of X is smooth and irreducible;*

2. *p is an isomorphism between $Y\backslash Y'$ and $X\backslash X'$ and also an isomorphism between $f^{-1}(t)$ and X_t;*

3. *For every critical value c_i, $i = 1, \ldots, r$ of f, the hyperplane H_{c_i} has a unique tangency of order two with X which lies out of A. The other hyperplanes H_c, $c \notin C$ are transverse to X;*

4. *The projection $f : Y \to \overline{\mathbb{C}}$ has $r = deg(\check{X})$ nondegenerate critical points $p_1, \ldots, p_r$ in $Y\backslash Y'$ such that $f(p_i) = c_i$'s are distinct values in $\overline{\mathbb{C}}$.*

Now we have the critical fiber bundle map $f : (Y, Y') \to \overline{\mathbb{C}}$. Note that $f\mid_{Y'}$ has no critical points. We conclude that the natural function $f : X\backslash A \to \overline{\mathbb{C}}$ is a fiber bundle map over $\overline{\mathbb{C}}\backslash C$.

Definition 2.3.1. We can view $f : X \to \overline{\mathbb{C}}$ as a meromorphic function on X. f is called the Lefschetz meromorphic function. The foliation induced by the pencil $\{H_t\}_{t\in\overline{\mathbb{C}}}$ is called the Lefschetz foliation.

Proposition 2.3.4. *The pencil $\{H_t\}_{t\in G}$ is in general position with respect to X if and only if*

1. *Choosing a good paramerization of G, $(f)_\infty = f^{-1}(\infty)$ and $(f)_0 = f^{-1}(0)$ are smooth irreducible varieties and intersect each other transversally and ;*

2. *f has only nondegenerate critical points with distinct images.*

Proof: If $\{H_t\}_{t\in G}$ is in general position with respect to X, then the axis of the pencil, A, intersects X transversally. Knowing that

$$(f)_0 = H_0 \cap X,\ (f)_\infty = H_\infty \cap X,\ H_0 \cap H_\infty = A$$

we conclude that $(f)_0$ and $(f)_\infty$ intersect each other transversally. The second statement is Proposition 2.3.3, 4.
Now suppose that f satisfies 1 and 2. Suppose that G is not in general position with respect to $\check{X}$ at $x \in G \cap \check{X}$. We can disitinguish two cases

1. x is a smooth point of $\check{X}$ and G is tangent to $\check{X}$ at x;

Let H_s be a hyperplane which passes through x, contains G and is tangent to $\check{X}$ at x. By Duality Theorem the information

$$x \in G \subset H_s,\quad H_s \text{ is tangent to } \check{X} \text{ at } x$$

can be translated to

$$s \in A \subset H_x,\quad H_x \text{ is tangent to } X \text{ at } s$$

But this contradicts the first statement.

2. x is a singular point of $\check{X}$;

This case also cannot happen. By the argument used in the proof of 1.6.4 of [11], we have: x is a smooth point of $\check{X}$, if and only if, H_x has a unique tangency point of order two with X. □

Theorem 2.3.1. *Suppose that the pencil $\{H_t\}_{t\in G}$ is in general position with respect to X and Let a be a point in $\overline{\mathbb{C}}\backslash C$. Then for every $b \in \overline{\mathbb{C}}\backslash C$*

1. $H_i(L_b) \simeq H_i(X\backslash H_a),\ i \neq n, n-1$

2. *If $H_{n-1}(X\backslash H_a) = 0$, then a distinguished set of vanishing cycles related to the critical values $c_1, c_2, \ldots, c_r$ generates the group $H_{n-1}(L_b)$.*

Proof: This is a direct consequence of Theorem 2.2.1 and Theorem 2.1.2 and the long exact sequence (4). □

Blow up: Fix the point $b \in \mathbb{C}P(n)$. All lines through b in $\mathbb{C}P(n)$ form a projective space of dimension $n-1$, namely P. Define

$$\mathbb{C}\tilde{P}(n) = \{(x, y) \in \mathbb{C}P(N) \times P \mid x \in y\}$$

$\mathbb{C}\tilde{P}(n)$ is a smooth subvariety of $\mathbb{C}P(N) \times P$. We have two natural projections

$$\mathbb{C}P(n) \xleftarrow{i} \mathbb{C}\tilde{P}(n) \xrightarrow{f} P$$

The reader can check that f is an isomorphism between $i^{-1}(b)$ and P and i is an isomorphism between $\mathbb{C}P(n)\backslash\{b\}$ and $\mathbb{C}\tilde{P}(n)\backslash i^{-1}(b)$. $\mathbb{C}\tilde{P}(n)$ is called the blow up of $\mathbb{C}P(n)$ at the point b. Roughly speaking, we delete the point b from $\mathbb{C}P(n)$ and substitute it by a projective space of dimension $n-1$.

Theorem 2.3.2. *Suppose that the pencil $\{H_t\}_{t\in\overline{\mathbb{C}}}$ is in general position with respect to X and let $b \in \overline{\mathbb{C}}\backslash C$, then*

1. *For every two Lefschetz vanishing cycles δ_0 and δ_1 in X_b there exists a closed path λ in $\overline{\mathbb{C}}\backslash C$ with initial and end point b and such that*

$$h_\lambda(\delta_0) = \pm\delta_1$$

   where h_λ is the monodromy along the path λ;

2. *If $H_{n-1}(X\backslash H_a) = 0$ for some $a \in \overline{\mathbb{C}}\backslash C$ and $H_{n-1}(X_b) \neq 0$ then for every Lefschetz vanishing cycle δ in L_b, the action of the monodromy group on δ generates $H_{n-1}(L_b)$.*

Proof: The first statement and its proof can be found in 7.3.5 of [11]. But we can give a rather short proof for it as follows:
Let us consider the pencil $\{H_t\}_{t\in G}$ as the projective line G in $\mathbb{C}\check{P}(n)$. Let δ_0 and δ_1 vanish along the paths λ_0 and λ_1 which connect b to critical values c_0 and c_1 in G, respectively. The subset $Z \subset \check{X}$ consisting of all points x such that the line through x and b is not in general position with respect to $\check{X}$ is a proper and algebraic subset of $\check{X}$. Since $\check{X}$ is an irreducible variety and $c_0, c_1 \in \check{X}\backslash Z$, there is a path w in $\check{X}\backslash Z$ from c_0 to c_1. Denote by G_s the line through b and $w(s)$. After blow up at the point b and using the Ehresmann's theorem, we conclude that:
There is an isotopy $H : [0, 1] \times G \to \cup_s G_s$ such that

1. $H(0, .)$ is the identity map;

2. for all $s \in [0, 1]$, $H(s, .)$ is a C^∞ isomorphism between G and G_s which sends points of $\check{X}$ to $\check{X}$;

3. For all $s \in [0, 1]$, $H(s, b) = b$ and $H(s, c_1) = w(s)$.

Let $\lambda'_s = H(s, \lambda_0)$. In each Lefschetz pencil $\{H_t\}_{t\in G_s}$ the cycle δ_0 in X_b vanishs along the path λ'_s in $w(s)$, therefore δ_0 vanishes along λ'_1 in $c_1 = w(1)$. Consider λ_1 and λ'_1 as the paths which start from b and end in a point b_1 near c_1 and put $\lambda = \lambda'_1 - \lambda_1$. By uniqueness of the Lefschetz vanishing cycle along a fixed path we can see that the path λ is the desired path.

Now let us prove the second part. Unfortunately the above argument is true for the fiber X_b and not L_b. Therefore by Theorem 2.3.1 we can only conclude that the action of the monodromy on a vanishing cycle generates $H_{n-1}(X_b)$. Since $H_{n-1}(X_b) \neq 0$, there is no homologous to zero vanishing cycle in X_b. Let us prove that the intersection matrix $[<\delta_i, \delta_j>]_{r\times r}$ of vanishing cycles is connected i.e., for any two vanishing cycles δ and δ' there exists a chain $\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_e}$ of vanishing cycles with the following properties:

$$\delta = \delta_{i_1}, \delta' = \delta_{i_e} \qquad <\delta_{i_k}, \delta_{i_{k+1}}> \neq 0, \quad k = 1, 2, \ldots, e-1$$

If δ' is not connected to δ as above then by Picard-Lefschetz formula δ' has intersection zero with all cycles obtained by the action of the monodromy on δ. But the action of the monodromy on δ generates $H_{n-1}(X_b)$. X_b is compact and so $\delta' = 0$ in X_b which is a contradiction. Now using Picard-Lefschetz formula in L_b we see that the action of the monodromy on a vanishing cycle generates any other vanishing cycle in L_b. By Theorem 2.3.1, vanishing cycles generate $H_{n-1}(L_b)$ and so the second statement is proved. □

3 Topology of Integrable Foliations

In this section we will combine the results of the sections 1 and 2 to generalize Theorem 2.3.1 and Theorem 2.3.2 for the foliation $\mathcal{F}(pGdF - qFdG)$. Note that the first Integral of $\mathcal{F}$ has the critical fibers $\{F = 0\}$ and $\{G = 0\}$, if $p > 1$ and $q > 1$ respectively, which don't appear in the Lefschetz foliation. Homologies are considered in an arbitrary field except in the mentioned cases.

3.1 Integrable Foliations and Lefschetz Pencil

Let $\mathcal{F}(pGdF - qFdG)$ be an integrable foliation satisfying the generic conditions of Proposition 0.1. Put

$$D_1 = \{F = 0\},\ D_2 = \{G = 0\},\ L_b = \left(\frac{F^p}{G^q}\right)^{-1}(b)\backslash\mathcal{R},\ X_b = L_b \cup \mathcal{R}$$

Consider the reduced normal crossing divisor $D = D_1 + D_2$ and the positive integer numbers q, p such that

$$deg(F) = qd,\ deg(G) = pd,\ g.c.d.(p, q) = 1$$

It is a well-known fact that the fundamental group of the complement of any smooth hypersurface V in $\mathbb{C}P(n)$ is isomorphic to $\mathbb{Z}_{deg(V)}$ and therefore

$$\pi_1(\mathbb{C}P(n)\backslash D_1)_q = \mathbb{Z}_q\ ,\ \pi_1(\mathbb{C}P(n)\backslash D_2)_p = \mathbb{Z}_p$$

By Theorem 1.2.1 there exists a degree pq ramification map

$$\tau : \mathbb{C}\tilde{P}(n) \to \mathbb{C}P(n) \tag{5}$$

with divisor D and ramification multiplicities q and p in D_1 and D_2, respectively. We can view the polynomials F, G and the coordinates x_i, $i = 0, \ldots, n-1$, as meromorphic functions with the pole divisor H_∞, the hyperplane at infinity. We have

$$div(\tilde{F}) = q(\tilde{D}_1 - d.\tilde{H}_\infty)$$

$$div(\tilde{G}) = p(\tilde{D}_2 - d.\tilde{H}_\infty)$$

therefore $\tilde{F}^{\frac{1}{q}}$ and $\tilde{G}^{\frac{1}{p}}$ are well-defined meromorphic functions on $\mathbb{C}\tilde{P}(n)$. Define

$$j : \mathbb{C}\tilde{P}(n)\backslash\tilde{H}_\infty \to \mathbb{C}^2, \quad j(x) = (\tilde{F}^{\frac{1}{q}}, \tilde{G}^{\frac{1}{p}})$$

The following proposition shows that the different sheets of $\mathbb{C}\tilde{P}(n)$ are due to the different values of $\tilde{F}^{\frac{1}{q}}$ and $\tilde{G}^{\frac{1}{p}}$.

Proposition 3.1.1. *For any $x \in \mathbb{C}P(n)\backslash H_\infty$ the map j takes distincts values in $\tau^{-1}(x)$. (If $x \in H_\infty$ choose another hyperplane as the hyperplane at infinity).*

Proof: The set

$$S = \{x \in \mathbb{C}P(n) \mid \exists a, b \in \mathbb{C}\tilde{P}(n)\ s.t.\ \tau(a) = \tau(b) = x,\ a \neq b,\ j(a) = j(b)\}$$

is an open closed subset of $\mathbb{C}P(n)$, because the values of $\tilde{F}^{\frac{1}{q}}$ ($\tilde{G}^{\frac{1}{p}}$) in $\tau^{-1}(x)$ are the same up to multiplication by some q-th (p-th) root of the unity. Choosing normalizing coordinates like in Definition 1.1.2 around the points $a \in D_1 \cap D_2$ and $\tau^{-1}(a)$, we have

$$(x_1, x_2) \xleftarrow{j} (x_1, x_2, y) \xrightarrow{\tau} (x_1^q, x_2^p, y) \tag{6}$$

τ has the degree pq and so S has not any point near a, therefore S is empty. □

The foliation $\tilde{\mathcal{F}} = \tau^*(\mathcal{F})$ in $\mathbb{C}\tilde{P}(n)$ is also integrable and has the first integral $\frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}}$ with divisor

$$div(\frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}}) = \tilde{D}_1 - \tilde{D}_2$$

For every $\tilde{b} \in \overline{\mathbb{C}}$, let

$$\tilde{L}_{\tilde{b}} = (\frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}})^{-1}(\tilde{b})\backslash\tilde{\mathcal{R}},\ \tilde{X}_{\tilde{b}} = \tilde{L}_{\tilde{b}} \cup \tilde{\mathcal{R}},\ \tilde{\mathcal{R}} = \tilde{D}_1 \cap \tilde{D}_2$$

The following proposition states the relations between the leaves of $\mathcal{F}$ and $\tilde{\mathcal{F}}$.

Proposition 3.1.2. *The following statements are true:*

1. *τ maps $\tilde{\mathcal{R}}$ to $\mathcal{R}$ biholomorphically;*
2. *$\tau\ |_{\tilde{L}_0}$: $\tilde{L}_0 \to L_0$ ($\tau\ |_{\tilde{L}_\infty}$: $\tilde{L}_\infty \to L_\infty$) is a finite covering map of degree q (repectively p);*
3. *For any $c \neq 0, \infty$, τ maps $\tilde{L}_c$ to $L_{c^{pq}}$ biholomorphically.*

Proof: The first and second statements are the results obtained in Proposition 1.1.3. For the third it is enough to prove that $\tau\ |_{\tilde{L}_c}$ is one to one.

If $\tau(x) = \tau(y)$ and $\frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}}(x) = \frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}}(y)$ then $\frac{\tilde{F}^{\frac{1}{q}}(x)}{\tilde{F}^{\frac{1}{q}}(y)} = \frac{\tilde{G}^{\frac{1}{p}}(x)}{\tilde{G}^{\frac{1}{p}}(y)}$ is a constant which is p-th and q-th root of the unity, but $g.c.d.(p, q) = 1$ and so $\tilde{F}^{\frac{1}{q}}(x) = \tilde{F}^{\frac{1}{q}}(y)$ and $\tilde{G}^{\frac{1}{p}}(x) = \tilde{G}^{\frac{1}{p}}(x)$. By Proposition 3.1.1 we conclude that $x = y$. □

Define $v : \mathbb{C}\tilde{P}(n) \to \mathbb{C}P(N)$ by

$$v(A) = [\ldots ; x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n}; \ldots ; x_n^d; \tilde{F}^{\frac{1}{q}}(A); \tilde{G}^{\frac{1}{p}}(A)],\ i_0 + \cdots + i_n = d$$

$N-2$ is the number of monomials of degree d with the variables $x_0, x_1, \ldots, x_n$.

Proposition 3.1.3. *v is an embedding.*

Proof: Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}P(n) & \stackrel{v_d}{\to} & \mathbb{C}P(N-2) \\ \tau \uparrow & & i \uparrow \\ \mathbb{C}\tilde{P}(n) & \stackrel{v}{\to} & \mathbb{C}P(N) \end{array} \qquad (7)$$

where v_d is the well-known veronese embedding and i is the projection on the first $N-1$ coordinates.

1. v is one to one;

If $a, b \in \tilde{\mathbb{C}P}(N)$ and $v(a) = v(b)$ then $v_d(\tau(a)) = v_d(\tau(b))$ and so $\tau(a) = \tau(b)$ and by Proposition 3.1.1 we conclude that $a = b$.

2. v is locally embedding;

For any $a \in \tilde{\mathbb{C}P}(n)$ choose normalizing coordinates around a and $\tau(a)$. For example , if $a \in D_1 \cap D_2$ the diagram (7) has the form

$$\begin{array}{ccc} (x_1^q, x_2^p, y) & \xrightarrow{v_d} & v_d(x_1^q, x_2^p, y) \\ \uparrow & & \uparrow \\ (x_1, x_2, y) & \xrightarrow{v} & (v_d(x_1^q, x_2^p, y), x_1, x_2) \end{array} \tag{8}$$

we have to prove that the bottom map is an embedding at 0.

$$Dv(0) = \begin{bmatrix} * & * & \frac{\partial v_d}{\partial y} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

v_d is the veronese embedding and so $Dv(0)$ has the maximal rank rank n. For other points the proof is similar. □

The foliation $\tilde{\mathcal{F}}$ is obtained by hyperplane sections of the following Lefschetz pencil

$$\{H_t\}_{t\in\overline{\mathbb{C}}},\ H_t = \{[x; x_N; x_{N+1}] \in \mathbb{C}P(N) \mid x_N = t x_{N+1}\}$$

$\tilde{D}_1$ and $\tilde{D}_2$ intersect each other transversally in $\tilde{\mathcal{R}} = \tilde{D}_1 \cap \tilde{D}_2$, and $\frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}}$ has nondegenerate critical points with distinct images, therefore $\{H_t\}_{t\in\overline{\mathbb{C}}}$ is in general position with respect to $X = v(\tilde{\mathbb{C}P}(n))$.
Now consider the following commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{C}P}(N) & \xrightarrow{\tau} & \mathbb{C}P(n) \\ \frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}} \downarrow & & \frac{F^p}{G^q} \downarrow \\ \overline{\mathbb{C}} & \xrightarrow{i} & \overline{\mathbb{C}} \end{array} \tag{9}$$

where $i(z) = z^{pq}$. Let $\tilde{C}$ denote the set of critical values of $\frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}}$, then by Proposition 3.1.2, we conclude that

Corollary 3.1.1. *$\frac{F^p}{G^q}$ and $\frac{\tilde{F}^{\frac{1}{q}}}{\tilde{G}^{\frac{1}{p}}}$ are fiber bundle maps over $\overline{\mathbb{C}}\backslash(C \cup A)$ and $\overline{\mathbb{C}}\backslash\tilde{C}$, respectively.*

Corollary 3.1.2. *Let $b \in \overline{\mathbb{C}}$ be a regular value of $\frac{F^p}{G^q}$. Then for every two Lefschetz vanishing cycles δ_1 and δ_2 in X_b there is a mondromy h_λ such that*

$$h_\lambda(\delta_1) = \pm\delta_2$$

Proof: Fix a point $\tilde{b} \in i^{-1}(b)$. By diagram 3.2, we have the following commutative diagram

$$\begin{array}{ccccc} \pi_1(\overline{\mathbb{C}}\backslash\tilde{C}, \tilde{b}) & \times & H_*(\tilde{X}_{\tilde{b}}) & \to & H_*(\tilde{X}_{\tilde{b}}) \\ i_* \downarrow & & \tau_* \downarrow & & \tau_* \downarrow \\ \pi_1(\overline{\mathbb{C}}\backslash(C \cup A), b) & \times & H_*(X_b) & \to & H_*(X_b) \end{array} \tag{10}$$

$\tilde{\delta}_i = \tau^*(\delta_i)$, $i = 1, 2$ are two Lefcshetz vanishing cycles in $\tilde{X}_{\tilde{b}}$. By Theorem 2.3.2, there exists a path $\tilde{\lambda} \in \pi_1(\overline{\mathbb{C}}\backslash\tilde{C}, \tilde{b})$ such that the related monodromy takes $\tilde{\delta}_1$ to $\pm\tilde{\delta}_2$. We can assume that this path doesn't pass through 0 and ∞. Now by the above diagram the path $i(\tilde{\lambda})$ is the desired path. □

3.2 More About the Topology of Integrable Foliations

Here we want to prove a theorem similar to Theorem 2.2.1 for the foliation $\mathcal{F}(pGdF - qFdG)$.

Let $\tau : \mathbb{C}\tilde{P}(n) \to \mathbb{C}P(n)$ be a ramification map with simple divisor $D = \{G = 0\}$ and multiplicity p at D.

$$div(\tilde{G}) = p(\tilde{D} - d.\tilde{H}_\infty)$$

Therefor $\tilde{G}^{\frac{1}{p}}$ is a well-defined meromorphic function on $\mathbb{C}\tilde{P}(n)$. We denote by $\tilde{C}$ the set of critical values of $\frac{\tilde{F}}{\tilde{G}^{\frac{q}{p}}}$ in $\overline{\mathbb{C}}\backslash\{\infty\}$. Also

$$\tilde{\mathcal{R}} = \{\tilde{F} = 0\} \cap \{\tilde{G} = 0\}$$

The foliation $\tilde{\mathcal{F}} = \tau^*(\mathcal{F})$ has the first integral $\frac{\tilde{F}}{\tilde{G}^{\frac{q}{p}}}$. Note that $0 \in \overline{\mathbb{C}}$ is no more a critical point of $\frac{\tilde{F}}{\tilde{G}^{\frac{q}{p}}}$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}\tilde{P}(n) & \xrightarrow{\tau} & \mathbb{C}P(n) \\ \frac{\tilde{F}}{\tilde{G}^{\frac{q}{p}}} \downarrow & & \frac{F^p}{G^q} \downarrow \\ \overline{\mathbb{C}} & \xrightarrow{i} & \overline{\mathbb{C}} \end{array}$$

where $i(z) = z^p$. Like before we have

Proposition 3.2.1. *The following statements are true:*

1. $\frac{\tilde{F}}{\tilde{G}^{\frac{q}{p}}}$ *is a fiber bundle map over* $\overline{\mathbb{C}} \backslash (\tilde{C} \cup \{\infty\})$;

2. τ *maps* $\tilde{\mathcal{R}}$ $(\tilde{L}_\infty)$ *to* $\mathcal{R}$ *(respectively* L_∞*) biholomorphically;*

3. $\tau \mid_{\tilde{L}_0}: \tilde{L}_0 \to L_0$ *is a finite covering map of degree* q;

4. *For any* $c \neq 0, \infty$, τ *maps* $\tilde{L}_c$ *to* L_{c^p} *biholomorphically.*

Theorem 3.2.1. *If* $n = 2$ *then a distinguished set of Lefschetz vanishing cycles related to the critical points in the set* $\tilde{C}$ *generates the first homology group of a regular fiber* $\tilde{L}_b$ *of* $\frac{\tilde{F}}{\tilde{G}^{\frac{q}{p}}}$.

Proof: (n=2) By Theorem 2.3.1 it is enough to prove that $H_{n-1}(\mathbb{C}\tilde{P}(n)\backslash\tilde{D}) = 0$. According to Proposition 1.1.1, $\tau_* : H_{n-1}(\mathbb{C}\tilde{P}(n)\backslash\tilde{D}, \mathbb{Z}) \to H_{n-1}(\mathbb{C}P(n)\backslash D, \mathbb{Z})$ is one to one, and we also know that $H_{n-1}(\mathbb{C}P(n)\backslash D, \mathbb{Z}) = \mathbb{Z}_{deg(G)}$, which implies that $H_{n-1}(\mathbb{C}P(n)\backslash D) = 0$ in an arbitrary field. These facts imply what we want. □

Corollary 3.2.1. *Let* b *be a regular value of* $\frac{F^p}{G^q}$ *and* Δ *be a set of distinguished Lefschetz vanishing cycles related to the critical points in the set* C. *Let also* h *be the monodromy around the critical value* 0. *Then the set*

$$\Delta \cup h(\Delta) \cup \cdots \cup h^{p-1}(\Delta)$$

generates $H_{n-1}(L_b)$.

Proof: Let $\tilde{\Delta}$ be a distinguished set of Lefschetz vanishing cycles as in Theorem 3.2.1. We can see easily that $\tau(\tilde{\Delta}) = \Delta \cup h(\Delta) \cup \cdots \cup h^{p-1}(\Delta)$. □

The fiber L_b does not contain the points of $\{F = 0\} \cap \{G = 0\}$, so this corollary partially claims that the cycle around a point of $\{F = 0\} \cap \{G = 0\}$ is a rational sum of vanishing cycles. In the initial steps of this article my objective was to prove the following corollary.

Corollary 3.2.2. *Suppose that* $n = 2$ *and the generic fiber of* $\frac{F^p}{G^q}$ *has genus greater than zero. Then the action of the monodromy group on a Lefschetz vanishing cycle generates* $H_{n-1}(L_b)$. *Let* ω_1 *be a meromorphic 1-form in the* *projective space of dimension two whose pole divisor is a union of some fibers of* $\frac{F^p}{G^q}$*. If*

$$\int_{\delta_t} \omega_1 = 0$$

for a continuous sequence δ_t *of vanishing cycles, then* ω_1 *restricted to the closure of each fiber of* $\frac{F^p}{G^q}$ *is exact.*

We recall that in the above corollary we have assumed the generic conditions of Proposition 0.1.

Proof: The first part is a direct consequence of Theorem 2.3.2 and Proposition 3.1.2. For the second part it is enough to prove that

$$\int_{\delta_t} \omega_1 = 0$$

For all 1-cycles in the fibers of $\frac{F^p}{G^q}$.

Using the ramification map τ, the reader can verify that:

Proposition 3.2.2. *Let* D_0 *be a small disk around* 0 *and* l *be the straight line which connects 0 to* b_0*, a point in* ∂D_0*, then*

1. (L_l, L_{b_0}) *is a strong deformation retract of* (L_{D_0}, L_{b_0})*;*

2. *There is a* C^∞ *function* $\phi : l \times L_b \to L_l$ *such that* ϕ *is a fiber bundle trivialization on* $l\backslash\{0\}$ *and the restriction of* ϕ *to* $\{0\} \times L_{b_0}$*, namely* g*, is a finite covering map of degree* p *from* L_{b_0} *to* L_0 *;*

3. *There is a monodromy* $h : L_{b_0} \to L_{b_0}$ *around* 0 *such that for every* $x \in L_{b_0}$ *we have*

$$g^{-1}(g(x)) = \{x, h(x), \cdots, h^{p-1}(x)\}$$

in particular $h^p = I$ *and* $g \circ h = g$.

A Generic Properties

Here we will prove Proposition 0.1. The main tool is the transversality theorem which appears both in Algebraic Geometry and Differential Topology. We will work in the category of algebraic varieties but the whole of this discussion can be done in the C^∞ category of manifolds.

In the sequel by TX we denote the tangent bundle of the variety X and by $(TX)_0$ we denote the image of the zero section of the vector bundle TX. For any $x \in X$ we have

$$T_{0_x}(TX) = T_{0_x}(TX)_0 \oplus T_{0_x}(T_xX)$$

and so we can define

$$d : T_{0_x}(TX) \to T_{0_x}(T_xX)$$

d is the projection on the second coordinate. We will essentially use the following transversality theorem in algebraic geometry:

Theorem 2. *Let $f : X \to Z$ and $\pi : X \to A$ be morphisms (C^∞ functions) between smooth varieties (resp. C^∞ manifolds) and W be a smooth subvariety (resp. submanifold) of Z. Also assume that π is surjective and f is transverse to W, then there exists an open dense subset U of A such that $f\mid_{\pi^{-1}(\alpha)}$ is transverse to W for every $\alpha \in U$.*

Proof: This theorem is a consequence of Bertini's theorems (see [17]). For more information about the transversality theorem the reader is referred to [18] and [1].

Recall that $f : X \to Z$ is transverse to W if for every $x \in X$ with $y = f(x) \in W$, we have $T_yW + (T_xf)(T_x(X)) = T_y(Z)$. This is equivalent to this fact that $f^{-1}(W)$ is empty or is a smooth subvariety of X of dimension $dim(X) - dim(Z) + dim(W)$. The following well-known proposition will be used.

Proposition A.0.3. *Let $f : X \to Z$ be a morphism between two smooth varieties and $dim(Z) = 1$. Then the critical points of f are nondegenerate, if and only if, $Tf : TX \backslash (TX)_0 \to TZ$ is transverse to $(TZ)_0$.*

Let

$$X = \{(F, G, x) \in \mathcal{P}_a \times \mathcal{P}_b \times \mathbb{C}P(n) \mid F(x) \neq 0,\ G(x) \neq 0\}$$

$$g : X \to \overline{\mathbb{C}},\ g(F, G, x) = \frac{F^p}{G^q}(x) = f(x)$$

and $\tilde{T}X$ be the subvector bundle of TX whose fiber $\tilde{T}_{(F,G,x)}X$ is the tangent space of $\{(F, G)\} \times \mathbb{C}P(n)$. Let also $\tilde{T}g$ be the restriction of Tg to $\tilde{T}X$ and $\pi : \tilde{T}X \to \mathcal{P}_a \times \mathcal{P}_b$ be the projection on the parameter (F, G).

Proposition A.0.4. *For a generic pair (F, G), the critical points of $\frac{F^p}{G^q}$ in $\mathbb{C}P(n)\backslash(\{F=0\}\cup\{G=0\})$ are nondegenerate.*

Proof: According to the transversality theorem and Proposition A.0.3 it is enough to prove that

$$\tilde{T}g : \tilde{T}X \to T\overline{\mathbb{C}}$$

is transverse to $(T\overline{\mathbb{C}})_0$. In a local coordinate around $(F, G, x', v) \in \tilde{T}X$ we have

$$\tilde{T}g(F, G, x', v) = (\frac{F^p}{G^q}(x'), D(\frac{F^p}{G^q})(x')(v))$$

$$B = d \circ T_{(F,G,x',v)}(\tilde{T}g)(\bar{F}, \bar{G}, u, w) =$$

$$D^2 f(x')(v)(u) + Df(x')(w) + pD(f\frac{\bar{F}}{F})(x')(v) - qD(f\frac{\bar{G}}{G})(x')(v)$$

If $\tilde{T}g$ is not transverse to $(T\overline{\mathbb{C}})_0$ at (x', v) with $v \neq 0$, then $B = 0$ for all $u, w, \bar{F}, \bar{G}$. Putting $\bar{F} = \bar{G} = 0$ we get

$$Df(x') = 0, \; D^2 f(x')(v) = 0$$

Let $x' = (x'_1, x'_2, \ldots x'_n)$ and $v' = (v_1, v_2, \ldots v_n)$ then for all $i = 1, 2, \ldots, n$ putting $\bar{F} = x - x'_i$, $\bar{G} = 0$, we obtain $v_i = 0$. This implies that $v = 0$ which is a contadiction. □

The next step is to prove that generically the images of the critical points of $\frac{F^p}{G^q}$ are distinct in $\overline{\mathbb{C}}$. I did not succeed to get this generic property by using the transversality theorem, therefore I will prove it in the projective space of dimension two, by an elementary arguments in algebraic geometry. The proof in higher dimensions is the same. The following lemmas will be used:

Lemma A.0.1. *Let $\phi : \mathbb{C}^n \to \mathbb{C}^m$ be a linear map and A be a subvariety of $\mathbb{C}^m$. Then $A \cap Im(\phi) = A_1 \cup A_2 \cup \cdots$ is the decomposition of $A \cap Im(\phi)$ into irreducible components, if and only if, $\phi^{-1}(A) = \phi^{-1}(A_1) \cup \phi^{-1}(A_2) \cup \cdots$ is the decomposition of $\phi^{-1}(A)$ into irreducible components.*

Proof: This is due to the fact that we can choose a basis for the vector space $\mathbb{C}^n$ such that $\phi : \mathbb{C}^n = \mathbb{C}^{n-m'} \times \mathbb{C}^{m'} \to Im(\phi) = \mathbb{C}^{m'}$ is the projection on the second coordinate. □

Let

$$A' = \{x \in \mathbb{C}^6 \mid px_4x_2 - qx_1x_5 = px_4x_3 - qx_1x_6 = 0\}$$
$$A'_r = \{x \in \mathbb{C}^6 \mid x_4 = x_1 = 0\}$$
$$A'_c = \{x \in \mathbb{C}^6 \mid px_4x_2 - qx_1x_5 = px_4x_3 - qx_1x_6 = x_2x_6 - x_3x_5 = 0\}$$

Lemma A.0.2. *The following statements are true:*

1. *A' has two irreducible components A'_r and A'_c;*
2. *$A' \times A'$ has four irreducible components $A'_{ij} = A'_i \times A'_j,\ i, j = r, c$;*
3. *For any linear subspace V of $\mathbb{C}^{12}$ of dimension greater than 8, $A' \times A' \cap V$ has also four irreducible components $A'_{ij} \cap V,\ i, j = r, c$.*

In fact, from this lemma we only need to the fact that, for any linear subspace V of $\mathbb{C}^{12}$ of dimension greater than 8, $A'_{cc} \cap V$ is irreducible.

Consider an affine open set $\mathbb{C}^2 \subset \mathbb{C}P(2)$ and let $0 = (0,0),\ 1 = (0,1)$. Define

$$A = \{\omega = pGdF - qFdG \mid (F, G) \in \mathcal{P}_a \times \mathcal{P}_b \ \&\ \omega \text{ has singularity at 0 and 1}\}$$

Lemma A.0.3. *The variety A has exactly four irreducible components A_{rr}, A_{rc}, A_{cr}, A_{cc}. The component A_{rc} contains all 1-forms in A which have a radial singularity at 0 and a center singularity at 1. In the same way other components are defined.*

Proof: For any $p \in \mathbb{C}^2$ define the linear map

$$\phi_p : \mathcal{P}_a \times \mathcal{P}_b \to \mathbb{C}^6,\ \phi_p(F, G) = (F(p), F_x(p), F_y(p), G(p), G_x(p), G_y(p))$$

where the partial derivatives are considered in the fixed affine coordinate. Also we define

$$\phi : \mathcal{P}_a \times \mathcal{P}_b \to \mathbb{C}^{12},\ \phi = (\phi_{(0,0)}, \phi_{(0,1)})$$

We can assume that $deg(F) \geq 2$ and $deg(G) \geq 1$. With this hypotheses the reader can check that $dim(Im(\phi)) \geq 8$. Now our assertion is the direct consequence of Lemmas A.0.1, A.0.2.

Proof of Proposition 0.1: According to Proposition A.0.4, it is enough to prove that generically the image of nondegenerate critical points are distinct. Let

$$S = \{(F, G) \in A_{cc} \mid \frac{F^p}{G^q}(0) = \frac{F^p}{G^q}(1)\}$$

Let $(F, G) \in \mathcal{P}_a \times \mathcal{P}_b$ and $\frac{F^p}{G^q}$ have r nondegenerate critical points $p_1, \cdots, p_r$. There is an small perturbation $(\bar{F}, \bar{G})$ of (F, G) such that $\frac{\bar{F}^p}{\bar{G}^q}$ has r distinct critical values. Suppose that this is not true, then we can assume that $\frac{F^p}{G^q}$ has maximal number r' of critical values in some neighbourhood of (F, G) and $r' < r$. There exist two critical points p_1, p_2 of $\frac{F^p}{G^q}$ such that $\frac{F^p}{G^q}(p_1) = \frac{F^p}{G^q}(p_2)$ and for any $(\bar{F}, \bar{G})$ near (F, G) with corresponding critical point $\bar{p_1}, \bar{p_2}$ near p_1 and p_2, respectively, we have

$$\frac{\bar{F}^p}{\bar{G}^q}(\bar{p_1}) = \frac{\bar{F}^p}{\bar{G}^q}(\bar{p_2})$$

Let L be the linear automorphism of $\mathbb{C}P(2)$ which sends 0 and 1 to $\bar{p_1}$ and $\bar{p_2}$, respectively. In some neighbourhood U of $(F \circ L, G \circ L)$ in $\mathcal{P}_a \times \mathcal{P}_b$ we have $A_{cc} \cap U \subset S \cap U$. Since A_{cc} is an irreducible variety we conclude that $A_{cc} \subset S$ which is contradiction because

$$(xy^{a-1} + y^a, \frac{q}{p} - a + x + ay) \in A_{cc} \backslash S \qquad \square$$

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