Special components of Noether-Lefschetz loci

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Abstract

We take a sum $C_1 + rC_2,\ r \in \mathbb{Q}$ of a line C_1 and a complete intersection curve C_2 of type $(3,3)$ inside a smooth surface of degree 8 and with $C_1 \cap C_2 = \emptyset$. We gather evidences to the fact that for all except a finite number of r, the Noether-Lefschetz loci attached to the cohomology classes of $C_1 + rC_2$ are distinct 31 codimensional subvarieties intersecting each other in a 32 codimensional subvariety of the ambient space. The maximum codimension for components of the Noether-Lefschetz locus in this case is 35, and hence, we provide a conjectural description of a counterexample to a conjecture of J. Harris. The methods used in this paper also produce in a rigorous way an infinite number of general components passing through the point representing the Fermat surface of degree ≤ 9, and many non-reduced components for such degrees.

1 Introduction

In the parameter space $\mathsf{T}_{\rm full}$ of smooth surfaces of degree $d \geq 4$ in $\mathbb{P}^3$ the Noether-Lefschetz locus NL_d is a union of enumerable subvarieties of $\mathsf{T}_{\rm full}$ and its points parameterize surfaces with Picard number ≥ 2. A component of NL_d of codimension equal to (resp. strictly less than) $h^{20} = \binom{d-1}{3}$ is called general (resp. special). It is known that general components are dense in $\mathsf{T}_{\rm full}$ in both usual and Zariski topology, see [CHM88], [Voi03, §5.3.4] and [CL91], and special components of codimension $d-3$ and $2d-7,\ d \geq 5$ are unique and parameterize respectively surfaces with a line and conic, see [Gre88, Gre89, Voi88, Voi89]. This implies that for $d = 5$ we have only two special components. J. Harris in 1980's conjectured that the number of special components must be finite. C. Voisin in [Voi91] found counterexamples to this for a large d, however, the conjecture in lower degrees remains open. In [Voi90] it is proved that for $d = 6, 7$ the number of reduced special components is finite, and so, it is expected that Harris' conjecture is true in these cases. However, for $d = 8$ it is widely open. In this article we describe a conjectural description of an infinite number of reduced special components of NL_8.

Let X_0 be a smooth surface in $\mathbb{P}^3$ of degree $d \geq 4$. We assume that the Picard number $\rho(X_0)$ of X_0 is bigger than or equal to 3, and hence, X_0 has two curves C_1 and C_2 whose cohomology classes are linearly independent in the second primitive cohomology of X_0. We consider a one dimension family $[C_1] + r[C_2] \in H_2(X_0, \mathbb{Q}),\ \ r \in \mathbb{Q}$ and the corresponding family of Noether-Lefschetz loci $V_{[C_1]+r[C_2]}$ inside the parameter space $\mathsf{T}_{\rm full}$ of smooth surfaces in $\mathbb{P}^3$, see §2 for the definition. It is equipped with an analytic scheme structure and its underlying analytic variety is a union of branches of NL_d near $0 \in \mathsf{T}$. In the present paper we are looking for a special pencil of Noether-Lefschetz locus $V_{[C_1]+r[C_2]}$.

Definition 1. We say that $V_{[C_1]+r[C_2]}$ is a special pencil if 1. for all $r \in \mathbb{Q}$, $\mathrm{codim}\mathbf{T}_0 V_{[C_1]+r[C_2]} < h^{20} := \binom{d-1}{3}$, 2. there is no inclusion between the tangent spaces $\mathbf{T}_0 V_{[C_1]}$ and $\mathbf{T}_0 V_{[C_2]}$ and 3. for all $r \in \mathbb{Q}$ except a finite number, $V_{[C_1]+r[C_2]}$ is smooth as an analytic scheme (and hence reduced). If instead of the last property, the N-th infinitesimal Noether-Lefschetz locus $V^N_{[C_1]+r[C_2]}$ is the N-jet of a smooth variety then we call it an N-th infinitesimal special pencil. If at least for one $r \in \mathbb{Q}$ we have $\mathrm{codim}\mathbf{T}_0 V_{[C_1]+r[C_2]} = h^{20}(X_0)$ and the condition 2 as above is satisfied then $V_{[C_1]+r[C_2]}$ is automatically smooth and we call it a general pencil.

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If a special pencil exists, it gives us an infinite number of special reduced components of NL_d passing through a point, and hence, a counterexample to Harris' conjecture. We focus on the following class of examples. Let $d, d_1, d_2, s_1, s_2, m_1, m_2$ be integers with

$$1 \le d_1 \le d_2 \le \frac{d}{2}, \ 1 \le s_1, s_2 \le \frac{d}{2}, \quad 0 \le m_1 \le \min\{d_1, s_1\} \ \ 0 \le m_2 \le \min\{d_2, s_2\}. \tag{1}$$

Let also $f = f_1 f_3 f_5 f_7 + f_2 f_4 f_6 f_8 \in \mathbb{C}[x]_d := \mathbb{C}[x_0, x_1, x_2, x_3]_d$ with

$$f_1 \in \mathbb{C}[x]_{m_1}, \quad f_2 \in \mathbb{C}[x]_{m_2}, \quad f_3 \in \mathbb{C}[x]_{d_1-m_1}, f_4 \in \mathbb{C}[x]_{d_2-m_2},$$

$$f_5 \in \mathbb{C}[x]_{s_1-m_1}, f_6 \in \mathbb{C}[x]_{s_2-m_2}, \quad f_7 \in \mathbb{C}[x]_{d-d_1-s_1+m_1}, f_8 \in \mathbb{C}[x]_{d-d_2-s_2+m_2}.$$

We consider the surface $X_0 \in \mathbb{P}^3$ given by $f = 0$ and two algebraic curves

$$C_1 \ : \ f_1 f_3 = f_2 f_4 = 0, \tag{2}$$

$$C_2 \ : \ f_1 f_5 = f_2 f_6 = 0. \tag{3}$$

Our main example is the Fermat surface given by $f = x_0^d + x_1^d + x_2^d + x_3^d$ and

$$f_1 := \prod_{i=0}^{m_1-1} (x_0 - \zeta_{2d}^{2i+1} x_1), \quad f_3 := \prod_{i=m_1}^{d_1-1} (x_0 - \zeta_{2d}^{2i+1} x_1), \quad f_5 := \prod_{i=d_1}^{d_1+s_1-m_1-1} (x_0 - \zeta_{2d}^{2i+1} x_1), \tag{4}$$

$$f_2 := \prod_{i=0}^{m_2-1} (x_2 - \zeta_{2d}^{2i+1} x_3), \quad f_4 := \prod_{i=m_2}^{d_2-1} (x_2 - \zeta_{2d}^{2i+1} x_3), \quad f_6 := \prod_{i=d_2}^{d_2+s_2-m_2-1} (x_2 - \zeta_{2d}^{2i+1} x_3), \tag{5}$$

and f_7, f_8 are the rest of the factors in the factorization of $x_0^d + x_1^d$ and $x_2^d + x_3^d$. In this paper we prove the following.

Theorem 1. *Let us consider the Fermat surface of degree $d = 4, 5, 6, 7, 8$ and a choice of integers in (1) except the case*

$$d = 8, \quad \{(d_1, d_2), (s_1, s_2)\} = \{(3, 3), (1, 1)\} \quad (m_1, m_2) = (0, 0). \tag{6}$$

Assume that $\mathrm{codim} \mathbf{T}_0 V_{[C_1]+r[C_2]} < \binom{d-1}{3}$ and there is no inclusion between $\mathbf{T}_0 V_{[C_1]}$ and $\mathbf{T}_0 V_{[C_2]}$. Moreover

$$r := \frac{r_2}{r_1}, \quad r_1, r_2, \in \mathbb{Z}, \quad 1 \le r_1 \le 10, \quad 0 \le |r_2| \le 10 \tag{7}$$

The Noether-Lefschetz locus $V_{[C_1]+r[C_2]}$ with

$$3 \le r_1, \quad \text{or} \quad 3 \le |r_2|, \tag{8}$$

is singular as an analytic scheme (as an analytic variety this means that either it is singular at the Fermat point 0 or its defining ideal is non-reduced).

For further non-reducedness statements see [Mac05, Proposition 1], [Dan17, Theorem 1.2], [Mov19, Theorem 18.3]. The number of cases such that the hypothesis of Theorem 1 is satisfied is the difference of # with the sum of 'General' and 'Inclusion' in Table 1. For instance for $d = 5$ we have $10 = 61 - (47 + 4)$ such cases. The upper bound for $r_1, |r_2|$ in (7) is due to our computational methods, and so, the above theorem suggests that $V_{[C_1]+r[C_2]}$ is not a special pencil except for (6). In this exceptional case we have all the properties of a special pencil except the last one. We expect this case provides a special pencil. In order to provide evidences for this missing property we consider the following deformation of the Fermat surface:

$$X_t : \ x_0^8 + x_1^8 + x_2^8 + x_3^8 - \sum t_i x^i = 0, \tag{9}$$

where the sum runs through the following collection of 32 monomials

$$x_1^6x_3^2, x_1^6x_2x_3, x_1^5x_3^3, x_0x_1^4x_3^3, x_1^6x_2^2, x_1^5x_2x_3^2, x_0x_1^4x_2x_3^2, x_1^4x_3^4, x_0x_1^3x_3^4, x_0^2x_1^2x_3^4,$$

$$x_1^5x_2^2x_3, x_0x_1^4x_2^2x_3, x_1^4x_2x_3^3, x_0x_1^3x_2x_3^3, x_0^2x_1^2x_2x_3^3, x_0^3x_3^5, x_0x_1^2x_2x_3^4, x_1^3x_2x_3^4, x_1^4x_2^2x_3^2, x_0x_1^3x_2^2x_3^2,$$
$$x_0^2x_1^2x_2^2x_3^2, x_0^2x_1x_2x_3^4, x_0x_1^2x_2^2x_3^3, x_1^3x_2^2x_3^3, x_0^2x_2^2x_3^4, x_1^2x_2x_3^5, x_0x_1x_2^3x_3^3, x_0x_1^5x_3^2, x_0^2x_1^3x_3^3, x_0^3x_1x_3^4,$$
$$x_0^2x_1x_2^2x_3^3, x_0^2x_2^3x_3^3.$$

This deformation is chosen in such a way that $V_{[C_1]} \cap V_{[C_2]} = \{0\}$, see §4. We prove that

Theorem 2. *For* (6) *the infinitesimal Noether-Lefschetz locus* $V^5_{[C_1]+r[C_2]}$ *in the parameter space of the deformation* (9) *and with* (7) *is the* 5*-jet of a smooth variety.*

In our way to prove Theorem 1 and Theorem 2, we have found many general pencils. By definition members of such a pencil are reduced and smooth at the Fermat point 0.

Theorem 3. *For* $d = 4, 5, 6, 7, 8, 9$*, a general pencil* $V_{[C_1]+r[C_2]}$ *exists and the number of such pencils are listed under the column 'General' in Table 1. For* $d = 10, 11$ *general pencils do not exist.*

In order to prove Theorem 1, Theorem 2 and Theorem 3 we have produced Table 1 which contains more data than what is announced in these theorems. Let us explain this table for the row $d = 6$. The number of pairs (C_1, C_2) in this case is $355 = 212 + 15 + 79 + 49$. Among these we have 212 general pencils. The number of pairs with an inclusion between $\mathbf{T}_0V_{[C_i]}, \quad i = 1, 2$ is 15. In the remaining cases we have analyzed the algebraic cycles $r_1C_1 + r_2C_2$ with (7). Note that if we set $r := \frac{r_1}{r_2}$ then we have $V_{r_1[C_1]+r_2[C_2]} = V_{[C_1]+r[C_2]}$ as the Noether-Lefschetz loci is unchanged if we multiply the algebraic cycle by a rational number. In these cases we have analyzed $V_{[C_1]+r[C_2]}$ in the parameter space which is described in §4. For $N = 2, 3, 4, 5, 6$ the number of pairs (C_1, C_2) such that at least for one (r_1, r_2), $V_{[C_1]+r[C_2]}$ is not N-smooth but it is M-smooth for all (r_1, r_2) as above and $M < N$, is respectively 79, 49, 0, 0 and 0. The only exceptional case is $d = 8$ and the two cases (6). In these cases $V_{[C_1]+r[C_2]}$ is 5-smooth for all (r_1, r_2) in (7) and the author was not able to verify the 6-smoothness. The number 299 under NT refers to the number of cases such that at least for one r as in (7), $\mathbf{T}_0V_{[C_1]+r[C_2]}$ is not transversal to the smaller deformation space described in §4. The numbers in Table 1 are hyperlinked to the author's webpage in which the reader can find the computer produced data. Except for the last column, the data is organized in the following way. It is a list of lists of the form:

```
[i]:
  [1]:
     d_1,d_2
  [2]:
     s_1,s_2
  [3]:
     m_1,m_2
  [4]:
     a_1,a_2,a_3,a_4
  [5]:
     [1]:
        r_1,r_2
     [2]:
        ...
```

where

(10)
$$(a_1, a_2, a_3, a_4) = (\mathrm{codim}(\mathbf{T}_0V_{[C_1]}),\ \mathrm{codim}(\mathbf{T}_0V_{[C_2]}), \mathrm{codim}(\mathbf{T}_0V_{[C_1]+r[C_2]}), \mathrm{codim}(\mathbf{T}_0V_{[C_1]} \cap \mathbf{T}_0V_{[C_2]}),$$

and the fifth item is the list of all (r_1, r_2) such that $V^N_{r_1[C_1]+r_2[C_2]}$ is the N-jet of a smooth variety (it does not exist for the second and third columns under 'General' and 'Inclusion'). In the case of last column, the fifth item of the data consists of all (r_1, r_2) with $\gcd(r_1, r_2) = 1$ such that $\mathbf{T}_0V_{[C_1]+r[C_2]}$ is not transversal to the smaller deformation space described in §4.

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d	#	General	Inclusion	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N\geq 7$	NT
4	61	54	7	0	0	0	0	0	0	7
5	61	47	4	0	5	0	0	5	0	39
6	355	212	15	79*	49	0	0	0	0	299
7	355	66	17	229	35	8	0	0	0	342
8	1220+113	113	45	1155	18	0	0	?	?	1319
9	1314+19	19								
10	3873	0								
11	3873	0								

Table 1: Number of general/special components etc.

2 Preliminaries

For $N = d, d-4$ let

$$I_N := \left\{(i_0, i_1, i_2, i_3) \in \mathbb{Z}^4 \middle| 0 \leq i_e \leq d-2, \quad i_0 + i_1 + i_2 + i_3 = N\right\}, \tag{11}$$

and for $N = 2d-4$ we define $\check{I}_N$ as above but with the stronger condition $i_0+i_1 = d-2,\ i_2+i_3 = d-2$. Let B_0, B_1 be subsets of $\{\zeta \in \mathbb{C} | \zeta^d + 1 = 0\}$ with cardinalities d_1, d_2, respectively. For $i \in \check{I}_{2d-4}$ we define the number

$$p_i := \left(\sum_{\zeta \in B_0} \zeta^{i_0+1}\right) \cdot \left(\sum_{\zeta \in B_1} \zeta^{i_2+1}\right). \tag{12}$$

For any other i which is not in the set $\check{I}_{2d-4}$, p_i by definition is zero. The complete intersection algebraic cycle

$$C : \prod_{\zeta \in B_0} (x_0 - \zeta x_1) = \prod_{\zeta \in B_1} (x_2 - \zeta x_2)$$

has the periods

$$\mathsf{p}_i([C]) := \int_C \text{Residue}\left(\frac{x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} \cdot \sum_{i=0}^3 (-1)^i x_i \widehat{dx_i}}{(x_0^d + x_1^d + x_2^d + x_3^d)^2}\right) = \frac{2\pi\sqrt{-1}}{d^2} p_i \tag{13}$$

see [MV19, Theorem 1]. Let $[\mathsf{p}_{i+j}]$ be the matrix whose rows and columns are indexed by $i \in I_{d-4}$ and $j \in I_d$, respectively, and in its (i,j) entry we have p_{i+j}.

We consider the family of surfaces $X_t \subset \mathbb{P}^3$ given by the homogeneous polynomial:

$$f_t := x_0^d + x_1^d + x_2^d + x_3^d - \sum_{j \in I_d} t_j x^j = 0, \tag{14}$$

where $t = (t_j)_{j \in I_d} \in (\mathsf{T}, 0)$. In a Zariski neighborhood of the Fermat variety, and up to linear transformations of $\mathbb{P}^3$, every surface can be written in this format. More precisely, the derivative of the canonical map $i : \mathrm{PGL}(4, \mathbb{C}) \times \mathsf{T} \to \mathsf{T}_{\text{full}}$ at $(\text{identity}, 0)$ is an isomorphism, and hence, i is etale at this point. By definition T is a Zariski open subset of the vector space $\mathbb{C}[x^{I_d}]$ generated by $x^i, \quad i \in I_d$ and it parameterizes smooth surfaces. Therefore, $f \in \mathsf{T}$ parametrizes a surface given by $x_0^d + x_1^d + x_2^d + x_3^d + f = 0$. In this way, $\mathbf{T}_0\mathsf{T} = \mathbb{C}[x^{I_d}]$. Any statement on Noether-Lefschetz locus for the full parameter space T_{full} which appears in the present article follows from the same statement for T, and from now on, we will only consider T.

A cycle $\delta_0 \in H_2(X_0, \mathbb{Q})$ satisfying

$$\int_{\delta_0} \omega = 0, \quad \forall \omega \in H^0(X_0, \Omega^2_{X_0})$$

is called a Hodge cycle. Let $\omega_1, \omega_2, \cdots, \omega_a, \;\; a = h^{20}(X)$ be sections of the bundle $H^0(X, \Omega^2_{X_t}),\ t \in (\mathsf{T}, 0)$ such that they form a basis at each fiber and $\delta_t \in H_n(X_t, \mathbb{Q})$ be the monodromy/parallel transport of δ_0 to X_t, see [Voi03, §5.3.2]. The analytic space V_{δ_0} with

$$\mathcal{O}_{V_{\delta_0}} := \mathcal{O}_{\mathsf{T},0} \Big/ \left\langle \int_{\delta_t} \omega_1, \int_{\delta_t} \omega_2, \cdots, \int_{\delta_t} \omega_a \right\rangle, \tag{15}$$

is called the Noether-Lefschetz locus passing through 0 and corresponding to δ_0. It might be non-reduced, see for instance [Voi03, Exercise 2, page 154]. The tangent space of the Noether-Lefschetz locus at the Fermat point is given by

$$\mathbf{T}_0 V_{\delta_0} = \ker([\mathsf{p}_{i+j}(\delta)]) := \left\{ \sum_{i \in I_d} v_i x^i \middle| [v_i][\mathsf{p}_{i+j}(\delta)]^{\mathsf{tr}} = 0 \right\}, \tag{16}$$

where $\mathsf{p}_i(\delta_0) := \int_{\delta_0} \omega_i, \;\; i \in I_{2d-4}$ are periods of δ_0. This follows from infinitesimal variation of Hodge structures introduced in [CGGH83]. For an easy proof of this see [Mov19, §16.5].

Let $\mathcal{M}_{\mathsf{T},0}$ be the maximal ideal of $\mathcal{O}_{\mathsf{T},0}$, that is, the set of germs of holomorphic functions in $(\mathsf{T}, 0)$ vanishing at 0. The N-th order infinitesimal scheme $V^N_{\delta_0}$ is the induced scheme by (15) in the infinitesimal scheme $\mathsf{T}^N := \mathrm{Spec}(\mathcal{O}_{\mathsf{T},0}/\mathcal{M}^{N+1}_{\mathsf{T},0})$. We denote by $\mathsf{X}^N/\mathsf{T}^N$ the N-th order infinitesimal deformation of X_0 induced by X/T. Let $\mathrm{cl}(Z_0) \in H^n_{\mathrm{dR}}(X_0)$ be the class of a divisor Z_0 in X_0. Let us consider the Gauss-Manin connection

$$\nabla : H^2_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H^2_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}).$$

It induces a connection in $H^2_{\mathrm{dR}}(\mathsf{X}^N/\mathsf{T}^N)$ which we call it again the Gauss-Manin connection. There is a unique section s of $H^2_{\mathrm{dR}}(\mathsf{X}^N/\mathsf{T}^N)$ such that $\nabla(\mathsf{s}) = 0$ and $\mathsf{s}_0 = \mathrm{cl}(Z_0)$. This is called the horizontal extension of $\mathrm{cl}(Z_0)$ or a flat section of the cohomology bundle. An equivalent definition for $V^N_{[Z_0]}$ is as follows.

Definition 2. The infinitesimal Noether-Lefschetz locus $V^N_{[Z_0]}$ is a subscheme of T^N given by the conditions

$$\nabla(\mathsf{s}) = 0, \tag{17}$$

$$\mathsf{s} \in F^1 H^2_{\mathrm{dR}}(\mathsf{X}^N/\mathsf{T}^N), \tag{18}$$

$$\mathsf{s}_0 = \mathrm{cl}(Z_0). \tag{19}$$

Definition 3. We say that $V_{[Z_0]}$ is N-smooth if $V^N_{[Z_0]}$ the N-jet of a smooth variety at 0. For a more computational and differential geometric approach to smoothness see [Mov19, §18.5].

3 General components

Using the following proposition we can produce many examples of general pencils.

Proposition 1. *Let X_0 be a smooth surface of degree d. Assume that X_0 has two Hodge cycles δ_1 and δ_2 such that*

1. V_{δ_1} *is very general, in the sense that* $\mathrm{codim}\mathbf{T}_0V_{\delta_1} = h^{20} := \binom{d-1}{3}$.

2. $\mathbf{T}_0V_{\delta_1} \not\subset \mathbf{T}_0V_{\delta_2} \neq \mathbf{T}_0\mathsf{T}$

Then the Noether-Lefschetz loci $V_{\delta_1+r\delta_2}$ *for all* $r \in \mathbb{Q}$ *except a finite number of them, are set theoretically different, smooth, reduced and very general.*

Proof. The function $r \mapsto \mathrm{codim}(\mathbf{T}_0V_{\delta_1+r\delta_2})$ is lower semi-continuous and it reaches its maximum at $r = 0$. This implies that for all except a finite number of $r \in \mathbb{Q}$ we have $\mathrm{codim}(\mathbf{T}_0V_{\delta_1+r\delta_2}) = \mathrm{codim}(\mathbf{T}_0V_{\delta_1})$. For two rational numbers r_1, r_2 with $r_1 \neq r_2$ we have

$$\mathbf{T}_0V_{\delta_1+r_1\delta_2} \cap \mathbf{T}_0V_{\delta_1+r_2\delta_2} = \mathbf{T}_0V_{\delta_1} \cap \mathbf{T}_0V_{\delta_2}$$

and the codimension of this vector space is bigger than h^{20}. This follows from our hypothesis $\mathbf{T}_0V_{\delta_1} \not\subset \mathbf{T}_0V_{\delta_2}$ and $\mathrm{codim}\mathbf{T}_0V_{\delta_1} = h^{20}$. This implies that the vector spaces $\mathbf{T}_0V_{\delta_1+r\delta_2}$ form a pencil with the axis $\mathbf{T}_0V_{\delta_1} \cap \mathbf{T}_0V_{\delta_2}$. Since $\mathrm{codim}(\mathbf{T}_0V_{\delta_1+r\delta_2})$ is also the number of equations defining $V_{\delta_1+r\delta_2}$, the statement follows. □

Proof. (of Theorem 3) We just need to check the hypothesis of Proposition 1 for all pairs (C_1, C_2) In Table 1 under the column 'General' we have the number of general pencils among all pencils described in the Introduction. Clicking at each number the reader can find the list of such pencils. □

Remark 1. The proof of Theorem 3 implies that Noether-Lefschetz locus $V_{[C_1]+r[C_2]}$ for

$$d_1 = d_2 = s_1 = s_2 = \left[\frac{d-1}{2}\right], \quad m_1 = m_2 = 0 \tag{20}$$

is general for all $4 \leq d \leq 8$. However, this is not true for $d = 9, 10$. In these cases we have $(a_1, a_2, a_3, a_4) = (46, 46, 50, 72)$, $h^{20} = 56$ and for $d = 9$ we have $(a_1, a_2, a_3, a_4) = (62, 62, 80, 114)$ and $h^{20} = 84$ for $d = 10$, where a_i's are defined in (10).

4 Deformation space

Let us take two Hodge cycles $\delta_1,\ \delta_2 \in H_2(X_0, \mathbb{Z})$. We would like to compute a vector space $W \subset \mathbf{T}_0\mathsf{T}$ such that $\mathbf{T}_0\mathsf{T}$ is a direct sum of $\mathbf{T}_0V_{\delta_1} \cap \mathbf{T}_0V_{\delta_2}$ and W, and $\mathbf{T}_0V_{\delta_1+r\delta_2}$ intersects W transversely, that is, the codimension of $\mathbf{T}_0V_{\delta_1+r\delta_2}$ in $\mathbf{T}_0\mathsf{T}$ is equal to the codimension of $\mathbf{T}_0V_{\delta_1+r\delta_2} \cap W$ in W. For this we consider the vertical concatenation A of $[\mathsf{p}_{i+j}(\delta_1)]$ and $[\mathsf{p}_{i+j}(\delta_2)]$. Its kernel is $\ker([\mathsf{p}_{i+j}(\delta_1)]) \cap \ker[\mathsf{p}_{i+j}(\delta_2)]$. We compute a $a \times a$ minor B of A such that $\det(B) \neq 0$ and a is the rank of A. Let I^* be the set of row indices of B. We also check that the submatrix of $[\mathsf{p}_{i+j}(\delta_1 + r\delta_2)]$, with rows indexed by I^* and all columns has the same rank as $[\mathsf{p}_{i+j}(\delta_1 + r\delta_2)]$. This implies that $\mathbf{T}_0V_{\delta_1+r\delta_2}$ intersects W transversely, where the vector space W is generated by monomials $x^i, i \in I^*$. In the new deformation space

$$X_t: \quad x_0^d + x_1^d + x_2^d + x_3^d - \sum_{i \in I^*} t_i x^i = 0, \quad t := (t_i,\ i \in I^*) \in \mathbb{C}^{\#I^*}, \tag{21}$$

we have $\mathbf{T}_0V_{\delta_1} \cap \mathbf{T}_0V_{\delta_2} = \{0\}$ and $\mathbf{T}_0V_{\delta_1+r\delta_2}$ form a pencil of vector spaces intersecting each other at $\{0\}$. The procedure `DeformSpace` is dedicated to the computation of the deformation space in (21).

5 The creation of a formula

In this section we compute the Taylor series of the integration of differential forms over monodromies of the rational curve

$$\text{(22)} \qquad \mathbb{P}^1: \begin{cases} x_0 - \zeta_1 x_1 = 0, \\ x_2 - \zeta_2 x_3 = 0, \end{cases} \qquad \zeta_1^d = \zeta_2^d = -1,$$

inside the Fermat surface $X_0 : x_0^d + x_1^d + x_2^d + x_3^d = 0$. The content of this section is a reformulation of [Mov19, §18.3]. For a rational number r let $[r]$ be the integer part of r, that is $[r] \le r < [r]+1$, and $\{r\} := r - [r]$. Let also $(x)_y := x(x+1)(x+2)\cdots(x+y-1)$, $(x)_0 := 1$ be the Pochhammer symbol. For $\beta \in \mathbb{N}_0^4$, $\bar{\beta} \in \mathbb{N}_0^4$ is defined by the rules:

$$0 \le \bar{\beta}_i \le d-1, \quad \beta_i \equiv_d \bar{\beta}_i.$$

Consider the family of surfaces in (14).

Theorem 4. *Let $\delta_t \in H_2(X_t, \mathbb{Z})$, $t \in (\mathsf{T}, 0)$ be the monodromy (parallel transport) of the cycle $\delta_0 := [\mathbb{P}^1] \in H_2(X_0, \mathbb{Z})$ along a path which connects 0 to t. For a monomial $x_0^{\beta_0} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$ of degree $d \cdot k - 4$ we have*

$$\text{(23)} \qquad \frac{-d^2 \cdot (k-1)!}{2\pi\sqrt{-1}} \int_{\delta_t} \mathrm{Resi}\left(\frac{x_0^{\beta_0} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \left(\sum_{i=0}^3 (-1)^i x_i \widehat{dx_i}\right)}{f_t^k} \right) =$$

$$\sum_{a: I_d \to \mathbb{N}_0} \left(\frac{1}{a!} \zeta_1^{\overline{(\beta+a^*)_0+1}} \cdot \zeta_2^{\overline{(\beta+a^*)_2+1}} \prod_{i=0}^{3} \left(\left\{ \frac{\check{\beta}_i + 1}{d} \right\} \right)_{\left[\frac{\check{\beta}_i+1}{d}\right]} \right) \cdot t^a,$$

where the sum runs through all $\#I_d$-tuples $a = (a_\alpha, \ \alpha \in I_d)$ of non-negative integers such that

$$\text{(24)} \qquad \left\{ \frac{(\beta + a^*)_{2e} + 1}{d} \right\} + \left\{ \frac{(\beta + a^*)_{2e+1} + 1}{d} \right\} = 1, \quad e = 0, 1,$$

and

$$\text{(25)} \qquad t^a := \prod_{\alpha \in I_d} t_\alpha^{a_\alpha}, \qquad a! := \prod_{\alpha \in I_d} a_\alpha!, \qquad a^* := \sum_\alpha a_\alpha \cdot \alpha.$$

6 Proof of Theorem 1, 2 and 3

We have written a computer code `code1` which for any C_1 and C_2 as in the Introduction performs the following computations. For the parameter space (14) it uses (13) and (16) in order to compute the numbers a_1, a_2, a_3, a_4 in (10) for $r = 11$. In order to be sure that $a_3 = \mathrm{codim}(\mathbf{T}_0 V_{[C_1]+r[C_2]})$ for generic $r \in \mathbb{Q}$, in a separate code called `code3` we have checked this equality for $a_3 + 2$ values $r = 2, 3, \cdots, a_3 + 3$. The number $a_4 := \mathrm{codim}(\mathbf{T}_0 V_{[C_1]} \cap \mathbf{T}_0 V_{[C_2]})$ is the rank of vertical concatenation of the matrices $[\mathsf{p}_{i+j}([C_1])]$ and $[\mathsf{p}_{i+j}([C_2])]$. Therefore, there is no inclusion between $\mathbf{T}_0 V_{[C_1]}$ and $\mathbf{T}_0 V_{[C_2]}$ if and only if $a_3 \neq a_4$. The main code `code1` verifies whether $a_3 = \binom{d-1}{3}$ & $a_3 \neq a_4$. This is the hypothesis of Proposition 1, and so, if the mentioned condition is satisfied we get a general pencil. All theses cases are gathered under the column 'General' in Table 1. Then `code1` verifies whether $a_3 = a_4$. These are collected under the column 'Inclusion'. The rest of the cases are $a_3 < \binom{d-1}{3}$ & $a_3 \neq a_4$. The verification of smoothness of $V_{[C_1]+r[C_2]}$ is an infinite number of polynomial equalities which at the present moment, the author does no know how to perform it by computer. However, the smoothness of $V^N_{[C_1]+r[C_2]}$ (equivalently N-smoothness of $V_{[C_1]+r[C_2]}$) is a finite number of polynomial equalities, see [Mov19, after Theorem 18.9]. For the parameter space (14), this verification is heavy even for $N=2$.

Remark 2. The case $d=5$ is the only instance in which the author was able to compute N-smoothness for $N=2,3,4$ for the family (14). In this case, it turns out that there are 10 Noether-Lefschetz loci $V_{[C_1]+r[C_2]}$ which are 2-smooth but not 3-smooth for all $r\in\mathbb{Q}$ as in Theorem 1. Note that according to Table 1 for the smaller parameter space (21) and for five of these 10 case we have to compute until 6-smoothness.

For the rest of the computation we use the parameter space $\check{\mathsf{T}}$ in (21). We have verified that $V_{[C_1]+r[C_2]}$ with r as in (7) is transversal in $\check{\mathsf{T}}$ at 0, that is, the codimension of $\mathbf{T}_0V_{[C_1]+r[C_2]}$ in $\mathbf{T}_0\mathsf{T}$ is equal to the codimension of $\mathbf{T}_0V_{[C_1]+r[C_2]}\cap\mathbf{T}_0\check{\mathsf{T}}$ in $\mathsf{T}_0\check{\mathsf{T}}$. All the non-transversal coprime pairs (r_1,r_2) are collected in the column 'NT' of Table 1. It turns out that such bad cases are included in the set $0\leq r_1\leq 2,\ |r_2|\leq 2$, and that is why we have excluded them in (8). For the preparation of this column we have used `code3`.

Remark 3. For fixed d we can make the set (7) bigger by looking the corresponding data under 'NT'. For instance, for $d=8$ we only need to exclude the cases $r_1,|r_2|=0,1$.

The transversality statement as above implies that $V^N_{[C_1]+r[C_2]}$ is not smooth in $(\mathsf{T},0)$ if its scheme theoretical intersection with $\check{\mathsf{T}}^N$ is not smooth. It follows that if $V_{[C_1]+r[C_2]}\cap\check{\mathsf{T}}$ is not smooth then $V_{[C_1]+r[C_2]}$ is not smooth too. Such non-smooth cases are gathered under the columns $N=2,3,\cdots$.

Remark 4. For the exceptional case in Theorem 2 the only non-transversal case is $(r_1,r_2)=(1,0)$. In this case $V_{[C_1]+r[C_2]}=V_{[C_1]}$ which is a branch of the Noether-Lefschetz locus parameterizing surfaces containing a line. It is known that $V_{[C_1]}$ for the parameter space (14) is smooth.

Remark 5. The most time consuming verification has been the proof of Theorem 2 using `code1`. It took more than 10 days which is mainly due to implementation of the Taylor series in §5. The verification of 3873 cases for $d=11$ in Table 1 and using `code2` and the fact that there is no general pencil in this case has taken several days, for further computational details see §A.

Remark 6. There are 7 exceptional cases in Table 1, $d=6, N=2$ and $(d_1,d_2,s_1,s_2,m_1,m_2)$ being:

$$(1,2,1,2,1,1),(2,2,2,2,1,2),(2,2,2,2,2,1), \tag{26}$$

$$(1,3,1,3,1,1),(2,3,2,3,2,1), \tag{27}$$

$$(1,3,1,3,1,2),(2,3,2,3,2,2), \tag{28}$$

which are among the 79 cases (that is why it is stared). For these cases $V^2_{[C_1]+r[C_2]}$ the 2-jet of a smooth variety except for $r=-1$ in (26), for $r=-1,\pm\frac{1}{2}$ in (27) and $r=1,\pm\frac{1}{2}$ in (28) for both parameter spaces (14) and (21). Therefore, we have to check the 3-smoothness. It turns out that $V^3_{[C_1]+r[C_2]}$ in the parameter space (21) the 3-jet of a smooth variety only for $r=0$.

Remark 7. For all the cases in Table 1 under the columns $N=2,3,\ldots$, except (6), $V_{[C_1]+r[C_2]}$ is possibly smooth, and hence a special component, for $r=0,\ \pm1$. These cases are not the focus of this paper, as they give at most a finite number of special components.

7 Higher dimensions

All the methods introduced in the present article can be used to investigate the Hodge locus for the full family X/T of smooth hypersurfaces of degree d and even dimension n. A Hodge locus V_δ is called general (resp. very general) if its codimension (codimension of its Zariski tangent space at a point) is the minimum of the Hodge number $h^{\frac{n}{2}+1,\frac{n}{2}-1}$ and the dimension of the moduli space of hypersurfaces $r := \binom{n+1-d}{d} - (n+2)^2$. For

$$(n,d) = (2,d), \quad d \geq 4, \quad (4,3),(4,4),(4,5),(6,3),(8,3), \tag{29}$$

such a minimum is reached by $h^{\frac{n}{2}+1,\frac{n}{2}-1}$ and this is not equal to r. In all these cases the Hodge numbers before $h^{\frac{n}{2}+1,\frac{n}{2}-1}$ are zero, and hence, the number of equations defining V_δ is exactly $h^{\frac{n}{2}+1,\frac{n}{2}-1}$. In these cases very general implies general. If the Zariski tangent space $\mathbf{T}_0 V_\delta$ has codimension $h^{\frac{n}{2}+1,\frac{n}{2}-1}$, since this is also the number of equations for V_δ, we conclude that V_δ is smooth, reduced and general. The vice versa is not true. Take for instance, a sum of two lines intersecting in a point and inside quintic surface. The tangent space at a generic point is of dimension $2d-7=3$, but the Noether-Lefschetz locus in this case is of dimension $h^{20} = 2d-6 = 4$. For (n,d) not in (29), we have $n \geq 4, \quad d > \frac{2(n+1)}{n-2}$ and $r \leq h^{\frac{n}{2}+1,\frac{n}{2}-1}$. A general Hodge cycle by definition satisfies

$$\mathrm{codim}(\mathbf{T}_0 V_\delta) = \binom{d+n+1}{n+1} - (n+2)^2, \tag{30}$$

which is the dimension of the moduli space of hypersurfaces of dimension n and degree d. This implies that a Hodge locus is just a branch of the orbit of $\mathrm{GL}(n+2,\mathbb{C})$ acting on $0 \in \mathsf{T}$. This means that most of the Hodge cycles of the Fermat variety cannot be deformed in the moduli of hypersurfaces, see [Mov19, §16.8] for further discussion on this. Proposition 1 is valid for arbitrary dimensions replacing h^{20} with $h^{\frac{n}{2}+1,\frac{n}{2}-1}$ in its announcement.

A The computer code and data

For the proofs and computation in the present article we have written `code1`, `code2` and `code3` which use many procedures from the author's library `foliations.lib` written in Singular, see [GPS01]. These codes and the computer data produced in Table 1 can be found in

`https://github.com/movasati/NoetherLefschetz/`

There are other two options in order to get these data. 1. The PDF file of the article is linked to the the `github` webpage, and clicking on the name of these codes one gets the corresponding code. In the first draft of the paper in `arxiv.com` the links takes the reader to the author's webpage. 2. at the bottom of the TEX file of the article in the `arxiv.com` one can find the codes. In order to check the computations of the present paper, we first get the library `foliation.lib` from the author's or `github` webpage as above. [2] Then we run the codes by doing paste copy, and of course changing the degree d and some other parameters if necessary. In order to learn about procedures, for instance `DeformSpace` used in §4 we run

```
LIB foliation.lib;
example DeformSapce;
```

For the computation in this paper we have used a computer with processor `Intel Core i7-7700`, 16 GB Memory plus 170 GB swap memory and the operating system `Ubuntu 16.04`. Note that we have increased the swap memory and this has been very useful for computations of the case $d = 8$.

[2] `http://w3.impa.br/~hossein/foliation-allversions/foliation.lib/`

For the convenience of the reader, we reproduce `code1` which produces Table 1 for $d = 8$. We have slightly modified it from the original one in order to make it printable here. Its output consists of `compo0, compo1, compo2, compo3, compo4` which contain the data of 113 (respectively $45, 1115, 18, 2$) cases for $d = 8$ in Table 1. We have to run the code with `d=8, tru=5` without the loop for `d1,d2,m1,m2,s1,s2` and by setting `d1=3; d2=3; s1=1; s2=1; m1=0; m2=0;` in order to check the 5-smoothness in Theorem 2.

```
LIB "foliation.lib";
int n=2; int d=8; int zb=10; int tru=4; int gene=11; //-a generic coefficient;
int dhalf= d div 2; int d1; int d2; int s1; int s2; int m1; int m2;
int i; int j; list ll; list Periods; list lcycles; intvec besh; list pkom;
list lMpij; int a1; int a2;intvec mlist=d;for(i=1;i<=n; i=i+1){mlist=mlist,d;}
ring r=(0,z), (x(1..n+1)),dp;
poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z); minpoly =number(cp);
list compo0; list compo1; list compo2; list compo3; list compo4;
intvec cods; list dta;  int count;
//-For checking case by case define  the following and jump the loop of
//-d1,d2,m1,m2,s1,s2                d1=3; d2=3; s1=1; s2=1; m1=0; m2=0;
for(d1=1;d1<=dhalf; d1=d1+1)
   {
   for(d2=d1;d2<=dhalf ; d2=d2+1)
      {
      for(m1=0;m1<=d1; m1=m1+1)
         {
         for(m2=0;m2<=d2; m2=m2+1)
           {
            if (m1==0){a1=1;}else{a1=m1;}
            if (m2==0){a2=1;}else{a2=m2;}
            for (s1=a1;s1<=dhalf ; s1=s1+1)
                {
                for (s2=a2;s2<=dhalf ; s2=s2+1)
                    {
                    count=count+1;
                    intvec(d1,d2), "*", intvec(s1,s2), "*",  intvec(m1,m2);
                    ll=TwoCI(n,d,intvec(d1,d2),intvec(s1,s2),intvec(m1,m2));
                    lcycles=ll[1]; besh=ll[2];
                    Periods=list();
                    for(i=1; i<=size(lcycles); i=i+1)
                        {
                        Periods=insert(Periods,
                        PeriodLinearCycle(mlist,lcycles[i][1],lcycles[i][2],
                        par(1)),size(Periods));
                        }
                    pkom=list();
                    for(i=1; i<=size(besh)-1; i=i+1)
                        {
                        for (j=besh[i]+1; j<=besh[i+1]-1; j=j+1)
                            {
                            Periods[besh[i]]=Periods[besh[i]]+Periods[j];
                            }
```

```
         pkom=insert(pkom, Periods[besh[i]], size(pkom));
         }
     Periods=pkom;
     lMpij=list();
     for(i=1; i<=size(Periods); i=i+1)
        {
        lMpij=
        insert(lMpij,Matrixpij(mlist,Periods[i]),size(lMpij));
        }
     cods=rank(lMpij[1]),rank(lMpij[2]),
     rank(lMpij[1]+gene*lMpij[2]),
     rank(concat(transpose(lMpij[1]), transpose(lMpij[2])));
     dta=intvec(d1,d2), intvec(s1,s2), intvec(m1,m2), cods;

     if(cods[3]==binomial(d-1,3) and cods[3]<>cods[4])
       {compo0=insert(compo0, dta, size(compo0));}
     if(cods[3]==cods[4])
        {compo1=insert(compo1, dta, size(compo1)); cods;}
     if(cods[3]<binomial(d-1,3) and cods[3]<>cods[4])
        {
        ll=SmoothReduced(mlist,tru, lcycles, intvec(1,-zb),
        intvec(zb,zb), besh,0);
        if(size(ll[1])<>zb*(2*zb+1))
           {ll[1]; dta= insert(dta, ll[1], size(dta));
            compo2=insert(compo2, dta, size(compo2));}
        else
           {
           "Checking tru+1 reducedness. tru=", tru+1;
           ll=SmoothReduced(mlist,tru+1, lcycles,
           intvec(1,-zb), intvec(zb,zb), besh,0);
           if (size(ll[1])<>zb*(2*zb+1))
              {ll[1]; dta= insert(dta, ll[1], size(dta));
               compo3=insert(compo3, dta, size(compo3));}
           else
              {
              "Checking tru+2 reducedness. tru=", tru+2;
               ll=SmoothReduced(mlist,tru+2, lcycles,
               intvec(1,-zb), intvec(zb,zb), besh,0);
               dta=insert(dta, ll[1], size(dta));
               compo4=insert(compo4, dta, size(compo4));
              }
          }
         }
}}}}}}//--closing the loops of d1,d2,m1,m2,s1,s2-----
```

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