Algebraic curves and foliations

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With an appendix by CLAUS HERTLING³

Abstract

Consider a field k of characteristic 0, not necessarily algebraically closed, and a fixed algebraic curve f = 0 defined by a tame polynomial $f \in k[x, y]$ with only quasihomogeneous singularities. We prove that the space of holomorphic foliations in the plane \mathbb{A}^2_k having f = 0 as a fixed invariant curve is generated as k[x, y]-module by at most four elements, three of them are the trivial foliations fdx, fdy and df. Our proof is algorithmic and constructs the fourth foliation explicitly. Using Serre's GAGA and Quillen-Suslin theorem, we show that for a suitable field extension K of k such a module over K[x, y] is actually generated by two elements, and therefore, such curves are free divisors in the sense of K. Saito. After performing Groebner basis for this module, we observe that in many well-known examples, K = k.

1 Introduction

The geometric analysis of one dimensional foliations in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, seen from a global perspective, is a wide open field of research. In particular, the study of foliations that admit an algebraic curve as an integral. In [CLS92] foliations whose limit set is an algebraic curve with hyperbolic holonomy are characterized as rational pull backs of linear foliations. Curves of high degree can be leaves of foliations of lower degree, and there is no general statement explaining this phenomenon. For example A. Lins Neto in [Lin02] considers a curve which is a union of nine lines supported by two foliations of projective degree 4. Studying the pencil generated by these two foliations he answers questions posed by Poincaré and Painlevé.

In this paper we explore the algebraic aspects of foliations supporting a fixed algebraic invariant curve. Let k be a subfield of \mathbb{C} and $\Omega^1_{\mathbb{A}^2_k}$ the space of 1-forms on \mathbb{A}^2_k . An element $\omega \in \Omega^1_{\mathbb{A}^2_{\mathbb{L}}}$ induces the foliation $\omega = 0$ in the affine variety $\mathbb{A}^2_{\mathbb{k}}$. Let $f \in \mathsf{k}[x, y]$ be a polynomial and C: f(x,y) = 0 the induced curve in \mathbb{A}^2_k . We assume that f is a tame polynomial.

Definition 1. (*Mov07*), *Mov19*, §10.6). A polynomial $f \in k[x, y]$ is called *tame* if the following property is satisfied. In the homogeneous decomposition of f

$$f = f_d + \dots + f_2 + f_1 + f_0,$$

into degree *i* homogeneous pieces f_i in the weighted ring k[x, y], $deg(x) = \alpha_1, deg(y) = \alpha_2$ for some $\alpha_1, \alpha_2 \in \mathbb{N}$, the last homogeneous piece $g := f_d$ has finite dimensional Milnor vector space $V_g := \frac{k[x,y]}{jacob(g)}$.

For $\alpha_1 = \alpha_2 = 1$ this is equivalent to say that g induces d distinct points in $\mathbb{P}^1_{\bar{k}}$, that is $g = \prod_{i=1}^{d} (x - a_i y)$, where the $a_i \in \bar{k}$ are pairwise different. In geometric terms this means that the line at infinity $\mathbb{P}^1_{\bar{k}} := \mathbb{P}^2_{\bar{k}} \setminus \mathbb{A}^2_{\bar{k}}$ intersects the curve induced by f = 0 in $\mathbb{P}^2_{\bar{k}}$ transversely.

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Let

(1)
$$E_f := \left\{ \omega \in \Omega^1_{\mathbb{A}^2_{\mathsf{k}}} \middle| df \wedge \omega = f\alpha, \text{ for some } \alpha \in \Omega^2_{\mathbb{A}^2_{\mathsf{k}}} \right\}.$$

This is the k[x, y]-module of differential 1-forms ω such that the foliation induced by $\omega = 0$ in \mathbb{A}^2_k leaves f = 0 invariant. By degree of $\omega = Pdx + Qdy$ we mean the affine degree $\deg(\omega) := \max\{\deg(P), \deg(Q)\}$. In this article we prove that:

Theorem 1. If all the singularities of f = 0 are quasi-homogeneous then there exists $\omega_f \in E_f$ such that fdx, fdy, df, ω_f generate the k[x, y]-module E_f .

The first three foliations fdx, fdy, df are obviously in E_f and we call them trivial foliations. Our proof of Theorem 1 is algorithmic and it computes ω_f . The proof of Theorem 1 only works for curves with quasi-homogeneous singularities. This follows from an argument due to C. Hertling, see Appendix A. The curve $x^5 + y^5 - x^2y^2 = 0$ (due to A'Campo) has a non quasi-homogeneous singularity at the origin and it turns out that the conclusion of Theorem 1 is false in this case. For further details see Example 1.

We have implemented the computation of ω_f in a computer. This 1-form is usually of high degree and it is not clear how to produce foliations of lowest possible degree in E_f . In order to investigate this problem, we write E_f in the standard Groebner basis and take its minimal resolution. It turns out that in many interesting examples, E_f is actually generated by two foliations $\omega_0 := P_0 dx + Q_0 dy$ and $\omega_\infty := P_\infty dx + Q_\infty dy$. This includes the curves studied by Lins Neto, Mendes-Pereira, and graphs of polynomial functions. For this we have prepared the Table 1, for more details regarding this table see §6.

For smooth curves it is easy to see that the foliations given by fdx, fdy, df generate E_f , see Proposition 3. For curves with only quasi-homogeneous singularities (including smooth curves) Quillen-Suslin theorem implies that $E_f \otimes_k \mathsf{K}$ is actually generated by two foliations ω_0 and ω_∞ , for some field extension K of k . Computing ω_0 and ω_∞ , and in particular the field extension K seems to be a new problem not treated in the literature. The arguments used in the proof of Theorem 1 are essentially valid in higher dimensions $\mathbb{A}^n_{\mathsf{k}}$. However, in this paper we are interested in holomorphic foliations, that is, in those $\omega \in E_f$ such that the integrability condition $\omega \wedge d\omega = 0$ holds. This trivially holds for n = 2, but for n > 2 it is a nonlinear identity in ω and it is not clear how to interpret results like Theorem 1 in this case.

The article is organized in the following way. In §2 we introduce a basis of monomials for the Milnor vector space of f. In §3 we recall quasi-homogeneous singularities and apply K. Saito's theorem to these singularities in order to get a local freeness statement. We prove Theorem 1 in §4 and observe that for a curve with A'Campo singularity Theorem 1 does not hold. The examples presented in Table 1 and even more, are discussed in §6. In §5 we use Quillen-Suslin theorem and Serre's GAGA in order to prove that $E_f \otimes_k K$ is actually free for some field extension $k \subset K$. We discuss this field extension in the case of a circle. The computer codes of the present paper are written in SINGULAR, see [GPS01]. For the computations in this paper we have written the procedures SyzFo1, MinFo1, BadPrV of foliation.lib which are available on the second author's webpage. 4

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 $^{^4}$ http://w3.impa.br/ \sim hossein/foliation-allversions/foliation.lib

Name $ \begin{pmatrix} f(x,y) = 0 \\ \begin{pmatrix} P_0 & Q_0 \\ P_{\infty} & Q_{\infty} \end{pmatrix} $	Figure
Riccati $\begin{pmatrix} f(x) \\ 1 & 0 \\ 0 & f(x) \end{pmatrix}$	
Quasi-homogeneous $\begin{pmatrix} f(x,y) \\ f_x & f_y \\ -\frac{\alpha_2}{d}y & \frac{\alpha_1}{d}x \end{pmatrix}$	
Graph $\begin{pmatrix} y - f(x) \\ ya_1(x) + a_2(x) & a_3(x) \\ yb_1(x) + b_2(x) & b_3(x) \end{pmatrix}$	
Lins Neto $(a = 3)$ $\begin{pmatrix} (x^a - 1)(y^a - 1)(x^a - y^a) \\ -y^{a+1} + y & x^{a+1} - x \\ -x^{a-1}(y^a - 1) & y^{a-1}(x^a - 1) \end{pmatrix}$	$(x^{3}-1)(y^{3}-1)(x^{3}-y^{3}) = 0$
Rose $k = \frac{1}{2}$ $\begin{cases} 4x^4 + 8x^2y^2 + 4y^4 - 4x^6 - 12x^4y^2 - 12x^2y^4 - 4y^6 - y^2 \\ 8 \cdot x^3 - xy^2 & 11 \cdot x^2y + 2 \cdot y^3 - y \\ 5 \cdot x^2y - 4 \cdot y^3 + 2 \cdot y & x^3 + 10 \cdot xy^2 - x \end{cases}$	
$2(2x^{2} - a^{2})^{2} - a(2y - a)^{2}(y + a)$ Lissajous $x = a\cos(3t), y = a\cos(4t)$ $\begin{pmatrix} -16 \cdot xy^{2} - 8a \cdot xy + 8a^{2} \cdot x & 12 \cdot x^{2}y + 6a \cdot x^{2} - 6a^{2} \cdot y - 3a^{3} \\ -16 \cdot x^{2}y - 8a \cdot y^{2} + 4a^{2} \cdot y + 4a^{3} & 12 \cdot x^{3} + 6a \cdot xy - 9a^{2} \cdot x \end{pmatrix}$	a = -1
Deltoid $ \begin{pmatrix} (x^2 + y^2)^2 + 8ax(x^2 - 3y^2) + 18a^2(x^2 + y^2) - 27a^4 \\ (x^2 - 3 \cdot y^2 + 6a \cdot x + 9a^2 & 4 \cdot xy - 6a \cdot y \\ -4 \cdot xy + 6a \cdot y & 3 \cdot x^2 - y^2 + 6a \cdot x - 9a^2 \end{pmatrix} $	

Table 1: Curves and foliations

2 Milnor vector space

Recall the definition of a tame polynomial in Introduction. In the following by degree of a polynomial we mean the weighted degree in k[x, y], $deg(x) = \alpha_1$, $deg(y) = \alpha_2$.

Proposition 1. Let $f \in k[x, y]$ be tame polynomial of weighted degree d in k[x, y], $deg(x) = \alpha_1$, $deg(y) = \alpha_2$. There is a basis of monomials $x^i y^j$ for the Milnor vector space

$$V_f := \frac{\mathsf{k}[x, y]}{\mathrm{jacob}(f)} \cong \frac{\Omega^2_{\mathbb{A}^2_{\mathsf{k}}}}{df \wedge \Omega^1_{\mathbb{A}^2}}$$

with weighted degree $\leq 2d - 2\alpha_1 - 2\alpha_2$, and among these monomials, only one monomial is of the highest usual degree $2d - 2\alpha_1 - 2\alpha_2$.

Proof. The proposition for f = g a homogeneous polynomial in a weighted ring k[x, y], deg $(x) = \alpha_1$, deg $(y) = \alpha_2$ is a classical fact due to V. Arnold and K. Saito, see [AGV85, Corollary 4, page 200]. Actually in this case, all the monomials of usual degree $> 2d - 2\alpha_1 - 2\alpha_2$ are zero in V_g . For an arbitrary f with the last homogeneous piece g in the same weighted ring, we first find a monomial basis with the desired property for V_g and it follows that the same set of monomials form a basis of V_f , see [Mov07, §6] or [Mov19, Proposition 10.7]. In this context, monomials $x^i y^j$ with $i\alpha_1 + j\alpha_2 > 2d - 2\alpha_1 - 2\alpha_2$ are not necessarily zero in V_f and we only know that they are equivalent in V_f to polynomials of degree $\leq 2d - 2\alpha_1 - 2\alpha_2$.

3 Quasi-homogeneous singularities

Let f = 0 be a germ of curve singularity given by a germ of a holomorphic function $f \in \mathcal{O}_{\mathbb{C}^2,0}$.

Theorem 2 (K. Saito [Sai80] page 270). The $\mathcal{O}_{\mathbb{C}^2,p}$ -module of holomorphic 1-forms tangent to f = 0 is freely generated if and only if it has two elements ω_0, ω_∞ such that $\omega_0 \wedge \omega_\infty = f dx \wedge dy$.

Quasi-homogeneous singularities are the main example of singularities satisfying Theorem 2.

Definition 2. A germ of curve singularity f = 0 is called quasi-homogeneous if there is a holomorphic change of coordinates (x, y) in $(\mathbb{C}^2, 0)$ such that (x(0), y(0)) = (0, 0) and $f(x, y) = a(x, y) \cdot g(x, y), \quad a \in \mathcal{O}_{\mathbb{C}^2,0}, a(0) \neq 0$ and g is a quasi-homogeneous polynomial in x, y, that is, there are $\alpha_1, \alpha_2 \in \mathbb{Z}$ such that g is homogeneous in the weighted ring $\mathbb{C}[x, y], \ \deg(x) = \alpha_1, \ \deg(y) = \alpha_2.$

Let $\eta := \frac{1}{\deg(g)}(\alpha_1 x dy - \alpha_2 y dx)$. We have $dg \wedge \eta = g dx \wedge dy$, and so by Theorem 2, the module of 1-forms tangent to g = 0 is freely generated by dg, η .

Theorem 3 (K. Saito [Sai71]). A germ of curve singularity f = 0 is quasi-homogeneous if and only if f belongs to the Jacobian ideal $\operatorname{jacob}(f) := \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u} \rangle$.

4 Proof of Theorem 1

In V_f we consider the linear map A_f defined by

$$A_f: V_f \to V_f, A_f([P]) = [Pf]$$

Let p(t) be the minimal polynomial of A_f . The critical values of f are exactly the zeros of p(t), see for instance [CLO05, Proposition 2.7 page 150] [Mov19, §10.9]. If $0 \in \bar{k}$ is not a critical value of f then $\{f = 0\}$ is a smooth curve. Suppose that f = 0 is singular, and hence, A_f has a non-trivial kernel. We write p(t) = tq(t). By definition of a minimal polynomial

(2)
$$fq(f) = 0, \ q(f) \neq 0 \text{ in } V_f.$$

The polynomial q(f) is of big degree. Proposition 1 implies that we can simplify q(f) in V_f and obtain a polynomial Θ_f of weighted degree $\leq 2d - 2\alpha_1 - 2\alpha_2$ which is equal to q(f) in V_f . We obtain

(3)
$$f\Theta_f dx \wedge dy = df \wedge \omega_f,$$

Note that we do not have any control on the degree of ω_f . Note also that Θ_f and ω_f depend on the monomial basis that we have chosen in Proposition 1.

The exponent of the affine curve f = 0 is the number m in $p(t) = t^m Q(t)$, $Q(0) \neq 0$. The exponent of a curve is also the maximum size of the Jordan blocks of A_f associated to the eigenvalue 0. The exponent of a critical point p of f with f(p) = 0 is the minimum number \tilde{m} such that $f^{\tilde{m}}$ is zero in the local Milnor vector space $V_p := \mathcal{O}_{\mathbb{A}^2_k, p}/\text{jacob}(f)$.

Proposition 2. Let k be an algebraically closed field of characteristic zero. We have an isomorphism

$$V_f \cong \bigoplus_{p \in P_f} \frac{\mathcal{O}_{\mathbb{A}^2_{\mathsf{k}}, p}}{\mathrm{jacob}(f)},$$

induced by canonical restriction, where p runs through the set P_f of critical points of f. In particular, each piece in the above summand is invariant under the multiplication by f map A_f . Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the critical values of f and m_i , $i = 1, 2, \ldots, k$ be the maximum of exponents of the singularities in $f(x, y) - \lambda_i = 0$. Then the minimal polynomial of A_f is $p(t) := (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$.

Proof. This is an immediate corollary of a well-known fact in commutative algebra, see [CLO05, Theorem 2.2]. It also follows from Max Noether's theorem, see [GH94, page 703]. For the surjectivity of the restriction map, we must modify the proof of Noether's theorem. Note that $A_f - f(p)$ Id restricted to $\frac{\mathcal{O}_{\mathbb{A}^2_k,p}}{\operatorname{jacob}(f)}$ is nilpotent of order which is the exponent of p.

Theorem 1 is a consequence of the following:

Theorem 4. We have

- 1. If there exists $\Theta \in \text{Kernel}(A_f)$ such that $\Theta \cdot V_f = \text{ker}(A_f)$ then E_f is generated by $fdx, fdy, df, \omega_{\Theta}$, where $f\Theta dx \wedge dy = df \wedge \omega_{\Theta}$.
- 2. If the Jordan blocks of A_f associated to the eigenvalue 0 have the same size then Θ_f satisfies $\Theta_f \cdot V_f = \ker(A_f)$.

3. If all the singularities of C are quasi-homogeneous then the Jordan blocks of A_f associated to the eigenvalue 0 have the size 1.

Proof. Proof of 1. For $\omega \in E_f$ we have $f\Theta_1 dx \wedge dy = df \wedge \omega$ for some polynomial $\Theta_1 \in \mathsf{k}[x, y]$. This implies that $\Theta_1 \in \ker(A_f)$ and by our hypothesis, we have $\Theta_1 = P\Theta$ in V_f for some $P \in \mathsf{k}[x, y]$. We write this as $\Theta_1 dx \wedge dy - P\Theta dx \wedge dy = df \wedge \eta$. After multiplication of this equality by f we get $(\omega - P\omega_f - f\eta) \wedge df = 0$ which implies the result. De Rham lemma, see for instance [Sai76] in the case of homogeneous polynomials and [Mov19, Proposition 10.3] for tame polynomials, implies that ω is of the desired format.

Proof of 2. It is enough to prove that the sequence

(4)
$$V_f \stackrel{q(A_f)}{\to} V_f \stackrel{A_f}{\to} V_f$$

is exact. For this we can assume that k is algebraically closed, as the exactness of a sequence of vector spaces is independent of whether k is algebraically closed or not. It follows from (2) that $\operatorname{Image}(q(A_f)) \subset \operatorname{Kernel}(A_f)$. The non-trivial assertion is $\operatorname{Kernel}(A_f) \subset$ $\operatorname{Image}(q(A_f))$. Let $V_f = V_1 \oplus V_2$ such that $A_f \mid_{V_1}$ is nilpotent and $A_f \mid_{V_2}$ is invertible. For this we simply write A_f in Jordan block format, V_1 is constructed from the blocks with eigenvalue 0 and V_2 from the remaining blocks. From $p(t) = tq(t) = t^m Q(t), \ Q(0) \neq 0$, it follows that $q(A_f) \mid_{V_1} = A_f^{m-1} \circ B_f$, where $B_f := Q(A_f) \mid_{V_1} : V_1 \to V_1$ is invertible. Therefore, we can assume that $p(t) = t^m, q(t) = t^{m-1}$ and Q = 1. Now, we use the following statement in linear algebra. Let A be an $n \times n$ matrix with entries in \bar{k} and all eigenvalues equal to zero. Let also m be the maximal size of its Jordan blocks. Then $\operatorname{Kernel}(A) = \operatorname{Image}(A^{m-1})$ if and only if all the Jordan blocks of A are of the same size. In order to see this fact, let

$$e_1^1, e_2^1, \dots, e_{k_1}^1, e_1^2, e_2^2, \dots, e_{k_2}^2, \dots, e_1^s, e_2^s, \dots, e_{k_s}^s, k_1 \le k_2 \le \dots \le k_s = m.$$

be a basis of \bar{k}^n such that A in each block $e_1^j, \ldots, e_{k_j}^j$ is a shifting map and $A_f(e_{k_j}^j) = 0$. By definition of m we have $\operatorname{Image}(A^{m-1}) \subset \operatorname{Kernel}(A)$. Moreover, $\operatorname{Kernel}(A) \subset \operatorname{Image}(A^{m-1})$ if and only if $k_1 = k_2 = \cdots = k_s$.

Proof of 3. The third part follows from Proposition 2 and the fact that the exponent of a quasi-homogeneous singularity is 1. $\hfill \Box$

Remark 1. Let $[\Theta_i]$, i = 1, 2, ..., a be a basis for the k-vector space ker (A_f) and write $df \wedge \omega_i = f \Theta_i dx \wedge dy$. The k[x, y]-module E_f is generated by $f dx, f dy, df, \omega_i, i = 1, 2, ..., a$. This is an immediate consequence of definitions.

Remark 2. A careful analysis of the proof of Theorem 1 shows that this theorem is true for curves f = 0 such that the multiplication by f map in its Milnor vector space V_f has Jordan blocks of the same size. Jordan blocks of size 1 correspond to quasi-homogeneous singularities. In Appendix A, C. Hertling proves that there is no curve singularity such that multiplication by f in its Milnor vector space has only Jordan blocks of size 2.

Proposition 3. A foliation which leaves a smooth curve $C \in \mathbb{P}^2_k$ invariant is of the form

$$\mathcal{F}(Pdf + f\omega), \ \omega \in \Omega^1_{\mathbb{A}^2_k}, \ P \in \Omega^0_{\mathbb{A}^2_k}$$

in an affine chart $\mathbb{A}^2_{\mathsf{k}} \subset \mathbb{P}^2_{\mathsf{k}}$. In other words the $\mathsf{k}[x, y]$ -module E_f defined in (1) is generated by fdx, fdy, df.

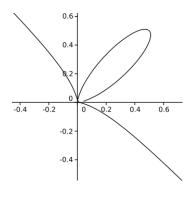


Figure 1: $x^5 + y^5 - x^2y^2 = 0$

Proof. We take a line in \mathbb{P}^2_k which intersects C transversally at deg(C) points, and hence, in the affine chart which is its complement, the curve C is given by the tame polynomial f(x,y) = 0. If $\omega \wedge df = fPdx \wedge dy$ then P is in the kernel of the map A_f . Since f = 0 is smooth, we conclude that P itself is zero in V_f , and hence, $Pdx \wedge dy = df \wedge \alpha$ for some $\alpha \in \Omega^1_{\mathbb{A}^2}$. Therefore, $df \wedge (f\alpha - \omega) = 0$. De Rham lemma for tame polynomials, see [Mov19, Proposition 10.3], implies that ω is of the desired format. \square

Example 1. The singularity (0,0) of the curve $f := x^5 + y^5 - x^2y^2 = 0$ is called A'Campo singularity and it is not quasi-homogeneous. The polynomial f has two critical values $s = 0, -\frac{16}{3125}$. Over the critical value s = 0, we have only the singularity $(0,0) \in \mathbb{A}^2_k$ of Milnor number 11. In this case A_f has 10 Jordan blocks, one is of size 2 and 9 of size 1. Over the critical value $s = -\frac{16}{3125}$ we have 5 singularities of Milnor number 1. In this case, A_f has 5 Jordan blocks of size 1. The Macaulay code gives us the following three generators of E_f :

loadPackage "VectorFields" R=QQ[x,y]; f=f=x^5+y^5-x^2*y^2; derlog(ideal (f)); -->image | -25x2y2+6xy -25x3v+5v3+4x2 -5x4+3xv2 | -25xy3+5x3+4y2 -25x2y2+6xy -5x3y+2y3 |

The columns $[P,Q]^{\text{tr}}$ of the output matrix are vector fields $P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$. Such three vector fields generate the $\mathbb{Q}[x,y]$ -module of vector fields tangent to f = 0. The corresponding 1-forms must be written as Pdy - Qdx. Let $\omega_1, \omega_2, \omega_3$ be such 1-forms. The minimal polynomial of A_f turns out to be $t^2(3125t + 16)$ and

$$\omega_f = -\frac{4}{25}x^2y\omega_1.$$

This implies that ω_1 is not in the module generated by df, fdx, fdy, ω_f . The real locus of f = 0 is depicted in Figure 1. For our computations we have used the following code.

LIB "foliation.lib";

ring r=0, (x,y),dp; poly f=x5+y5-x2*y2; matrix A=mulmat(f,f); list ll=jordan(A); print(jordanmatrix(ll));

vector v1=[-(-25xy3+5x3+4y2), -25x2y2+6xy]; vector v2=[-(-25x2y2+6xy),-25x3y+5y3+4x2]; vector v3=[-(-5x3y+2y3),-5x4+3xy2];

vector df=[diff(f,x),diff(f,y)]; vector fdy=[0,f]; vector fdx=[f,0]; module m=v1,v2,v3; division(df,m); division(fdx,m); division(fdy,m);

poly disc=Discrim(f); list l=factorize(disc); disc=x^2*(3125x+16);

5 A consequence of Quillen-Suslin theorem

In Theorem 1 we have not assumed that $\mathsf{k} \subset \mathbb{C}$ is an algebraically closed field. It turns out that if we use the algebraic closure of k then E_f is free of rank 2.

Proposition 4. Let $f \in k[x, y]$ be a tame polynomial and assume that f = 0 is smooth or at most it has quasi-homogeneous singularities. The $\bar{k}[x, y]$ -module $E_f \otimes_k \bar{k}$ is free of rank 2.

Proof. The k[x, y]-module E_f is not necessarily projective, however, $E_f \otimes_k \bar{k}$ turns out to be projective. The reason is as follows. We compactify \mathbb{A}_k^2 in \mathbb{P}_k^2 and consider the curve C induced by f = 0 in \mathbb{P}_k^2 . For $\alpha_1 = \alpha_2 = 1$ this curve intersects the line at infinity transversely, and hence, it has no singularity at infinity. However, for arbitrary α_i 's it might have singularities at infinity. We perform a desingularization of singularities of C at infinity and get a surface M defined over k and with a chart $\mathbb{A}_k^2 \subset M$. The curve given by f = 0 in \mathbb{A}_k^2 induces a curve (we call it again C) in M and it has now only smooth points in the compactification divisor $M - \mathbb{A}_k^2$. Note that for all these we do not need to assume that k is algebraically closed. Let M^{an} be the underlying complex manifold of M/k for a fixed embedding $k \subset \mathbb{C}$.

Let S^{alg} be the subsheaf of $\Omega^1_{M/k}$ containing differential 1-forms tangent to the curve given by C in M. In a similar way we define $S^{\text{an}} \subset \Omega^1_{M^{\text{an}}}$ in the holomorphic context. By definition S^{an} is the analytification of the algebraic sheaf $S^{\text{alg}} \otimes_k \mathbb{C}$. Theorem 2 applied to quasi-homogeneous singularities implies that S^{an} is a locally free sheaf. Note that here we use the desingularization process as above: the points at infinity of f = 0 are smooth. By Serre's GAGA, $S^{\text{alg}} \otimes_k \mathbb{C}$ is also a locally free sheaf. Now we look at $S^{\text{alg}} \otimes_k \mathbb{C}$ in the affine chart $\mathbb{C}^2 \subset M^{\text{an}}$ and conclude that $S^{\text{alg}} \otimes_k \bar{k}$ is a locally free sheaf in \mathbb{A}^2_k . For an algebraically closed field \bar{k} , locally free sheaves over $\mathbb{A}^2_{\bar{k}}$ are in one to one correspondence with projective modules: the correspondence is given by taking global sections. Now by Quillen-Suslin theorem, see for instance [Lan02, Theorem 3.7, page 850] projective $\bar{k}[x, y]$ modules are free. Note that for Quillen-Suslin theorem we do not need that the base field is algebraically closed, however, for the correspondence with locally free sheaves we need this fact.

Remark 3. In general, we do not know how to write the local generators of \mathcal{S}^{alg} around each point, however, for smooth curves this is as follows: In the chart $U_0 : f_x \neq 0$ (resp. $U_1 : f_y \neq 0$) it is freely generated by df, fdy (resp. df, fdx). For instance, $fdx = \frac{f}{f_{-}}df - \frac{f_y}{f_{-}}fdy$. If f = 0 is smooth, U_0, U_1 cover \mathbb{A}^2_k .

Remark 4. If the module E_f has two elements ω_0, ω_∞ with $\omega_0 \wedge \omega_\infty = f dx \wedge dy$ then they generate E_f as k[x, y]-module. This is as follows. By Theorem 2, the sheaf \mathcal{S}^{an} is locally free, and hence a similar argument as in Proposition 4 implies that $E_f \otimes_k \mathbb{C}$ is generated by two elements $\check{\omega}_0, \check{\omega}_\infty$. Moreover, these two elements generate the stalk of \mathcal{S}^{an} at any point of \mathbb{C}^2 . It follows that $\check{\omega}_0 \wedge \check{\omega}_\infty = f dx \wedge dy$. We write ω_0, ω_∞ in terms of $\check{\omega}_0, \check{\omega}_\infty$ and we get a 2×2 matrix which has non-zero constant determinant.

Let ω_0, ω_∞ be two generators of the $\bar{\mathsf{k}}[x, y]$ -module $E_f \otimes_{\mathsf{k}} \bar{\mathsf{k}}$. It turns out that we have a finite extension K of k such that ω_0 and ω_∞ are defined over K and $E_f \otimes_{\mathsf{k}} \mathsf{K}$ is freely generated by ω_0, ω_∞ . It is a natural question to bound $[\mathsf{K} : \mathsf{k}]$ in terms of some arithmetic invariants of the curve C. The case of a circle might be enlightening, see Example 2.

6 Examples

In this section we explain many examples of curves f = 0 such that E_f is generated by two elements (including those in Table 1). In all these examples E_f is generated by two foliations ω_0, ω_∞ with $\omega_0 \wedge \omega_\infty = f dx \wedge dy$. The polynomial 1-form ω_f is usually big and so we do not reproduce it here. All the figures in Table 1 are the real locus of f = 0, except for Lins Neto's example, whose figure is an artistic way to depict the arrangement of lines and it is taken from the original article [Lin02].

Example 2. The module of 1-forms tangent to f := xy - 1 = 0 is freely generated by

$$\omega_0 := ydf - fdy = y^2 dx + dy, \ \omega_\infty := fdx,$$

which satisfy $\omega_0 \wedge \omega_\infty = -f dx \wedge dy$. This follows from Proposition 3 and the identities:

(5)
$$df = x\omega_0 + y\omega_\infty, \quad fdy = f\omega_0 - y^2\omega_\infty.$$

Over $\mathbb{Q}(i)$ this curve is isomorphic to $f := x^2 + y^2 - 1 = 0$. In fact, the transformation $(x, y) \mapsto (x + iy, x - iy)$ sends the circle $x^2 + y^2 - 1 = 0$ to xy - 1 = 0. The pull-back of the above differential forms is

$$\omega_0 = \Re(\omega_0) + i\Im(\omega_0) := (x^2 - y^2 + 1)dx + 2xydy + i((x^2 - y^2 - 1)dy - 2xydx), \omega_\infty = \Re(\omega_\infty) + i\Im(\omega_\infty) = fdx + ifdy.$$

It would be interesting to prove that the submodule of $\Omega^1_{\mathbb{A}^2_{\mathbb{Q}}}$ tangent to $x^2 + y^2 - 1$ is not generated by two elements defined over \mathbb{Q} . For our computations in this example we have used

LIB "foliation.lib"; ring r=(0,2), (x,y), dp; minpoly=z^2+1; poly f=x*y-1; matrix P=transpose(MinFol(f,1)[1]); P; vector df=[diff(f,x),diff(f,y)]; MinFol(f,1, df); vector fdy=[0,f]; MinFol(f,1, fdy); poly g=x^2+y^2-1; vector v1=[(x^2-y^-2+1),2*x*y]; vector v1=[-2*x*y, (x^2-y^-2-1)]; vector v2=[g,0]; vector w2=[0,g]; module m=v1+z*w1,v2+z*w2; division(v1,m); division(v1,m); division(v2,m); division(w2,m); division(v1-z*w1,m); division(v2-z*w2,m); vector dg=[diff(g,x),diff(g,y)]; division(dg,m); MinFol(g,1, dg);

Example 3 (Riccati). Let us consider the case in which f does not depend on y, and hence, f = 0 is a union of d parallel to y axis lines. In this case $\omega_0 = dx$ and $\omega_{\infty} = f(x)dy$. The Riccati foliations given by $(p_2(x)y^2 + p_1(x)y + p_0(0))dx + f(x)dy$ are in E_f .

Example 4 (Quasi-homogeneous singularities). These are curves C : f = 0 given by a homogeneous tame polynomial f of weighted degree d in the weighted ring k[x, y], $deg(x) = \alpha_1$, $deg(y) = \alpha_2$, see Definition 1. This is also the algebraic version of Definition 2. In this case E_f is generated by $\omega_0 = df, \omega_\infty := \frac{\alpha_1}{d} x dy - \frac{\alpha_2}{d} y dx$.

Example 5. The arrangement of lines given by

$$f = (x^{a} - 1)(y^{a} - 1)(x^{a} - y^{a}).$$

for a = 3 has been studied by Lins Neto in [Lin02]. In this case E_f is generated by

$$\omega_0 := -(y^a - 1)x^{a-1})dx + (x^a - 1)y^{a-1}dy, \quad \omega_\infty := -(y^{a+1} - y)dx + (x^{a+1} - x)dy,$$

The birational $\mathbb{A}_{k}^{2} \to \mathbb{A}_{k}^{2}$ given by $(x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$ in the affine chart (x, y) maps ω_{0} to ω_{∞} and vice versa. For a = 3 one can prove that $\mathcal{F}(\omega_{0} + t\omega_{\infty})$, $t \in \mathbb{P}_{k}^{1}$ has a first integral if and only if t is a constant in $\mathbb{Q}(e^{\frac{2\pi i}{3}})$. The degree of such a first integral has been computed in [Med13]. Another example due to Lins Neto is $f = 4y^{2}(1 - 3x) - 4x^{3} + (3x^{2} + y^{2})^{2}$ which is up to a linear transformation is the deltoid in Table 1.

Example 6. The arrangement of lines given by

$$f = (x^2 - 1)(y^2 - 1)(x^2 - (2\tau + 1)^2)(y^2 - (2\tau + 1)^2) \cdot (x^2 - y^2)$$
$$\cdot (y + 1 + \frac{1}{\tau}(x+1))(y + 1 + \tau(x+1)) \cdot (y - 1 + \tau(x-1))(y - 1 + \frac{1}{\tau}(x-1))$$

with $\tau^2 - \tau - 1 = 0$ is studied in [MP05] and in this case E_f is generated by $\omega_0 := -(y^2 - (2\tau + 1)^2)(y^2 - 1)(y + (2\tau - 1)x)dx + (x^2 - (2\tau + 1)^2)(x^2 - 1)(x + (2\tau - 1)y)dy$ and ω_{∞} whose expression can be found in the mentioned reference. Another example from this reference is $f = -1728x^5 + 720x^3y - 80xy^2 + 64(5x^2 - y)^2 + y^3$. In [KN88] the authors describe the 1-forms $\omega_0, \omega_{\infty}$ in another coordinate system, and it turn out that these are defined over \mathbb{Q} .

Example 7. [Graph of a function] In this example we consider a curve of the form y = f(x) which is smooth and of genus zero. Let $d := \deg(f)$. For $d = 2, 3, \dots, 11$ we can verify the following conjecture: For a generic polynomial $f(x) \in k[x]$ of degree d (in a Zariski open subset of the parameter space of f), the k[x, y]-module E_f is generated by

$$\omega_0 := (ya_1(x) + a_2(x))dx + a_3(x)dy, \quad \omega_\infty := (yb_1(x) + b_2(x))dx + b_3(x)dy,$$

where for d even $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathsf{k}[x]$ are respectively of degree $\frac{d}{2} - 2, \frac{d}{2}, \frac{d}{2} - 1, \frac{d}{2} - 1, \frac{d}{2} - 1, \frac{d}{2}, \frac{d}{2$

$$a_1b_2 - a_2b_1 = -f', \quad fa_1 + a_2 + f'a_3 = 0, \quad fb_1 + b_2 + f'b_3 = 0.$$

From these three equalities we can derive

$$a_1b_3 - a_3b_1 = 1$$
, $a_2b_3 - a_3b_2 = -f$.

We can also verify the existence of ω_0 and ω_∞ for random choices of $f \in \mathbb{Q}[x, y]$ of higher degree. The three canonical foliations can be written in terms of generators:

$$(y-f)dx = b_3\omega_0 - a_3\omega_\infty,$$

$$(y-f)dy = -(yb_1 + b_2)\omega_0 + (ya_1 + a_2)\omega_\infty$$

$$d(y-f) = -b_1\omega_0 + a_1\omega_\infty.$$

For the computations in this example we have used the following code:

LIB "foliation.lib"; int d=8; ring r=(0,t(0..d-2)), (x,y), dp; int i=1; int j; poly f=-y+x^d; for (i=0; i<=d-2; i=i+1){f=f+t(i)*x^i;} //--Use the next command for a random choice of f. //--poly f=RandomPoly(list(x,y),d,-19,10); f=subst(f,y,1); f=f-y; matrix P=transpose(MinFol(f,1)[1]); poly co=det(P)/f; co*f=det(P); matrix Q1=diff(P,y); intmat degQ1[2][2]; matrix Q2=P-y*Q1; intmat degQ2[2][2]; for (i=1; i<=2; i=i+1) {for (j=1; j<=2; j=i+1){degQ1[i,j]=deg(Q1[i,j]); degQ2[i,j]=deg(Q2[i,j]);}} f; P; degQ1; degQ2;

Example 8 (Rose with $k = \frac{1}{2}$). In this case the foliation $\mathcal{F}(\omega_0)$ has the first integral $F := \frac{(x^2+y^2-1/3)^3}{36x^2+9y^2-4}$. Its generic fiber is smooth and has two singular points at infinity. It has three critical values $t = 0, \frac{1}{108}, \infty$. The fiber over $t = \frac{1}{108}$ is our initial curve f = 0 which has single non-degenerated singularity (Milnor number equal to one). Since f = 0

is a rational curve and all the fibers intersects the line at infinity in the same way, we conclude that the genus of a generic fiber of F is one. If we set

 $\check{\omega}_0 = (8x - y)dx + (11x + 2y - 1)dy, \quad \check{\omega}_\infty = (5xy - 4y^2 + 2y)dx + (x^2 + 10xy - x)dy,$ and $\pi : \mathbb{C}^2 \to \mathbb{C}^2 : (x, y) \mapsto (x^2, y^2)$ then we have

 $\pi^*\check{\omega}_0 = 2\omega_0, \ \pi^*\check{\omega}_\infty = 2xy\omega_\infty,$

This shows that this example is birational to the case of a quasi-homogeneous singularity. We can use Katz-Grothendieck conjecture for vector fields in order to investigate whether a foliation by curves has a first integral or not. For this we have written the code BadPrV which computes the bad and good primes of a vector field. For instance, before computing the first integral of $\mathcal{F}(\omega_0)$ by hand, we used this in order to be sure that such a foliation has a first integral.

LIB "foliation.lib"; ring r=(0,t), (x,y), dp; poly f=4*x²4+8*x²2*y²44*x⁴6-12*x⁴4*y²-12*x²*y⁴-4*y⁶-y²; matrix P=transpose(MinFol(f,1)[1]); P; poly l=(x²+ y²-1/3)³-(36*x² + 9*y²-4)*t; (P[1,1]*diff(1,y)-P[1,2]*diff(1,x))/l; list vf=P[1,2], -P[1,1]; BadPrV(vf, 40);

The 19th century has produced a lot of curves which are named after many engineers, astronomer and mathematicians. The website mathcurve.com contains a rather full list of such curves. Among all these curves f = 0, those with E_f generated by two elements, seem to be rare. The Lissajous and deltoid are among them, as shown in Table 1.

A Nonexistence of a curve singularity f where all Jordan blocks of the multiplication by f in its Milnor vector space have size 2 (By Claus Hertling)

Proposition 5. Let $f \in \mathbb{C}\{x, y\}$ be a holomorphic function germ with f(0) = 0 and with an isolated singularity at $0 \in \mathbb{C}^2$. Let A_f be the multiplication by f in its Milnor vector space $Q_f := \mathbb{C}\{x, y\}/\text{jacob}(f)$. Then A_f is nilpotent and all Jordan blocks of A_f have size 1 or 2. Not all Jordan blocks of A_f have size 2.

Let us call a singularity $f \in \mathbb{C}\{x, y\}$ such that all Jordan blocks of A_f have size 2 Jordan block extreme. Proposition 5 says that such singularities do not exist. If they would exist, one could allow in Theorem 1 that either all singularities on C are quasi-homogeneous or that all singularities on C are Jordan block extreme. This follows from Theorem 4.

The nonexistence of Jordan block extreme singularities is not really surprising. But it is also not trivial. The following proof uses the Gauss-Manin connection, the spectral numbers and the fact that Hertling's variance conjecture on the spectral numbers is true in the case of curve singularities [Bré04]. The idea is to show that a Jordan block extreme curve singularity would have spectral numbers which violate the variance conjecture.

Proof of Proposition 5: First we have to review the spectral numbers and their background. For this, we start with an arbitrary holomorphic function germ $f \in \mathcal{O}_{\mathbb{C}^{n+1},0} =$

 $\mathbb{C}\{x_0, ..., x_n\}$ with f(0) = 0 and with an isolated singularity at 0, here $n \ge 1$. Two references in book form for the following are [AGZV88] and [Her02].

Choose a good representative $\tilde{f} : X \to \Delta$, with $B_{\varepsilon} \subset \mathbb{C}^{n+1}$ a small ball around 0, $\Delta = \Delta_{\delta} \subset \mathbb{C}$ a very small disk around 0 (so $0 < \delta \ll \varepsilon \ll 1$) and $X := B_{\varepsilon} \cap f^{-1}(\Delta)$. Consider its *cohomology bundle* $\bigcup_{t \in \Delta^*} H^n(\tilde{f}^{-1}(t), \mathbb{C})$, here $\Delta^* := \Delta - \{0\}$. It is a flat vector bundle of rank $\mu \in \mathbb{N}$. The covariant derivative on it with respect to the coordinate vector field of the coordinate t on Δ is called ∂_t . We define

$$\begin{aligned} Q_f &:= \mathcal{O}_{\mathbb{C}^{n+1},0}/(\operatorname{jacob}(f)) & \text{Milnor vector space}, \\ \Omega_f &:= \Omega_{\mathbb{C}^{n+1},0}^{n+1}/df \wedge \Omega_{\mathbb{C}^{n+1},0}^n, \\ H_0'' &:= \Omega_{\mathbb{C}^{n+1},0}^{n+1}/df \wedge d\Omega_{\mathbb{C}^{n+1},0}^{n-1} & \text{Brieskorn lattice}, \\ H_0' &:= df \wedge \Omega_{\mathbb{C}^{n+1},0}^n/df \wedge d\Omega_{\mathbb{C}^{n+1},0}^{n-1}, \end{aligned}$$

$$\begin{split} V^{>-\infty} &:= \{ \text{germs at } 0 \text{ of sections in the cohomology bundle of moderate growth} \}, \\ C^{\alpha} &:= \{ \sigma \in V^{>-\infty} \, | \, (t\partial_t - \alpha)^{n+1}(\sigma) = 0 \}, \\ V^{\alpha} &:= \bigoplus_{\beta \in [\alpha, \alpha + 1]} \mathbb{C}\{t\} \cdot C^{\beta}, \\ V^{>\alpha} &:= \bigoplus_{\beta \in (\alpha, \alpha + 1]} \mathbb{C}\{t\} \cdot C^{\beta}. \end{split}$$

 Q_f and Ω_f are \mathbb{C} -vector spaces of dimension μ . $V^{>-\infty}$ is $\mathbb{C}\{t\}[t^{-1}]$ -vector space of dimension μ . It contains the spaces C^{α} , V^{α} , $V^{>\alpha}$, H''_0 and H'_0 . For any $\alpha \in \mathbb{R}$

$$V^{>-\infty} = \bigoplus_{\beta \in [\alpha, \alpha+1)} \mathbb{C}\{t\}[t^{-1}]C^{\beta},$$

 $V^{\alpha}, V^{>\alpha}, H_0''$ and H_0' are free $\mathbb{C}\{t\}$ -modules of rank μ . And V^{α} for $\alpha > -1, V^{>\alpha}$ for $\alpha \ge -1, H_0''$ and H_0' are free $\mathbb{C}\{\{\partial_t^{-1}\}\}$ -modules of rank μ ($\mathbb{C}\{\{\partial_t^{-1}\}\}$ is the ring of power series of Gevrey class 1 in ∂_t^{-1} .) The Brieskorn lattice H_0'' satisfies

$$V^{>-1} \supset H_0'' \supset V^{n-1}.$$

We have the \mathbb{C} -vector space isomorphisms

$$\begin{aligned} t: V^{\alpha} &\to V^{\alpha+1}, \\ t: V^{>\alpha} &\to V^{>\alpha+1}, \\ \partial_t^{-1}: V^{\alpha} &\to V^{\alpha+1} \text{ for } \alpha > -1, \\ \partial_t^{-1}: V^{>\alpha} &\to V^{>\alpha+1} \text{ for } \alpha \ge -1, \\ \partial_t^{-1}: H_0'' &\to H_0', \\ \partial_t^{-1}: H_0'' &\to H_0', \\ \Omega_f &\cong H_0''/H_0', \\ C^{\alpha} &\cong V^{\alpha}/V^{>\alpha} =: \operatorname{Gr}_V^{\alpha}V^{>-\infty} \end{aligned}$$

The space Ω_f inherits from $H_0'' \subset V^{>-1}$ a V-filtration, as follows,

$$V^{\alpha}\Omega_f := \frac{V^{\alpha} \cap H_0'' + H_0'}{H_0'} \subset \frac{H_0''}{H_0'} = \Omega_f,$$

with quotients

$$\operatorname{Gr}_{V}^{\alpha}\Omega_{f} = \frac{V^{\alpha}\Omega_{f}}{V^{>\alpha}\Omega_{f}} \cong \frac{\operatorname{Gr}_{V}^{\alpha}H_{0}''}{\operatorname{Gr}_{V}^{\alpha}H_{0}'} \cong \frac{H_{0}'' \cap V^{\alpha} + V^{>\alpha}}{H_{0}' \cap V^{\alpha} + V^{>\alpha}}.$$

The spectral numbers of the singularity f are the μ rational numbers $\alpha_1, ..., \alpha_{\mu}$ with

$$\alpha_1 \leq \ldots \leq \alpha_\mu$$
 and $\dim \operatorname{Gr}_V^{\alpha}\Omega_f = \sharp \{j \in \{1, \ldots, \mu\} \mid \alpha_j = \alpha \}.$

The inclusions $V^{>-1} \supset H_0'' \supset V^{n-1}$ and the isomorphism $\Omega_f = H_0''/\partial_t^{-1}H_0''$ show

$$-1 < \alpha_1 \le \dots \le \alpha_\mu < n.$$

Also the symmetry

$$\alpha_j + \alpha_{\mu+1-j} = n-1$$
, or, equivalenty, $\dim \operatorname{Gr}_V^{\alpha} \Omega_f = \dim \operatorname{Gr}_V^{n-1-\alpha} \Omega_f$

holds (one proof of it uses Varchenko's description via H_0'' of Steenbrink's mixed Hodge structure, another one uses K. Saito's higher residue pairings). The following *variance inequality* for the spectral numbers was conjectured by Hertling [Her02],

$$\mu^{-1} \sum_{j=1}^{\mu} (\alpha_j - \frac{n-1}{2})^2 \le \frac{\alpha_\mu - \alpha_1}{12}.$$

For curve singularities (i.e. n = 1), it was proved by Brélivet [Bré04].

The action of t on $V^{>-\infty}$ induces an action on Ω_f which is called t_{Ω_f} . It satisfies

$$t_{\Omega_f}: V^{\alpha}\Omega_f \to V^{\alpha+1}\Omega_f.$$

Therefore t_{Ω_f} is nilpotent. Because of $V^{\alpha_1}\Omega_f = \Omega_f$ and $V^{>\alpha_\mu}\Omega_f = 0$ and $\alpha_1 + (n+1) > \alpha_\mu$, t_{Ω_f} has at most Jordan blocks of size n + 1.

The μ -dimensional \mathbb{C} -vector spaces Q_f and Ω_f are not canonically isomorphic. But the choice of any volume form $u(x)dx_0...dx_n = u(x)dx$ (volume form: $u(0) \neq 0$, i.e. $u(x) \in \mathcal{O}^*_{\mathbb{C}^{n+1},0}$) induces an isomorphism

$$Q_f \to \Omega_f, \quad g \mapsto g \cdot [u(x) \mathrm{d}x],$$

which commutes with the multiplication by f respectively t,

$$\begin{array}{cccc} Q_f & \stackrel{\cong}{\to} & \Omega_f & g \mapsto g \cdot [u(x) \mathrm{d}x] \\ \downarrow A_f & \downarrow t_{\Omega_f} \\ Q_f & \stackrel{\cong}{\to} & \Omega_f & g \mapsto g \cdot [u(x) \mathrm{d}x] \end{array}$$

Therefore A_f and t_{Ω_f} have the same Jordan block structure. Now the review of the spectral numbers and their background is finished.

Now we suppose n = 1, so f is a curve singularity, and we suppose that f is Jordan block extreme, i.e. all Jordan blocks of A_f and t_{Ω_f} have size 2. We will come to a contradiction. μ is even. We have

$$V^{>\alpha_{\mu}-1}\Omega_{f}\subset \ker t_{\Omega_{f}}=\operatorname{im} t_{\Omega_{f}}=t_{\Omega_{f}}(\Omega_{f})=t_{\Omega_{f}}(V^{\alpha_{1}}\Omega_{f})\subset V^{\alpha_{1}+1}\Omega_{f}.$$

Together with $\alpha_1 + 1 > 0 > \alpha_\mu - 1$, so $V^{>\alpha_\mu - 1}\Omega_f \supset V^0\Omega_f \supset V^{\alpha_1 + 1}\Omega_f$, this shows

$$V^{>\alpha_{\mu}-1}\Omega_f = \ker t_{\Omega_f} = \operatorname{im} t_{\Omega_f} = V^{\alpha_1+1}\Omega_f$$

and

$$\{\alpha_1,...,\alpha_{\mu/2}\} \subset [\alpha_1,\alpha_\mu-1], \quad \{\alpha_{\mu/2+1},...,\alpha_\mu\} \subset [\alpha_1+1,...,\alpha_\mu]$$

This distribution of the spectral numbers is already strange. But for a contradiction to the variance inequality, we need more.

For any element $v \in \Omega_f - \{0\}$ denote by $\gamma(v) \in \{\alpha_1, ..., \alpha_\mu\}$ the unique number γ such that $v \in V^{\gamma}\Omega_f - V^{>\gamma}\Omega_f$, and then denote by $\operatorname{Gr}_V^{\gamma(v)}v \in \operatorname{Gr}_V^{\gamma(v)}\Omega_f - \{0\}$ the class of v in $\operatorname{Gr}_{V}^{\gamma(v)}\Omega_{f}.$

We choose elements $v_1, ..., v_{\mu/2} \in \Omega_f$ such that $\gamma(v_j) = \alpha_j$ and such that the classes $\operatorname{Gr}_{V}^{\alpha_{1}}v_{1}, \dots, \operatorname{Gr}_{V}^{\alpha_{\mu/2}}v_{\mu/2} \text{ form a } \mathbb{C} \text{-vector space basis of the sum of quotients } \sum_{\alpha \in \{\alpha_{1}, \dots, \alpha_{\mu/2}\}} Gr_{V}^{\alpha}\Omega_{f}.$ Then any linear combination $v = \sum_{j=1}^{\mu/2} \lambda_j v_j$ with $(\lambda_1, ..., \lambda_{\mu/2}) \neq 0$ is nonzero and has $\gamma(v) \in \{\alpha_1, ..., \alpha_{\mu/2}\}$, so it is not in $V^{>\alpha_{\mu}-1} = \ker t_{\Omega_f}$. Therefore the vector space $\sum_{j=1}^{\mu/2} \mathbb{C} \cdot v_j$ has dimension $\mu/2$, and

$$\Omega_f = \left(\bigoplus_{j=1}^{\mu/2} \mathbb{C} \cdot v_j\right) \oplus \ker t_{\Omega_f}.$$

Recall ker $t_{\Omega_f} = \operatorname{im} t_{\Omega_f}$ and that this subspace of Ω_f has dimension $\mu/2$. Therefore

$$\ker t_{\Omega_f} = \operatorname{im} t_{\Omega_f} = \bigoplus_{j=1}^{\mu/2} \mathbb{C} \cdot t_{\Omega_f}(v_j).$$

Obviously $\gamma(t_{\Omega_f}(v_j)) \geq \alpha_j + 1$, but equality does not necessarily hold, and the classes

 $\operatorname{Gr}_{V}^{\gamma(t_{\Omega_{f}}(v_{j}))} t_{\Omega_{f}}(v_{j}) \text{ for } j \in \{1, ..., \mu/2\} \text{ are not necessarily linearly independent.}$ The following claim replaces the basis $t_{\Omega_{f}}(v_{1}), ..., t_{\Omega_{f}}(v_{\mu/2})$ of $\operatorname{im} t_{\Omega_{f}}$ by a basis $w_{1}, ..., w_{\mu/2}$ of im t_{Ω_f} which fits better to the spectral numbers $\alpha_{\mu/2+1}, ... \alpha_{\mu}$.

Claim: (a) There is a lower triangular matrix $(a_{ij}) \in M_{\mu/2 \times \mu/2}(\mathbb{C})$ with $a_{ii} = 1$ such that the basis

$$(w_1, ..., w_{\mu/2}) = (t_{\Omega_f}(v_1), ..., t_{\Omega_f}(v_{\mu/2})) \cdot (a_{ij})$$

of im t_{Ω_f} satisfies the following: Write $\beta_j := \gamma(w_j)$. The classes $\operatorname{Gr}_V^{\beta_1} w_1, ..., \operatorname{Gr}_V^{\beta_{\mu/2}} w_{\mu/2}$ are linearly independent.

(b) Therefore they form a basis of $\bigoplus_{\alpha \in \{\alpha_{\mu/2+1}, \dots, \alpha_{\mu}\}} \operatorname{Gr}_{V}^{\alpha} \Omega_{f}$, and therefore there is a bijection $\sigma : \{1, ..., \mu/2\} \to \{\mu/2 + 1, ..., \mu\}$ with $\alpha_{\sigma(j)} = \beta_j$ for $j \in \{1, ..., \mu/2\}$. (c) $\beta_j \ge \alpha_j + 1$ for $j \in \{1, ..., \mu/2\}$.

Proof of the Claim: (a) The vectors w_i are constructed inductively in the order $w_{\mu/2}, w_{\mu/2-1}, ..., w_1$. The first step $w_{\mu/2} = t_{\Omega_f}(v_{\mu/2})$ is trivial. Suppose that the vectors $w_{j+1}, ..., w_{\mu/2}$ (and the corresponding entries a_{im}) have been constructed. One constructs w_j be a sequence of steps which give $w_j^{(0)} := t_{\Omega_f}(v_j), w_j^{(1)}, ..., w_j^{(k)} = w_j$ for some $k \ge 0$. Each of these vectors is in $t_{\Omega_f}(v_j) + \bigoplus_{i \ge j+1} \mathbb{C} \cdot w_i$. Suppose that $w_j^{(l)}$ has been constructed. If (1)

$$Gr_V^{\gamma(w_j^{(l)})} w_j^{(l)} \notin \bigoplus_{i: i \ge j+1, \beta_i = \gamma(w_j^{(l)})} \mathbb{C} \cdot \operatorname{Gr}_V^{\beta_i} w_i$$

then l = k and $w_j^{(l)} = w_j$. If

$$Gr_V^{\gamma(w_j^{(l)})}w_j^{(l)}\in \bigoplus_{i:\,i\geq j+1,\beta_i=\gamma(w_j^{(l)})}\mathbb{C}\cdot \mathrm{Gr}_V^{\beta_i}w_i,$$

one adds to $w_j^{(l)}$ a suitable linear combination $\sum_{i: i \ge j+1, \beta_i = \gamma(w_j^{(l)})} a_{ij} \cdot w_i$, such that the sum $w_j^{(l+1)}$ satisfies $\gamma(w_j^{(l+1)}) > \gamma(w_j^{(l)})$. This construction stops at some $w_j^{(k)} = w_j$. (b) This follows immediately from part (a).

(c) The construction in the proof of part (a) gives

$$\beta_j = \gamma(w_j) = \gamma(w_j^{(k)}) > \dots > \gamma(w_j^{(0)}) = \gamma(t_{\Omega_f}(v_j)) \ge \alpha_j + 1$$

(in the case k = 0 without the strict inequalities). This finishes the proof of the Claim. (\Box)

Now we can estimate the variance of the spectral numbers. It is

$$\mu^{-1} \sum_{j=1}^{\mu} \alpha_j^2 = \mu^{-1} \sum_{j=1}^{\mu/2} (\alpha_j^2 + \beta_j^2) = \mu^{-1} \sum_{j=1}^{\mu/2} \left((\alpha_j + \frac{1}{2} - \frac{1}{2})^2 + (\beta_j - \frac{1}{2} + \frac{1}{2})^2 \right)$$

$$= \mu^{-1} \sum_{j=1}^{\mu/2} \left((\alpha_j + \frac{1}{2})^2 + (\alpha_j + \frac{1}{2})(-1) + \frac{1}{4} + (\beta_j - \frac{1}{2})^2 + (\beta_j - \frac{1}{2}) \cdot 1 + \frac{1}{4} \right)$$

$$= \mu^{-1} \sum_{j=1}^{\mu/2} \left(\frac{1}{2} + (\alpha_j + \frac{1}{2})^2 + (\beta_j - \frac{1}{2})^2 + (\beta_j - \alpha_j - 1) \right)$$

$$\geq \mu^{-1} \sum_{j=1}^{\mu/2} \frac{1}{2} = \frac{1}{4} \qquad \text{(here part (c) of the claim is used).}$$

This does not fit to the variance inequality for curve singularities [Bré04],

$$\mu^{-1} \sum_{j=1}^{\mu} \alpha_j^2 \le \frac{\alpha_\mu - \alpha_1}{12} < \frac{2}{12} = \frac{1}{6}.$$

We arrive at a contradiction. A Jordan block extreme curve singularity does not exist. \Box

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