ON ELLIPTIC MODULAR FOLIATIONS, II

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ABSTRACT. We give an example of a one-dimensional foliation $\mathcal{F}$ of degree two in a Zariski open set of a four-dimensional weighted projective space which has only an enumerable set of algebraic leaves. These are defined over rational numbers and are isomorphic to modular curves $X_0(d)$, $d \in \mathbb{N}$, minus cusp points. As a by-product we get new models for modular curves for which we slightly modify an argument due to J. V. Pereira and give closed formulas for elements in their defining ideals. The general belief was that such formulas do not exist and the emphasis in the literature has been on introducing faster algorithms to compute equations for small values of d.

1. INTRODUCTION

One of the central challenges in the theory of holomorphic foliations in compact complex manifolds is to give a criterion or an algorithm which tells us when a foliation has a first integral. G. Darboux is the founding father of this problem, who in [Dar78a] proves that if a foliation in $\mathbb{P}^2$ has an infinite number of algebraic leaves then it has a first integral, and hence all its leaves are algebraic. Motivated by Darboux's result H. Poincaré in [Poi97] obtained partial results in this direction assuming certain type of singularities for the foliation. J. P. Jouanolou in [Jou79] generalizes Darboux's result, with some extra conditions, to codimension one foliations, and E. Ghys in [Ghy00] remarks that such conditions are not necessary. The spirit of all these results are that of Darboux: each algebraic leaf induces an element in a finite-dimensional vector space, and the existence of infinite number of such leaves result in many linear relations among such elements, which will eventually give us the first integral. Similar ideas can be applied for foliation with infinite number of invariant hypersurfaces, see [MC10]. The main topic of many recent works has been to find criterion for Darboux integrability of a foliation, see [Bos01], [PS19] and the references therein. Bounding the degree and genus of leaves of holomorphic foliations of fixed degree is another challenge, which is mainly attributed to P. Painlevé and it has attracted many research, see [Mou01], [PS19] and in particular the introduction of [LN04], [LN02] for the history of this problem. For

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an excellent expository text on both topics the reader is referred to J. V. Pereira's monograph [Per03].

Darboux's theorem in dimensions greater than 3 is false and the first counterexample is the E. Picard's "équation différentielle curieuse" in [Pic89, pages 298–299]. For a moduli space interpretation of Picard's differential equation see §10. Picard uses an evaluation of the Weierstrass $\wp$ function to give an infinite number of algebraic solutions to the Painlevé VI equation with a particular parameter, see also Corollary 2 and 3 in [Lor16]. The next class of examples is due to A. Lins Neto in [LN04], [LN02], where he constructs one-dimensional foliations in $\mathbb{P}^2 \times \mathbb{P}^1$ with a first integral which is the projection in $\mathbb{P}^1$ and for an enumerable subset E of $\mathbb{P}^1$ the foliations in $\mathbb{P}^2 \times \{t\}$, $t \in \mathbb{P}^1$, has a first integral of increasing degree and genus of fibers. For counterexamples in non-algebraic context see [Ghy00], [Win04]. There is no classification of foliations in higher dimensions without a first integral and with only an enumerable set of algebraic leaves with increasing degree and genus. In this article inspired by the concept of join of polynomials in singularity theory, see [AGZV88], we introduce the self-join of foliations and for one example we show that it gives us a new counterexample to Darboux's integrability and the problem of bounding the degree and genus of algebraic leaves. The main ingredient is the geometric interpretation of Ramanujan's differential equation using Gauss–Manin connection of elliptic curves, which appears first in N. Katz's article [Kat73] and is fully elaborated in [Mov08b], see also the lecture notes [Mov12b].

Let $\mathcal{F}(\mathbf{v})$ be the foliation in $(x_2,\, x_3,\, y_2,\, y_3) \in \mathbb{C}^4$ given by:

$$
\begin{aligned}
\mathbf{v} := {}& \left(2x_2 - 6x_3 + \frac{1}{6}(x_2 - y_2)x_2 \right) \frac{\partial}{\partial x_2} + \left(3x_3 - \frac{1}{3}x_2^2 + \frac{1}{4}(x_2 - y_2)x_3 \right) \frac{\partial}{\partial x_3} \\
& - \left(2y_2 - 6y_3 + \frac{1}{6}(y_2 - x_2)y_2 \right) \frac{\partial}{\partial y_2} - \left(3y_3 - \frac{1}{3}y_2^2 + \frac{1}{4}(y_2 - x_2)y_3 \right) \frac{\partial}{\partial y_3}. \quad (1)
\end{aligned}
$$

There is no x_1 variable and the indices for x_i and y_i are chosen because of their natural weights. The singular set of the foliation $\mathcal{F}(\mathbf{v})$ in the weighted projective space $\mathbb{P}^w := \mathbb{P}^{2,3,2,3,1}$ with the coordinate system $[x_2 : x_3 : y_2 : y_3 : y_1]$ consists of an isolated point and a rational curve:

$$
\mathrm{Sing}(\mathcal{F}(\mathbf{v})) = \{0\} \cup \left\{ 27x_3^2 - x_2^3 = 27y_3^2 - y_2^3 = x_3^{\frac{1}{3}} + y_3^{\frac{1}{3}} - 2y_1 = 0 \right\}. \quad (2)
$$

We parametrize the curve singularity of $\mathrm{Sing}(\mathcal{F}(\mathbf{v}))$ by

$$
g \colon \mathbb{P}^1 \to \mathbb{P}^w, \quad [t : s] \mapsto [3t^2 : t^3 : 3s^2 : s^3 : \tfrac{1}{2}(s + t)]. \quad (3)
$$

It can be easily checked that the following union of two hypersurfaces

$$
\Delta := \Delta_1 \cup \Delta_2, \quad (4)
$$

$$
\Delta_1 := \{27x_3^2 - x_2^3 = 0\}, \quad \Delta_2 := \{27y_3^2 - y_2^3 = 0\} \quad (5)
$$

is tangent to $\mathcal{F}(\mathbf{v})$. Recall the modular curve

$$
X_0(d) := \Gamma_0(d) \backslash \mathbb{H}^*, \quad d \in \mathbb{N}, \quad (6)
$$

where $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and

$$\Gamma_0(d) := \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathrm{SL}(2,\,\mathbb{Z}) \colon a_3 \equiv 0 \pmod{d} \right\}.$$

It has a structure of a compact Riemann surface and points of $\Gamma_0(d) \backslash (\mathbb{Q} \cup \{\infty\})$ are called its cusps.

Theorem 1. *The foliation $\mathcal{F}(\mathbf{v})$ has the following properties*:

(1) *For each d there is an algebraic curve $S_0(d)$ (defined over $\mathbb{Q}$) in $\mathbb{P}^w$, not contained in Δ and tangent to the vector field $\mathbf{v}$ in (1) such that $S_0(d) \backslash \Delta$ is isomorphic to $X_0(d)$ minus cusps. Moreover, these are the only algebraic leaves of $\mathcal{F}(\mathbf{v})$ in the complement of Δ in $\mathbb{P}^w$. The curve $S_0(d)$ intersects Δ only at the points $g([a : -b])$, $d = ab$, a, $b \in \mathbb{N}$ in the curve singularity of $\mathcal{F}(\mathbf{v})$.*

(2) *There is an $\mathcal{F}(\mathbf{v})$-invariant set $M_{\mathbb{R}} \subset \mathbb{P}^w$ such that $M_{\mathbb{R}} \cap \Delta = g(\mathbb{P}^1_{\mathbb{R}})$, and $M_{\mathbb{R}} \cap (\mathbb{P}^w \backslash \Delta)$ is a real analytic variety of dimension 5. It contains all $S_0(d)$ and*

$$M_{\mathbb{Q}} := \bigcup_{d=1}^{\infty} S_0(d) \tag{7}$$

is dense in $M_{\mathbb{R}}$. Moreover, the foliation $\mathcal{F}(\mathbf{v})$ has a real first integral $B : \mathbb{P}^w \backslash \Delta \to \mathbb{R}$ and $M_{\mathbb{R}}$ is inside $B^{-1}(1)$.

The topic of computing equations for modular curves has also a long history with fruitful applications in number theory, see [Yui78], [Gal96], [Yan06] and the references therein. It seems to the author that no expert in this area believes in the existence of models for modular curves over $\mathbb{Q}$ with closed formulas for its defining equations and therefore, the emphasis has been on computing them using q-expansion of modular forms. We can also give an explicit description of $S_0(d)$ in terms of modular forms for $\Gamma_0(d)$, see the map described in (35). However, this is not the main focus of the present text. The most surprising aspect of the vector field $\mathbf{v}$ in (1) is that it gives closed formulas for equations for $S_0(d)$ for arbitrary d. This is as follows. Let $\psi(d) := d \prod_p (1 + \frac{1}{p})$ be the Dedekind ψ function, where p runs through primes p dividing d, and for $i = 1$, 2, 3 let $\alpha_{i,j}$, $j = 1$, 2, $\ldots$, $m_{d,i}$, be the set of monomials:

$$y_i^{a_1} x_2^{a_2} x_3^{a_3}, \quad i \cdot \psi(d) = i a_1 + 2 a_2 + 3 a_3, \;\; a_1,\, a_2,\, a_3 \in \mathbb{N}_0. \tag{8}$$

For $i = 1$ let us consider such monomials with $y_1 = 1$. We consider $\mathbf{v}$ as a derivation from $\mathbb{Q}[x_2,\, x_3,\, y_2,\, y_3]$ to itself and define the matrix:

$$B_{d,i}(x_2,\, x_3,\, y_2,\, y_3) = \begin{bmatrix} \alpha_{i,1} & \alpha_{i,2} & \cdots & \alpha_{i,m_{d,i}} \\ \mathbf{v}(\alpha_{i,1}) & \mathbf{v}(\alpha_{i,2}) & \cdots & \mathbf{v}(\alpha_{i,m_{d,i}}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}^{m_{d,i}-1}(\alpha_{i,1}) & \mathbf{v}^{m_{d,i}-1}(\alpha_{i,2}) & \cdots & \mathbf{v}^{m_{d,i}-1}(\alpha_{i,m_{d,i}}) \end{bmatrix}. \tag{9}$$

The entries of $B_{d,i}$ are polynomials in $\mathbb{Z}[\frac{1}{6}][x_2,\, x_3,\, y_2,\, y_3]$.

Theorem 2. *The polynomials $J_{d,i} := \det B_{d,i}$, $i = 1$, 2, 3, restricted to $S_0(d)$ vanish.*

The polynomial $J_{d,i}$ is huge, and it might be difficult or impossible to use $J_{d,i}$ for computational purposes. However, we can also give defining equations for $S_0(d)$ with some conditions on an isogeny which can be verified computationally. This is as follows. Using Hecke operators, one can see that $S_0(d)$ in $\mathbb{P}^w \backslash \Delta$ is a complete intersection given by three polynomials of the form

$$Q_{d,i} := \sum_{j=1}^{m_{d,i}} c_{i,j} \alpha_{i,j}, \ \ i = 1, 2, 3, \ \ c_{i,j} \in \mathbb{Q} \tag{10}$$

and further we can assume that the coefficient of $y_i^{\psi(d)}$ in $Q_{d,i}$ is 1. Let $E_1 \colon y^2 = 4x^3 - t_2 x - t_3$ and $E_2 \colon y^2 = 4x^3 - s_2 x - s_3$ be two elliptic curves in the Weierstrass format and let $f \colon E_1 \to E_2$ be an isogeny with cyclic kernel of order d, all defined over $\mathbb{C}$. We can compute the numbers k, $k' \in \mathbb{C}$ from the equality

$$\begin{bmatrix} f^* \frac{dx}{y} & f^* \frac{x\,dx}{y} \end{bmatrix} = \begin{bmatrix} \frac{dx}{y} & \frac{x\,dx}{y} \end{bmatrix} \begin{bmatrix} kd^{\frac{1}{2}} & k'd^{-\frac{1}{2}} \\ 0 & k^{-1}d^{-\frac{1}{2}} \end{bmatrix}, \tag{11}$$

where $f^* \colon H^1_{\mathrm{dR}}(E_2) \to H^1_{\mathrm{dR}}(E_1)$ is the induced map in de Rham cohomologies, see for instance [Ked08] or [Mov12b].

Theorem 3. *We have*

(1) $[t_2 : t_3 : s_2 : -s_3 : k^{-1}k'] \in S_0(d) \backslash \Delta$ *and any point of this set is obtained in this way.*

(2) *Assume that for an isogeny of elliptic curves as above we have $k' \neq 0$ and define*

$$p := \left(t_2 k^2 k'^{-2}, \ t_3 k^3 k'^{-3}, \ s_2 k^2 k'^{-2}, \ -s_3 k^3 k'^{-3} \right) \in \mathbb{C}^4. \tag{12}$$

Let $C_{d,i}$ be the $m_{d,i} \times 1$ matrix with entries $c_{i,j}$, $j = 1, 2, \ldots, m_{d,i}$, which are coefficients of $Q_{d,i}$ in (10). It satisfies

$$B_{d,i}(p) C_{d,i} = 0.$$

If the matrix $B_{d,i}(p)$ has rank $m_{d,i} - 1$ then Theorem 3 determines $C_{d,i}$, and hence $Q_{d,i}$, uniquely (we have already normalized one of the coefficients of $Q_{d,i}$ to 1). This will give a simple formula for the coefficients of $Q_{d,i}$ in terms of the inverse of a $(m_{d,i} - 1) \times (m_{d,i} - 1)$ submatrix of $B_{d,i}$. If the isogeny for constructing the point p is defined over a number field k then we may use the fact that $c_{i,j}$'s are rational numbers and invent other formulas by decomposing $B_{d,i}(p)$ in a basis of $k/\mathbb{Q}$. In [LMF13] we can find many examples of isogenies which might be used for this discussion, see also [Hus04, page 96] for an explicit formula of a 2-isogeny.

2. Self Join of Foliations

Let Θ be a $\mathbb{C}$-vector space generated by global vector fields in a complex manifold T and assume that it is closed under Lie bracket. Let also $\mathcal{F} = \mathcal{F}(\Theta)$ be the induced holomorphic foliation in T. For any vector field $\mathbf{v} \in \Theta$ we attach two vector fields v_i, $i = 1, 2$, in $\mathsf{T} \times \mathsf{T}$. For instance, v_1 is uniquely determined by the fact that it is tangent to $\mathsf{T} \times \{x\}$, $x \in \mathsf{T}$ and under $\mathsf{T} \times \{x\} \cong \mathsf{T}$ it is identified with v. The self join $\mathcal{F} + \mathcal{F}$ of $\mathcal{F}$ is a foliation in $\mathsf{T} \times \mathsf{T}$ which is given by vector fields $v_1 + v_2$ for all $v \in \Theta$. If $\mathcal{F}$ is of dimension c then its self join is also of dimension c, however, note that the codimension of the self join $\mathcal{F} + \mathcal{F}$ is twice the codimension of $\mathcal{F}$. The self join leaves the diagonal of $\mathsf{T} \times \mathsf{T}$ invariant. The concept of self join is inspired from a similar definition in singularity theory, see [AGZV88]. A self join of a foliation with trivial dynamics might have complicated dynamics. In this text we will consider self join of foliations $\mathcal{F}$ with a left action $\bullet$ of an algebraic group G on T and hence we can identify $\mathrm{Lie}(\mathsf{G})$ with a $\mathbb{C}$-vector space of global vector fields in T. We assume that $\mathrm{Lie}(\mathsf{G}) \subset \Theta$ and hence the action of G leaves the leaves of $\mathcal{F}$ invariant. It turns out that the diagonal action of G on $\mathsf{T} \times \mathsf{T}$

$$(\mathsf{T} \times \mathsf{T}) \times \mathsf{G} \to \mathsf{T} \times \mathsf{T}, \quad ((t, s), \mathsf{g}) \mapsto (t \bullet \mathsf{g}, s \bullet \mathsf{g}) \tag{13}$$

acts also on each leaf of $\mathcal{F} + \mathcal{F}$, and hence, it gives us a foliation $\mathcal{F}^*$ in the quotient $(\mathsf{T} \times \mathsf{T})/\mathsf{G}$, which by abuse of notation we call again the self join of $\mathcal{F}$, being clear in the context whether we mean $\mathcal{F}^*$ or $\mathcal{F} + \mathcal{F}$. Our main example for this is the foliation $\mathcal{F}$ in $(t_1, t_2, t_3) \in \mathbb{C}^3$ given by three vector fields

$$\begin{aligned} f &= -(t_1^2 - \tfrac{1}{12}t_2)\tfrac{\partial}{\partial t_1} - (4t_1t_2 - 6t_3)\tfrac{\partial}{\partial t_2} - (6t_1t_3 - \tfrac{1}{3}t_2^2)\tfrac{\partial}{\partial t_3}, \\ h &= -6t_3\tfrac{\partial}{\partial t_3} - 4t_2\tfrac{\partial}{\partial t_2} - 2t_1\tfrac{\partial}{\partial t_1}, \quad e = \tfrac{\partial}{\partial t_1}, \end{aligned} \tag{14}$$

which has just one leaf which is $\mathbb{C}^3 \setminus \{27t_3^2 - t_2^3 = 0\}$ and its complement is the singular set of $\mathcal{F}$. The $\mathbb{C}$-vector space generated by these vector fields equipped with the classical bracket of vector fields is isomorphic to the Lie Algebra $\mathfrak{sl}_2$:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \tag{15}$$

see for instance [Gui07, §3]. The vector field $\mathsf{R} = -f$ is mainly attributed to Ramanujan and it is the main ingredient of the geometric theory of quasi-modular forms, see [Mov08a] and Section 5. Its dynamics and arithmetic properties are fairly described in [Mov08b]. In order to describe the self join of $\mathcal{F}$, we consider two copies of $\mathcal{F}$ in $(t_1, t_2, t_3) \in \mathbb{C}^3$ and $(s_1, s_2, s_3) \in \mathbb{C}^3$ with the corresponding vector fields e_1, f_1, h_1 and e_2, f_2, h_2, respectively. The self join $\mathcal{F} + \mathcal{F}$ is the foliation in $(t, s) \in \mathbb{C}^6$ given by $e_1 + e_2$, $f_1 + f_2$, $h_1 + h_2$. For this example we consider the algebraic group

$$\mathsf{G} := \left\{ \begin{bmatrix} k & k' \\ 0 & k^{-1} \end{bmatrix} : \ k' \in \mathbb{C}, \ k \in \mathbb{C} - \{0\} \right\} \tag{16}$$

and its action on $\mathbb{C}^3$ is given by

$$t \bullet \mathsf{g} := (t_1 k^{-2} + k'k^{-1}, \ t_2 k^{-4}, \ t_3 k^{-6}),$$

$$t = (t_1, t_2, t_3) \in \mathsf{T}, \quad \mathsf{g} = \begin{bmatrix} k & k' \\ 0 & k^{-1} \end{bmatrix} \in \mathsf{G}.$$

In the quotient space $\mathbb{P}^w := \mathbb{C}^3 \times \mathbb{C}^3/\mathsf{G}$ with $w = (2, 3, 2, 3, 1)$ we have the affine coordinate system

$$(x_2, x_3, y_2, y_3) := \left(\frac{t_2}{(t_1 - s_1)^2}, \frac{t_3}{(t_1 - s_1)^3}, \frac{s_2}{(s_1 - t_1)^2}, \frac{s_3}{(s_1 - t_1)^3} \right). \qquad (17)$$

We make derivations of x_i and y_i variables along the vector fields and divide them over $t_1 - s_1$ and conclude that the foliation $\mathcal{F}^*$ in the affine chart $(x_2, x_3, y_2, y_3) \in \mathbb{C}^4$ is given by the quadratic vector field $\mathbf{v}$ in (1), and so, $\mathcal{F}^* = \mathcal{F}(\mathbf{v})$. Note that in the case of y_2 and y_3 we factor out the term $s_1 - t_1$ and so the corresponding terms in $\mathbf{v}$ have negative sign.

3. Generalized Period Domain

The generalized period domain in the case of elliptic curves

$$\Pi := \left\{ x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : x_i \in \mathbb{C}, \ x_1 x_4 - x_2 x_3 = 1, \ \mathrm{Im}(x_1 \bar{x}_3) > 0 \right\} \qquad (18)$$

was first introduced in [Mov08b], [Mov08a], see also [Mov13] for this notion in connection with arbitrary projective varieties and Griffiths period domain. The discrete group $\mathrm{SL}(2, \mathbb{Z})$ (resp. G in (16)) acts from the left (resp. right) on Π by usual multiplication of matrices. We also call $\mathsf{U} := \mathrm{SL}(2, \mathbb{Z})\backslash\Pi$ the generalized period domain. We consider the following family of elliptic curves

$$E_t : y^2 - 4(x - t_1)^3 + t_2(x - t_1) + t_3 = 0, \quad t \in \mathsf{T} := \mathbb{C}^3\backslash\{27t_3^2 - t_2^3 = 0\}. \qquad (19)$$

with $\left[\frac{dx}{y} \right], \left[\frac{x\,dx}{y} \right] \in H^1_{\mathrm{dR}}(E_t)$. This is the universal family of triples (E, α, ω), where $\alpha, \omega \in H^1_{\mathrm{dR}}(E)$, α is holomorphic in E and $\mathrm{Tr}(\alpha \cup \omega) = 1$, for details see [Mov12b, §5.5]. It turns out that the period map

$$\mathsf{pm} \colon \mathsf{T} \to \mathsf{U}, \quad t \mapsto \left[\frac{1}{\sqrt{2\pi i}} \begin{bmatrix} \int_\delta \frac{dx}{y} & \int_\delta \frac{x\,dx}{y} \\ \int_\gamma \frac{dx}{y} & \int_\gamma \frac{x\,dx}{y} \end{bmatrix} \right].$$

is a biholomorphism of complex manifolds and it respects the right action of G on both sides, see [Mov08b, Proposition 2]. Here, $[\cdot]$ means the equivalence class and $\{\delta, \gamma\}$ is a basis of the $\mathbb{Z}$-module $H_1(E_t, \mathbb{Z})$ with $\langle \delta, \gamma \rangle = -1$. From now on we identify T with U under this map and use the same notations in both sides. The push-forward of the vector fields f, e, h by the period map are respectively given by

$$f = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}, \qquad\qquad\qquad (20)$$

$$e = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}, \qquad\qquad\qquad (21)$$

$$h = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}. \qquad\qquad (22)$$

This follows from the computation of the Gauss–Manin connection of three parameter family of elliptic curves E_t, see for instance [Sas74] or [Mov08b, Proposition 1]. It turns out that in order to study the self joins $\mathcal{F} + \mathcal{F}$ and $\mathcal{F}^*$ in the algebraic side $\mathsf{T} \times \mathsf{T}$, it is enough to study it in the holomorphic side $\mathsf{U} \times \mathsf{U}$, for which we use

the coordinate system $(x, y) \in \Pi \times \Pi$. It can be easily checked that the local first integral of the foliation $\mathcal{F} + \mathcal{F}$ constructed in Section 2 is the function

$$F \colon \mathsf{U} \times \mathsf{U} \to \mathrm{Mat}(2 \times 2, \mathbb{C}),$$
$$F(x, y) := y \cdot x^{-1}. \tag{23}$$

Note that F is a multi-valued function. We have an action of $\mathrm{SL}(2, \mathbb{Z})$ on Π. This means that analytic continuation of F will result in the multiplication of F both from the left and right by elements of $\mathrm{SL}(2, \mathbb{Z})$.

Proposition 1. *The space of leaves of $\mathcal{F} + \mathcal{F}$ is*

$$\mathrm{SL}(2, \mathbb{Z}) \backslash \big(\mathrm{SL}(2, \mathbb{C}) - \{ A \in \mathrm{SL}(2, \mathbb{C}) \colon \mathrm{Re}(A) = 0 \} \big) \big/ \mathrm{SL}(2, \mathbb{Z}). \tag{24}$$

Proof. The space of leaves of $\mathcal{F} + \mathcal{F}$ is defined to be $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{Im}(F) / \mathrm{SL}(2, \mathbb{Z})$ and so we have to determine $\mathrm{Im}(F)$. We have

$$\mathsf{G} \times \mathbb{H} \cong \Pi, \quad (\mathsf{g}, \tau) \mapsto \begin{bmatrix} \tau & -1 \\ 1 & 0 \end{bmatrix} \cdot \mathsf{g},$$

where $\mathbb{H}$ is the upper half plane. We use this for x and y and we have

$$y \cdot x^{-1} = \begin{bmatrix} a^{-1} - b\tau_2 & -a^{-1}\tau_1 + a\tau_2 + b\tau_1\tau_2 \\ -b & a + b\tau_1 \end{bmatrix},$$

where (a, b, τ_1, τ_2) varies in $(\mathbb{C} - \{0\}) \times \mathbb{C} \times \mathbb{H} \times \mathbb{H}$. For $b = 0$, $(a^{-1}, -a, -a^{-1}\tau_1 + a\tau_2)$ only avoids pure imaginary vectors. For $b \neq 0$, the vector $(b, a^{-1} - b\tau_2, -a - b\tau_1)$ also avoids only pure imaginary vectors. Note that in this case we can neglect the $(2, 2)$-entry of this matrix as its determinant is 1. $\square$

Remark 1. If we replace the period maps $x := \mathsf{pm}(t)$ and $y := \mathsf{pm}(s)$ in the definition of F, we conclude that there are three quadratic combinations of elliptic integrals which are constant along the leaves of the foliation given by the vector field $\mathbf{v}$. This is the main reason for the naming "elliptic modular foliation" used in this article and [Mov08b]. Modular mainly refers to either that such foliations live in moduli spaces or they have solutions in term of modular forms.

4. Real First Integral and Levi Flat

We have the following global real first integral for $\mathcal{F} + \mathcal{F}$:

$$B \colon \mathsf{U} \times \mathsf{U} \to \mathbb{R},$$
$$B(x, y) := \frac{1}{2} \mathrm{Tr}(F \bar{F}^{-1}) = \frac{1}{2} \mathrm{Tr}(y \cdot x^{-1} \cdot \bar{x} \cdot \bar{y}^{-1}).$$

Note that by taking $F\bar{F}^{-1}$, the right action of $\mathrm{SL}(2, \mathbb{Z})$ on F is killed, and by taking the trace the left action also (recall that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for two matrices A and B). We have also the pencil of real surfaces $\mathrm{Re}(F(x)) + a \, \mathrm{Im}(F(x)) = 0$, $a \in \mathbb{R} \cup \{\infty\}$, which are invariant under both left and right action of $\mathrm{SL}(2, \mathbb{Z})$. Since $\det(F) = 1$, only for $a = 0, \infty$ we might get non-empty set. By Proposition 1 this set for $a = 0$ is also empty. Therefore, we define

$$M_{\mathbb{R}} := \{ p \in \mathsf{U} \times \mathsf{U} \colon \mathrm{Im}(F(p)) = 0 \}, \tag{25}$$

which is called a Levi-flat hypersurface, see [CLN11] for the general definition. Note that this set is inside the fiber $B^{-1}(1)$ of B over 1. We also define

$$M_{\mathbb{Q}} := \left\{ p \in \mathsf{U} \times \mathsf{U} \colon F(p) \in \sqrt{\mathbb{Q}^+}\mathrm{GL}^+(2,\,\mathbb{Q}) \right\}$$
$$= \bigcup_{d=1}^{\infty} V_d,$$

where

$$V_d := \{p \in \mathsf{U} \times \mathsf{U} \colon F(p) = A_d\}, \quad A_d := \begin{bmatrix} d^{\frac{1}{2}} & 0 \\ 0 & d^{-\frac{1}{2}} \end{bmatrix}. \tag{26}$$

The second equality is modulo the actions of $\mathrm{SL}(2,\,\mathbb{Z})$ from the left and right on F. Note that $\mathrm{SL}(2,\,\mathbb{Z})\backslash\mathrm{Mat}_d(2,\,\mathbb{Z})/\mathrm{SL}(2,\,\mathbb{Z})$ is naturally isomorphic to the set (up to isomorphism) of finite abelian groups of order d and generated by at most two elements, and so, each element in this quotient has a unique representative of the form $\begin{bmatrix} d_1 d_2 & 0 \\ 0 & d_2 \end{bmatrix}$. We have the following immersion of the three-dimensional non-compact version of modular curves inside $\mathsf{U} \times \mathsf{U}$:

$$\Gamma_0(d)\backslash\Pi \hookrightarrow \mathsf{U} \times \mathsf{U}, \quad x \mapsto (x,\, A_d \cdot x) \tag{27}$$

whose image is exactly V_d defined above and it is a connected fiber of F, and hence a leaf of $\mathcal{F} + \mathcal{F}$.

Proposition 2. *The only closed leaves of $\mathcal{F} + \mathcal{F}$ in $\mathsf{U} \times \mathsf{U}$ are V_d's for $d \in \mathbb{N}$.*

Proof. For A in the double quotient (24) the leaf $F^{-1}(A)$ is closed in $\mathsf{U} \times \mathsf{U}$ if and only if the double quotient

$$\#\mathrm{SL}(2,\,\mathbb{Z})\backslash(\mathrm{SL}(2,\,\mathbb{Z}) \cdot A \cdot \mathrm{SL}(2,\,\mathbb{Z})) < \infty \tag{28}$$

is finite. This happens if and only if the matrix A, up to multiplication by a constant, has rational entries. Since $\det(A) = 1$, we conclude that such a leaf is in $M_{\mathbb{Q}}$. $\qquad\square$

Remark 2. In the algebraic side $\mathsf{T} \times \mathsf{T}$, it is easy to verify that the affine variety $V_d \subset \mathsf{T} \times \mathsf{T}$ is the locus of isogenies

$$f \colon E_t \to E_s, \quad f^* \frac{dx}{y} = d^{\frac{1}{2}} \cdot \frac{dx}{y}, \quad f^* \frac{x\,dx}{y} = d^{-\frac{1}{2}} \cdot \frac{x\,dx}{y} \tag{29}$$

Here, $f^* \colon H^1_{\mathrm{dR}}(E_s) \to H^1_{\mathrm{dR}}(E_t)$ is the map induced in de Rham cohomologies.

5. Eisenstein Series and Halphen Property

Recall the Eisenstein series:

$$E_{2k}(\tau) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n \geqslant 1} \sigma_{2k-1}(n)q^n, \quad k = 1,\,2,\,3, \ \ \tau \in \mathbb{H}, \tag{30}$$

where $q = e^{2\pi i \tau}$ and $\sigma_i(n) := \sum_{d|n} d^i$ and B_i's are the Bernoulli numbers, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, ... S. Ramanujan in [Ram16, page 181] proved that

$$g = (g_1,\, g_2,\, g_3) = (a_1 E_2,\, a_2 E_4,\, a_3 E_6), \tag{31}$$

with $(a_1, a_2, a_3) = (\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3)$ is a solution of the vector field $\mathsf{R} := -f$ in (14) with derivation with respect to τ, and so f is mainly known as Ramanujan relation (or differential equation) between Eisenstein series. Thirty years before Ramanujan, G. Halphen calculated the Ramanujan differential equation and apparently he did not know about Eisenstein series (see [Hal86, page 331]). He had even a generalization of this differential equation into a three parameter family, which we will discuss it in §9. A fundamental but simple observation due to Halphen is the following. For a holomorphic function defined in $\mathbb{H}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C})$, $m \in \mathbb{N}$ let

$$\begin{aligned} (f|_m^0 A)(\tau) &:= (c\tau + d)^{-m} f(A\tau), \\ (f|_m^1 A)(\tau) &:= (c\tau + d)^{-m} f(A\tau) - c(c\tau + d)^{-1}. \end{aligned} \tag{32}$$

If ϕ_i, $i = 1, 2, 3$, are the coordinates of a solution of the Ramanujan differential equation R then $\phi_1|_2^1 A$, $\phi_2|_4^0 A$, $\phi|_6^0 A$ are also coordinates of a solution of R for all $A \in \mathrm{SL}(2, \mathbb{C})$. This is known as the Halphen property. The subgroup of $\mathrm{SL}(2, \mathbb{C})$ which fixes the solution given by Eisenstein series is $\mathrm{SL}(2, \mathbb{Z})$.

Remark 3. By Halphen property and the construction of the vector field $\mathbf{v}$ in §2 it follows that a general solution of this vector field in $\mathbb{P}^w - \{\Delta = 0\}$, up to change of coordinate system in τ, is given by

$$\mathbb{H} \to \mathbb{P}^w, \tag{33}$$

$$\tau \mapsto \left[g_2(\tau) : g_3(\tau) : (g_2|_4^0 A)(\tau) : -(g_3|_6^0 A)(\tau) : g_1(\tau) - (g_1|_2^1 A)(\tau) \right]. \tag{34}$$

Note that in the final step of the construction of the vector field $\mathbf{v}$ we have the division over $t_1 - s_1$. This implies that the map (35) is not necessarily a solution of the vector field and it is only tangent to it.

6. Proof of Theorem 1

For the proof of Theorem 1 we note that $\mathcal{F}(\mathbf{v}) = \mathcal{F}^* := (\mathcal{F} + \mathcal{F})/\mathsf{G}$, where $\mathcal{F}$ is the one leaf foliation in U given by the vector fields (20) and we have the diagonal action of G on $\mathsf{U} \times \mathsf{U}$ as in (13). For simplicity, we have used the same notation for objects in $\mathsf{T} \times \mathsf{T}$, $\mathsf{U} \times \mathsf{U}$ or their quotients by G. In this and the next section we use $\Gamma := \mathrm{SL}(2, \mathbb{Z})$.

Proof of assertion (1). Let us consider the holomorphic map

$$\mathfrak{s} \colon X_0(d) \to \mathbb{P}^w, \quad d \in \mathbb{N}, \tag{35}$$

which in the affine chart $\mathbb{C}^4 \subset \mathbb{P}^w$ is given by

$$\tau \mapsto \left(\frac{g_2(\tau)}{(g_1(\tau) - d \cdot g_1(d \cdot \tau))^2}, \frac{g_3(\tau)}{(g_1(\tau) - d \cdot g_1(d \cdot \tau))^3}, \right.$$
$$\left. \frac{d^2 \cdot g_2(d \cdot \tau)}{(d \cdot g_1(d \cdot \tau) - g_1(\tau))^2}, \frac{d^3 \cdot g_3(d \cdot \tau)}{(d \cdot g_1(d \cdot \tau) - g_1(\tau))^3} \right), \tag{36}$$

where g_i's are the Eisenstein series given in (31). The fact that (g_1, g_2, g_3) is a solution of the Ramanujan differential equation implies the same statement for $(d \cdot g_1(d \cdot \tau), d^2 g_2(d \cdot \tau), d^3 g_2(d \cdot \tau))$. By our construction of the vector field (1) in

Section 2, we conclude that the image of the map (35) is tangent to the vector field $\mathbf{v}$ in (1). The map (33) for $A = A_d$ gives us the map $\mathfrak{s}$ after taking the quotient by $\Gamma_0(d)$, where A_d is defined in (26). The curve $S_0(d)$ is just the quotient of the three-dimensional variety V_d by G. The fact that these are the only algebraic solutions follows from Proposition 2. Chow theorem implies that the image of the map (35) is an algebraic curve. However, this theorem does not give any information about the field of definition. In §7 we give the description of $S_0(d)$ as a curve over $\mathbb{Q}$.

For A_d defined in (26), the equivalent classes in the quotient $\Gamma \backslash \Gamma A_d \Gamma$ give the same map (35). Each class is represented uniquely by an element of the from

$$d^{-\frac{1}{2}} \begin{bmatrix} a & e \\ 0 & b \end{bmatrix}, \quad d = ab, \ a, \ b, \ e \in \mathbb{N}, \ 0 \leqslant e \leqslant b - 1, \tag{37}$$

but not all these matrices are in $\Gamma A_d \Gamma$. The cardinality of $\Gamma \backslash \Gamma A_d \Gamma$ is $\psi(d)$ and the cardinality of matrices in (37) is $\sigma_1(d)$. The map (33) with A as in (37) is

$$\tau \mapsto \left[g_2(\tau) : g_3(\tau) : a^2 b^{-2} g_2(\tfrac{a\tau + e}{b}) : -a^3 b^{-3} g_3(\tfrac{a\tau + e}{b}) : g_1(\tau) - ab^{-1} g_1(\tfrac{a\tau + e}{b}) \right] \tag{38}$$

This evaluated at $\tau = i\infty$ (equivalently $q = 0$) is

$$\left[12 : 8 : 12(ab^{-1})^2 : -8(ab^{-1})^3 : 1 - (ab^{-1}) \right] = g([2 : -2a^{-1}b]) = g([a : -b]).$$

The fact that $\mathfrak{s}$ is a birational map follows from the classical statement that the map $X_0(d) \to \mathbb{P}^1 \times \mathbb{P}^1$, $\tau \mapsto (j(\tau), j(d\tau))$, where $j = \frac{g_2^3}{27 g_3^2 - g_2^3}$ is the j-invariant, is a birational map between $X_0(d)$ and its image. Since $X_0(d)$ minus cusps and $S_0(d) \backslash \Delta$ are both smooth, it follows that $\mathfrak{s}$ is an isomorphism between them. $\square$

Remark 4. It seems that for any decomposition $d = ab$, $a, b \in \mathbb{N}$, we have $\varphi(\gcd(a, b))$ smooth branches of $S_0(d)$ crossing the point $g([a : -b])$ in the curve singularity of $\mathcal{F}(\mathbf{v})$, where φ is the Euler's totient function. This is compatible with the well-known fact that the cardinality of the set of cusps $\Gamma_0(d) \backslash (\mathbb{Q} \cup \{\infty\})$ is $\sum_{a|d} \varphi(\gcd(a, \frac{d}{a}))$.

Proof of assertion (2). The set $M_{\mathbb{R}}$ announced in the theorem is just the quotient of $M_{\mathbb{R}}$ in (25) by the diagonal action of G and the first integral B is just the first integral in (23) after taking the quotient. $\square$

Remark 5. The eigenvalues of the linear part of the isolated singularity $0 \in \mathbb{C}^4$ of $\mathcal{F}$ are $2, 3, -2, -3$, which is resonant and hence it does not lie in the Poincaré domain. Therefore, we may not be able to do a holomorphic change of coordinates in $(\mathbb{C}^4, 0)$ such that $(\mathcal{F}, 0)$ is equivalent to its linear part, which has the first integrals

$$\frac{(x_2 + 6x_3)^3}{x_3^2} = \text{const}_1, \quad \frac{(y_2 + 6y_3)^3}{y_3^2} = \text{const}_2, \tag{39}$$

see [IY08, Theorem 4.3].

Remark 6. The curve singularity of $\mathcal{F}$ intersects the weighted projective space at infinity $\{y_1 = 0\} \cong \mathbb{P}^{2,3,2,3}$ at the point $g([1 : -1])$. Therefore, an algebraic leaf $S_0(d)$ of $\mathcal{F}$ intersects the curve singularity at infinity if and only if d is a square.

7. Proof of Theorems 2, 3

Let us define

$$P_f(x) := \prod_{A \in \Gamma \backslash \Gamma A_d \Gamma} \left(x - f|_k^0 A \right), \qquad (f, k) = (g_2, 4), (g_3, 6),$$

$$P_f(x) := \prod_{A \in \Gamma \backslash \Gamma A_d \Gamma} \left(x - f + f|_k^1 A \right), \quad (f, k) = (g_1, 2),$$

where $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ and the slash operators are defined in (32). Using Hecke operators one can prove that P_{g_i}, $i = 1, 2, 3$, is a homogeneous polynomial of degree $i\psi(d)$ in

$$\mathbb{Q}[x, g_2, g_3], \quad \deg(x) = m, \ \deg(g_2) = 2, \ \deg(g_3) = 3,$$

This is classical for $i = 2, 3$, but less well-known for $i = 1$. In this case we use the following equalities for $f = g_1$:

$$f|_2^1 A|_2^0 B = f|_2^1 AB + c'(c'\tau + d')^{-1}, \tag{40}$$

$$f|_2^0 B = f + c'(c'\tau + d')^{-1} \quad \forall A, \ B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C}), \tag{41}$$

and conclude that the coefficient x^i of P_{g_1} in x is a modular form of weight $2(\psi(d)-i)$ for Γ and defined over $\mathbb{Q}$, and hence can be written as a polynomial of in g_2, g_3 with $\mathbb{Q}$ coefficients. For further details see [Mov15, Proposition 6]. Note that $\Gamma \backslash \Gamma A_d \Gamma$ is isomorphic to the fiber of the map $\Gamma \backslash \mathrm{Mat}_d(2, \mathbb{Z}) \to \Gamma \backslash \mathrm{Mat}_d(2, \mathbb{Z})/\Gamma$ over over the matrix $\begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}$, and in [Mov15] we have used the latter set in order to define P_{g_i}'s. In general, we can define P_f for any quasi-modular form of weight k and differential order n for Γ. For examples of polynomials P_{g_i}, see [Mov15, page 440].

The conclusion is that we have three homogeneous polynomials $Q_{d,1}(y_1, x_2, x_3)$, $Q_{d,2}(y_2, x_2, x_3)$, $Q_{d,3}(y_3, x_2, x_3)$ of degrees respectively $\psi(d)$, $2\psi(d)$, $3\psi(d)$ in the ring $\mathbb{Q}[x_2, x_3, y_1, y_2, y_3]$, $\deg(x_i) = \deg(y_i) := i$ such that

$$Q_{d,1}(d \cdot g_1(d \cdot \tau) - g_1(\tau), g_2(\tau), g_3(\tau)) = 0, \tag{42}$$

$$Q_{d,2}(d^2 \cdot g_2(d \cdot \tau), g_2(\tau), g_3(\tau)) = 0, \tag{43}$$

$$Q_{d,3}(d^3 \cdot g_3(d \cdot \tau), g_2(\tau), g_3(\tau)) = 0 \tag{44}$$

and hence they give three equations for $S_0(d) \subset \mathbb{P}^w$.

Proof of Theorem 2. The polynomial $Q_{d,i}$ is a linear combination of the monomials (8) as in (10). We prove that $\det B_{d,i}$ restricted to $S_0(d)$ is identically zero. Since $\mathbf{v}$ is tangent to the curve $S_0(d)$, we know that for all $r \in \mathbb{N}_0$ we have $\mathbf{v}^r(\sum_{j=1}^{m_{d,i}} c_j \alpha_{i,j}) = \sum_{j=1}^{m_{d,i}} c_j \mathbf{v}_d^r(\alpha_{i,j})$ restricted to $S_0(d)$ is zero. This in turn implies that the matrix $B_{d,i}$ restricted to points of $S_0(d)$ has non-zero kernel and so its determinant restricted to V_d is zero. The last part of our proof is a slight generalization to higher dimensions of an argument due to J. V. Pereira, see [Per01, Proposition 1, page 1390]. □

Proof of Theorem 3, *assertion* (2). Once we know a point $p \in S_0(d)$ then we can repeat the proof of Theorem 2 and conclude that $B_{d,i}(p)C_{d,i} = 0$. $\qquad\square$

Proof of Theorem 3, *assertion* (1). Let us write our elliptic curves $E_1 := E_{0,t_2,t_3}$ and $E_2 := E_{0,s_2,s_3}$ in the notation of the three parameter Weierstrass format (19). The equality (11) can be written as

$$\begin{bmatrix} f^* \frac{dx}{y} & f^* \frac{x\,dx}{y} \end{bmatrix} = \begin{bmatrix} \frac{dx}{y} & \frac{x\,dx}{y} \end{bmatrix} \mathsf{g} \cdot A_d,$$

where A_d is the matrix (26) and g is in the algebraic group G defined in (16). This follows, for instance, from [Mov15, Proposition 1]. Define $s_1 := 0$ and s_2, s_3 as before and redefine

$$(t_1,\, t_2,\, t_2) := (0,\, t_2,\, t_3) \bullet \mathsf{g} = (k'k^{-1},\, t_2 k^{-4},\, t_3 k^{-6}),$$

where we have used the action of the algebraic group G in (16). The point $(t,\, s) \in \mathsf{T} \times \mathsf{T}$ lies in the algebraic set V_d defined in Remark 2. After taking the quotient by the diagonal action of G on $\mathsf{T} \times \mathsf{T}$ we get the point (12) of $S_0(d)$. $\qquad\square$

8. Computing at a Cusp

We can also state a theorem similar to Theorem 3 without the input of an isogeny and using a point in the curve singularity of $\mathcal{F}(\mathbf{v})$ which corresponds to cusp points of $S_0(d)$. Note that at these points the vector field $\mathbf{v}$ vanishes and the matrix $B_{d,i}$ evaluated there has all its lines equal to zero except the first one. Therefore, we have to use second order approximation of modular curves at these points.

Proposition 3. *Let* $d = ab$ *with* $a,\, b \in \mathbb{N}$, $b < a$, $r := \frac{a}{b}$ *and* $p := g([r : -1])$. *Let also* $C_{d,i}$ *be the* $m_{d,i} \times 1$ *matrix with entries* $c_{i,j}$, $j = 1,\, 2,\, \ldots,\, m_{d,i}$ *which are coefficients of* $Q_{d,i}$ *in* (10). *We have*

$$\left(\frac{\partial B_{d,i}}{\partial x_2}(p) \cdot (6 - 5r)(1 - r) + \frac{\partial B_{d,i}}{\partial x_3}(p) \cdot (7r - 6) + \frac{\partial B_{d,i}}{\partial y_2}(p) \cdot r^2(1 - r) \right.$$

$$\left. - \frac{\partial B_{d,i}}{\partial x_3}(p) \cdot r^3 \right) C_{d,i} = 0 \quad (45)$$

Proof. Since the image $S_0(d)$ of the map $\mathfrak{s}$ in (35) is tangent to $\mathbf{v}$ and $Q_{d,i}$ restricted to it vanishes, we conlude that the pull-back of $B_{d,i}$ by $\mathfrak{s}$ satisfies $B_{d,i}(\mathfrak{s}(\tau))C_{d,i} = 0$. The theorem follows from derivating this equality with respect to a variable Q and then setting $Q = 0$, which we explain below.

The map (38) is just a different parametrization of $S_0(d)$ using the action of Γ on $\mathbb{H}$. Let us write $r = \frac{a'}{b'}$ with $\gcd(a',\, b') = 1$. The g_i's in this map have Fourier expansions in terms of $Q := e^{\frac{2\pi i}{b'}}$. We have $g_i(\tau) = * + *Q^{b'} + \cdots$ and $g_i(r\tau + \frac{e}{b}) = * + *Q^{a'} + \cdots$, where $*$'s are constants. Since $b' < a'$, we need only the constant term of $g_i(r\tau + \frac{e}{b})$. If we write $g_i = a_i(1 + b_i q + \cdots)$ we get

$$\mathfrak{s}(\tau) = g([r : -1])$$

$$+ \left(\frac{a_2(-b_2 r + b_2 - 2b_1)}{a_1^2(1 - r)^3},\, \frac{a_3(-b_3 r + b_3 - 3b_1)}{a_1^3(1 - r)^4},\, \frac{-2r^2 a_2 b_1}{a_1^2(1 - r)^3},\, \frac{3r^3 a_3 b_1}{a_1^3(1 - r)^4} \right) Q + \cdots,$$

where $\cdots$ means higher order terms. Multiplying the coefficient of Q by $-\frac{a_1^3(1-r)^4}{3a_3b_1}$ we get the desired constants in r which are used in (45). Note that $(a_1, a_2, a_3) = (1, 12, 8)$ (up to $\frac{2\pi i}{12}$ factors, which do not affect this computation) and $(b_1, b_2, b_3) = (-24, 240, -504)$. $\qquad\square$

A computer implementation of the matrix $B_{d,i}(p)$ for $p := g([d : -1])$ and small values of d shows that the rank of the matrix $B_{d,i}(p)$ is much below $m_{d,i} - 1$, and hence, $C_{d,i}$ is not uniquely determined by the equality (45). For instance, for $d = 2, 3, 4, 5$ and $i = 2$ we have $m_{d,i} = 5, 7, 12, 12$ and the rank of $B_{d,i}$ is respectively 1, 3, 3, 3. [1]

9. Self Join of Halphen Differential Equation

We can carry out the self join of foliations introduced in Section 2 for the Halphen differential equation:

$$\text{H:} \begin{cases} \dot{t}_1 = (1 - \alpha_1)(t_1 t_2 + t_1 t_3 - t_2 t_3) + \alpha_1 t_1^2, \\ \dot{t}_2 = (1 - \alpha_2)(t_2 t_1 + t_2 t_3 - t_1 t_3) + \alpha_2 t_2^2, \\ \dot{t}_3 = (1 - \alpha_3)(t_3 t_1 + t_3 t_2 - t_1 t_2) + \alpha_3 t_3^2, \end{cases} \tag{46}$$

see [Hal81], with $\alpha_i \in \mathbb{C} \cup \{\infty\}$ (if for instance $\alpha_1 = \infty$ then the first row is replaced with $-t_1 t_2 - t_1 t_3 + t_2 t_3 + t_1^2$). It turns out that if we consider two copies of H in (t_1, t_2, t_3) and (s_1, s_2, s_3) variables, but with the same parameters $\alpha_1, \alpha_2, \alpha_3$, and compute the derivation of the new variables

$$(x_1, x_2, y_1, y_2) := \left(\frac{t_2 - t_1}{s_1 - t_1}, \frac{t_3 - t_1}{s_1 - t_1}, \frac{s_2 - s_1}{t_1 - s_1}, \frac{s_3 - s_2}{t_1 - s_1} \right)$$

and divide the result over $s_1 - t_1$, then we arrive at:

$$\begin{cases} \dot{x}_1 = x_1(\alpha_2 x_1 + (2 - \alpha_2 - \alpha_1)x_2 - (1 - \alpha_1)(x_1 x_2 - y_1 y_2) - 1), \\ \dot{x}_2 = x_2(\alpha_3 x_2 + (2 - \alpha_3 - \alpha_1)x_1 - (1 - \alpha_1)(x_1 x_2 - y_1 y_2) - 1), \\ \dot{y}_1 = -y_1(\alpha_2 y_1 + (2 - \alpha_2 - \alpha_1)y_2 - (1 - \alpha_1)(y_1 y_2 - x_1 x_2) - 1), \\ \dot{y}_2 = -y_2(\alpha_3 y_2 + (2 - \alpha_3 - \alpha_1)y_1 - (1 - \alpha_1)(y_1 y_2 - x_1 x_2) - 1). \end{cases} \tag{47}$$

The Halphen vector field for $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is a pull-back of the Ramanujan vector field with a degree 6 map, see [Mov12b, page 335], and we have a similar relation between (47) and $\mathbf{v}$ in this case. Therefore, in Theorem 1 we have classified all algebraic solutions of (47) with $\alpha = 0$ in the Zariski open set $x_1 x_2(x_1 - x_2) \neq 0$, $y_1 y_2(y_1 - y_2) \neq 0$. Some important ingredients of the proof of Theorem 1 in the case of (47) have been worked out in [Mov12a]. This includes the relation of (46) and the period map, and the explicit computation of the stabilizer Γ of a solution of (46) as an explicit subgroup of $\mathrm{SL}(2, \mathbb{C})$. The most critical part of the generalization would be to classify the set

$$C(\Gamma) := \{ A \in \mathrm{SL}(2, \mathbb{C}) \colon |\Gamma \backslash \Gamma A \Gamma| < \infty \}, \tag{48}$$

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[1]See https://w3.impa.br/~hossein/WikiHossein/files/Singular%20Codes/2018-06-EllipticModularFoliationII.txt.

which we need in Proposition 2. Since we have actions of Γ from both the left and right on $C(\Gamma)$, we are actually interested to classify the double quotient $\Gamma \backslash C(\Gamma)/\Gamma$, which for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is isomorphic to $\mathbb{N}$ through $d \mapsto A_d$, where A_d is the matrix in (26). In a personal communication A. Salehi Golsefidy pointed out that $C(\Gamma)$ contains the commensurability group, and even for this we do not know much beyond their Zariski-closure. For a particular examples of α, the group Γ is the subgroup of $\mathrm{SL}(2, \mathbb{R})$ generated by

$$\gamma_1 = \begin{pmatrix} 2\cos(\frac{\pi}{m_1}) & 1 \\ -1 & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 2\cos(\frac{\pi}{m_2}) \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 1 & 2\cos(\frac{\pi}{m_1}) + 2\cos(\frac{\pi}{m_2}) \\ 0 & 1 \end{pmatrix}$$
$$(49)$$

for some $m_1, m_2 \in \mathbb{N}$, $\frac{1}{m_1} + \frac{1}{m_2} < 1$. This is a triangle group of type (m_1, m_2, ∞). The generalization of Eisenstein series in this case are done in [DGMS13]. In this case it seems that $C(\Gamma)$ is an infinite enumerable set if and only if Γ is arithmetic. Note that we have a finite number of arithmetic triangle groups which are classified by Takeuchi in [Tak77].

10. Picard's Curious Example

Let T be the moduli space of triples (E, P, ω), where E is an elliptic curve over $\mathbb{C}$ and by definition it comes together with a point O, $P \neq O$ is another point in E and ω is a meromorphic differential 1-form in E with poles only at O and P and with the order of pole equal to one at both points. Moreover, the residue of ω at P and O is respectively $+1$ and -1.

Proposition 4. *We have*

$$\mathsf{T} \simeq \mathbb{P}^{1,2,3,4} \backslash \{[s : a : b : c] \in \mathbb{P}^{1,2,3,4} : \Delta = 0\},$$

where $\Delta := 27(-b^2 + 4a^3 - ca)^2 - c^3$, and the universal family over T is given by

$$E = E_{a,b,c} : y^2 = 4x^3 - cx + b^2 - 4a^3 + ca,$$
$$\omega = \omega_{s,a,b} := \frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y} + s\frac{dx}{y}, \quad P = (a, b). \tag{50}$$

Proof. We choose Weierstrass coordinates x, y on E. These are rational functions on E with pole of order 2 and 3 at O, respectively. In this way we can write E in the Weierstrass format $E_{c,\check{c}} : y^2 = 4x^3 - cx - \check{c}$ with $\Delta := 27\check{c}^2 - c^3 \neq 0$. In these coordinates we write $P = (a, b)$ and it follows that any triple of the moduli space T is isomorphic to a triple in (50) for some $(s, a, b, c) \in \mathbb{C}^4$. Note that $\check{c} = 4a^3 - ca - b^2$ and so it can be discarded. For $k \in \mathbb{C}^*$ we have

$$f : E_{k^{-4}c, k^{-6}\check{c}} \simeq E_{c,\check{c}},$$
$$f(x, y) = (k^2x, k^3y),$$
$$f_*\omega_{k^{-1}s, k^{-2}a, k^{-3}b} = \omega_{s,a,b}.$$

and so (s, a, b, c) and $(k^{-1}s, k^{-2}a, k^{-3}b, k^{-4}c)$ represents the same point in T. $\square$

Let us consider the affine chart $s = 1$ for the moduli space T. Let $\delta \in H_1(E_{a,b,c}, \mathbb{Z})$ be a continuous family of cycles. A simple, but long, calculus computation[2] gives us the following:

$$d \int_\delta \left(\frac{1}{2} \frac{y+b}{x-a} \frac{dx}{y} + \frac{dx}{y} \right) = \frac{\alpha_1}{\Delta} \cdot \int_\delta \frac{x \, dx}{y} + \frac{\alpha_2}{\Delta} \cdot \int_\delta \frac{dx}{y}, \tag{51}$$

where d is the differential of holomorphic functions in $(a, b, c) \in \mathbb{C}^3$, $\alpha_i = \alpha_{1i} \, da + \alpha_{2i} \, db + \alpha_{3i} \, dc$, $i = 1, 2$,

$\alpha_{11} = 3c^2 - 36ca^2 + 45cab - 108a^3b + 27b^3$,

$\alpha_{12} = -\frac{1}{2}(9c^2a + 3c^2b - 144ca^3 + 54ca^2b + 9cb^2 + 432a^5 - 216a^4b - 108a^2b^2 + 54ab^3)$,

$\alpha_{21} = 2c^2 - 30ca^2 + 6cb + 72a^4 - 18ab^2$,

$\alpha_{22} = -(2c^2a - 30ca^3 + 9cab + 3cb^2 + 72a^5 - 36a^3b - 18a^2b^2 + 9b^3)$,

$\alpha_{31} = -\frac{1}{2}(3ca + 3cb - 36a^3 + 9b^2)$,

$\alpha_{32} = \frac{1}{4}(c^2 - 18ca^2 + 9cab + 72a^4 - 36a^3b - 18ab^2 + 9b^3)$.

The matrix $\alpha := (\alpha_{ij})$ has rank two and the vector field

$$\mathbf{v} := (2c - 24a^2 + 6ab + 6b)\frac{\partial}{\partial a} - (3c - 36a^2 + 36ab - 9b^2)\frac{\partial}{\partial b}$$
$$+ (12ca + 12cb - 144a^3 + 36b^2)\frac{\partial}{\partial c} \tag{52}$$

is in the kernel of α_1 and α_2 and generates it. This implies that along the solutions of the vector field $\mathbf{v}$ in T, the integral in the left hand side of (51) is constant for all continuous family of cycles.

The foliation $\mathcal{F}(\mathbf{v})$ in T has infinite number of algebraic leaves $S_1(N)$, $N = 2, 3, \ldots$. The leaf $S_1(N)$ parameterizes the triples $(E, \frac{1}{N}\frac{df_N}{f_N}, P)$, where P is a torsion point of order N and f_N is a rational function in E with $\mathrm{div}(f_N) = N \cdot (P - O)$. One can give a parametrization of $S_1(N)$ by modular forms as follows. We consider the complex torus $E := \frac{\mathbb{C}}{\mathbb{Z}\tau + \mathbb{Z}}$ and its embedding in $\mathbb{P}^2$ using $z \mapsto [\wp(z, \tau) : \wp'(z, \tau) : 1]$, where $\wp(z, \tau)$ is the Weierstrass $\wp$ function and its derivation means with respect to z. We also consider the torsion point $P = \frac{1}{N}$ in E. The following function

$$F_N(\tau) := \frac{1}{N}\frac{f_N'}{f_N} - \frac{1}{2}\frac{\wp'(z, \tau) + \wp'(\frac{1}{N}, \tau)}{\wp(z, \tau) - \wp(\frac{1}{N}, \tau)}$$

is holomorphic on the torus and hence it is independent of z. Here, $f_N(z)$ is a double periodic function in z with period 1 and τ and it has a zero (resp. pole) of order N at $z = \frac{1}{N}$ (resp. $z = 0$) and $'$ means derivation with respect to z. The compactification of the curve $S_1(N)$ in $\mathbb{P}^{1,2,3,4}$ is birational to the modular curve $X_1(N) := \Gamma_1(N)\backslash\mathbb{H}^*$, where

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \colon a_3 \equiv 0, \ a_1 \equiv a_4 \equiv 1 \pmod{N} \right\}.$$

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[2]See https://w3.impa.br/~hossein/WikiHossein/files/Singular%20Codes/2018-10-Picard CuriousDifferentialEquation.txt.

Such a birational map is given by

$$\Gamma_1(N)\backslash\mathbb{H}^* \to \mathbb{P}^{1,2,3,4}, \ \ \tau \mapsto \left[F_N(\tau) : \wp\left(\tfrac{1}{N}, \tau\right) : \wp'\left(\tfrac{1}{N}, \tau\right) : 60G_4(\tau)\right].$$

The four functions involved in the above parameterization are modular forms for $\Gamma_1(N)$. The precise comparision of our Picard's differential equation given by **v** in (52) and the Picard's differential equation in [Pic89, pages 298–299] is left to the reader. For the line bundle $L := \mathcal{O}(P - O)$ on $E = E_{a,b,c}$ with its global meromorphic section s such that $\operatorname{div}(s) = P - O$, we can associate the holomorphic connection $\nabla\colon L \to \Omega_E \otimes L,\ s \mapsto \omega_{s,a,b} \otimes s$. Isomonodromic deformations of (E, ∇) is the same as deformations with constant integrals in the left hand side of (51). This and [Lor16, Corollary 2 and 3] have been the starting point of our reformulation of Picard's example.

11. Final Comments

In the present text we have avoided the arithmetic of modular curves, which is a vast territory of research with fruitful applications such as arithmetic modularity theorem. There are few topics which would fit perfectly into this article and we mention them briefly. Using geometric Hecke operators for (quasi) modular forms, see for instance [Mov15, page 432], one can prove that $Q_{d,i}$ have coefficients in $\mathbb{Z}[\tfrac{1}{6}]$ which might be used for mod p study of modular curves[3]. For arithmetic purposes such as those in [DR73] it would be essential to classify the bad primes of $S_0(d)$ and the classification of fibers over bad primes. One might use a desingularization of $\mathcal{F}$ along the curve singularity so that one gets smooth models of modular curves. We started our article from a problem posed by Darboux in [Dar78a] and elaborated a counterexample to this problem. Surprisingly, one main ingredient of this counterexample is the differential equation (46) with $\alpha = 0$, which Darboux himself derived in the article [Dar78b]; both articles being published in 1878.

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