Michigan Math. J. **71 (2022), 705–724**
https://doi.org/10.1307/mmj/20205851

Convolution of Picard–Fuchs Equations

Hossein Movasati & Stefan Reiter

Abstract. We determine explicit generators for a cohomology group constructed from a solution of a Fuchsian linear differential equation and describe its relation with cohomology groups with coefficients in a local system. In the parametrized case, this yields into an algorithm which computes new Fuchsian differential equations from those depending on multi-parameters. This generalizes the classical convolution of solutions of Fuchsian differential equations.

1. Introduction

Explicit expressions for Picard–Fuchs equations (or Gauss–Manin connections in a general context) attached to families of algebraic varieties are usually huge even if the corresponding family is simple, for examples see the first author's book [Mov]. However, there are some families of algebraic varieties for which such expressions are small enough to fit into a mathematical paper, but one is not able to calculate them through the Dwork–Griffiths method (see for instance [Gri69]) or its modification in the context of Brieskorn modules, see [Mov], see also [Lai16] for another variant of this (we call this algebraic method). For such families, we first compute a period and then the corresponding Picard–Fuchs equation, see for instance [A+10] (we call this transcendental method). The main reason why computing Picard–Fuchs equations fails through the algebraic method is that in this way we produce huge polynomials and the Groebner basis algorithm fails to work. The transcendental method is restricted to very particular families of algebraic varieties.

In this article, we propose a new method which uses the internal fibration structure of algebraic varieties in order to perform Picard–Fuchs equation computations. It involves only solving linear equations, and it is a generalization of the classical convolution of solutions of Fuchsian differential equations and Deligne's work on the cohomology with coefficients in a local system, see [Del70]. One of our main motivations for the present work is the increasing need for explicit expressions of Picard–Fuchs equations in topological string theory and in particular in the B-model of mirror symmetry, see for instance [C+91]. We are also inspired by a personal communication of the first author with Ch. Doran a few years ago, in which he expressed the importance of iterative construction of Picard–Fuchs equations in the case of Calabi–Yau manifolds. Meanwhile, in the paper [DM15] he and Malmendier realized this in the case of 14 families of Calabi–Yau threefolds classified in [DM06].

Received January 7, 2020. Revision received April 14, 2021.

Let us give:

$$
\begin{tikzcd}
 & X_1 \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1_y & & \mathbb{P}^1_x \\
 & X_2 \arrow[ul] \arrow[ur] &
\end{tikzcd}
\tag{1}
$$

and a global meromorphic section ω_i, $i = 1, 2$, of the n_ith cohomology bundle of $X_i \to \mathbb{P}^1_x \times \mathbb{P}^1_y$. Here, X_i, $i = 1, 2$, are two algebraic varieties over $\mathbb{C}$, $\mathbb{P}^1_*$, $* = x, y$, is the projective line with the coordinate system $*$, and all the arrows are morphisms of algebraic varieties. The convolution of the above data in the framework of algebraic geometry is simply the fiber product

$$X \to \mathbb{P}^1_y, \qquad X := \cup_{y\in\mathbb{P}^1_y} X_{1,y} \times_{\mathbb{P}^1_x} X_{2,y}, \qquad \omega := dx \wedge \omega_1 \wedge \omega_2.$$

Here, $X_{i,y}$, $i = 1, 2$, is the fiber of $X_i \to \mathbb{P}^1_y$ over the point $y \in \mathbb{P}^1_y$ and ω gives us a global meromorphic section of the $(n_1 + n_2 + 1)$-the cohomology bundle of $X \to \mathbb{P}^1_y$. Let $\delta_{i,x,y}$, $i = 1, 2$, be a continuous family of cycles in the fibers of $X_i \to \mathbb{P}^1_x \times \mathbb{P}^1_y$. Knowing the linear differential system (Gauss–Manin connection) of $X_i \to \mathbb{P}^1_x \times \mathbb{P}^1_y$ and, in particular, the Picard–Fuchs equation

$$L_i := p_{0,i}\partial_x^{n_i} + \cdots + p_{n_i-1,i}\partial_x + p_{n_i,i}, \quad p_{j,i} \in \mathsf{k}[x, y], i = 1, 2, \tag{2}$$

of the periods $I_i(x, y) := \int_{\delta_{i,x,y}} \omega_i$, and under certain irreducibility condition (see 3.5), we give an algorithm for computing the Picard–Fuchs equation

$$L := q_0\partial_y^n + \cdots + q_{n-1}\partial_y + q_n, \quad q_j \in \mathsf{k}[y], \tag{3}$$

of

$$I(y) := \int_\delta I_1(x, y)I_2(x, y)\, dx, \tag{4}$$

where δ is any closed path in the x-plane such that I_1 and I_2 along δ are one-valued. This integral can be written as the integration of ω over a cycle $\tilde{\delta}_y \in H_{n_1+n_2+1}(X_y, \mathbb{Z})$, where X_y is the fiber of $X \to \mathbb{P}^1_y$ over y.

2. Cohomology with Coefficients in a Local System

In this section we remind some basic facts on local systems and connections with regular singularities. For further details, the reader is referred to [Del70]. We fix a field k of characteristic zero and not necessarily algebraically closed and work over the category of algebraic varieties over k. If k is a subfield of $\mathbb{C}$ or $\mathbb{C}(t)$, where t is a multi-parameter, then for an algebraic variety M over k we use the same letter M to denote the underlying complex variety or a family of varieties; being clear in the text which we mean.

2.1. Flat Connections

Let M be a smooth variety, E be a vector bundle over M. We consider a flat regular connection

$$\nabla : E \to \Omega^1(E).$$

We use the same notation E for both the vector bundle and the sheaf of its sections. We have the induced maps

$$\nabla_i : \Omega^i(E) \to \Omega^{i+1}(E), \quad \nabla_i(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla(e),$$

and the integrability is by definition $\nabla_1 \circ \nabla_0 = 0$. It implies that $\nabla_{i+1} \circ \nabla_i = 0$, and so we have the complex $(\Omega^i(E), \nabla_i)$. According to the comparison theorem of Grothendieck, see for instance Deligne's notes [Del70] Theorem 6.2, we have canonical isomorphisms

$$H^*(M, \mathcal{O}(E)) \to \mathbb{H}^*(M^{an}, \Omega^*(E)) \leftarrow \mathbb{H}^*(M, \Omega^*(E)) \tag{5}$$

of $\mathbb{C}$-vector spaces. Here, M^{an} is the underlying complex variety of M, $\mathcal{O}(E)$ is the sheaf of constant sections of E, and $H^*(M, \mathcal{O}(E))$ is the Cech cohomology with coefficients in $\mathcal{O}(E)$. The first $\mathbb{H}$ is the hypercohomology in the complex context, and the second one is the algebraic hypercohomology. Note that $\Omega^*(E)$ is an algebraic sheaf, and so its sections have poles of finite order along a compactification of M. If M is an affine variety, then $H^i(M, \Omega^*(E)) = 0$ for $i > 0$, and so

$$\mathbb{H}^i(M, \Omega^*(E)) \cong \frac{\ker(H^0(M, \Omega^i(E)) \to H^0(M, \Omega^{i+1}(E)))}{\mathrm{Im}(H^0(M, \Omega^{i-1}(E)) \to H^0(M, \Omega^i(E)))} \tag{6}$$

see [Del70] Corollary 6.3.

2.2. Logarithmic Differential Forms

Let us now consider a meromorphic connection ∇ on X with poles along a normal crossing divisor $S \subset X$, and hence, it induces a holomorphic connection on $M := X \backslash S$. We denote by $\Omega^1_X\langle S\rangle$ the sheaf of meromorphic differential forms in X with only logarithmic poles along S. The sheaf $\Omega^p_X\langle S\rangle$ is a p-times wedge product of $\Omega^1_X\langle S\rangle$. If ∇ has only logarithmic poles along S, see [Del70] page 78, then ∇ induces $\Omega^p_X\langle S\rangle(E) \to \Omega^{p+1}_X\langle S\rangle(E)$, and we have an isomorphism

$$\mathbb{H}^*(X, \Omega^*_X\langle S\rangle(E)) \cong \mathbb{H}^*(X, \Omega^*_X(E)) \tag{7}$$

induced by inclusion and then restriction to M provided that the residue matrix of ∇ along the irreducible components of S does not have eigenvalues in $\mathbb{N}$ see [Del70] Corollary 3.15. These conditions will appear later in Theorem 1 and Theorem 2. If X is an affine variety, then we conclude that in (6) every element is represented by a logarithmic differential, see also [Del70] Corollary 6.10.

The hypercohomology groups (5) and (7) are finite dimensional k-vector space, see [Del70] Proposition 6.10 and [Dim04] Proposition 2.5.4. However, explicit bases for these cohomology groups and algorithms that compute an element as a linear combination of the basis are not the main focus of [Del70; Dim04]. A regular connection may not have logarithmic poles along S, and one has to modify it in order to get such a property, see Manin's result in [Del70] Proposition 5.4. In our terminology this is the same as to write any regular differential equation in the Okubo format, see Section 4.1. All these together lead us to the fact that the theoretical approach in [Del70] is not applicable to our main problem posed in Introduction.

2.3. *Relation with Integrals*

Let us now consider meromorphic global sections $e_1, e_2, \dots, e_n$ of E such that for points x in some open Zariski subset of M, $e_i(x), i = 1, 2, \dots, n$, form a basis of E_x. Replacing M with this Zariski subset, we can assume that this property is valid for all $x \in M$. In this way E becomes a trivial bundle. Let $e = [e_1, e_2, \dots, e_n]$ and

$$\nabla(e^{\mathrm{tr}}) = A \cdot e^{\mathrm{tr}},$$

where A is an $n \times n$ matrix whose entries are regular differential forms in M (with poles along the complement of M in its compactification). We identify $(\Omega^i_M)^n$ with $\Omega^i(E)$ through the map $\omega \mapsto \omega \cdot e^{\mathrm{tr}}$, and we get

$$\nabla_i : (\Omega^i_M)^n \to (\Omega^{i+1}_M)^n, \qquad \nabla_i \omega = d\omega + (-1)^i \omega A,$$

and so

$$\mathbb{H}^i(M, \Omega^*(E)) \cong \frac{\ker(H^0(M, \Omega^i_M)^n \to H^0(M, \Omega^{i+1}_M)^n)}{\mathrm{Im}(H^0(M, \Omega^{i-1}_M)^n \to H^0(M, \Omega^i_M)^n)}.$$

Let $\check{E}$, $\check{\nabla} : \check{E} \to \Omega^1(\check{E})$, and $\check{e}_i$ be the dual bundle to E, the dual connection, and the dual basis, respectively. We have $\nabla \check{e}^{\mathrm{tr}} = -A^{\mathrm{tr}} \check{e}^{\mathrm{tr}}$. For a flat section I of $\check{E}$, we write $I = \check{e} \cdot f$, where $f = [f_1, f_2, \dots, f_n]^{\mathrm{tr}}$ and f_is are holomorphic functions in a small open set U in M. We have $0 = \nabla I = (\nabla \check{e}) f + \check{e}\, df = \check{e}(df - Af)$, and so we get a system

$$L : dY = AY$$

with the solution f. Let us define

$$H^0(M, \Omega^i_M)^n_f := \{\omega \in H^0(M, \Omega^i_M)^n \mid \omega \cdot f = 0\},$$

and let H^i_f be the ith cohomology group of the complex $(H^0(M, \Omega^i_M)^n_f, \nabla_i)$. We also define

$$H^i_{\mathrm{dR}}(M, L) := \frac{\{\omega = \sum_{k=1}^n f_k \omega_k \mid \omega_k \in H^0(M, \Omega^i_M), d\omega = 0\}}{\{d(\sum_{k=1}^n f_k \omega_k) \mid \omega_k \in H^0(M, \Omega^{i-1}_M)\}}. \tag{8}$$

This depends on f; however, for simplicity we have not used f in its notation. We have the exact sequence

$$H^i_f \to \mathbb{H}^i(M, \Omega^*(E)) \to H^i_{\mathrm{dR}}(M, L) \to H^{i+1}_f, \tag{9}$$

which is the part of the long exact sequence of the short exact sequence $0 \to H^0(M, \Omega^i_M)^n_f \xrightarrow{j} H^0(M, \Omega^i_M)^n \to \text{cokernel}(j) \to 0$.

From now on we use the cohomology group $H^i_{\mathrm{dR}}(M, L)$. The advantage of this is that we can integrate its elements. Let δ be a topological i-cycle in M such that the restriction of the analytic continuations of f_ks to δ is one-valued. For ω in the right-hand side (8), the integration $\int_\delta \omega$ is well defined. One of our motivations in the present text is to study this integral. Later we will see that for M the punctured line $H^1_{\mathrm{dR}}(M, L)$ is finite dimensional $\mathbb{C}$-vector space, and so the $\mathbb{C}$ vector space generated by

$$\int_\delta f_i\omega, i = 1, 2, \ldots, n, \quad \omega \in H^0(M, \Omega^i_M)$$

for a fixed δ is of finite dimension.

3. Cohomology of Linear Differential Equations

In this section we translate the machinery introduced in the previous section for the case of the punctured line, that is, M is $\mathbb{P}^1$ minus a finite number of points. We consider local systems given by Fuchsian differential equations, and we give an explicit set of generators for the corresponding cohomology groups. For simplicity, we work with $\mathsf{k} \subset \mathbb{C}$. The case $\mathsf{k} \subset \mathbb{C}(t)$ is reduced to the previous one by taking t as a collection of algebraically independent transcendental numbers in $\mathbb{C}$.

3.1. The Case of the Punctured Projective Line

Let

$$L : Y' = AY, \quad ' = \partial_x \tag{10}$$

be a Fuchsian differential system of dimension n with a solution $f = [f_1, f_2, \ldots, f_n]^{\mathrm{tr}}$. We assume that the entries of A are in $\mathsf{k}(x)$. Let also $S \subset \mathbb{P}^1$ be the set of singularities of L. We define

$$H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L) := \frac{\langle pf_i\, dx \mid p \in \mathsf{k}(x), \mathrm{pol}(p) \subset S, i = 1, 2, \ldots, n\rangle_{\mathsf{k}}}{\langle d(pf_i) \mid p \in \mathsf{k}(x), \mathrm{pol}(p) \subset S, i = 1, 2, \ldots, n\rangle_{\mathsf{k}}}, \tag{11}$$

which is the same as in (8) for $\mathsf{k} = \mathbb{C}$. For a Fuchsian differential operator

$$L := p_0\partial_x^n + \cdots + p_{n-1}\partial_x + p_n, \quad p_i \in \mathsf{k}[x] \tag{12}$$

with a solution f, that is, $Lf=0$, we can attach the linear differential system (10) with

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -\frac{p_n}{p_0} & -\frac{p_{n-1}}{p_0} & -\frac{p_{n-2}}{p_0} & \cdots & -\frac{p_1}{p_0}\end{pmatrix}, \tag{13}$$

and so

$$\begin{aligned} &H^1_{\mathrm{dR}}(\mathbb{P}^1-S,L)\\ &:=\frac{\langle pf^{(i)}\,dx \mid p\in \mathsf{k}(x), \mathrm{pol}(p)\subset S, i=0,1,\dots,n-1\rangle_{\mathsf{k}}}{\langle d(pf^{(i)}) \mid p\in \mathsf{k}(x), \mathrm{pol}(p)\subset S, i=0,1,\dots,n-1\rangle_{\mathsf{k}}}. \end{aligned} \tag{14}$$

3.2. *Indicial Equation*

In order to study the Fuchsian differential equation (12) $Ly=0$ near a point $t\in\mathbb{P}^1$, it is useful to compute its Riemann-scheme [Beu07, Section 3]. Let

$$a_{i,t}:=\lim_{x\to t}(x-t)^i\frac{p_i}{p_0}\in\mathsf{k},\qquad t\in\mathsf{k},$$
$$a_{i,\infty}:=\lim_{x\to\infty}(-x)^i\frac{p_i}{p_0}\in\mathsf{k}.$$

Since L is Fuchsian, we have

$$\frac{p_i}{p_0}=\frac{a_{i,t}}{(x-t)^i}+R_{i,t},\qquad \mathrm{ord}_{x=t}R_{i,t}\geq -i+1$$

for any finite $t\in\mathsf{k}$, and for any $l\geq n$,

$$x^l\frac{p_i}{p_0}=(-1)^ia_{i,\infty}x^{l-i}+P_{i,l,\infty}(x)+R_{i,l,\infty}(x),$$

where $P_{i,l,\infty}(x)$ is a polynomial in x of degree $<l-i$ and $R_{i,l,\infty}$ is a sum over all finite singularities $t_j\in\bar{\mathsf{k}}$ of L, of polynomials in $1/x-t_j$ of degree $\leq i$. Note that both $R_{i,l,\infty}$ and $P_{i,l,\infty}$ are defined over k. We conclude that the differential operator L can be also written in the format

$$\partial_x^{(n)}+\sum_{i=1}^n\frac{a_{i,t}}{(x-t)^i}\partial_x^{(n-i)}+\sum_{i=1}^n R_{i,t}\partial_x^{(n-i)}, \tag{15}$$

$$x^l\partial_x^{(n)}+\sum_{i=1}^n a_{i,\infty}x^{l-i}\partial_x^{(n-i)}+\sum_{i=1}^n(P_{i,l,\infty}+R_{i,l,\infty})\partial_x^{(n-i)}. \tag{16}$$

Furthermore the indicial equation I_t at t is given by

$$\begin{aligned} I_t&=X(X-1)\cdots(X-n+1)+a_{1,t}X(X-1)\cdots(X-n+2)\\ &\quad+\cdots+a_{n,t}, \end{aligned} \tag{17}$$

$$\begin{aligned} I_\infty&=X(X+1)\cdots(X+n-1)+a_{1,\infty}X(X+1)\cdots(X+n-2)\\ &\quad+\cdots+a_{n,\infty}. \end{aligned} \tag{18}$$

The Riemann scheme of L at a point $t \in \mathsf{k} \cup \{\infty\}$ is the set of the roots of I_t.

3.3. *Explicit Set of Generators,* $\mathsf{k} = \bar{\mathsf{k}}$

We are now in a position to describe an explicit set of generators for the cohomology group $H^1(\mathbb{P}^1 - S, L)$, where $S = \{t_1, t_2, \ldots, t_r, \infty\}$ is the set of singularities of L.

THEOREM 1. *Let* k *be an algebraically closed subfield of* $\mathbb{C}$. *If the Fuchsian differential operator* L *has no integer exponent* $\geq n$ *in the Riemann-scheme at a finite point and no positive integer exponent at* ∞, *then* $H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L)$ *is generated by*

$$(x - t_j)^{-1} f^{(i)}\, dx, \quad j = 1, 2, \ldots, r, i = 0, 1, \ldots, n-1, \tag{19}$$

and so it is of dimension at most $n \cdot r$.

Proof. All the qualities that follow are in the cohomology group $H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L)$ that is obviously generated by

$$f^{(i)} x^l\, dx, \frac{f^{(i)}}{(x - t_j)^l}\, dx, \quad l \in \mathbb{N}, i = 0, 1, \ldots, n-1, j = 1, 2, \ldots, r.$$

At first we show that $f^{(i)}/(x - t_j)^l\, dx, l \in \mathbb{N}, i = 0, 1, \ldots, n-1$, is in the k-vector space V generated by (19). Since

$$0 = d\left(\frac{f^{(i)}}{(x - t_j)^l}\right) = \frac{f^{(i+1)}}{(x - t_j)^l}\, dx + (-l)\frac{f^{(i)}}{(x - t_j)^{l+1}}\, dx,$$

we get

$$\begin{aligned} \frac{f^{(i)}}{(x - t_j)^l}\, dx &= \frac{1}{l-1} \frac{f^{(i+1)}}{(x - t_j)^{l-1}}\, dx = \cdots \\ &= \begin{cases} \frac{1}{(l-1)\cdots(l-n+i)} \frac{f^{(n)}}{(x-t_j)^{l-n+i}}\, dx, & i + l \geq n+1, \\ \frac{1}{(l-1)!} \frac{f^{(i+l-1)}}{x - t_j}\, dx, & i + l < n+1. \end{cases} \end{aligned} \tag{20}$$

Now, we use induction on $i + l$. For $i + l < n + 1$, the claim follows from the second case in (20). Thus, by the first case in (20), we can assume $i = n$. We have

$$\begin{aligned} \frac{f^{(n)}}{(x - t_j)^l}\, dx &\overset{(15)}{=} -\left(\sum_{i=1}^{n} a_{i,t_j} \frac{f^{(n-i)}}{(x - t_j)^{i+l}}\right) dx - \left(\sum_{i=1}^{n} \frac{R_{i,t_j}}{(x - t_j)^l} f^{(n-i)}\right) dx \\ &\overset{(20)}{=} -\left(\sum_{i=1}^{n} \frac{a_{i,t_j}}{l(l+1)\cdots(l+i-1)}\right) \frac{f^{(n)}\, dx}{(x - t_j)^l} \\ &\quad - \left(\sum_{i=1}^{n} \frac{R_{i,t_j}}{(x - t_j)^l} f^{(n-i)}\right) dx. \end{aligned}$$

In $R_{i,t_j}/(x - t_j)^l f^{(n-i)}\, dx$ there appear only terms $f^{(n-i)}/(x - t_j)^{l+k}\, dx$ with $k \leq i - 1$ and terms $f^{(n-i)}/(x - t_{j'})^i\, dx$ for $j' \neq j$. Using the first case in (20), the former terms are by induction in V and the latter terms by the second case in (20). By assumption we get

$$1+\left(\sum_{i=1}^{n}\frac{a_{i,t_j}}{l(l+1)\cdots(l+i-1)}\right)\overset{(17)}{=}\frac{I_{t_j}(l+n-1)}{l\cdots(l+n-1)}\neq 0,\quad \forall l\in\mathbb{N}.$$

Hence $f^{(n)}/(x-t_j)^l\,dx\in V$.

Similarly, we prove that $x^l f^{(i)}\,dx$ is in the vector space V. If $l-i<0$, then we have

$$x^l f^{(i)}\,dx=-(l-1)x^{l-1}f^{(i-1)}\,dx=\cdots=0.$$

For $l-i\geq 0$, we use induction on $l-i$, and we have

$$\begin{aligned} x^l f^{(i)}\,dx&=\frac{1}{-(l+1)}x^{l+1}f^{(i+1)}\,dx=\cdots\\ &=\frac{(-1)^{n-i}}{(l+1)\cdots(l+n-i)}x^{l+n-i}f^{(n)}\,dx, \end{aligned}\tag{21}$$

and so we can assume that $i=n$. Now, for $l\geq n$, we have by (21) and (16)

$$\begin{aligned} x^l f^{(n)}\,dx=&-\sum_{i=1}^{n}a_{i,\infty}x^{l-i}f^{(n-i)}\,dx-\sum_{i=1}^{n}P_{i,l,\infty}f^{(n-i)}\,dx\\ &-\sum_{i=1}^{n}R_{i,l,\infty}f^{(n-i)}\,dx. \end{aligned}$$

The second sum is by hypothesis of induction in V and the third by (20). The first sum is by (21)

$$\begin{aligned} &-\left(\sum_{i=1}^{n}\frac{(-1)^i a_{i,\infty}}{(l-i+1)(l-i+2)\cdots l}\right)x^l f^{(n)}\,dx\\ &\overset{(18)}{=}\left(1-\frac{I_\infty(l-n+1)}{l\cdots(l-n+1)}\right)x^l f^{(n)}\,dx. \end{aligned}$$

Therefore, since $l-n+1$ is not an exponent at ∞, we get $x^l f^{(n)}\,dx$ is in V. □

REMARK 1. Since $x^i f^{(n)}\,dx=0$ in $H^1_{\mathrm{dR}}(\mathbb{P}^1-S,L)$ for $i=0,\ldots,n-1$, we obtain n k-linear relations between the generators (19) of $H^1_{\mathrm{dR}}(\mathbb{P}^1-S,L)$.

REMARK 2. If ∞ is no singularity, then the exponents at ∞ are $0,-1,\ldots,-n+1$. Thus the condition that the exponent is positive is compatible with the condition that the exponents are $\geq n$ at the finite singularities.

REMARK 3. Without the hypothesis on indicial equations of L, we have to add the following elements:

$$\begin{aligned} &\frac{f^{(n)}}{(x-t_j)^l}\,dx,\quad \text{if } I_{t_j}(l+n-1)=0,\\ &x^l f^{(n)}\,dx,\quad \text{if } I_\infty(l-n+1)=0, l\geq n \end{aligned}$$

to the set (19) in order to get a set of generators.

3.4. Explicit Set of Generators, $\mathsf{k} \neq \bar{\mathsf{k}}$

In case $\mathsf{k} \neq \bar{\mathsf{k}}$ and for computational purposes, we modify Theorem 1 and reprove it over k. For this we proceed as follows. Let

$$\Delta = \prod_{i=1}^{r} (x - t_i),$$

where $t_i \in \bar{\mathsf{k}}$ are the finite singular points of $L = 0$ (without repetition). Since L is defined over k, the Galois group of $\bar{\mathsf{k}}$ over k acts on t_js, and so $\Delta \in \mathsf{k}[x]$. Thus we can write L in the following way:

$$L = \sum_{i=0}^{n} \Delta^i \tilde{p}_{n-i}(x)\partial_x^i, \quad \tilde{p}_i \in \mathsf{k}[x], \deg \tilde{p}_i \leq i(r-1), \tilde{p}_0 = 1, \tag{22}$$

see for instance [I+91, I. Prop. 4.2].

THEOREM 2. *If L has no integer exponent $\geq n$ in the Riemann-scheme at a finite point and no positive integer exponent at ∞, then $H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L)$ is generated by*

$$\frac{x^j f^{(i)}\, dx}{\Delta}, \quad j = 0, 1, \ldots, r-1, i = 0, 1, 2, \ldots, n-1. \tag{23}$$

Proof. The $\bar{\mathsf{k}}$-vector space $H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L) \otimes_{\mathsf{k}} \bar{\mathsf{k}}$ has a set of generators (19), and for fixed $i \in \mathbb{N}$ we have

$$\left\langle \frac{f^{(i)}\, dx}{x - t_j} \,\middle|\, j = 1, \ldots, r \right\rangle_{\bar{\mathsf{k}}} = \left\langle \frac{x^k f^{(i)}\, dx}{\Delta} \,\middle|\, k = 0, \ldots, r-1 \right\rangle_{\bar{\mathsf{k}}},$$
$$i = 0, 1, 2, \ldots, n-1. \qquad \square$$

In order to implement the above proof in a computer, one has to introduce new variables t_j for each singularity, and so it does not give an effective algorithm which writes an element of $H^1(\mathbb{P}^1 - S, L)$ in terms of the generators (23). We give a second proof which is algorithmic and does not use $\bar{\mathsf{k}}$.

By the extended Euclidean algorithm, there are polynomials $a, b \in \mathsf{k}[x]$ such that

$$1 = a\Delta + b\Delta'.$$

For $t \in \mathsf{k}$ we define

$$c_t := 1 + \sum_{i=1}^{n} \frac{\tilde{p}_i b^i}{t(t+1)\cdots(t+i-1)} \in \mathsf{k}[x]. \tag{24}$$

LEMMA 1. *For fixed $t \in \mathsf{k}$, we have $gcd(c_t, \Delta) = 1$ if and only if $t + n - 1$ is not an exponent of L at finite singularities t_j, $j = 1, 2, \ldots, r$.*

Proof. Let t_j be a root of Δ. We have

$$1 = b(t_j) \cdot \Delta'(t_j), \quad \Delta'(t_j) = \left(\frac{\Delta}{x - t_j}\right)(t_j),$$

and so

$$a_{i,t_j} := \lim_{x \to t_j} \frac{\tilde{p}_i(x)(x - t_j)^i}{\Delta^i} = p_i(t_j)\left(\frac{1}{\Delta'(t_j)}\right)^i = \tilde{p}_i(t_j) b(t_j)^i.$$

Thus multiplying c_t by $t(t+1)\cdots(t+n-1)$ and evaluating x at t_j gives the value of the indicial equation I_{t_j} evaluated at $t+n-1$. □

Second Proof of Theorem 2. All the qualities that follow are in the cohomology group $H^1_{\mathrm{dR}}(\mathbb{P}^1 - \Delta, L)$. Obviously it is generated by

$$f^{(i)} x^l \, dx, \frac{f^{(i)} x^k}{\Delta^l} \, dx, \quad l \in \mathbb{N}, i = 0, 1, \ldots, n-1, k = 0, 1, \ldots, r-1.$$

Note that, by division over Δ, it is enough to consider $0 \le k < r$. Let V be the k-vector space generated by (23). At first we show that $f^{(j)} x^k / \Delta^l \, dx \in V$. Again for $p \in \mathsf{k}[x]$ we have

$$0 = d\left(\frac{f^{(j)} p}{\Delta^l}\right) = \frac{(f^{(j)} p)'}{\Delta^l} \, dx + (-l) \frac{f^{(j)} p \Delta'}{\Delta^{l+1}} \, dx.$$

For $l \in \mathbb{N}$ we get

$$\begin{aligned} \frac{f^{(j)} x^k}{\Delta^{l+1}} \, dx &= \frac{f^{(j)} x^k (a\Delta + b\Delta')}{\Delta^{l+1}} \, dx \\ &= \frac{f^{(j)} x^k a}{\Delta^l} \, dx + \frac{f^{(j)} x^k b \Delta'}{\Delta^{l+1}} \, dx \\ &= \frac{f^{(j)} x^k a}{\Delta^l} \, dx + \frac{1}{l} \frac{(f^{(j)} x^k b)'}{\Delta^l} \, dx \\ &= \frac{f^{(j)} x^k a}{\Delta^l} \, dx + \frac{1}{l} \frac{f^{(j)} (x^k b)'}{\Delta^l} \, dx + \frac{1}{l} \frac{f^{(j+1)} x^k b}{\Delta^l} \, dx. \end{aligned} \tag{25}$$

Hence if $j < n-1$, then we can reduce the pole order. It remains to show that for $j = n-1$ we can reduce the pole order. Let $q \in k[x]$. Then

$$\begin{aligned} \frac{q f^{(n)}}{\Delta^l} \, dx &= \frac{\Delta^n q f^{(n)}}{\Delta^{l+n}} \, dx \\ &= -\sum_{i=1}^{n} \frac{q \tilde{p}_i f^{(n-i)}}{\Delta^{l+i}} \, dx \\ &\overset{(24)}{=} -q(c_l - 1) \frac{f^{(n)}}{\Delta^l} \, dx + \text{lower pole orders terms}, \end{aligned} \tag{26}$$

where the last equality follows by (24) and (25). Since $gcd(c_l, \Delta) = 1$, we have polynomials $A, B \in \mathsf{k}[x]$ such that $Ac_l + B\Delta = 1$. Hence the pole order of

$$\frac{x^k f^{(n)}}{\Delta^l} \, dx = \frac{(x^k A) c_l f^{(n)}}{\Delta^l} \, dx + \frac{x^k B f^{(n)}}{\Delta^{l-1}} \, dx$$

can be reduced to $l-1$ using (26) with $q = x^k A$. Reducing the pole order of Δ may yield also terms $x^j f^{(i)}\, dx$. However, by the same arguments as in the proof of Theorem 1, we have $x^j f^{(i)}\, dx \in V$. $\square$

REMARK 4. In Theorem 2, without the hypothesis on indicial equations of L, we have to add the following finite number of elements:

$$\begin{aligned} &\frac{x^k f^{(n)}}{\Delta^l}\, dx, \quad \text{if } 0 \le k < \deg(gcd(\Delta, c_l)), \\ &x^l f^{(n)}\, dx, \quad \text{if } I_\infty(l-n+1) = 0, l \ge n \end{aligned}$$

to the set (19) in order to get a set of generators.

3.5. Cohomologies over Function Fields

In this section we turn to the main problem posed in Introduction, that is, how to compute the linear differential equation of (4). Let us assume that $\mathsf{k} = \tilde{\mathsf{k}}(y)$, where y is a variable and $\tilde{\mathsf{k}}$ is a subfield of $\mathbb{C}$, and so we have the derivation $\partial_y : \mathsf{k} \to \mathsf{k}$. Let

$$dY_i = A_i Y_i, i = 1, 2, \quad A_i \in \mathrm{Mat}_{n_i \times n_i}(\tilde{\mathsf{k}}(x, y)\, dx + \tilde{\mathsf{k}}(x, y)\, dy) \tag{27}$$

be the Gauss–Manin connection of the family $X_i \to \mathbb{P}^1_x \times \mathbb{P}^1_y, i = 1, 2$. We make the Kronecker product of these two systems and obtain the system

$$dY = M \cdot Y, \tag{28}$$

where $M = A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$ and I_{n_i} is the $n_i \times n_i$ identity matrix. A solution of (28) is given by $Y = Y_1 \otimes Y_2$. If we write $M = Adx + Bdy, A, B \in \mathrm{Mat}_{n\times n}(\tilde{\mathsf{k}}(x, y))$, then the two-dimensional system (28) in x, y variables is equivalent to $\partial_x Y = AY, \partial_y Y = BY$. It is integrable, and hence $dM = -M \wedge M$ or equivalently $\partial_x B - \partial_y A = BA - AB$.

The first entry f, and in general any $\mathsf{k}(x)$-linear combination of the entries, of Y satisfies a linear differential equation $L = 0$, $L \in \mathsf{k}[x, \partial_x]$, with respect to the variable x. From the integrability condition we conclude that a solution of $L = 0$ depends holomorphically on both x, y. We need ∂_y to induce a well-defined map

$$\partial_y : H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L) \to H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L). \tag{29}$$

Note that if we use system (10) and definition (11), then (29) is well defined; however, for linear differential equations with definition (14), (29) is not necessarily well defined. In order to get map (29), we assume that the differential system $\partial_x Y = AY$ is irreducible over $\mathsf{k} = \tilde{\mathsf{k}}(y)$, that is, there is no nonzero ∂_x invariant proper subspace of the $\mathsf{k}(x)$ vector space generated by the entries of Y. This may not be the case in general, for instance when the two systems in (27) are the same. In this case we have to find the decomposition of (28) into irreducible components. This irreducibility condition is satisfied in many examples in which one of the systems in (27), say $i = 1$, is trivial, that is, $n_1 = 1$ and $A_1 = 0$, and so $M = A_2$. Therefore, we have to assume that (27) for $i = 2$ is irreducible. Our main examples in Section 5 are of this form. We conclude that the $\mathsf{k}(x)$ vector space generated by the entries of Y is the same as the $\mathsf{k}(y)$ vector space generated by $f, \partial_x f, \partial_x^2 f, \ldots$. This implies that if we set $X := \left[f, \partial_x f, \ldots, \partial_x^{n-1} f\right]^{\mathrm{tr}}$ and write

$$X = CY, \quad C \in \mathrm{Mat}_{n\times n}(\tilde{\mathsf{k}}(x, y)),$$

then C is invertible. The matrix C can be computed in the following way. We have $\partial_x^m Y = A_m Y$ with

$$A_{m+1} = \partial_x A_m + A_m \cdot A, \quad A_1 := A, \tag{30}$$

and the ith row of C is the first row of A_i. It follows that

$$\partial_y X = D \cdot X, \qquad D := \partial_y C \cdot C^{-1} + C \cdot B \cdot C^{-1}. \tag{31}$$

Let

$$\omega = \left[\frac{f}{\Delta}, \frac{\partial_x f}{\Delta}, \ldots, \frac{\partial_x^{n-1} f}{\Delta}, \ldots \frac{x^j f}{\Delta}, \frac{x^j \partial_x f}{\Delta}, \ldots, \frac{x^j \partial_x^{n-1} f}{\Delta}, \ldots\right]^{\mathrm{tr}}$$

be the $nr \times 1$ matrix containing elements (23). We write (29) in ω:

$$\partial_y \omega = E \cdot \omega. \tag{32}$$

The matrix E can be computed in the following way. We have

$$\partial_y \left(\frac{x^j \partial_x^i f}{\Delta}\right) = \frac{x^j \partial_y \partial_x^i f}{\Delta} - \frac{x^j \partial_y \Delta \cdot \partial_x^i f}{\Delta^2}. \tag{33}$$

The first term can be written in the basis ω using (31). For the second term, we have to use pole order reduction as in the proof of Theorem 2. The differential system

$$\partial_y W = E \cdot W \tag{34}$$

is satisfied by $W = \int \omega$, where the integration takes place over a fixed closed path in the x-domain such that the entries of ω are one-valued. Let $I_i, i = 1, 2$, be as in Introduction. By convention I_i is the first entry of Y_i, and hence $I_1 I_2$ is the first entry of Y is (28). In order to compute the Picard–Fuchs equation of integral (4), we have to write $f = I_1 I_2$ as a $\tilde{\mathsf{k}}(y)$-linear combination of the entries of ω. Then we compute the Picard–Fuchs equation of the same linear combination of the entries of W using system (34).

REMARK 5. The output linear differential equation L of our algorithm is not necessarily the differential equation of minimal order annihilating W. The fact that (23) might not form a basis of the cohomology $H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L)$ might result in this kind of phenomena. As far as the authors are aware, this defect is also present in almost all algorithms in the literature for computing Picard–Fuchs equations. One has to use other algorithms in order to decompose L into irreducible factors and check the minimality.

Algorithm 1: Computation of the convolution of two PF equations.

Data: Two matrices A_1 and A_2 with entries in $\tilde{\mathsf{k}}(x, y)\,dx + \tilde{\mathsf{k}}(x, y)\,dy$ representing systems (27).

Result: The matrix $[q_0, q_1, \cdots, q_n]$ with entries in $\tilde{\mathsf{k}}[y]$ representing Picard–Fuchs equation (3).

begin

Compute the Kronecker product $M := A\,dx + B\,dy$ of A_1 and A_2 as in (28);

Compute the linear differential equation L, represented by $[p_0, p_1, \ldots, p_m]$ of the first entry of f of Y in $\partial_x Y = AY$. This can be done for instance by `sysdif` from `foliation.lib`, see [Mov] ;

Compute A_ms through recursion (30) and then C whose ith row is the first row of A_i ;

Compute $D := \partial_y C \cdot C^{-1} + C \cdot B \cdot C^{-1}$ in (31);

Apply the algorithm in Theorem 2 and write $\frac{x^j \partial_y \Delta \cdot \partial_x^i f}{\Delta^2}$ (and f itself) in terms of the generators ω. This involves computing $\tilde{p}_i$s in (22). In the case of f, let us denote the coefficients by Q ;

Compute E in (32) using (33). ;

Compute the Picard–Fuchs equation L, represented by the matrix $[q_0, q_1, \cdots, q_n]$, of the linear combination of the entries of system (34) with coefficients coming from Q;

return L;

4. The Classical Convolution

In this section we remind the classical convolution. Any Fuchsian differential equation can be transformed into a Fuchsian system in Okubo format, see [Koh99, Chapter 4.1]; however, this requires working over algebraically closed fields, which makes its computer implementation useless from a practical point of view. Once this is done, we get a closed formula for a Fuchsian system that is satisfied by the classical convolution of solutions of two Fuchsian systems. We argue that the material of the present paper is a generalization of this concept. The only difference is that our approach gives an algorithm for computing the differential equation of the convolution, whereas we have a closed formula for the differential equation of classical convolution, and it is only for Fuchsian differential equations in the particular format of Okubo systems.

Recall our geometric framework in Introduction. Let $f_i : Y_i \to \mathbb{P}^1_x$, $i = 1, 2$, be two one-parameter families of algebraic varieties, $X_i := Y_i \times \mathbb{P}^1_y$, and consider (1) in which the two left arrows are projections in y, the top right arrow is $(\cdot, y) \mapsto f_1(\cdot)$, and the bottom right arrow is $(\cdot, y) \mapsto y - f_2(\cdot)$. By abuse of notation we have used y as the projection map on the second coordinate. We can see easily that $I_1(x, y) = I_1(x)$ and $I_2(x, y) = I_2(y - x)$, and so integral (4) turns out to be of the format $I(y) := \int_\delta I_1(x) I_2(y - x)\,dx$. The input matrices in Algorithm 1 are respectively $B_1(x)\,dx$, $B_2(x-y)\,dx - B_2(x-y)\,dy$, where $B_1(x)$, $B_2(x)$ are respectively the Gauss–Manin connections matrices of f_1 and f_2. An even more particular case of this situation leads us to the classical convolution for which we do have closed formulas for the differential equation of $I(y)$, see Theorem 4. The Gauss–Manin connections of the families $f_i : Y_i \to \mathbb{P}^1_x$ are replaced with two Okubo systems.

4.1. Okubo System

A linear differential system of the format

$$\begin{aligned}&(xI_n - T)Y' = AY,\\ &\qquad T = \operatorname{diag}(t_1 I_{n_1}, \dots, t_r I_{n_r}), \sum n_i = n, T, A \in \operatorname{Mat}_{n\times n}(\mathbb{C})\end{aligned} \tag{35}$$

is called an Okubo system (in normal form), and it is a useful format for doing computations, such as convolution. For a Fuchsian system

$$D_a : Y' = \sum_{i=1}^{r} \frac{a_i}{x-t_i} Y, \quad a_i \in \operatorname{Mat}_{n\times n}(\mathbb{C}), a = (a_1, a_2, \dots, a_r),$$

we introduce the following special Okubo system:

$$\begin{aligned}&D_{c_\mu(a)} : (xI_{nr} - T)X' = c_\mu(a)X,\\ &c_\mu(a) := \begin{pmatrix} a_1 & \dots & a_r \\ \vdots & \vdots & \vdots \\ a_1 & \dots & a_r \end{pmatrix} + \mu I_{nr}, \mu \in \mathbb{C},\end{aligned} \tag{36}$$

where $n_i = n, i = 1, \dots, r$. Via the following procedure we see that any Fuchsian system is a factor system of an Okubo system. Let $f(x)$ be a solution of D_a. Then

$$\tilde{f}(x) := \begin{pmatrix} f(x)(x-t_1)^{-1} \\ \vdots \\ f(x)(x-t_r)^{-1} \end{pmatrix} \tag{37}$$

satisfies the Okubo system $D_{c_{-1}}(a)$. If k is not algebraically closed, then we work with the following equivalent Okubo system defined over k. Let L be a Fuchsian system of dimension n

$$L : Y' = \sum_{i=1}^{r} \frac{x^{i-1}}{\Delta} \tilde{a}_i Y$$

with $\tilde{a}_i \in \operatorname{Mat}_{n\times n}(\mathsf{k})$ and $\Delta = \sum_{i=0}^{r} b_{r-i} x^i \in \mathsf{k}[x]$, $b_0 = 1$. If f is a solution of L, then

$$(f/\Delta, xf/\Delta, \dots, x^{r-1} f/\Delta)^{\mathrm{tr}}$$

satisfies the Okubo system

$$(xI_{rn} - \tilde{T})Y' = \tilde{A}Y,$$

where

$$\tilde{T} = \begin{pmatrix} 0 & I_n & 0\cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & I_n \\ -b_r I_n & \cdots & -b_2 I_n & -b_1 I_n \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ \tilde{a}_1 & \cdots & \tilde{a}_r \end{pmatrix} - I_{rn}.$$

4.2. Cohomology of Okubo System

For the matrix A in (35), we consider its submatrices $A = [A_{ij}]$ according to the partition $n = n_1 + n_2 + \cdots + n_r$ and in particular its $n_i \times n_i$ submatrices A_{ii} lying in the diagonal of A.

THEOREM 3. *Let L be Okubo system* (35) *with solution f. If*

$$\det(A + mI_{n\times n}) \neq 0, \det(A_{ii} - mI_{n_i\times n_i}) \neq 0, \quad \forall m \in \mathbb{N} \tag{38}$$

then $H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L)$ is generated by

$$(x - t_i)^{-1} f_j \, dx, \quad i = 1, 2, \ldots, r, j = 1, 2, \ldots, n, \tag{39}$$

and so it is of dimension at most $n \cdot r$.

Proof. In $H^1_{\mathrm{dR}}(\mathbb{P}^1 - S, L)$ and for $m \neq -1$, we have

$$\begin{aligned}(x - t_i)^m f_j \, dx &= -(m+1)^{-1}(x - t_i)^{m+1} f_j' \, dx \\ &= -(m+1)^{-1}(x - t_i)^{m+1}(x - t_j)^{-1} \sum_{k=1}^{n} a_{jk} f_k \, dx,\end{aligned}$$

where $A = (a_{jk})$. If m is negative and $t_i \neq t_j$, then we have reduced the pole order. If m is negative and $t_i = t_j$, in order to reduce the pole order, we need $A_{ii} + (m + 1)I_{n_i\times n_i}$ to be invertible, and if m is positive or zero, we need $A + (m + 1)I_{n\times n}$ to be invertible. □

REMARK 6. For a linear differential equation of an entry of the Okubo system, conditions (38) imply the conditions in Theorem 2. Without these conditions a similar observation as in Remark 3 is valid.

4.3. Convolution of Okubo Systems

Given two solutions of two Okubo systems we can easily determine the Okubo system that is satisfied by their convolution.

THEOREM 4. *Let $f_i(x)$ be a solution of the Okubo system $(xI_{n_i} - T_i)Y_i' = A_i Y_i$, $i = 1, 2$, $A_i \in \mathrm{Mat}_{n_i\times n_i}(\mathbb{C})$. Then*

$$\int f_1(x) \otimes f_2(y - x) \, dx,$$

where the integration is over a path in the $x \in \mathbb{C}$ plane such that the integrand is one-valued, is a solution matrix for the Okubo system

$$(yI_{n_1n_2} - T_1 \otimes I_{n_2} - I_{n_1} \otimes T_2)Y' = (A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2 + I_{n_1n_2})Y. \tag{40}$$

The proof of the above theorem is similar to [DR07, Lemma 4.2]. Note that system (40) has singularities at $t_i^1 + t_j^2, i = 1, \dots, r_1, j = 1, \dots, r_2$, and possibly at infinity.

4.4. Convolution of Fuchsian Systems

In the case of Fuchsian systems we proceed as follows. Let

$$D_{a^i} : Y' = \sum_{j=1}^{r_i} \frac{a_j^i}{x - t_j^i} Y, \quad a_j^i \in \mathrm{Mat}_{n_i \times n_i}(\mathbb{C}), i = 1, 2,$$

be two Fuchsian systems with solutions f_1, f_2 resp. Then the Okubo system

$$\begin{aligned}(yI_{n_1r_1n_2r_2} &- (T_1 \otimes I_{n_2r_2} + I_{n_1r_1} \otimes T_2))Y' \\ &= (c_0(a^1) \otimes I_{n_2r_2} + I_{n_1r_1} \otimes c_0(a^2) - I_{n_1r_1n_2r_2})Y\end{aligned}$$

has $\int \tilde{f}_1(x) \otimes \tilde{f}_2(y - x)\, dx$ as solution with $\tilde{f}_1, \tilde{f}_2$ as in (37).

5. Examples

In this section we discuss some examples of families of algebraic varieties whose Picard–Fuchs equation can be computed through the methods introduced in this article. In the first two examples, we consider the case in which we have only the family X_1 (take X_2 the product of some variety with $\mathbb{P}_x^1 \times \mathbb{P}_t^1$). In this case we want to use the Gauss–Manin connection of the two-parameter family $X_1 \to \mathbb{P}_x^1 \times \mathbb{P}_t^1$ and integrate it over the variable x and obtain the Picard–Fuchs equation of the one-parameter family $X_1 \to \mathbb{P}_t^1$ obtained by the composition $X_1 \to \mathbb{P}_x^1 \times \mathbb{P}_t^1 \to \mathbb{P}_t^1$, where the second map is the projection. For this class of examples and for the input of the Algorithm 1, the matrix A_2 is the 1×1 zero matrix and A_1 is the Gauss–Manin connection matrix of $X_1 \to \mathbb{P}_x^1 \times \mathbb{P}_t^1$. A well-known example for this situation is the following.

EXAMPLE 1 (Legendre family of elliptic curve). This is given by

$$y^2 = x(x-1)(x-t),$$

where t is a parameter. We can compute the Picard–Fuchs equation of $\int \frac{dx}{y}$ either by direct methods or by the methods introduced in this article:

$$I + (8t - 4)I' + (4t^2 - 4t)I'' = 0.$$

In the second case we consider x and t as a parameter, and so we get a two-parameter family of zero dimensional varieties with two points. The 1×1 matrices A_1 and A_2 are respectively $[P_t/P\, dt + P_x/P\, dx]$ and 0, where $P = x(x-1)(x-t)$.

EXAMPLE 2 (A rank 19 family of K3 surfaces). Let us consider the rank 19 family of K3 surfaces given in the affine coordinates (x, y, w) by the equation $P = 0$, where

$$\begin{aligned}
&P := y^2w - 4x^3 + 3axw^2 + bw^3 + cxw - (1/2)(dw^2 + w^4) = 0,\\
&a = (16+t)(256+t), \qquad b = (-512+t)(-8+t)(64+t),\\
&c = 0, \qquad d = 2{,}985{,}984t^3,
\end{aligned}$$

and t is a parameter, see [D+14, Section 6.7]. Here, we would like to compute the Picard–Fuchs equation of the holomorphic 2-form given by $\omega = \frac{dx\wedge dy\wedge dw}{dP}$. The generic member of the family has two isolated singularities, and so one cannot apply the Griffiths–Dwork method or its modification using Brieskorn modules. In order to apply the methods introduced in this article, we consider $P = 0$ as a two-parameter family of elliptic curves depending on (t, w). In this case we know the explicit expression of Gauss–Manin connection, see for instance [D+14, Section 6]. Using this we can compute the following differential equations for the elliptic integral $f(t, w) := \int \frac{dx\wedge dy}{dP}$:

$$\begin{aligned}
L := A_1 f + A_2\partial_w f + A_3\partial_w^2 f = 0,\\
B_1 f + B_2\partial_t f + B_3\partial_t^2 f = 0,
\end{aligned}$$

where A_i, B_is are explicit polynomials in w, t with rational coefficients:

$$\begin{aligned}
A_1 = (&1{,}283{,}918{,}464{,}548{,}864t^9w - 133{,}116{,}666{,}404{,}426{,}219{,}520t^9\\
&+ 1{,}486{,}016{,}741{,}376t^8w^2 - 585{,}466{,}819{,}834{,}281{,}984t^8w\\
&+ 72{,}814{,}820{,}327{,}424t^7w^2 - 37{,}469{,}876{,}469{,}394{,}046{,}976t^7w\\
&+ 1{,}719{,}926{,}784t^6w^3 - 2{,}077{,}451{,}404{,}443{,}648t^6w^2\\
&+ 336{,}571{,}521{,}970{,}697{,}404{,}416t^6w - 784{,}286{,}613{,}504t^5w^3\\
&+ 298{,}249{,}504{,}061{,}128{,}704t^5w^2 - 50{,}194{,}343{,}264{,}256t^4w^3\\
&+ 24{,}931{,}223{,}849{,}681{,}289{,}216t^4w^2 - 144t^3w^5 - 474{,}771{,}456t^3w^4\\
&+ 450{,}868{,}486{,}864{,}896t^3w^3 + 65{,}664t^2w^5 + 4{,}202{,}496tw^5 + 77w^6\\
&- 37{,}748{,}736w^5),\\
A_2 = (&3{,}851{,}755{,}393{,}646{,}592t^9w^2 - 1{,}916{,}879{,}996{,}223{,}737{,}561{,}088t^9w\\
&+ 2{,}972{,}033{,}482{,}752t^8w^3 - 1{,}756{,}400{,}459{,}502{,}845{,}952t^8w^2\\
&+ 145{,}629{,}640{,}654{,}848t^7w^3 - 112{,}409{,}629{,}408{,}182{,}140{,}928t^7w^2\\
&+ 1{,}719{,}926{,}784t^6w^4 - 3{,}138{,}467{,}357{,}786{,}112t^6w^3\\
&+ 1{,}009{,}714{,}565{,}912{,}092{,}213{,}248t^6w^2 - 497{,}664t^5w^5\\
&- 784{,}286{,}613{,}504t^5w^4 + 596{,}499{,}008{,}122{,}257{,}408t^5w^3\\
&- 24{,}385{,}536t^4w^5 - 50{,}194{,}343{,}264{,}256t^4w^4
\end{aligned}$$

$$
\begin{aligned}
&+49{,}862{,}447{,}699{,}362{,}578{,}432t^4w^3-432t^3w^6\\
&-119{,}439{,}360t^3w^5+450{,}868{,}486{,}864{,}896t^3w^4+196{,}992t^2w^6\\
&-99{,}883{,}155{,}456t^2w^5+12{,}607{,}488tw^6-8{,}349{,}416{,}423{,}424tw^5\\
&+144w^7-113{,}246{,}208w^6),\\
A_3={}&36w^2(-w^2+2{,}985{,}984t^3)(-w^4+(4t^3-1{,}824t^2-116{,}736t\\
&+1{,}048{,}576)\cdot w^3+(6{,}912t^5+338{,}688t^4-7{,}299{,}072t^3\\
&+1{,}387{,}266{,}048t^2+115{,}964{,}116{,}992t)\cdot w^2+(11{,}943{,}936t^6\\
&-5{,}446{,}434{,}816t^5-348{,}571{,}828{,}224t^4+3{,}131{,}031{,}158{,}784t^3)\cdot w\\
&-8{,}916{,}100{,}448{,}256t^6),\\
B_1={}&-\frac{1}{36}A_3\cdot(-w^2+2{,}985{,}984t^3)^{-1}\cdot(2{,}985{,}984t^4w^2-3{,}456t^3w^3\\
&+2{,}842{,}656{,}768t^3w^2-34{,}560t^2w^3+73{,}383{,}542{,}784t^2w^2-7tw^4\\
&+2{,}211{,}840tw^3-952w^4+905{,}969{,}664w^3),\\
B_2={}&-13{,}824t^3w^2+9{,}746{,}251{,}776t^3w-138{,}240t^2w^2\\
&+293{,}534{,}171{,}136t^2w-24tw^3+8{,}847{,}360tw^2-3{,}264w^3\\
&+3{,}623{,}878{,}656w^2,\\
B_3={}&8(t+256)(t+16)(2{,}985{,}984t^3-w^2).
\end{aligned}
$$

We use the second equality in order to compute the action of ∂_t on the cohomology group constructed from L. This data is enough to compute the Picard–Fuchs equation of the integral $g(t)=\int f(t,w)\,dw$ using the techniques introduced in this article. Note that we have to use Theorem 2 together with Remark 4 because the differential equation L has the apparent singularities $-w^2+2{,}985{,}984t^3$. The end result has a factor

$$
\tilde{L} := 1+(26t+512)\partial_t+(36t^2+1{,}536t)\partial_t^2+(8t^3+512t^2)\partial_t^3, \tag{41}
$$

where $\tilde{L}g=0$. This differential operator is obtained by direct computations as in [Mov, Chapters 10,12]. The function g can be written as the period of $\frac{dx\wedge dy\wedge dw}{dP}$ over two-dimensional cycles living in the $K3$-surface, for further details see [D+14]. Note that the generic fiber of $P=0$ is singular, and in order to apply the algorithms in [Mov], in [D+14] we have used a new parameter s and computed the Gauss–Manin connection of the five-parameter family of K3 surfaces $P-s=0$. For this we had to run our computer for a few hours and the outcome data of the Gauss–Manin connection is more than 4 mega bytes. Despite the fact we have not computed (41) by methods introduced in this paper, we believe that it is faster as it computes Picard–Fuchs equation by increasing the dimension one by one, and the available algorithms for higher dimensional families, despite being correct, do not work in practice. In this example the input matrices in Algorithm 1 are respectively the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -\frac{A_1}{A_3} & -\frac{A_2}{A_3} \end{pmatrix} dw + \begin{pmatrix} 0 & 1 \\ -\frac{B_1}{B_3} & -\frac{B_2}{B_3} \end{pmatrix} dt$ and the 1×1 zero matrix.

Example 3 (Join of two polynomials). This example fits into the framework introduced at the beginning of Section 4. Let $f_1 \in \mathbb{C}[x_1, x_2, \dots, x_{n_1+1}]$, $f_2 \in \mathbb{C}[y_1, y_2, \dots, y_{n_2+1}]$ be two polynomials in $n_1 + 1$ and $n_2 + 1$ variables, respectively. We get $f_i : \mathbb{C}^{n_i+1} \to \mathbb{C}_x, i = 1, 2$, which are one-parameter families of affine varieties. We define $X_i := \mathbb{C}^{n_i} \times \mathbb{C}_x$. Note that we are ignoring the fiber at infinity $\infty = \mathbb{P}^1_x \backslash \mathbb{C}_x$. The geometric convolution, which is basically a fiber product, in this case gives us the fibration given by $f : \mathbb{C}^{n_1+n_2+1} \to \mathbb{C}_y$, $f(x_1, \dots, x_{n_1+1}, y_1, \dots, y_{n_2+1}) = f_1(x_1, \dots, x_{n_1+1}) + f_2(y_1, \dots, y_{n_2+1})$. In [AGV88], f is called the join of f_1 and f_2. If f_1 and f_2 are tame polynomials in the sense of [Mov], then we have the Brieskorn module methods for computing the Gauss–Manin connections of f_1, f_2, f. This is similar to the Griffiths–Dwork method, see [Gri69]. We consider the problem of computing the Picard–Fuchs equation of $I(y) := \int dx_1 \wedge \cdots \wedge dx_{n_1+1} \wedge dy_1 \wedge \cdots \wedge dy_{n_2+1}/df$, which can be also written as $\int I_1(x) I_2(y-x)$, where $I_1(x) = \int dx_1 \wedge \cdots \wedge dx_{n_1+1}/df_1$ and $I_2(x) = \int dy_1 \wedge \cdots \wedge dy_{n_2+1}/df_2$, see [Mov, Section 13.8] for further details. The application of available algorithms directly for f and I usually fails. For instance, we can compute the desired Picard–Fuchs equation for $f_1(x_1) = x_1^d - x_1$, $f_2(y_1) = y_1^d - y_1$, only for $d \leq 6$. However, the Picard–Fuchs of $I_i, i = 1, 2$, can be computed for $d \leq 20$, and we expect that the application of algorithms developed in this article will give us the differential equation of I for many other $d \geq 7$. For the computer code used in this example, see the first author's webpage.

References

[A+10] G. Almkvist, C. van Enckevort, D. van Straten, and W. Zudilin, *Tables of Calabi–Yau equations,* Preprint, 2010, arXiv:math/0507430.

[AGV88] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Monodromy and asymptotics of integrals Vol. II,* Monogr. Math., 83, Birkhäuser, Boston, 1988.

[Beu07] F. Beukers, *Gauss' hypergeometric function,* Arithmetic and geometry around hypergeometric functions. Lecture notes of a CIMPA summer school held at Galatasaray, University, Istanbul, Turkey, June 13–25, 2005, pp. 23–42, Birkhäuser, Basel, 2007.

[C+91] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory,* Nuclear Phys. B 359 (1991), no. 1, 21–74.

[Del70] P. Deligne, *Équations différentielles à points singuliers réguliers,* Lecture Notes in Math., 163, Springer, Berlin, 1970.

[DR07] M. Dettweiler and S. Reiter, *Middle convolution of Fuchsian systems and the construction of rigid differential systems,* J. Algebra 318 (2007), no. 1, 1–24.

[Dim04] A. Dimca, *Sheaves in topology,* Universitext, Springer, Berlin, 2004.

[DM15] C. F. Doran and A. Malmendier, *Calabi–Yau manifolds realizing symplectically rigid monodromy tuples,* arXiv e-prints (2015), arXiv:1503.07500.

[DM06] C. F. Doran and J. W. Morgan, *Mirror symmetry and integral variations of Hodge structure underlying one-parameter families of Calabi–Yau threefolds,* Mirror symmetry. V, AMS/IP Stud. Adv. Math., 38, pp. 517–537, Am. Math. Soc., Providence, 2006.

[D+14] C. Doran, H. Movasati, U. Whitcher, and A. Harder, Humbert surfaces and the moduli of lattice polarized k3 surfaces. *Proceedings of Symposia in Pure Mathematics, String-Math conference proceeding 2014,* 93, 2016.

[Gri69] P. A. Griffiths, *On the periods of certain rational integrals. I, II,* Ann. of Math. (2) 90 (1969), 460–495; ibid. (2), 90:496–541, 1969.

[I+91] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida, *From Gauss to Painlevé: a modern theory of special functions,* Aspects Math., E16, Friedr. Vieweg & Sohn, Braunschweig, 1991.

[Koh99] M. Kohno, *Global analysis in linear differential equations,* Math. Appl., 471, Kluwer Academic, Dordrecht, 1999.

[Lai16] P. Lairez, *Computing periods of rational integrals,* Math. Comp. 85 (2016), no. 300, 1719–1752.

[Mov] H. Movasati, A Course in Hodge Theory: with Emphasis on Multiple Integrals, Available at ⟨http://w3.impa.br/~hossein/myarticles/hodgetheory.pdf⟩, 2019.

H. Movasati
Instituto de Matemática Pura e Aplicada, IMPA,
Estrada Dona Castorina, 110,
22460-320, Rio de Janeiro, RJ
Brazil

hossein@impa.br

S. Reiter
Mathematisches Institut
Universität Bayreuth
95440 Bayreuth
Germany

Stefan.Reiter@uni-bayreuth.de