Preface

The guiding principal in this book is to give a detailed and historical exposition of the theory of holomorphic foliations in the projective space of dimension two. Our emphasis is the algebraic aspects of such a theory and so, we would like to rise the need for working with arbitrary fields instead of the field of complex numbers. This makes our text different from the available texts in the literature such as Camacho-Sad’s monograph [CS87] which emphasizes local aspects, Brunella’s monograph [Bru00] which emphasizes the classification of holomorphic foliations similar to classification of two dimensional surfaces, Lins Neto-Scárdua’s book [LNS] and Ilyashenko-Yakovenko’s book [IY03] which both emphasize analytic and holomorphic aspects. We have in mind an audience with a basic knowledge of Complex Analysis in one variable and Algebraic Geometry of curves in the two dimensional projective space. The text is mainly written for two primary target audiences: undergraduate students who want to have a flavor of an important class of holomorphic foliations and algebraic geometers who want to learn how the theory of holomorphic foliations can be written in the framework of Algebraic Geometry.

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Frequently used notations

\((X, x)\) For a topological space \(X\) and \(x \in X\) we denote by \((X, x)\) an small neighborhood of \(x\) in \(X\).

\(\mathcal{O}_{\mathbb{C}^2, p}\) The ring of holomorphic functions in a neighborhood of \(p\) in \(\mathbb{C}^2\).

\(k, \bar{k}\) A field of characteristic zero and its algebraic closure.

\(k[x, y]\) The ring of polynomials in \(x, y\) with coefficients in \(k\).

\(k(x, y)\) The quotient field of \(k[x, y]\) which is the same as the field of rational functions in \(x, y\) with coefficients in \(k\).

\(M^t\) The transpose of a matrix \(M\). We also write \(M = [M_{ij}]\), where \(M_{ij}\) is the \((i, j)\) entry of \(M\). The indices \(i\) and \(j\) always count the rows and columns, respectively.

\(V^\vee\) The dual of an \(R\)-module \(V\), where \(R\) is usually the ring \(\mathbb{Z}\) or the field \(k\). We always write a basis of a free \(R\)-module of rank \(r\) as a \(r \times 1\) matrix. For a basis \(\delta\) of \(V\) and \(\alpha\) of \(V^\vee\) we denote by

\[ [\delta, \alpha]^t := [\alpha_j(\delta_i)]_{i,j} \]

the corresponding \(r \times r\) matrix.

\(d\) The differential operator or a natural number, being clear in the text which one we mean.

\(X(k)\) or \(X_k\) The set of \(k\)-rational points of \(X\) defined over the field \(k\). In particular for \(k \subseteq \mathbb{C}\), \(X(\mathbb{C})\) is the underlying complex manifold of \(X\). Sometimes, for simplicity we write \(X = X(\mathbb{C})\), being clear in the context that \(X\) is a complex manifold.

\(\mathcal{F}_\mathbb{R}, \mathcal{F}_\mathbb{C}\) The foliation in \(\mathbb{R}^2\) and \(\mathbb{C}^2\), respectively.
Chapter 1
Introduction

The present text has been arisen from the lecture notes of the author during the academic years 2015-2016 at IMPA. Its objective is to introduce the reader with a basic knowledge in holomorphic foliations. Our approach is purely algebraic and we avoid many transcendental arguments in the literature. For our purpose we take foliations in the affine variety $\mathbb{A}^2_k$ with $k = \mathbb{R}$ or $\mathbb{C}$ and given by polynomial vector fields. The most famous problem for such foliations is the centennial Hilbert 16-th problem, H16 for short. Our aim is not to collect all the developments and theorems in direction of H16 (for this see for instance [Ily02]), but to present a way of breaking the problem in many pieces and observing the fact that even such partial problems are extremely difficult to treat. Our point of view is algebraic and we want to point out that the both real and complex Algebraic Geometry would be indispensable for a systematic approach to the H16. Here is Hilbert’s announcement of the problem:

16. Problem of the topology of algebraic curves and surfaces

The maximum number of closed and separate branches which a plane algebraic curve of the n-th order can have has been determined by Harnack. There arises the further question as to the relative position of the branches in the plane. As to curves of the 6-th order, I have satisfied myself-by a complicated process, it is true-that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4-th order in three dimensional space can really have.

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as
to the maximum number and position of Poincar’s boundary cycles (cycles limites) for a differential equation of the first order and degree of the form

\[ \frac{dy}{dx} = \frac{Y}{X}, \]

where \( X \) and \( Y \) are rational integral functions of the \( n \)-th degree in \( x \) and \( y \). Written homogeneously, this is

\[ X(ydz/dt - zdy/dt) + Y(zdx/dt - xdz/dt) + Z(xdy/dt - ydx/dt) = 0 \]

where \( X, Y, \) and \( Z \) are rational integral homogeneous functions of the \( n \)-th degree in \( x, y, z \), and the latter are to be determined as functions of the parameter \( t \).
Chapter 2
Hilbert’s sixteen problem

In this chapter we introduce limit cycles of polynomial differential equations in $\mathbb{R}^2$ and state the well-known Hilbert 16-th problem. Despite the fact that this is not the main problem of the present text, it must be considered the most important unsolved problem related to the topic of the present text.

2.1 Real foliations

What we want to study is the following ordinary differential equation:

\[
\begin{align*}
\dot{x} &= P(x,y) \\
\dot{y} &= Q(x,y)
\end{align*}
\]  

(2.1)

where $P, Q$ are two polynomials in $x$ and $y$ with coefficients in $\mathbb{R}$ and $\dot{x} = \frac{dx}{dt}$. We will assume that $P$ and $Q$ do not have common factors. Its solutions are the trajectories of the vector field:

\[X := P(x,y) \frac{\partial}{\partial x} + Q(x,y) \frac{\partial}{\partial y}\]

(we will also write $X = (P, Q)$). For now, the reader may use the notations defined by

\[\frac{\partial}{\partial x} := (1,0), \quad \frac{\partial}{\partial y} := (0,1)\]

This is introduced in order to distinguish between points and vectors in $\mathbb{R}^2$.

Let us first recall the first basic theorem of ordinary differential equations.

Theorem 1 For $A \in \mathbb{R}^2$ if $X(A) \neq 0$ then there is a unique analytic function

\[\gamma : (\mathbb{R}, 0) \to \mathbb{R}^2\]

such that

\[\gamma : (\mathbb{R}, 0) \to \mathbb{R}^2\]
\( \gamma(0) = A, \ \gamma = X(\gamma(t)) \)

Proof. Let us write formally

\[
\gamma = \sum_{i=0}^{\infty} \gamma_i t^i, \ \gamma_i \in \mathbb{R}^2, \ \gamma_0 := A
\]

and substitute it in \( \dot{\gamma} = X(\gamma) \). It turns out that \( \gamma \) can be written in a unique way in terms of \( \gamma_j, \ j < i \). This guarantee the existence of a unique formal \( \gamma \). Note that if \( X(A) = 0 \) then \( \gamma_i = 0 \) for all \( i \geq 1 \) and so \( \gamma \) is the constant map \( \gamma(t) = A \).

Exercise 1 Recover the proof of convergence of \( \gamma \) from classical books on ordinary differential equations.

In Theorem \([1]\) we have even claim that \( \gamma \) depends on \( A \) analytically, that is, there is a small neighborhood \((\mathbb{R}^2, A)\) of \( A \) in \( \mathbb{R}^2 \) and an analytic function

\[
\Gamma : (\mathbb{R}^2, A) \times (\mathbb{R}, 0) \to (\mathbb{R}^2, A)
\]

such that \( \Gamma(B, \cdot) \) for all \( B \in (\mathbb{R}^2, A) \) is the solution in Theorem \([1]\) crossing the point \( B \). In this way we may reformulate the following theorem:

**Theorem 2** For \( A \in \mathbb{R}^2 \) if \( X(A) \neq 0 \) then there is an analytic isomorphism \( F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, A) \) such that the push-forward of \( \frac{\partial}{\partial x} \) by \( F \) is \( X \).

Proof. The push forward of the vector field \( \frac{\partial}{\partial x} \) by \( F \) is \( X \). This is equivalent to

\[
\begin{pmatrix}
\frac{\partial F_1}{\partial x}(0,0) & \frac{\partial F_1}{\partial y}(0,0) \\
\frac{\partial F_2}{\partial x}(0,0) & \frac{\partial F_2}{\partial y}(0,0)
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix} = \begin{pmatrix}
P(A) & 0 \\
0 & Q(A)
\end{pmatrix}
\]

where \( F = (F_1, F_2) \). By a rotation around \( A \), we may assume that \( P(A) \neq 0 \). In a similar way as in Theorem \([1]\) we have a unique solution \((F_1, F_2)\) to the above differential equation with

\[
(F_1(0,y), F_2(0,y)) = A + (0,y).
\]

We have

\[
\begin{pmatrix}
\frac{\partial F_1}{\partial x}(0,0) & \frac{\partial F_1}{\partial y}(0,0) \\
\frac{\partial F_2}{\partial x}(0,0) & \frac{\partial F_2}{\partial y}(0,0)
\end{pmatrix} = \begin{pmatrix}
P(A) & 0 \\
0 & Q(A)
\end{pmatrix}
\]

and so \( F = (F_1, F_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, A) \) is an analytic isomorphism.

**Exercise 2** Describe the trajectories of the following differential equations:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x
\end{align*}
\]

**Example 1** The trajectories of the differential equation

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x
\end{align*}
\]
Fig. 2.1 A limit cycle crossing \((x, y) \sim (-1.79, 0)\)

\[
\begin{aligned}
\dot{x} &= 2y + \frac{x^2}{2} \\
\dot{y} &= 3x^2 - 3 + 0.9y
\end{aligned}
\]  

(2.2)

are depicted in Figure (2.1).

The collection of the images of the solutions of (2.1) gives us us an analytic singular foliation \(F = F(X)_\mathbb{R} = F(X) = F_R\) in \(\mathbb{R}^2\). Therefore, when we are talking about a foliation we are not interested in the parametrization of its leaves(trajectories). It is left to the reader to verify that:

**Exercise 3** For a polynomial \(R \in \mathbb{R}[x,y]\) the foliation associated to \(X\) and \(RX\) in \(\mathbb{R}^2\{R = 0\}\) are the same.

For this reason from the beginning we have assumed that \(P\) and \(Q\) have no common factors. Being interested only on the foliation \(F(X)\), we may write (2.1) in the form

\[
\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)},
\]

\(\omega = 0\), where \(\omega = Pdy - Qdx \in \Omega^1\mathbb{R}^2\).

In the second case we use the notation \(\mathcal{F} = \mathcal{F}(\omega)_\mathbb{R} = \mathcal{F}(\omega)\). In this case the foliation \(\mathcal{F}\) is characterized by the fact that \(\omega\) restricted to the leaves of \(\mathcal{F}\) is identically zero. A systematic definition of differential 1-forms will be done in §3.

**Definition 1** The singular set of the foliation \(\mathcal{F}(Pdy - Qdx)\) is defined in the following way:

\[
\text{Sing}(\mathcal{F}) = \text{Sing}(\mathcal{F})_\mathbb{R} := \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = Q(x, y) = 0\}.
\]

By our assumption \(\text{Sing}(\mathcal{F})\) is a finite set of points. The leaves of \(\mathcal{F}\) near a point \(A \in \text{Sing}(\mathcal{F})\) may be complicated.

**Exercise 4** Using a software which draws the trajectories of vector fields, describe the solutions of (2.2) near its singularities.

By Bezout theorem we have
\#Sing(\mathcal{F}) \leq \deg(P)\deg(Q)

The upper bound can be reached, for instance by the differential equation \( \mathcal{F}(Pdy - Qdx) \), where

\[ P = (x-1)(x-2)\cdots(x-d), \quad Q = (y-1)(y-2)\cdots(y-d'). \]

### 2.2 Poincaré first return map

From topological point of view a leaf \( L \) of \( \mathcal{F} = \mathcal{F}(\omega) \) is either homeomorphic to \( \mathbb{R} \) or to the circle \( S^1 := \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \). In the second case \( L \) is called a closed solution of \( \mathcal{F} \) (but not yet a limit cycle).

**Exercise 5** For a foliation \( \mathcal{F} = \mathcal{F}(\omega)_R \) the curve \( \{ R = 0 \} \), where \( d\omega = Rdx \wedge dy \), intersects all closed leaves of \( \mathcal{F} \).

We consider a point \( p \in L \) and a transversal section \( \Sigma \) to \( \mathcal{F} \) at \( p \). For any point \( q \) near enough to \( p \), we can follow the leaf of \( \mathcal{F} \) in the anti-clockwise direction and since \( L \) is closed we will encounter a new point \( h(q) \in \Sigma \). We have obtained an analytic function

\[ h : \Sigma \to \Sigma, \]

which is called the Poincaré first return map. Later in the context of holomorphic foliations we will call it the holonomy map. Usually we take a coordinate system \( z \) in \( \Sigma \) with \( z(p) = 0 \) and write the power series of \( h \) at 0:

\[ h(z) = \sum_{i=0}^{\infty} \frac{h^{(n)}(0)}{n!} z^n \]

**Definition 2** \( h'(0) \) is called the multiplier of the closed solution \( L \). If the multiplier is 1 then we say that \( h \) is tangent to the identity. In this case the tangency order is \( n \) if

\[ h^{(i)}(0) = 0, \quad h^{(n)}(0) \neq 0. \]

A closed solution \( L \) of \( \mathcal{F} \) is called a limit cycle if its Poincaré first return map is not identity. In case the Poincaré first return map is identity then the leaves of \( \mathcal{F} \) near \( L \) are also closed. In this case we can talk about the continuous family of cycles \( \delta_z, \ z \in \Sigma \), where \( \delta_z \) is the leaf of \( \mathcal{F} \) through \( z \).

**Exercise 6** Prove that the multiplier and order of tangency do not depend on the coordinate system \( z \) in \( \Sigma \).

**Proposition 1** In the above situation, we have

\[ h'(0) = \exp\left(-\int_{\delta} \frac{d\omega}{\omega}\right). \]
2.3 Hilbert 16-th problem

It is natural to ask whether a foliation $\mathcal{F}(Pdy + Qdx)$ has a finite number of limit cycles. This is in fact the first part of Hilbert 16-th problem:

**Theorem 3** Each polynomial foliation $\mathcal{F}(Pdy + Qdx)$ has a finite number of limit cycles.

The above theorem was proved by Yu. Ilyashenko and J. Ecalle independently around 80’s. We have associated to each foliation $\mathcal{F}$ the number $N(\mathcal{F})$ of its limit cycles. It is natural to ask how $N(\mathcal{F})$ depends on the ingredient polynomial $P$ and $Q$ of $\mathcal{F}$.

2.1. (Hilbert 16’th problem) Fix a natural number $n \in \mathbb{N}$. Is there some natural number $N(n) \in \mathbb{N}$ such that each foliation $\mathcal{F}(Pdx - Qdy)$ with $\deg(P), \deg(Q) \leq n$ has at most $N(n)$ limit cycles.

Of course, it would be of interest to give an explicit description of $N(n)$ and more strongly determine the nature of

$$N(n) := \max \{ N(\mathcal{F}(\omega)) \mid \omega = Pdy - Qdx, \deg(P), \deg(Q) \leq n \}.$$ 

One of the objective of the present text is to explain the fact that Hilbert 16’th problem is a combination of many unsolved difficult problems. We note that even the case $n = 2$ is open.

2.3.1 Algebraic curves invariant by foliations

Let $f \in \mathbb{R}[x,y]$. An algebraic curve over $\mathbb{R}$ is defined to be

$$\{f = 0\} := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}.$$

It can happen that such an algebraic curve is empty, for instance take $f = x^2 + y^2 + 1$, or it is a point, for instance take $f = x^2 + y^2$. For a moment assume that $\{f = 0\}$ at a point is really look like a smooth curve, for instance take $f = x^2 + y^2 - 1$ for which the curve is a circle of radius 1.

Let $\mathcal{F}(X), \ X = P \partial / \partial x + Q \partial / \partial y$ be a foliation in $\mathbb{R}^2$ as before. We would like to see when the smooth part of $f$ is a part of trajectories of $X$ (leaves of $\mathcal{F}(X)$). The gradient vector

$$\frac{\partial f}{\partial x} \frac{\partial}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y}$$

is perpendicular to the curve and so if we have
\[
\frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = f \cdot R. \quad (2.3)
\]
then \( \{ f = 0 \} \) in neighborhood of \( p \) is a part of a leaf of \( \mathcal{F} \). If the equality (2.3) occurs then we say that the algebraic curve \( \{ f = 0 \} \) is \( \mathcal{F} \)-invariant.
Chapter 3
Darboux’s theorem

In this chapter we will state and prove a theorem due to Darboux. It says that if an algebraic foliation in $\mathbb{A}^2_k$ has infinite number of algebraic leaves then it must have a first integral. Here, $k$ is an algebraically closed field of characteristic zero.

3.1 Some algebraic notations

The set of polynomial differential 1-forms

$$\Omega^1_{\mathbb{A}^2_k} := \{Pdy - Qdx \mid P, Q \in k[x, y]\},$$
and differential two forms

$$\Omega^2_{\mathbb{A}^2_k} = \{Pdx \wedge dy \mid P \in k[x, y]\}.$$

One usually defines:

$$\Omega^0_{\mathbb{A}^2_k} := k[x, y].$$

The wedge product is defined in the following way:

$$(P_1dx + Q_1dy) \wedge (P_2dx + Q_2dy) = (P_1Q_2 - P_2Q_1)dx \wedge dy.$$

**Exercise 7** Verify that for all $\omega_1, \omega_2 \in \Omega^1_{\mathbb{A}^2_k}$ we have $\omega_1 \wedge \omega_1 = 0$ and $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$.

We have the differential maps:

$$d_0 : \Omega^0_{\mathbb{A}^2_k} \to \Omega^1_{\mathbb{A}^2_k}, \quad d_0(P) = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy.$$
\[ d_1 : \Omega^1_{\mathbb{A}_k} \rightarrow \Omega^2_{\mathbb{A}_k}, \quad d_1(Pdx + Qdy) = dP \wedge dx + dQ \wedge dy. \]

**Exercise 8** Show that \( d_1 \circ d_0 = 0 \).

We will usually drop the sub index 0 and 1 and simply write \( d = d_0, d = d_1 \).

**Exercise 9** If \( d\omega = 0 \) for some \( \omega \in \Omega^1_{\mathbb{A}_k} \) then there is a \( f \in \Omega^0_{\mathbb{A}_k} \) such that \( \omega = df \).

Is this true for \( \text{char}(k) \neq 0 \)? As a hint take \( \omega = -2x^{p-1}yd\frac{y}{x} + x^{p-2} dy \), where \( p := \text{char}(k) \). Can you classify all \( \omega \)'s which do not satisfy the mentioned property.

An easier statement is that if \( df = 0 \) for \( f \in \Omega^0_{\mathbb{A}_k} \) then \( f \) is a constant, that is, \( f \in k \). This is false if the characteristic of \( k \) is not zero. For instance, in a field of characteristic \( p \) we have \( dx^p = px^{p-1}dx = 0 \) but \( x^p \) is not a constant.

Stokes formula. Let \( \delta \) be a closed anti-clockwise oriented path in \( \mathbb{R}^2 \) which does not intersect itself. Let also \( \Delta \) be the region in \( \mathbb{R}^2 \) which \( \delta \) encloses. Then

\[ \int_{\delta} \omega = \int_{\Delta} d\omega. \]

**Exercise 10** Give a proof of Stokes formula using the classical books in calculus.

Let \( k = \mathbb{R} \) or \( \mathbb{C} \). Let also \( \gamma = (x(t), y(t)) : (k, 0) \rightarrow \mathbb{R}^2 \) be an analytic map and \( \omega = Pdx + Qdy \in \Omega^1_{\mathbb{A}_k} \). The pull-back of \( \omega \) by \( \gamma \) is defined to be

\[ (\gamma^* \omega) := (P(x(t), y(t))) \frac{dx(t)}{dt} + (Q(x(t), y(t))) \frac{dy(t)}{dt} \]

**Exercise 11** Show that \( \gamma^* \omega = 0 \) is independent of the parametrization \( t \), i.e if \( a : (k, 0) \rightarrow (k, 0) \) is an analytic map and \( \gamma^* \omega = 0 \) then \((\gamma \circ a)^* \omega = 0 \).

If \( \gamma^* \omega = 0 \) then we say that \( \omega \) restricted to the image of \( \gamma \) is zero. We denote by

\[ k(x,y) := \{ \frac{P}{Q} \mid P, Q \in \mathbb{A}[x,y] \} \]

the field of rational (meromorphic) functions in \( \mathbb{A}^2_k \). The set of meromorphic differential \( i \)-forms is denoted by \( \Omega^i_{\mathbb{A}_k}(\ast) \) (instead of \( k[x,y] \) we have used \( k(x,y) \)).

**Exercise 12** Show that if for \( \omega_1, \omega_2 \in \Omega^1_{\mathbb{A}_k}(\ast) \) we have \( \omega_1 \wedge \omega_2 = 0 \) then \( \omega_2 = R\omega_1 \) for some \( R \in k(x,y) \). Show that if for \( \omega_1 = Pdy - Qdx \) and \( \omega_2 \in \Omega^1_{\mathbb{A}_k} \) we have \( \omega_1 \wedge \omega_2 \) and \( P \) and \( Q \) are relatively prime then \( \omega_2 = R\omega_1 \) for some \( R \in k(x,y) \). Is this exercise true for \( \text{char}(k) \neq 0 \)?

For \( \Omega \in \Omega^2_{\mathbb{A}_k} \) and \( \omega \in \Omega^1_{\mathbb{A}_k} \) we denote by \( \frac{\Omega}{\omega} \) any meromorphic differential 1-form \( \alpha \) such that
Exercise 13  Show that such an \( \alpha \) exists and is defined up to addition by an element in \( K(x,y) \omega \).

3.2 Invariant algebraic sets and first integrals

In this section assume that \( f \in k[x,y] \) is irreducible.

**Definition 3** We say that a curve \( \{ f = 0 \} \) is \( \mathcal{F}(\omega) \)-invariant if

\[
\omega \wedge df = f \eta, \quad \text{for some } \eta \in \Omega^1_{\mathbb{R}^2}.
\]  

(3.1)

The geometric description of the equality (3.1) is as follows. Let us write \( \omega = Pdy - Qdx \) and \( X = \partial X / \partial x + Q \partial X / \partial y \) as usual. We know that

\[
\omega \wedge df = (Pdy - Qdx) \wedge (\partial f / \partial x dx + \partial f / \partial y dy) = (X \cdot \nabla f) dx \wedge dy = f dx \wedge dy
\]  

(3.2)

where \( \eta = Rdx \wedge dy \). Note that at the points where \( f = 0 \) we have that \( X \cdot \nabla f = 0 \), but since \( \nabla f \) is perpendicular to the level curve of \( f \) we have that \( X \) is tangent to \( \{ f = 0 \} \).

**Definition 4** We say that \( f \in k(x,y) \) is a (rational) first integral of the foliation \( \mathcal{F}(\omega) \) if

\[
\omega \wedge df = 0.
\]  

(3.3)

In other words, there is \( g \in k(x,y) \) such that

\[
\omega = gd f.
\]

If this is the case we say that \( \mathcal{F}(\omega) \) has a first integral.

Let us assume that \( f = \frac{F}{G} \) where \( F, G \in k[x,y] \). We have \( df = \frac{G \frac{dF}{dx} - F \frac{dG}{dx}}{G^2} \) and so \( \omega \wedge (G \, dF - F \, dG) = 0 \) and so \( \mathcal{F}(G \, dF - F \, dG) \) has the first integral \( \frac{F}{G} \).

**Proposition 2** Let us assume that the foliation \( \mathcal{F}(\omega) \) has the first integral \( \frac{F}{G} \) as above. The algebraic curves \( F - cG = 0, c \in k \) are \( \mathcal{F}(\omega) \)-invariant.

**Proof.** We have to show that \( \omega \wedge d(F - cG) \) is divisible by \( F - cG \).

**Theorem 4** (Darboux) If the foliation \( \mathcal{F} \) has infinite number of invariant algebraic curves then \( \mathcal{F} \) has a rational first integral.

Recall that by definition two algebraic curves \( \{ f_1 = 0 \}, \{ f_2 = 0 \} \) are the same if \( f_1 = c \cdot f_2 \) for some \( c \in k \).
Proof. The proof is classical and can be found in [LNS] page 92. Let us assume that \( \mathcal{F}(\omega) \) has infinite number of invariant algebraic curves \( \{ f_i = 0 \}, \ i \in \mathbb{N} \). By definition \( \omega \wedge \partial f_i = f_i \eta_i, \ \eta_i \in \Omega^2_{\mathbb{A}^2_k} \). We rewrite this

\[
\omega \wedge \frac{df_i}{f_i} = p_i dx \wedge dy \quad \text{where} \quad p_i \in k[x, y]
\]

We make the observation that \( \text{deg} \{ p_i \} \) is independent of the degree of \( f_i \). To see this fact we write

\[
(Pdx + Qdy) \wedge (\frac{\partial f_i}{\partial x} dx + \frac{\partial f_i}{\partial y} dy) = f_i, p, dx \wedge dy
\]

then

\[
P \frac{\partial f_i}{\partial x} - Q \frac{\partial f_i}{\partial y} = f_i, p
\]

Let \( d := \text{Max} \{ \text{deg}P, \text{deg}Q \} \). Then

\[
\text{deg}p_i + \text{deg}f_i = \text{deg}(f_i, p_i) = \text{deg}(P \frac{\partial f_i}{\partial x} - Q \frac{\partial f_i}{\partial y}) \leq d + \text{deg}f_i - 1
\]

and so \( \text{deg}p_i \leq d - 1 \). The vector space \( k[x, y]_{\leq n} = \{ f \in k[x, y] | \text{deg}f \leq n \} \) is finite dimensional and in fact

\[
\text{dim} k[x, y]_{\leq n} = \binom{n + 2}{2}
\]

We set \( n = d - 1 \) and define \( a_n \) to be the dimension of the \( k \)-vector space generated by \( p_i \)'s. We have

\[
a_n \leq \binom{n + 2}{2}
\]

We choose a basis \( p_1, p_2, \ldots, p_{a_n} \) for such a vector space. The element \( p_{a_n + 1} \) is linearly dependent with the element of such a basis, that is, there are \( r_i \in k, i = 1, \ldots, a_n + 1 \) such that

\[
\sum_{i=1}^{a_n+1} r_i p_i = 0
\]

and \( r_{a_n+1} \neq 0 \). In other words

\[
\omega \wedge \sum_{i=1}^{a_n+1} r_i \frac{df_i}{f_i} = \sum_{i=1}^{a_n+1} r_i (\omega \wedge \frac{df_i}{f_i}) = 0
\]

Let \( \alpha = \sum_{i=1}^{a_n+1} r_i \frac{df_i}{f_i} \) and so \( d\alpha = 0 \) because

\[
\alpha = \sum_{i=1}^{a_n+1} r_i (\ln f_i)
\]
3.2 Invariant algebraic sets and first integrals

We repeat the same argument for \( p_1, p_2, \ldots, p_{a_n}, p_{a_n+2} \)

\[
\sum_{i=1, i\neq a_n+1}^{a_n+2} \tilde{r}_i, p_i = 0
\]

for some \( \tilde{r}_i \in \mathcal{K} \) and so

\[
\omega \wedge \left( \sum_{i=1, i\neq a_n+1}^{a_n+2} \tilde{r}_i \frac{df_i}{f_i} \right)
\]

Let \( \beta = \sum_{i=1, i\neq a_n+1}^{a_n+2} \tilde{r}_i \frac{df_i}{f_i} \) and so we have

\[
\omega \wedge \alpha = \omega \wedge \beta = 0.
\]

For this we conclude that \( \alpha = f\beta \) for some non-constant function \( f \in \mathcal{K}(x, y) \) (see Exercise 12). Since \( d\alpha = 0 \) we conclude that \( df \wedge \beta = 0 \) and so \( f \) is a first integral of \( F(\omega) \). Note that \( f \) is non-constant because in the expression of \( \alpha \) and \( \beta \) we have respectively the terms \( \frac{df_{a_n+1}}{f_{a_n+1}} \) and \( \frac{df_{a_n+2}}{f_{a_n+2}} \).

It is possible to derive refinements of the Darboux’s theorem by analyzing its proof.

**Theorem 5** If the foliation \( F(\omega) \), \( \omega = Pdy - Qdx \), \( \max(\deg(P), \deg(Q)) = d \has (d+1) + 2 \) number of invariant algebraic curves then \( F \) has a rational first integral.

We observe that we have a new examples of foliations appearing in the proof of Darboux’s theorem.

**Definition 5** A holomorphic foliation \( F(\omega) \) has a logarithmic first integral if there are polynomials \( f_1, f_2, \ldots, f_s \in \mathcal{K}(x, y) \) and \( \lambda_1, \lambda_2, \ldots, \lambda_s \in \mathcal{K} \) such that

\[
\omega \wedge \left( \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i} \right) = 0
\]

For \( \mathcal{K} = \mathbb{R} \) or \( \mathbb{C} \), the level surfaces of the multi-valued functions \( f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_s^{\lambda_s} \) are tangent to the foliations \( F(\omega) \). We call this a logarithmic first integral of \( F(\omega) \).

**Theorem 6** If the foliation \( F(\omega) \), \( \omega = Pdy - Qdx \), \( \max(\deg(P), \deg(Q)) = d \has (d+1) + 1 \) number of invariant algebraic curves then \( F \) has a logarithmic first integral.

**Exercise 14** Where exactly in this chapter, we need that \( \mathcal{K} \) is algebraically closed and its characteristic is zero? For instance, discuss Darboux’s theorem over a field of non-zero characteristic.
In this chapter we will do two main things. First, we will consider the foliation $\mathcal{F}(\omega)$ in $\mathbb{C}^2$ instead of $\mathbb{R}^2$. This will be the beginning of the theory of holomorphic foliations on complex manifolds.

### 4.1 Complexification

**Exercise 15** All the discussions in §2 is valid replacing $\mathbb{R}$ with $\mathbb{C}$. In this way, we replace analytic with holomorphic etc.

In particular,

**Theorem 7** For $A \in \mathbb{C}^2$ if $X(A) \neq 0$ then there is a biholomorphism $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, A)$ such that the push-forward of $\frac{\partial}{\partial x}$ by $F$ is $X$.

The images of the complex solutions of the vector field $X$ give us a holomorphic foliation $\mathcal{F} = \mathcal{F}(\omega)_\mathbb{C} = \mathcal{F}_\mathbb{C}$ in $\mathbb{C}^2$. The leaves of $\mathcal{F}_\mathbb{C}$ are two dimensional real manifolds embedded in a real four dimensional space. If $P, Q \in \mathbb{R}[x, y]$ we will denote by $\mathcal{F}_\mathbb{R} = \mathcal{F}(\omega)_\mathbb{R}$ the corresponding real foliation in $\mathbb{R}^2$. Note that $\mathbb{R}^2 \subset \mathbb{C}^2$ and

$$\mathcal{F}_\mathbb{R} = \mathbb{R}^2 \cap \mathcal{F}_\mathbb{C}$$

i.e. the intersection of a leaf of $\mathcal{F}_\mathbb{C}$ with $\mathbb{R}^2$ is a union of leaves of $\mathcal{F}_\mathbb{R}$. Note that $\mathcal{F}_\mathbb{C}$ may has more singularities.

### 4.2 Imagining curves and leaves in a correct way!

We consider the curves

$$C : x^2 + y^2 = 1, \quad D : xy - 1 = 0.$$
The curve $C(\mathbb{R})$ is the circle of radius 1 and $D(\mathbb{R})$ is a hyperbola and they are not isomorphic topological spaces because the first one has one connected component, whereas the second one has two. However, over complex numbers these two curves are the same and the isomorphism is given by

$$C(\mathbb{C}) \to D(\mathbb{C}), \quad (x,y) \mapsto (x+iy, x-iy)$$

where $i = \sqrt{-1}$. The curve $D(\mathbb{C})$ is parameterized in the polar coordinates by

$$x = re^{2\pi i \theta}, \quad y = r^{-1}e^{-2\pi i \theta}, \quad r \in \mathbb{R}^+, \quad \theta \in [0,1]. \quad (4.1)$$

Since the bijection $\mathbb{R}^+ \to \mathbb{R}^+, \ x \mapsto x^{-1}$ sends 0 to $\infty$, both curves $C(\mathbb{C})$ and $D(\mathbb{C})$ are cylinders with two infinities, let us say $-\infty$ and $+\infty$. A cycle $\delta$ travels from $-\infty$ to $+\infty$ and it covers the whole cylinder. We would like to make a correct intuition of this travel. This is fairly easy in the case of $C(\mathbb{C})$. This cycle is in the real four dimensional space $\mathbb{C}^2$. In a certain time it fully lies in the two dimensional space $\mathbb{R}^2 \subset \mathbb{C}^2$ which is seen as a circle of radius 1 and center $0 \in \mathbb{R}^2$. It disappears from the two dimensional world and continues its travel toward $-\infty$, see Figure 4.1, A. The case of $D(\mathbb{C})$ is a little bit tricky as the first reasonable intuition turns out to be false. First of all we have to identify two connected components of the hyperbola $D(\mathbb{R})$ inside the cylinder $D(\mathbb{C})$. These are just two lines in $D(\mathbb{C})$ coming from $-\infty$ and going to $+\infty$ without touching each other. Our cycle touches each of these lines at exactly one point and it seems to make the intuition in Figure 4.1, B. However, a simple check of the intuition with the parametrization (6.5) gives us the intuition in Figure 4.1, B, that is, the cycle $\delta$ near $-\infty$ is stretched along the $y$-axis and as it goes to $+\infty$ it becomes stretched along the $x$-axis.

![Fig. 4.1 Correct intuition](image)
Chapter 5
Holonomy I

5.1 Integrating Form

In this section we will work with a foliation $\mathcal{F}(\omega)$ in the complex manifold $M = (\mathbb{C}^2, 0)$ of dimension two, where $\omega$ is a holomorphic 1-form on the manifold $M$. We denote by $L_p$ the leaf through $p \in M$.

**Theorem 8** Assume that $0$ is not a singularity of $\mathcal{F}(\omega)$, that is, $\omega(0) \neq 0$. There are holomorphic functions $f, g \in \mathcal{O}_M$ such that

$$\omega = g \cdot df$$

Further, $g(0) \neq 0$, $f(0) = 0$ and $f$ is regular at $0$, that is, the derivation of $f$ at zero is not zero.

**Proof.** The proof follows from Theorem 2. Latex the proof presented in the class.

**Definition 6** In Theorem 8, we call $f$ a first integral of $\mathcal{F}(\omega)$ and we call $g$ an integrating factor of $\mathcal{F}(\omega)$.

**Definition 7** In Theorem 8, we some times need to take another $\tilde{f} \in \mathcal{O}_{\mathbb{C}^2, 0}$ such that $(\tilde{f}, f)$ evaluated at $p$ is $(0,0)$ and the determinant of its derivation at $p$ is non-zero. There is an open subset $U$ of $\mathbb{C}^2$ and $V$ of $0 \in \mathbb{C}^2$ such that $(\tilde{f}, f) : U \to V$ is a biholomorphism. We call this a local chart of $\mathcal{F}$ around $p$.

Now, we would like to discuss the issue of different choices of the pair $(f, g)$.

**Proposition 3** In Theorem 8 let us consider two pairs $(f_i, g_i), \ i = 1, 2$ such that

$$\omega = g_1 df_1 = g_2 df_2.$$ 

There is a biholomorphism $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$f_2 = h(f_1), \ g_2 = \frac{g_1}{h'(f_1)}.$$
Proof. Write down the details for the proof presented in the class.

5.2 Transversal section

In this section we describe a transversal section to a foliation in a point to a foliation. It has always a coordinate system given by a first integral.

Definition 8 Let $\mathcal{F} = \mathcal{F}(\omega)$ be a foliation in $\mathbb{C}^2$ and let $p$ be a regular point of $\mathcal{F}$. A transversal section to $\mathcal{F}$ at $p$ is

$$\Sigma_p := \{ q \in (\mathbb{C}^2, p) \mid \tilde{f}(q) = 0 \}$$

where $\tilde{f} \in \mathcal{O}_{\mathbb{C}^2, p}$ together with a first integral $f \in \mathcal{O}_{\mathbb{C}^2, p}$ form a coordinate system around $p$. The transversal section $\Sigma_p$ has always the coordinate system given by the image of $f$.

Proposition 4 Let $(\tilde{f}, f) : U \to V$ be a local chart for $\mathcal{F}$ as in Definition 7, $p, q \in U$ be two points in the same leaf and $\Sigma_p, \Sigma_q$ be two transversal sections to $\mathcal{F}$ at $p$ and $q$, respectively. There is a unique biholomorphism

$$h : (\Sigma_p, p) \to (\Sigma_q, q)$$

which is characterized by the fact that $z \in (\Sigma_p, p)$ and $h(z) \in (\Sigma_q, q)$ are in the same leaf of $\mathcal{F}$ in $U$.

The map $h$ is called a local holonomy of $\mathcal{F}$.

Proof. Latex the proof presented in the class.
5.3 Holonomy

Let $\delta : [0, 1] \to L$ be a path in a leaf $L$ of the foliation $\mathcal{F}$ with initial point $p$ and end point $q$. Assume that $\delta$ has a finite number of self intersecting points and take two transversal sections $\Sigma_p$ and $\Sigma_q$ at $p$ and $q$, respectively. We cover the image of $\delta$ with local charts for $\mathcal{F}$ and since $[0, 1]$ is compact we can do this by a finite number of local charts:

$$U_i, i = 0, 1, 2, 3, \ldots, n$$

Further, we can assume that $U_i \cap U_{i-1} \neq \emptyset$. We also take a transversal section $\Sigma_i$ at some point $p_i$ of the path $\delta$ in $U_{i-1} \cap U_i$. By convention, we set

$$\Sigma_0 := \Sigma_p, \quad \Sigma_{n+1} := \Sigma_q, \quad p_0 := p, \quad p_{n+1} := q.$$ 

Using Proposition 4 we get biholomorphisms

$$h_i : (\Sigma_i, p_i) \to (\Sigma_{i+1}, p_{i+1}), \quad i = 0, 1, 2, \ldots, n$$

**Definition 9** The holonomy map from $\Sigma_p$ to $\Sigma_q$ is defined to be

$$h := h_n \circ \cdots \circ h_1 \circ h_0 : (\Sigma_p, p) \to (\Sigma_q, q).$$

The following discussion may help to have a better geometric picture of the notion of holonomy.

There is a neighborhood $U_{\delta}$ of the path $\delta$ such that for every $t \in [0, 1]$ and $z \in U_{\delta}$ near $\delta(t)$, the lifting path $\tilde{\delta}_{k(z)}$ of $\delta_{[0,t]}$ in the leaf $L_z$ is well-defined. Roughly speaking, the path $\tilde{\delta}_{k(z)}$, in the leaf $L_z$, connects $k(z) \in \Sigma_p$ to $z$ in the direction of the path $\delta_{[0,t]}$. In Figure 6.2 we have shown that in the self intersecting points of $\delta$, depending on the choice of $t$, we can choose non-homotop $\tilde{\delta}_{k(z)}$’s. These paths are depicted by dash-dot-dot lines. Let $U_{\delta}$ be the set of all homotopy classes $[\delta_{k(t)}]_z$ in an small neighborhood $U_{\delta}$ of $\delta$. The reader can easily verify that $U_{\delta}$ is a complex manifold and the natural map $\tau : U_{\delta} \to U_{\delta}$ may not be one to one near the self intersecting points of $\delta$ (see Figure 6.2). All functions, for example $k(z)$, that we define on the set $U_{\delta}$ are multivalued near such points and are one valued in $U_{\delta}$. For simplicity, we will work with $U_{\delta}$ instead of $U_{\delta}$.

Let $q = \delta(t_1), 0 \leq t_1 \leq 1$, be a point of $\delta$ and $\Sigma_q$ be a small transverse section at $q$ to $\mathcal{F}$. For any point $z \in \Sigma_q$, the lifting $\tilde{\delta}_{k(z)}$ of $\delta_{[0,t_1]}$ defines the holomorphic function $k : \Sigma_q \to \Sigma_p$. The function

$$h = k^{-1} : \Sigma_p \to \Sigma_q$$

is called the holonomy of $\mathcal{F}$ along $\delta$ from $\Sigma_p$ to $\Sigma_q$. If $\delta$ is a closed path, $q = \delta(1)$ and $\Sigma_p = \Sigma_q$, we have the holomorphic germ

$$h : \Sigma_p \to \Sigma_p.$$
$h$ is called the holonomy of $\mathcal{F}$ along $\delta$ in $\Sigma_p$.

### 5.4 A formula for integrating Factor

Let $\mathcal{F}(\omega)$ be a holomorphic foliation in $\mathbb{C}^2$ and let $p$ be a regular point of $\mathcal{F}$. There is a Zariski neighborhood $U$ of $p$ and a regular 1-form $\alpha$ defined in $U$ such that

$$d\omega = \omega \wedge \alpha$$

For example, if $\mathcal{F}$ is given by a 1-form $Pdy - Qdx$, then we can define $\alpha$ as follows:

$$\alpha = -\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} dx.$$  

This is defined in the Zariski open set $P \neq 0$. For every two such 1-forms $\alpha_1$ and $\alpha_2$, we have:

$$d\omega = \omega \wedge \alpha_1 = \omega \wedge \alpha_2 \Rightarrow \omega \wedge (\alpha_1 - \alpha_2) = 0 \Rightarrow \alpha_1 \mid_L = \alpha_2 \mid_L \text{ for any leaf $L$ of $\mathcal{F}$}$$

Therefore, the $\alpha_i$’s coincide in the leaves of $\mathcal{F}$. They define a holomorphic 1-form on each leaf $L$. By

$$\frac{d\omega}{\omega}$$

we mean the collection of all 1-Forms $\alpha$. The 1-form $\alpha := \frac{d\omega}{\omega}$ can be considered as a multivalued holomorphic 1-form on $\mathbb{C}^2 \setminus \text{Sing}(\mathcal{F})$ which is one valued on the leaves of $\mathcal{F}$ and satisfies:

$$d\omega = \omega \wedge \alpha$$

By definition, if two 1-forms $\omega$ and $\omega'$ induce the same foliation $\mathcal{F}$, then there is a rational function $f = f(x,y)$ such that $\omega' = f\omega$ and therefore:

$$d\omega' = d(f\omega) = df \wedge \omega + f d\omega = \omega' \wedge (-\frac{df}{f} + \omega_1) \Rightarrow$$

$$\alpha' = \alpha - \frac{df}{f} \quad (5.1)$$

### 5.5 Formulas for integrating factors

Define the integrating factor $g_\omega$ as follows:

$$g_\omega : U_\delta \to \mathbb{C}$$
5.5 Formulas for integrating factors

\[ g_\omega(z) = \exp\left( \int_{\delta_k(z)} -\frac{d\omega}{\omega} \right) \]

**Proposition 5** The following statements are true:
1. \( d\left( \frac{\omega}{g_\omega} \right) = 0 \) in the definition domain \( U_\delta \) of \( g_\omega \);
2. Let \( a = \delta(t_1), b = \delta(t_2), 0 \leq t_1 \leq t_2 \leq 1 \), and \( h: \Sigma_a \to \Sigma_b \) be the holonomy along the path \( \delta' = \delta \mid_{[t_1,t_2]} \), then

\[ h^* \left( \frac{\omega}{g_\omega} \bigg|_{\Sigma_b} \right) = \frac{\omega}{g_\omega} \bigg|_{\Sigma_a} \quad (5.2) \]

where \( h^*a \) is the pull-back of the differential form \( a \) by \( h \).

**Proof.** Let us prove the mentioned facts in a local coordinates system. Let \( q \in \delta \) and \((U, (x,y))\) be a foliation chart around \( q \) such that in this coordinates system the leaf \( L_p \) is given by \( y = 0 \) and

\[ \omega = f dy, \quad \omega_1 = A(x,y) dx \]

Fix two points \( c = (x_1,0) \) and \( d = (x_2,0) \) in \( \delta \cap U \). For any point \( z \) in \( U \), let \( r \) be the intersection point of the leaf \( L_z \) and \( \Sigma_c \) (Figure 5.2). Define

\[ g(z) = \int_r^z \omega_1 \]

In this coordinate we have

\[ g(z) = \int_{k(z)}^z \omega_1 + \int_r^z \omega_1 = s(y) + \int_{x_1}^z A(\xi,y) d\xi \]

where \( s(y) \) only depends on \( y \) and so

\[ dg = s'(y)dy + Adx + \left( \int_{x_1}^z \frac{dA}{dy} \right) dy \Rightarrow dg \wedge \omega = dg \wedge fdy = \omega_1 \wedge \omega \]

By definition, \( g_\omega = e^g \) and so

\[ d\left( \frac{\omega}{g_\omega} \right) = d(e^{-g} \omega) = e^{-g}(-dg \wedge \omega + d\omega) = e^{-g}(-\omega_1 \wedge \omega + d\omega) = 0 \]

and here the proof of the first statement finishes.

In the coordinate \((x,y)\) we can write

\[ \frac{\omega}{g_\omega} = Gdy \]

The first part of the lemma implies that \( \frac{dG}{dx} = 0 \) or equivalently \( G = G(y) \) does not depend on the variable \( x \).
Let \((U_i, (x_i, y_i))\), \(i = 0, 1, 2, \ldots, n\) be foliation charts which cover the path \(\delta_{[t_1, t_2]}\). Let also \(\Sigma_c^i\) and \(\Sigma_d^i\) be small transverse sections at \(c_i, d_i \in U_i\) to \(\mathcal{F}\) such that
\[
c_0 = p, \quad c_i = d_{i-1}, \quad \Sigma_c^i = \Sigma_d^{i-1}, \quad i = 1, 2, \ldots, n - 1, \quad d_n = b
\]
Now our affirmation is obtained by the combination of (5.3)’s in each chart \(U_i\).

Now we fix two transverse section section
\[
\Sigma_1 := \Sigma_p, \Sigma_2 := \Sigma_{p_1}, p_1 = \delta(1).
\]
Recall that \(I_\omega\) restricted to \(\Sigma_1\) is identically 1.

**Corollary 1** We have
\[
h^*\left(\frac{\omega}{g_\omega} | \Sigma_2\right) = h^*\left(Gd\gamma | \Sigma_2\right) = h^*\left(G | \Sigma_2\right) d(h^*\gamma | \Sigma_2) = Gd\gamma | \Sigma_2 = \frac{\omega}{g_\omega} | \Sigma_2 \Rightarrow
\]
\[
h^*\left(\frac{\omega}{g_\omega} | \Sigma_2\right) = \frac{\omega}{g_\omega} | \Sigma_2
\]
(5.3)

In particular, if we choose the coordinates system \(z\) in \(\Sigma_1\) and \(\tilde{z}\) in \(\Sigma_2\) such that
\[
\omega | \Sigma_1 = dz, \quad \omega | \Sigma_2 = d\tilde{z}, \quad \delta(p) = 0,
\]
then
\[
h'(z) = \exp\left(-\int_{\delta(z)} \frac{d\omega}{\omega}\right)
\]

**Proof.** We have:
\[
I_\omega | \Sigma_1 = 1, \quad I_\omega | \Sigma_2 = \exp\left(\int_{\delta(z)} \omega\right)
\]
we have
5.5 Formulas for integrating factors

\[ h^*(e^{-\int_{\delta_{\omega_1}} \omega}) = e^{-\int_{\delta_{\omega_1}} \omega}. \]

Putting these equalities in (5.2), our affirmation is proved.

**Corollary 2** (Poincaré formula) Let \( \delta \) be a closed path in a leaf \( L \) of the foliation \( \mathcal{F} \), \( \Sigma \) be a transverse section at \( p \in \delta \) to the foliation and \( h : \Sigma \to \Sigma \) be the holonomy along \( \delta \). Then

\[ h'(p) = \exp\left(\int_{\delta} - \frac{d\omega}{\omega}\right) \quad (5.4) \]

**Proof.** This is a direct consequence of the previous corollary.

The ideas of Theorem (5) come from the author’s first course in complex dynamical system with S. Shahshahani in Iran.
In this chapter we collect some local aspect of holomorphic foliations. We would like to study
\[ F(\omega), \quad \omega = P(x,y)dy - Q(x,y)dx, \quad P, Q \in \mathcal{O}(\mathbb{C}^2, 0). \]
with \( P(0) = Q(0) = 0 \). This study will be important for the algebraic aspects of holomorphic foliations. For instance, the fact many holomorphic foliations do not have algebraic invariant curves is closely related to the analysis of their singularities.

### 6.1 Singularities of multiplicity one

The following discussion can be found partially in [CS87] page 40–44. Let \( \omega = P(x,y)dy - Q(x,y)dx \), with \( P, Q \in \mathcal{O}(\mathbb{C}^2, 0) \), be a germ of a holomorphic foliation at \( 0 \in \mathbb{C}^2 \). We assume that \( 0 \) is a singularity of \( F(\omega) \), this is, \( P(0) = Q(0) = 0 \).

Writing the Taylor series of \( \omega \) at 0 we get
\[ \omega = \omega_m + \omega_{m+1} + \ldots \]
with \( \omega_i = P_i(x,y)dy - Q_i(x,y)dx \) such that \( P_i, Q_i \) are homogeneous polynomials of degree \( i \). The number \( m \) is called the multiplicity of \( \omega \) at \( 0 \in \mathbb{C}^2 \). If \( m = 1 \) then we say that \( \omega_1 \) is the linear part of \( \omega \).

In this section we are mainly interested in the germ of holomorphic foliations with a non-zero linear part. We can use Jordan canonical form for a \( 2 \times 2 \) matrix with complex coefficients and get the following result.

**Proposition 6** Let \( F(X) \) be a germ of holomorphic foliation at 0 and let 0 be a singularity of \( F(X) \). Then, up to biholomorphisms \( h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \), \( X \) can be written in one of the following formats:

1. \( y \frac{\partial}{\partial x} + \ldots \)
2. \((x + y) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \ldots\)
3. \(ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \ldots\) with \((a, b) \neq (0, 0)\)

**Exercise 16** State and prove a similar proposition as Proposition 6 over the field of real numbers. One have to use the Jordan canonical form of two times two matrices over real numbers.

We are interested in foliations \(F(X)\) such that the linear part of \(X\) is of the form \(X_1 = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y}\). Let us analyze the foliation \(F(X_1)\). The corresponding ordinary differential equation and its solution passing through \((x_0, y_0)\) is

\[
\dot{x} = ax \quad x(t) = x_0 e^{at}
\]
\[
\dot{y} = by \quad y(t) = y_0 e^{bt}
\]

**Exercise 17** For a leaf \(L\) of \(F(X_1)\), describe the topological closure \(\overline{L}\) of \(L\) (Hint: See [CS87] pages 44-46)

Let us calculate some holonomies. From the above equation we see that the \(x\) and \(y\) axis are leaves of \(F(X_1)\). We’ll name them \(L_1\) and \(L_2\), respectively. Let \(p \neq 0\) be in \(L_1\), that is \(p = (x_0, 0)\). Let also \(\delta\) be the circle through \(p\) turning around 0 in \(L_1\) anticlockwise and \(\Sigma = \{(x, y) \in \mathbb{C}^2 | x = x_0\}\). We take a point \(z = (x_0, y) \in \Sigma\) and would like to compute the action of holonomy on \(z\). We know that \(\delta(s) = (x_0 e^{2\pi is}, 0)\). The analytic continuation of the leaf \(L\) of \(F(X)\) passing through \(z\) and along \(\delta\) is of the form \(\tilde{\delta}(s) = (x_0 e^{2\pi is}, ye^{2\pi is})\).

For \(s = 1\) we get the holonomy map

\[
h : \Sigma \rightarrow \Sigma
\]
\[
(x_0, y) \mapsto (x_0, ye^{2\pi i})
\]

If we parametrize \(\Sigma\) by \(y\) this is simply

\[
(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)
\]
\[
y \mapsto e^{2\pi i y}
\]

(6.1)

Since \(d(x^{bc}y^c) = cx^{bc-1}y^{-a-c-1}(ady - bydx)\) for \(c \neq 0\), the foliation \(F(axdy - bydx)\) has the first integral \(x^{bc}y^{ac}\) and the integrating factor \(c^{-1}x^{1+bc}y^{1-ac}\).

### 6.2 Poincaré theorem

**Definition 10** We say that a foliation \(F(ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \ldots)\) belongs to the Poincaré domain if

1. \(\frac{a}{b} \notin \mathbb{R}^+\)
2. \( \frac{a}{b} \notin \{2, 3, 4, \ldots, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \)

**Theorem 9** Let us assume that the holomorphic foliation \( \mathcal{F}(X) \), \( X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \ldots \) is in the Poincaré domain. Then there exists a biholomorphism \( h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) such that the pull-back of \( X \) by \( h \) is its linear part \( ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} \).

**Proof.** The theorem can also be stated for vector fields in \((\mathbb{C}^n, 0)\) (see [Arnold]). For this reason, we adopt the notation \((x_1, x_2) = (x, y)\) and \((a, b) = (\lambda_1, \lambda_2)\). Let

\[
h = (u_1, u_2) = (x_1 + \xi_1(x_1, x_2), x_2 + \xi_2(x_1, x_2)) \tag{6.2}
\]

where \( \xi_1, \xi_2 \) are two formal power series \( \xi_j = \sum_{|n| \geq 2} \xi_{j,n} x^n \), where \( n = (n_1, n_2) \) is a multi index, \( |n| = n_1 + n_2 \) and \( x^n = x_1^{n_1} x_2^{n_2} \). We first prove the theorem formally, that is, there is \( h \) as before such that

\[
\dot{u}_j = \lambda_j u_j + f_j(u_1, u_2) \tag{6.3}
\]

with \( X = (\lambda_1 x_1 + \phi_1(x_1, x_2)) \frac{\partial}{\partial x_1} + (\lambda_2 x_2 + \phi_2(x_1, x_2)) \frac{\partial}{\partial x_2} \). This is the same as to say that the pull-back of \( X \) is \( \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2} \). The equalities (6.2) and (6.3) imply that

\[
x_j + \frac{\partial \xi_j}{\partial x_1} x_1 + \frac{\partial \xi_j}{\partial x_2} x_2 = \lambda_j (x_j + \phi_j) + \phi_j (x_1 + \xi_1, x_2 + \xi_2)
\]

we have \( \dot{x}_j = \lambda_j x_j \) and so

\[
\sum_{|n| \geq 2} (\lambda_j - n_1 \lambda_1 - n_2 \lambda_2) \xi_{j,n} x^n = -\phi_j (x_1 + \xi_1, x_2 + \xi_2). \tag{6.4}
\]
The fact that \( \mathcal{F}(X) \) is in the Poincaré domain implies that \( \lambda_j - n_1 \lambda_1 - n_2 \lambda_2 \neq 0 \) \( \forall (n_1, n_2) \in \mathbb{N}^2 \) with \( n_1 + n_2 \geq 2 \). The equality in (6.4) is a recursion in \( \xi_j, x \). It follows that we can determine the coefficients of \( \xi_1, \xi_2 \) uniquely.

Now, let us check that these series are in fact convergent. Given two series \( A(x_1, x_2) \) and \( B(x_1, x_2) \) with positive coefficients we say that \( A < B \) if \( A_n < B_n \) \( \forall n \in \mathbb{N} \). We denote by \( \hat{C}(x_1, x_2) \) the series \( C(x_1, x_2) \) replacing its coefficients by their norm and by \( \hat{\hat{C}}(x, x) \) the series \( \hat{C}(x_1, x_2) \) by taking \( x_1 = x_2 = x \). We know that \( \hat{C} \) is convergent in \( |x_1| < R \) if \( \hat{\hat{C}} \) is convergent for \( |x| < R \). Let us prove now that \( \xi_1 + \xi_2 \) is convergent. The fact that \( \mathcal{F}(X) \) is in the Poincaré domain implies that there exists a \( \delta > 0 \) such that

\[
\delta < |\lambda_j - n_1 \lambda_1 - n_2 \lambda_2|, \quad \forall |n| \geq 2.
\]

(6.5)

From (6.4) we get

\[
\delta \xi_j < \phi_j(x_1 + \xi_1, x_2 + \xi_2)
\]

\[
\Rightarrow \xi_1 + \xi_2 < \delta^{-1}[\phi_1(x + \xi_1 + \xi_2) + \phi_2(x + \xi_1 + \xi_2)]
\]

(6.6)

Our problem is reduced to the following one. Let \( F(x) \in \mathcal{O}(\mathbb{C}, 0) \) be a convergent series with positive coefficients and assume that its multiplicity at \( x = 0 \) is \( \geq 2 \). By implicit function theorem, there exists a holomorphic function \( y(x) \in \mathcal{O}(\mathbb{C}, 0) \) such that

\[
y(x) = F(x + y(x))
\]

(6.7)

and the multiplicity of \( y \) at \( x = 0 \) is \( \geq 2 \). The coefficients of the Taylor series of \( y(x) \) can be determined uniquely by the above equality, and moreover, one can check that they are positive numbers. Let \( z(x) \) be another formal power series such that

\[
z(x) \leq F(x + z(x))
\]

(6.8)

Then \( z(x) \leq y(x) \). This follows by induction on the \( n \)-the coefficient of the inequality \( z(x) \leq y(x) \). The case \( n = 2 \) follows from (6.8). Note that the the coefficient of \( x^2 \) in \( y(x) \) is the same as the coefficient of \( x^2 \) in \( F(x) \). Assuming the \( m \)-th step of the induction for all \( m < n \), we again realize the \( n \)-th coefficient of \( z \) is \( \leq \) a polynomial combination with coefficients in \( \mathbb{N} \) of the \( m \)-th coefficients of \( z(x) \) and \( F(x) \) for \( m \leq n \). Replacing the coefficient of \( z(x) \) with the corresponding coefficients of \( y(x) \) we get a bigger quantity which is the \( n \)-the coefficient of \( y(x) \).

**Exercise 18** Show that the condition (6.5) is equivalent to the fact that \( \mathcal{F}(X) \) is in the Poincaré domain.

**Theorem 10** (Dulac) let \( \mathcal{F}(X), X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \cdots \) be a holomorphic foliation in \( (\mathbb{C}^2, 0) \) with \( a = nb, n \in \mathbb{N} \) and \( n \geq 2 \). There is a unique biholomorphic function \( h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \) tangent to the identity such that the pull-back of \( X \) by \( h \) is

\[
(ax + cy^n) \frac{\partial}{\partial x} + (by) \frac{\partial}{\partial y}
\]

(6.9)
6.3 Siegel domain

Exercise 19 The proof is similar to the proof of the Poincaré theorem. Explain the details?

Exercise 20 Use a computer and draw $\mathcal{F}(X)$ with $X$ as in (6.10) for $a = n = 2, b = 1$.

Definition 11 A biholomorphism $h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is tangent to the identity if $h = (x + \zeta_1(x,y), y + \zeta_2(x,y))$ where the multiplicity of $\zeta_1, \zeta_2$ at 0 is $\geq 2$.

6.3 Siegel domain

In the Poincaré theorem we have excluded a class of holomorphic foliations and it is natural to ask whether they are also linearizable. Let us start with the definition of such a class.

Definition 12 The foliation $\mathcal{F}(X), X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \cdots$ is in the Siegel domain if

$$ab \neq 0, \frac{a}{b} \in \mathbb{R}^-$$

We say that $\mathcal{F}(X)$ has resonance if $\frac{a}{b} \in \mathbb{Q}^-$. 

For a holomorphic foliation $\mathcal{F}(X)$ in the Siegel domain and without resonance, we have still the formal power series $h$ conjugating $X$ with its linear part. However, it can happen that this formal power series is not convergent.

Definition 13 We say that $\mathcal{F}(X)$ is of type $(c, v), c, v > 0$ if

$$\left\{ \frac{|a - n_1a - n_2b|}{|b - n_1a - n_2b|} > \frac{c}{(n_1 + n_2)} \forall n_1, n_2 \geq 0, n_1, n_2 \in \mathbb{N} \right\}$$

Theorem 11 (Siegel) If $\mathcal{F}(X)$ is of type $(c, v)$ then there exist a local biholomorphism $h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that the pulled backed of $X$ by $h$ is the linear part of $X$, that is, $ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y}$.

Proof. See [Arn80].

It is natural to define the sets:

$$SD := \{(a, b) \in \mathbb{C}^2 | \frac{a}{b} \in \mathbb{R}^- \}$$

$$\tilde{SD} := \{(a, b) \in SD | (a, b) \text{ of type } (c, v) \text{ for some } (c, v), c, v > 0 \}$$

Exercise 21 Is $\tilde{SD}$ dense in $SD$? Give examples of elements of $\tilde{SD}$ and $SD \setminus \tilde{SD}$. See [CS87] and the references therein.
6 Singularities of holomorphic foliations

Fig. 6.2 Poincaré linearization theorem

6.4 Separatrix

Let $\mathcal{F}(X), X := P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ and $\omega := P(x,y)dy - Q(x,y)dx$ be a germ of holomorphic foliation in $(\mathbb{C}^2, 0)$, and let $0 \in \mathbb{C}^2$ be a singularity of $\mathcal{F}(\omega)$.

Definition 14 For $f \in O_{(\mathbb{C}^2,0)}, f(0) = 0, f = 0$ is a separatrix of $\mathcal{F}(\omega)$, if $\omega \wedge df = f. \eta$ for some $\eta \in \Omega^2((\mathbb{C}^2,0))$

In the global (algebraic) context we say that $\{f = 0\}$ is $\mathcal{F}(\omega)$—invariant.

Theorem 12 (Camacho-Sad) The germ of any holomorphic foliation $F(\omega)$ in $(\mathbb{C}^2, 0)$ has a separatrix.

This will be proved after introducing the note of ”blow up” of singularities. We have already proved it for the following particular cases.

1. If $\mathcal{F}(\omega)$ is in the Poincaré domain or it’s in the Siegel domain and it is of type $(c,v)$. In this case it has more then two separatrices. Note that $\{x = 0\}$ and $\{y = 0\}$ are separatrices of $\mathcal{F}(X_1)$ and so $\{h_1 = 0\}$ and $\{h_2 = 0\}$ are two separatrices of $\mathcal{F}(X)$.

2. In Dulac’s theorem $\mathcal{F}(X)$ has at lest one separatrix. In Dulac’s theorem we have conjugated $X$ with $(ax + cy^n)\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$, $a = nb, n \geq 2$ and $\{y = 0\}$ is a separatrix.

6.5 Singularities with resonance

Recall the definition of a germ of holomorphic foliation $\mathcal{F}(X)$ with resonance in Definition 12. In the resonance case note that if we write $\frac{a}{b} = -\frac{n}{m} (m,n) = 1, n,m \in \mathbb{N}$ then we have
In this case, the coefficients \( \xi_{1,(m+1,n)}, \xi_{2,(m+1,n)} \) cannot be determined in the recursion given in the proof of Theorem 9.

**Theorem 13** Let \( \mathcal{F}(X), X := ax + \frac{\partial}{\partial x} + by + \frac{\partial}{\partial y} + ... \) be a germ of holomorphic foliation in \((\mathbb{C}^2, 0)\) and assume that \( \frac{a}{b} \in \mathbb{Q}^- \) (the resonance case). Then there is a biholomorphism \( h := (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \) such that the pull-back of \( X \) by \( h \) is of the format

\[
\bar{X} := (ax + xyA(x,y)) \frac{\partial}{\partial x} + (by + xyB(x,y)) \frac{\partial}{\partial y}
\]

where \( A, B \in \mathcal{O}(\mathbb{C}^2, 0) \) with \( A(0) = B(0) = 0 \).

In the above theorem the foliation \( \mathcal{F}(X) \) has at least two separatrices because \( \mathcal{F}(\bar{X}) \) has two Separatrics \( \{x = 0\} \) and \( \{y = 0\} \).

**Proof.** Proceeding as in theorem 6.1 we write \( X \) as

\[
\dot{u}_j = \lambda_j u_j + \phi_j(u_1, u_2) \tag{6.11}
\]

where

\[
u_j = x_j + \xi_j(x_1, x_2) \tag{6.12}
\]

We need to find \( \xi_j(x_1, x_2) \) such that \( h^*X \) is of the form \( x_j = \lambda_j x_j + \psi_j(x_1, x_2) \) where \( \psi_j(x_1, x_2) \in (x_1 \cdot x_2) \). Here, \( (x_1 \cdot x_2) \) denotes the ideal of analytic functions generated by \( x_1 x_2 \). Making the same substitutions we get that

\[
\sum_{|n| \geq 2} (n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j) \xi_{j,n} x^n + \sum_{|n| \geq 2} \psi_{j,n} x^n = \phi_j(x_1 + \xi_1, x_2 + \xi_2) - \frac{\partial \xi_j}{\partial x_1} \psi_1 - \frac{\partial \xi_j}{\partial x_2} \psi_2 \tag{6.13}
\]

We define

- if \( x^n \notin (x_1 \cdot x_2) \) take \( \psi_{j,n} = 0 \)
- if \( x^n \in (x_1 \cdot x_2) \) take \( \xi_{j,n} = 0 \)

If \( x^n \notin (x_1 \cdot x_2) \) then the coefficient \( n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j \neq 0 \). It follows that we can calculate \( \xi_1, \xi_2 \) formally. To see that they are convergent we claim that if \( x^n \notin (x_1 \cdot x_2) \) then \( \exists \delta > 0 \) such that \( |n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j| > \delta \). From this the calculation of \( \xi_{j,n} \) is done by

\[
\sum_{|n| \geq 2} (n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j) \xi_{j,n} x^n = \phi_j(x_1 + \xi_1, x_2 + \xi_2) \mod(x_1 \cdot x_2)
\]

and so \( \delta \xi_j < \phi_j(x_1 + \xi_1, x_2 + \xi_2) \), which imply

\[
\delta (\xi_1 + \xi_2) < \phi_1(x + \xi_1 + \xi_2) + \phi_2(x + \xi_1 + \xi_2)
\]

and so we proceed as in theorem 6.1.
Chapter 7
Projective spaces

In algebraic geometry many theorems are stated for compact/complete varieties. A typical example is the Bezout theorem on the number of intersections of two curves. Curves in $\mathbb{A}^2_k$ may not intersect each other at all, even if we assume that $k$ is an algebraically closed field. In this case there are many intersection points at infinity, and we are going to explain what means infinity in this case. Holomorphic foliations are also best viewed in a compactification of $\mathbb{A}^2_k$.

### 7.1 Projective spaces as complex manifolds

The projective space of dimension $n$ as a complex manifold is defined as follows:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\sim$$

where $a, b \in \mathbb{C}^{n+1} - \{0\}$, $a \sim b \iff a = kb$, for some $k \in \mathbb{C} - \{0\}$.

For the purpose of the present text, we will mainly use $\mathbb{P}^1$ and $\mathbb{P}^2$. The projective space of dimension one $\mathbb{P}^1$ is covered by two charts $x, x'$ biholomorphic to $\mathbb{C}$ and the transition map is given by

$$x' = \frac{1}{x}.$$  

The projective space of dimension two $\mathbb{P}^2$ is covered by three charts $(x, y), (u, v), (u', v')$ biholomorphic to $\mathbb{C}^2$ and the transition maps are given by

$$v = \frac{y}{x}, u = \frac{1}{x}, \quad v' = \frac{x}{y}, u' = \frac{1}{y}.$$  

Considering the chart $(\mathbb{C}^2, (x, y))$, $\mathbb{P}^2$ becomes a compactification of $\mathbb{C}^2$.  

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7.2 Projective spaces as schemes

In this section we define the projective space of dimension two $\mathbb{P}^2_k$ over an arbitrary field. We also explain the main idea behind the definition $\mathbb{P}^2_k$ as a scheme.

By the affine scheme $\mathbb{A}^2_k$, we simply think of the polynomial ring $k[x, y]$. Open subsets of $\mathbb{A}^2_k$ are given by the localization of $k[x, y]$. We will need two open subsets of $\mathbb{A}^2_k$ given respectively by

$$k[x, y, \frac{1}{y}] \text{ and } k[x, y, \frac{1}{x}]$$

By the projective scheme $\mathbb{P}^2_k$ we mean three copies of $\mathbb{A}^2_k$, namely

$$k[x, y], \ k[x, z], \ k[y, z]$$

together with the isomorphism of affine subsets:

$$k[x, y, \frac{1}{y}] \cong k[x, z, \frac{1}{z}], \ x \mapsto \frac{x}{z}, \ y \mapsto \frac{1}{z} \tag{7.1}$$

$$k[x, y, \frac{1}{x}] \cong k[y, z, \frac{1}{z}], \ x \mapsto \frac{1}{z}, \ y \mapsto \frac{y}{z}$$

$$k[x, z, \frac{1}{x}] \cong k[y, z, \frac{1}{y}], \ x \mapsto \frac{1}{y}, \ z \mapsto \frac{z}{y}$$

The best way to see these isomorphisms is the following. We look at an element of $k[x, y]$ as a function on the $k$-rational points $k^2$ of the first chart and for $(a, b) \in k^2$, we use the identities

$$[a; b; 1] = \left[ \frac{a}{b}; 1; \frac{1}{b} \right] = \left[ 1; \frac{b}{a}; \frac{1}{a} \right].$$

Let $C$ be a curve in $\mathbb{A}^2_k$ given by the polynomial $f(x, y) \in k[x, y]$. It induces a curve $\overline{C}$ in $\mathbb{P}^2_k$ in the following way. Let us define $f_1 := f$ and

$$f\left(\frac{x}{z}, \frac{1}{z}\right) = z^{-d}f_2(x, z), \ f\left(\frac{1}{z}, \frac{y}{z}\right) = z^{-d}f_3(y, z)$$

We think of the the curve $\overline{C}$ in the same way as $\mathbb{P}^2_k$, but replacing $k[x, y]$ with $k[x, y]/\langle f_1 \rangle$ and so on. Here, $\langle f_1 \rangle$ is the ideal $k[x, y]$ generated by a single element $f_1$. We can also think of $C$ in the same way as $\mathbb{P}^2_k$ but with the following additional relations between variables:

$$f_1(x, y) = 0 \text{ in } k[x, y]$$

$$f_2(x, z) = 0 \text{ in } k[x, z]$$

and

$$f_3(y, z) = 0 \text{ in } k[y, z].$$
The above discussion does not use the fact that $k$ is a field. In fact, we can use an arbitrary ring $R$ instead of $k$. In this way, we say that we have an scheme $C$ over the ring $R$. The function field of the projective space $\mathbb{P}^2_k$ is defined to be

$$k(\mathbb{P}^2_k) := k(x, y) \cong k(x, z) \cong k(y, z).$$

where the isomorphisms are given by (7.1). The field of rational functions on the curve $C$ is the field of fractions of the ring $k[x,y]/(f_1)$. Using the isomorphism (7.1), this definition does not depend on the chart with $(x, y)$ coordinates. We can also think of $k(C)$ as $k(x, y)$ but with the relation $f_1(x, y) = 0$ between the variables $x, y$. Any $f \in k(C)$ induces a map

$$C(k) \to k$$

that we denote by the same letter $f$.

### 7.3 Foliations in projective spaces

A foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$ extends to a holomorphic foliation in $\mathbb{P}^2_k$. For instance, in the chart $(u, v)$ we have

$$\omega = P\left(\frac{1}{u}, \frac{v}{u}\right)du - Q\left(\frac{1}{u}, \frac{v}{u}\right)dv = \frac{\tilde{P}(u,v)dv - \tilde{Q}(u,v)du}{u^{d+2}},$$

(7.2)

$\tilde{P}, \tilde{Q} \in k[u,v]$.

**Definition 15** The smallest number $d$ in the equality (7.2) is called the (projective) degree of the foliation $\mathcal{F}(\omega)$.

It is also natural to define the (affine) degree of $\mathcal{F}(\omega)$:

$$\deg(\mathcal{F}) := \max\{\deg(P), \deg(Q)\}.$$

These two notions if degree are different. Working with foliations in $\mathbb{P}^2_k$ it is useful to use the projective degree.

**Proposition 7** A line in $\mathbb{P}^2$ which does not cross any singularity of $\mathcal{F}$ has a fixed number $d$ (counted with multiplicity) of tangency points with the foliation $\mathcal{F}$. In particular, for a generic line we have exactly $d$ simple tangency points. This number $d$ is the projective degree of $\mathcal{F}$.

**Proposition 8** A foliation of the projective degree $d$ in the affine coordinate $\mathbb{A}^2_k \subset \mathbb{P}^2_k$ is given by the differential form:

$$Pdx + Qdy + g(xdy - ydx)$$
where either \( g \) is a non-zero homogeneous polynomial of degree \( d \) and \( \deg(P) \), \( \deg(Q) \leq d \) or \( g \) is zero and \( \max\{\deg(P), \deg(Q)\} = d \). In the first case the line at infinity is not invariant by \( \mathcal{F} \) and in the second case it is invariant by \( \mathcal{F} \).

We may redefine \( \mathcal{F}(d) \) to be the set of holomorphic foliations of projective degree \( d \) in \( \mathbb{P}^2 \). The spaces \( \mathcal{F}(d) \) corresponding to two different definitions of the degree have different aspects. For instance, a generic foliation of projective degree \( d \) does not have an algebraic solution and a generic foliation of (affine) degree leaves the line at infinity invariant.

### 7.4 Riccati foliations

Another compactification of \( \mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k \) is \( \mathbb{P}^1 \times \mathbb{P}^1 \) which is useful for studying the Riccati foliations is given by:

\[
\omega = q(x) dy - (p_0(x) + p_1(x) y + p_2(x) y^2) dx, \quad p_0, p_1, p_2, q \in k[x].
\]

Substituting \( y = \frac{1}{y} \) we have

\[
\omega = \frac{1}{y^2} (-q(x) dy - (p_0(x) y^2 + p_1(x) y + p_2(x)) dx)
\]

and so all the projective lines \( \{a \in \mathbb{C} \mid q(a) \neq 0\} \times \mathbb{P}^1 \) are transversal to the foliation. This will be later used to define the global holonomy of Ricatti foliations.

### 7.5 Minimal set

For a holomorphic foliation in \( \mathbb{P}^2_k \) one may formulate many problems related to the accumulation of its leaves. The most simples one which is still open is the following:

**Problem 7.1.** Is there a foliation \( \mathcal{F} \) in \( \mathbb{P}^2 \) with a leaf \( L \) which does not accumulate in the singularities of \( \mathcal{F} \).

For instance the above problem for Jouanolou foliation is proved numerically for \( d \leq 4 \) and it is still open for general \( d \).

Let us suppose that such an \( \mathcal{F} \) and \( L \) exist and set \( M := \overline{L} \), where the closure is taken in \( \mathbb{P}^2 \). It follows that \( M \) is a union of leaves of \( \mathcal{F} \). We may suppose that \( M \) does not contain a proper \( \mathcal{F} \)-invariant subset. In this case we call \( M \) a minimal set.

**Proposition 9** A foliation in \( \mathbb{P}^2 \) with algebraic leaf has not a minimal set.

For many other useful statement on minimal sets see [CLNS88]. For local theory of holomorphic foliations see [CS87].
Chapter 8
Camacho-Sad theorem

In this chapter we explain one of the main index theorems in holomorphic foliations, namely the Camacho-Sad index theorem. This together with Baum-Bott index theorem and a local analysis of holomorphic foliations around singularities, are our main tools in order to study the non-existence of invariant algebraic curves for holomorphic foliations.

8.1 Camacho-Sad index

Let $\mathcal{F}(\omega), \omega := Pdy - Qdx, P, Q \in \mathcal{O}(\mathbb{C}^2, 0)$ be a germ of holomorphic foliation in $(\mathbb{C}^2, 0)$ and assume that $0 \in \mathbb{C}^2$ is an isolated singularity of $\mathcal{F}$, that is, $P(0) = Q(0) = 0$ and $P$ and $Q$ do not have common factors. Let also $f \in \mathcal{O}(\mathbb{C}^2, 0)$ and $\{f = 0\}$ is a separatrix of $\mathcal{F}$, that is,

$$df \wedge \omega = f.\eta \text{ where } \eta \in \Omega^2(\mathbb{C}^2, 0).$$

Proposition 10 There exist holomorphic functions $g, h \in \mathcal{O}(\mathbb{C}^2, 0)$ and $\eta \in \Omega^1(\mathbb{C}^2, 0)$ such that $h$ is not divisable by $f$ and

$$g\omega = h \cdot df + f.\eta.$$

Proof. Since $f = 0$ is a separatrix, we have $df \wedge \omega = f.\eta$ and so $f_x.P + f_y.Q = fS$ for some $S \in \mathcal{O}(\mathbb{C}^2, 0)$. Then

$$f_x.\omega = f_x(Pdy - Qdx) = (f_x.P)dy - (f.\omega)dx - f_y.Pdx = Pdx - f(Sdx)$$

The same statement is true replace $\mathcal{O}(\mathbb{C}^2, 0)$ with $k[x, y]$, and ”separatrix” with ”invariant algebraic curve”.
Theorem 14 (Puiseux parametrization) Let \( C = \{ f(x, y) = 0 \}, f \in \mathcal{O}(\mathbb{C}^2, 0) \) be a germ of a curve in \((\mathbb{C}^2, 0)\). There is a holomorphic map \( \gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0) \) such that \( f(\gamma(t)) = 0 \) and \( \gamma \) is a bijection between \((\mathbb{C}, 0)\) and \( \{ f(x, y) = 0 \}\).

We will prove this theorem later when we introduce the notion of a blow-up. For now, we only mention that the above theorem is trivial for smooth curves. If \( \{ f = 0 \} \) is smooth at 0, that is \( (\frac{\partial f}{\partial x}(0), \frac{\partial f}{\partial y}(0)) \neq (0, 0) \) then one can find \( \gamma \) using implicit function theorem. Another example is the singular curve given by \( f = y^2 - x^3 \). It has the parametrization given by \( \gamma(t) = (t^2, t^3) \). From now on let \( \gamma \) be a path in \( C = \{ f = 0 \} \) which is the image of a path in \((\mathbb{C}, 0)\) turning around 0 anti-clockwise and under the map \( \gamma \). Recall the definition of \( \frac{d\omega}{\omega} \) from \([5,4]\).

Definition 16 The Camacho-Sad index of \((\mathcal{F}, C, 0)\) is

\[
I(F,C,0) := \frac{-1}{2\pi i} \int_{\gamma} \frac{\bar{\eta}}{h}.
\]

Note that \( \lambda = e^{2\pi i I(F,C)} \) is the multiplier of the holonomy \( h \) of \( \mathcal{F}(\omega) \) along the path \( \gamma \).

We can reinterpret Proposition \([10]\) in the following way. There is a meromorphic 1-form \( \Omega \) in \((\mathbb{C}^2, 0)\) which induces the foliation \( \mathcal{F} \) and

\[
\bar{\eta} := \Omega - \frac{df}{f}
\]

has no poles along \( f = 0 \). Actually, this 1-form \( \bar{\eta} \) is unique restricted to \( f = 0 \). With the notation of Proposition \([10]\) we write \( g\omega = h \cdot df + f \cdot \eta \) and we have \( \Omega = \frac{\bar{\eta}}{h}, \bar{\eta} = \frac{n}{h} \). Note also that if we define \( \bar{\omega} = f \Omega = df + f \bar{\eta} \) then

\[
\frac{d\bar{\omega}}{\bar{\omega}} = \bar{\eta}, \text{ restricted to } f = 0.
\]
This follows from
\[
\frac{d\omega}{\omega} = \frac{d(df + f\eta)}{\omega} = \frac{d(f\eta)}{\omega} = (df \wedge \eta) + f \cdot d(\eta) = (df \wedge \eta) + f \cdot d\eta =\]
\[
(d f \wedge \eta) + f \cdot d\eta =\]
\[
\frac{d(f \wedge \eta)}{\omega} = \frac{(df + f\eta) \wedge \eta}{\omega} = \frac{(df + f\eta) \wedge \eta}{\omega} = \eta.
\]

For (1) we restrict to \( f = 0 \). Note that it makes sense to say that the restriction of \( f \frac{d\eta}{\omega} \) to \( \{ f = 0 \} \) is zero because \( \eta \) has no poles along \( f = 0 \).

If we take \( \omega = Pdy - Qdx \) then we have
\[
\frac{d\omega}{\omega} = -\frac{\partial P}{\partial x} dx - \frac{\partial Q}{\partial y} dy = -\frac{\partial P}{\partial x} dx + \frac{\partial Q}{\partial y} dy
\]
\[(8.1)\]

The second equality is valid when it is restricted to the leaves of \( \mathcal{F}(\omega) \). Note that the residue of \( \frac{d\omega}{\omega} \) in a separatrix differs from the Camacho-Sad index by an integer. However, if we take the differential 1-form
\[
\hat{\omega} = \frac{1}{P} \omega = dy - \frac{Q}{P} dx
\]
\[(8.2)\]

then we have
\[
\frac{d\hat{\omega}}{\hat{\omega}} = -\frac{\partial (\frac{Q}{P})}{\partial y} dx
\]
\[(8.3)\]

and

**Proposition 11** If the curve \( f = 0 \) is smooth and it is not tangent to the y axis at 0 then the Camacho-Sad index can be computed using \( \frac{d\hat{\omega}}{\hat{\omega}} \), that is,
\[
I(F,C,0) := \frac{-1}{2\pi i} \int_\gamma \frac{d\hat{\omega}}{\hat{\omega}}.
\]

**Proof.** From the hypothesis it follows that \( f_y \) has not zeros in \( (\mathbb{C}^2, 0) \). From another side we have \( \hat{\omega} = f_y \hat{\omega} \) which follows from the explicit construction of \( \eta \) in Proposition 10. Therefore,
\[
\frac{d\hat{\omega}}{\hat{\omega}} = -\frac{d f_y}{f_y} + \frac{d\hat{\omega}}{\hat{\omega}}
\]
and the proof follows.

**Exercise 22** Let \( \omega = Pdy - yQdx \) and so \( y = 0 \) is a separatrix of \( \mathcal{F}(\omega) \) calculate \( I(F,0) \).

Sometimes we write \( I(F,C) = I(F,C,0) \), being clear in the context which singularity we are dealing with.
8.2 Residue formula

The notion of a residue is purely algebraic and we can avoid integrals in its definition, see for instance [Tat68] and Serre’s book in this article. Therefore, the Camacho-Sad index can be defined for foliations in $\mathbb{P}^2_k$ for arbitrary field $k$.

The residue formula for smooth curves.

**Theorem 15** Let $C \subset \mathbb{P}^2_k$ be a smooth curve and let $\omega$ be a meromorphic differential 1-form in $C$. We have

$$\sum_{p \in C} \text{residue}_p(\omega) = 0.$$  

**Proof.** We prove this for $k = \mathbb{C}$. The curve $C$ over $\mathbb{C}$ is naturally a Riemann surface. By definition $\text{residue}_p(\omega) = \frac{1}{2\pi i} \oint_p \omega$. Since $d\omega = 0$, by the Stokes theorem

$$\sum_{p_i \in p} \oint_{p_i} \omega = \int \int_{X \setminus \bigcup_{i=1}^n D_i} d\omega = 0$$

8.3 Camacho-Sad theorem

**Theorem 16** Let $\mathcal{F}$ be a holomorphic foliation in $\mathbb{P}^2_k$ and let $C$ be a smooth algebraic $\mathcal{F}$-invariant curve of degree $d$ in $\mathbb{P}^2_k$, then

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap C} I(\mathcal{F}, C, p) = d^2.$$  

**Proof.** First of all note that can we choose a line $\mathbb{P}^1_k \subset \mathbb{P}^2_k$ and we can write the foliation $\mathcal{F}(\omega)$, $\omega = P(x, y)dy - Q(x, y)dx$, in the coordinates $(x, y)$ of the affine chart $\mathbb{A}_k^2 = \mathbb{P}^2_k \setminus \mathbb{P}^1_k$ such that

1. The smooth algebraic curve $C \subset \mathbb{P}^2$ intersects $\mathbb{P}^1_k$ transversely in $d$ points.
2. In the affine chart $\mathbb{A}^2_k$, the vertical lines $x = c$ are either transversal or have tangencies of order two with the curve $C$. In addition, all the tangent points are regular points of the foliation $\mathcal{F}$. By the Bezout theorem the number of such tangency points is $d(d - 1)$.

Now consider the differential form (8.2) which induces the foliation $\mathcal{F}$ and let $\eta$ be the differential 1-form in (8.3) multiplied with $-1$. The poles of the 1-form $\eta$ restricted to $C$ are divided in three groups: 1. Singularities of $\mathcal{F}$ in $C$ 2. The tangency points of the curve with vertical lines 3. The intersections of $C$ with the line at infinity. We compute the residue of $\eta$ around all these points and use the residue formula in Theorem 15 and we get the proof.

For a singular point $p \in C$ of $\mathcal{F}$, by definition we have $\text{Residue}(\eta, p) = I(\mathcal{F}, C, p)$. Therefore, we do not need to compute it. For tangency points, we can locally parameterized a leaf tangent to a vertical line by $L : x = g(y) = t y^2 + \cdots$. For simplicity we assume that such a tangency point is at $(0,0)$. Since $\frac{P(x,y)}{Q(x,y)} = \frac{dx}{dy}$, we have $\frac{P(g(y),y)}{Q(g(y),y)} = g'(y)$. Therefore,

$$\frac{\partial (\frac{P(g(y),y)}{Q(g(y),y)})}{\partial y} = - \frac{g''}{(g')^2}.$$  

From this we get

$$\eta|_{L} = \frac{\partial (\frac{P(g(y),y)}{Q(g(y),y)})}{\partial y} g'(y) dy = - \frac{g''}{g'} dy$$

This has residue $-1$ at the tangency point $p$ and so in total we get $-d(d - 1)$. 

Fig. 8.3 Tangency
Now let us calculate the residue of $\eta$ for a point $p \in C$ in the third group. The differential form (8.2) has a pole order $-1$ at infinity. Using the formula (5.1), we conclude that the residue of $\frac{d\omega}{\omega} = -\eta$ at $p$ is $+1$ and so in total we get $-d$ for residues of $\eta$ for the third group. Finally, by residue formula we have
\[
\sum_p I(\mathcal{F}, C, p) - d(d - 1) - d = 0.
\]

**Exercise 23** Discuss the Camacho-Sad index and theorem for arbitrary field $k$ instead of $\mathbb{C}$.

**Exercise 24** If we use $\frac{da}{\omega} = \frac{-\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ in the definition of Camacho-Sad index what would be the corresponding Camacho-Sad theorem. Repeat the same proof as in Theorem 16.
Chapter 9
Baum-Bott index

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Theorem 9.1. (Baum-Bott in \( \mathbb{C}P(2) \)). Let \( \text{Im} \) be a foliation of degree \( k \) in \( \mathbb{C}P(2) \) with isolated singularities. Then

\[
\sum_{p \in \text{Sing}(\text{Im})} \text{BB}(\text{Im}, p) = (k + 2)^2
\]

(9.1)

Proof. We can consider that \( \text{Sing}(\text{Im}) \subset P_0 \). Let \( X_0 = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} \) be a polynomial vector field that represents \( \text{Im} \) in \( E_0 \) and \( \omega_0 = Pdy - Qdx \) its dual. Given \( \phi_{10} = (\frac{1}{u}, \frac{y}{u}) = (x, y) \) the change of coordinates between \( E_0 \) and \( \omega_0 \).

Then \( \phi_{10}^* (\omega_0) = u^{-1} \omega_1 \), where \( \omega_1 \) represents \( \text{Im} \) in \( E_1 \). Then its easily checked that also \( \phi_{20}^* (\omega_0) = \frac{y}{x} \omega_2 \).

Then we can say that in the intersection \( E_i \cap E_j \) we have \( \omega_i = f_{ij} \omega_j \) where \( f_{ij} = \frac{1}{f_{ji}} \) with

\[
\begin{align*}
&f_{01}|_{E_0} = x^{k + 2}, \\&f_{02}|_{E_0} = y^{k + 2}, \\&f_{01}|_{E_0} = \frac{y^{k + 2}}{x^{k + 2}}. 
\end{align*}
\]

By these we see that \( d f_{ij} = 0 \) \( \forall i, j, k \in \{1, 2, 3\} \).

Now let us consider \( \eta_j \) such that \( d \omega_j = \eta_j \wedge \omega_j \). Then

\[
\eta_i \wedge \omega_i = d \omega_i = d (f_{ij} \omega_j) = d f_{ij} \omega_j + f_{ij} d \omega_j = \left( d f_{ij} + \eta_j \right) \wedge \omega_i \]

(9.2)

and so exists a \( C^\infty \) function \( g_{ij} \) in \( E_i \cap E_j \setminus \text{Sing}(\text{Im}) \) such that \( \eta_i = \eta_j + d f_{ij} \wedge g_{ij} \omega_i \).

Now let us consider \( \theta_i \) such that \( d \theta_i = \theta_i \wedge \omega_i \). Then \( \theta_i = \eta_i \wedge d \eta_i \). We get

\[
\theta_i = \theta_j + d f_{ij} \wedge \eta_j + (\eta_i + d g_{ij}) \wedge \theta_i \]

(9.3)

Using these and the fact that \( \text{Sing}(\text{Im}) \subset P_0 \) we have that

\[
4\pi^2 \sum_{p \in \text{Sing}(\text{Im})} \text{BB}(\text{Im}, p) = \int_{\partial P_0} \Theta_0 + \int_{\partial P_1} \Theta_1 + \int_{\partial P_3} \Theta_3
\]

(9.4)

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But recalling how we named those sets we can write

\[ 4\pi^2 \sum_{p \in \text{Sing} (\text{Im})} BB(\text{Im}, p) = \int_{p_{01}} \Theta_0 + \int_{p_{02}} \Theta_0 + \int_{p_{00}} \Theta_1 + \int_{p_{22}} \Theta_1 + \int_{p_{20}} \Theta_2 + \int_{p_{21}} \Theta_2 \]

\[ = \int_{p_{01}} (\Theta_0 - \Theta_1) + \int_{p_{12}} (\Theta_1 - \Theta_2) + \int_{p_{20}} (\Theta_2 - \Theta_0) \]

and so by Stoke’s theorem we get

\[ 4\pi^2 \sum_{p \in \text{Sing} (\text{Im})} BB(\text{Im}, p) = \int_T \alpha \]  \hspace{1cm} (9.5)

where

\[ \alpha = \eta_1 \wedge \frac{df_{01}}{f_{01}} - \eta_0 \wedge \alpha_{01} + \eta_2 \wedge \frac{df_{12}}{f_{12}} - \eta_1 \wedge \alpha_{12} + \eta_0 \wedge \frac{df_{20}}{f_{20}} - \eta_2 \wedge \alpha_{20} \]  \hspace{1cm} (9.6)

Now let us take, as stated before, that \( \eta_i = \eta_0 + \frac{df_{0i}}{f_{0i}} + \alpha_0 \) and by replacing we obtain

\[ \alpha = (\eta_0 + \frac{df_{01}}{f_{01}} + \alpha_0) \wedge \frac{df_{01}}{f_{01}} - \eta_0 \wedge \alpha_{01} + (\eta_0 + \frac{df_{20}}{f_{20}} + \alpha_2) \wedge \frac{df_{12}}{f_{12}} - \eta_1 \wedge \alpha_{12} - \eta_0 \wedge \alpha_0 \]

\[ = \frac{df_{20}}{f_{20}} \wedge \frac{df_{12}}{f_{12}} + \alpha_0 \wedge \frac{df_{01}}{f_{01}} + \alpha_2 \wedge \frac{df_{12}}{f_{12}} - \frac{df_{01}}{f_{01}} \wedge \alpha_{01} - \frac{df_{20}}{f_{20}} \wedge \alpha_{20} \]

Now we just have to note that

\[ \alpha_{01} \wedge \frac{df_{01}}{f_{01}} + \alpha_{20} \wedge \frac{df_{12}}{f_{12}} - \frac{df_{01}}{f_{01}} \wedge \alpha_{12} - \frac{df_{20}}{f_{20}} \wedge \alpha_{20} \]

\[ = \alpha_{20} \wedge \left( \frac{df_{12}}{f_{12}} + \frac{df_{20}}{f_{20}} \right) + \alpha_{01} \wedge \frac{df_{01}}{f_{01}} - \frac{df_{10}}{f_{10}} \wedge \alpha_{12} \]

\[ = \alpha_{20} \wedge \left( - \frac{df_{01}}{f_{01}} \right) + \alpha_{01} \wedge \frac{df_{01}}{f_{01}} - \frac{df_{10}}{f_{10}} \wedge \alpha_{12} \]

\[ = \frac{df_{01}}{f_{01}} \wedge (\alpha_{01} + \alpha_{12} + \alpha_{20}) = 0 \]

Hence \( \alpha = \frac{df_{20}}{f_{20}} \wedge \frac{df_{12}}{f_{12}} = (k + 2)^2 \frac{dx}{x} \wedge \frac{dy}{y} \) and by parametrizing \((x, y) = (e^{i\theta}, e^{i\psi})\) with \(\theta, \psi \in (0, 2\pi)\) we obtain that

\[ 4\pi^2 \sum_{p \in \text{Sing} (\text{Im})} BB(\text{Im}, p) = \int_T \alpha = -(k + 2)^2 \int_T \frac{dx}{x} \wedge \frac{dy}{y} = 4\pi^2 (k + 2)^2 \]
which is what we wanted to proof.
Chapter 10
Jouanolou foliation

The holomorphic foliation $\mathcal{F}_d$ defined in $\mathbb{C}^2$ by the 1-form
\[
\omega := (y^d - x^{d+1})dy - (1-x^d)dx
\]
is called the Jouanolou foliation of degree $d$. Consider the group
\[
G := \{ \epsilon \in \mathbb{C} \mid \epsilon^{d^2 + d + 1} = 1 \}.
\]
It acts on $\mathbb{C}^2$ discontinuously in the following way:
\[
(\epsilon, (x, y)) \rightarrow (\epsilon^{d+1}x, \epsilon y) \quad \epsilon \in G, \ (x, y) \in \mathbb{C}^2
\]
It has a fixed point $p_1 = (0, 0)$ at $\mathbb{C}^2$ (and two other fixed points $p_2 = [0 : 1 : 0], p_3 = [1 : 0 : 0]$ at infinity). For each $\epsilon \in G$ we have $\epsilon^* (\omega) = \epsilon^{d+1} \omega$ and so $G$ leaves $\mathcal{F}_d$ invariant. We have
\[
\text{Sing}(\mathcal{F}_d) \subset \{ (\epsilon, \epsilon^{-d}) \mid \epsilon \in G \}
\]
(there is no singularity at infinity) and $G$ acts on $\text{Sing}(\mathcal{F}_d)$ transitively.

For pictures of Jouanolou foliation see [MV09].
References


LNS. Alcides Lins Neto and Bruno Scárdua. *Introdução à Teoria das Folhações Algébricas Complexas*. Available online at IMPA’s website.
