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PRODUCT FORMULAS FOR WEIGHT TWO NEWFORMS

FÓRMULAS DE PRODUTO PARA NOVAS-FORMAS DE PESO DOIS

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Resumo

Consideremos uma nova-forma f de peso dois decorrente de uma curva elíptica definida sobre números racionais $\mathbb{Q}$, e a escrevemos da forma $f = q\prod_{n=1}^{\infty}(1-q^n)^{g_n},\ g_n \in \mathbb{Z}$. Observaremos que para alguns casos especiais de curvas elípticas g_n é uma sequência crescente dos inteiros positivos.

Palavras-chave: nova-forma de peso dois.

Abstract

For a weight two newform f attached to an elliptic curve E defined over rational numbers we write $f = q\prod_{n=1}^{\infty}(1-q^n)^{g_n},\ g_n \in \mathbb{Z}$ and we observe that for some special elliptic curves g_n is an increasing sequence of positive integers.

Keywords: weight two newform.

MSC2010: 14J15,14J32,11Y55.

1 Introduction

Mathematical experiments are cheap provided that one knows which kind of experiment to do. In this note we report such an experiment which arose from the first author's work on quasi-modular forms, see [10, 11], the following simple equality

$$\frac{q\frac{\partial}{\partial q}\eta}{\eta} = \frac{1}{24}E_2\,, \tag{1}$$

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where

$$\eta(q) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \quad E_2(q) := \frac{1}{24} - \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}\,, \tag{2}$$

and K. Ono and Y. Martin's classification of weight 2 cusp forms which are expressible using the η function, see [9]. The most celebrated application of modular forms is the so called (arithmetic) modularity of elliptic curves. This was originally named as Shimura-Taniyama conjecture, and it states that for an elliptic curve E defined over $\mathbb{Z}$, there is a weight two newform $f = \sum_{n=1}^{\infty} f_n q^n$ such that for a good prime p the number of $\mathbb{F}_p$-rational points of E is $p - f_p$. In the case of semi-stable elliptic curves this was proved by A. Wiles in [13] which enabled him to complete the proof of Fermat's last theorem. For arbitrary elliptic curves it was proved by C. Breuil, B. Conrad, F. Diamond, and R. Taylor in [2]. Tables of the pair (E, f) was produced much before the proof of arithmetic modularity theorem, see for instance the Cremona's book [5] and the webpage [8]. In this note we use the latter source and formulate a conjecture. The present text was written in 2016 and it was distributed among few experts in the area. Since then there have been some developments and we knew of some other related works, see [1]. However, its main conjecture is still open, and so we decided to publish it as it is.

2 Product formulas

Any weight two newform $f := \sum_{i=1}^{\infty} f_n q^n$ can be represented by a product formula in the form

$$f(q) = q \prod_{n=1}^{\infty}(1-q^n)^{g_n}, \tag{3}$$

where g_n's are integer constants. This can be verified by a direct expansion of $(1-q^n)^{g_n}$ (thanks to B. Conrad for this observation). In [9], Martin and Ono give a list of all weight 2 newforms $f(q)$ that are products and quotients of the Dedekind eta-function, i.e,

$$f(q) = \prod_{i=1}^{s} \eta^{r_i}(q^{t_i})\,, \quad s, t_1, \ldots, t_s \in \mathbb{N} \; and \; r_1, r_2, \ldots, r_s \in \mathbb{Z}\,.$$

Therefore, in this case g_n is a repeating sequence of numbers $r_1, r_2, \ldots, r_s$.

Our method of computing g_n (see below) is by using the logarithmic derivative of f which in some sense misleading because one does not see the integrality of g_n's. Let us suppose that we can write f as in (3) . Then we take the logarithmic derivative of f and find

$$E_f := \frac{q\frac{\partial f}{\partial q}}{f} = q\frac{\partial}{\partial q}\ln f = 1 - \sum_{n=1}^{\infty} g_n \cdot n \cdot \frac{q^n}{1-q^n} \ . \tag{4}$$

We have written a code in `Singular`, see [6], which computes g_n's. This can be found in the library `foliation.lib` in the first author's webpage. The function E_f is a quasi-modular form of weight two and differential order 1, see [7] and [10]. For examples of E, f, E_f see Table 1. These examples have a nice property that we explain it in the next section.

3 Building blocks of weight two newforms

Conjecture 1. *There is an enumerable set of weight two newforms of the form $\eta_i^{\check{r}_i}(q^{\check{t}_i})$ for some $\check{r}_i, \check{t}_i \in \mathbb{N}$ such that*

$$\eta_i(q) := q^{\frac{1}{\check{r}_i\check{t}_i}} \prod_{n=1}^{\infty}(1-q^n)^{a_{i,n}}, \quad a_{i,n} \in \mathbb{N} \tag{5}$$

and $a_{i,n}$ for fixed i is an increasing sequence of positive integers and with greatest common divisor equal to one. Further, any other weight two newform f can be written as

$$f(q) = \prod_{i=1}^{s} \eta_i^{r_i}(q^{t_i}), \quad s \in \mathbb{N}, \tag{6}$$

for some $r_i \in \mathbb{Z}$ and $t_i \in \mathbb{N}$.

The classical Dedekind eta function is just one of the η_i functions in (5). For this and other examples of η_i see Table 1. If we apply the logarithmic derivative to the f given in (6), then we find

$$\frac{q\frac{\partial f}{\partial q}}{f} = 1 - \sum_{m=1}^{\infty}\left(\sum_{t_i|m} r_i \cdot a_{i,\frac{m}{t_i}}\right) m \cdot \frac{q^m}{1-q^m} \ . \tag{7}$$

Note that $f = q + O(q^2)$ and so the constant term of the equality (7) implies that:

$$\sum_{i=1}^{s} \frac{r_i t_i}{\check{r}_i \check{t}_i} = 1 \ . \tag{8}$$

Moreover, weight$(f) = 2$ and weight$(\eta_i)\check{r}_i = 2$ and so

$$\sum_{i=1}^{s} \frac{r_i}{\check{r}_i} = 1 \ . \tag{9}$$

In Table 1 by quintuple $[a_1, a_2, a_3, a_4, a_5]$ we mean the elliptic curve given by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Table 1: Weight two newforms with a single η_i

N	$[a_1, a_2, a_3, a_4, a_5]$	$\check{r}$	$\check{t}$	$g_n, \quad 1 \leq n \leq 12$
36	$[0, 0, 0, 0, 1]$	4	6	$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$
37	$[0, 0, 1, -1, 0]$	2	1	$1, 2, 3, 8, 16, 41, 97, 242, 598, 1532, 3898, 10067, \ldots$
43	$[0, 1, 1, 0, 0]$	1	1	$2, 3, 4, 12, 22, 52, 114, 268, 608, 1448, 3418, 8210, \ldots$
53	$[0, 0, 1, -1, 0]$	1	1	$1, 3, 4, 7, 13, 31, 57, 123, 259, 559, 1195, 2624, \ldots$
61	$[1, 0, 0, -2, 1]$	1	1	$1, 2, 3, 7, 10, 20, 38, 77, 149, 314, 626, 1295, \ldots$
79	$[1, 1, 1, -2, 0]$	1	1	$1, 1, 2, 5, 6, 11, 18, 36, 61, 118, 213, 400, \ldots$
83	$[1, 1, 1, 1, 0]$	1	1	$1, 1, 2, 4, 5, 11, 16, 31, 53, 97, 174, 330, \ldots$
88	$[0, 0, 0, -4, 4]$	1	2	$3, 6, 19, 48, 163, 506, 1683, 5618, 19123, 65634, 228102, 797858, \ldots$
89	$[1, 1, 1, -1, 0]$	1	1	$1, 1, 2, 3, 4, 10, 13, 25, 43, 79, 135, 246, \ldots$
92	$[0, 0, 0, -1, 1]$	1	2	$3, 5, 18, 43, 138, 426, 1371, 4428, 14683, 48882, 164970, 560368, \ldots$
101	$[0, 1, 1, -1, -1]$	1	1	$0, 2, 2, 2, 4, 7, 10, 18, 30, 52, 84, 152, \ldots$
243	$[0, 0, 1, 0, -1]$ $[0, 0, 1, 0, 20]$	1	3	$2, 5, 10, 32, 80, 234, 668, 1988, 5888, 17840, 54284, 166950, \ldots$
256	$[0, 0, 0, -2, 0]$ $[0, 0, 0, 8, 0]$	1	4	$4, 9, 36, 129, 516, 2041, 8516, 35780, 153252, 663305, 2901860, 12795009, \ldots$
288	$[0, 0, 0, -12, 0]$ $[0, 0, 0, 3, 0]$	2	4	$2, 3, 13, 46, 166, 593, 2266, 8712, 34147, 135033, 540090, 2176712, \ldots$
389	$[0, 1, 1, -2, 0]$	1	1	$2, 3, 4, 11, 20, 51, 110, 259, 582, 1395, 3262, 7822, \ldots$
675	$[0, 0, 1, 0, -169]$ $[0, 0, 1, 0, 6]$	1	3	$2, 5, 10, 20, 56, 129, 362, 945, 2590, 7093, 19772, 55306, \ldots$
2304	$[0, 0, 0, -72, 0]$ $[0, 0, 0, 18, 0]$	1	4	$4, 6, 16, 42, 132, 381, 1220, 3851, 12532, 40994, 135908, 453455, \ldots$

Source: Authorial

A general method to verify Conjecture 1 is as follow. We take two of η_i's in

Table 1, let us say η_1 and η_2, with the corresponding conductor N_1 and N_2. Now we search for examples of weight two newforms $f = \eta_1^{r_1}(q^{t_1})\eta_2^{r_2}(q^{t_2})$ with explicit $r_1 \in \mathbb{Z}$ and $t_i \in \mathbb{N}$ satisfying (8) and (9). The conductor of f might be $N_1N_2t_1t_2$ or a product of powers of its primes. Similar methods as in [9] can be used in order to classify all such newforms.

In our way to analyze Conjecture 1 we found some relations with the Ramanujan theta functions. Ramanujan's two-variable theta function $f(a, b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2} = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty, \quad \text{for } |ab| < 1\,, \tag{10}$$

where $(a, q)_n := \prod_{i=0}^{n-1}(1 - aq^i)$ is the q-Pochhammer symbol. The second identity is called the Jacobi triple product identity. Let us denote the newform associated with the conductor $N = 256$ given in Table 1 by $f(q) = \eta_{256}(q^4)$. We get

$$\eta_{256} = q^{\frac{1}{4}}(1-4q-3q^2-4q^3-2q^4+11q^6-4q^7+12q^9-10q^{10}+12q^{11}-7q^{12}+\ldots)\,, \tag{11}$$

and hence we find

$$\eta_{256}^2(q^2) = q-8q^3+10q^5+16q^7+37q^9+40q^{11}+50q^{13}-80q^{15}-30q^{17}-40q^{19}-128q^{21}+\ldots$$

which must be an eigenform of weight 4, because its coefficients are multiplicative. It is in a close relationship with [12, $A228072$]. We have

$$q^{-\frac{1}{2}}\eta_{256}^2(q) = \varphi^2(q^2)\psi^2(-q^2)(\varphi^4(q^2) - 8q\psi^4(-q^2))\,,$$

where $\varphi(q) = f(q, q) = \theta_3(q^2)$ and $\psi(q) = f(q, q^3) = \frac{1}{2}q^{-1/4}\theta_2(q)$. We also found that

$$\eta_{256}^2(q) = \frac{\eta^{12}(-q^2) - 8q\eta^{12}(q^4)}{\eta^2(-q^2)\eta^2(q^4)}.$$

Finally, it is worth to mention that in the literature we have the Borcherds lift, see [3, Theorem 14.1], in which we have a class of meromorphic modular forms for some character of $\mathrm{SL}(2, \mathbb{Z})$, of integral weight, leading coefficient one, whose coefficients are integers, all of whose zeros and poles are either cusps or imaginary quadratic irrationals and have a product formula. Moreover, for such a modular form f in [4, Theorem 5] the authors relates the exponents in the product formula of f to certain values of j_n functions over the zeros of f. The intersection of this class with the class of weight two newforms does not seem to be big, and both cases might be generalized into a general framework in which the product formulas are explained in a uniform way.

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