Integral Hodge conjecture for Fermat varieties

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Abstract

We describe an algorithm which verifies whether linear algebraic cycles of the Fermat variety generate the lattice of Hodge cycles. A computer implementation of this confirms the integral Hodge conjecture for quartic and quintic Fermat fourfolds. Our algorithm is based on computation of the list of elementary divisors of both the lattice of linear algebraic cycles, and the lattice of Hodge cycles written in terms of vanishing cycles, and observing that these two lists are the same.

1 Introduction

The integral Hodge conjecture was formulated by Hodge in 1941 [Hod41], and it states that every integral Hodge cycle is algebraic. In 1962, Atiyah and Hirzebruch [AH62] showed that not every torsion cycle is algebraic, providing the first counterexamples to integral Hodge conjecture and restating it over rational numbers. For a survey on rational Hodge conjecture, which is still an open problem, see [Lew99]. Even for many varieties for which the Hodge conjecture is known to be true, the integral Hodge conjecture remains open. Kollár in [Kol92] shows that integral Hodge conjecture fails for very general threefold in $\mathbb{P}^4$ of high enough degree. Beside surfaces, integral Hodge conjecture is known to be true for cubic fourfolds, see [Voi13, Theorem 2.11]. For improvements in Atiyah-Hirzebruch approach see [Tot97] and [SV05], see also [CTV12] and [Tot13] for counterexamples in the direction of Hassett-Tschinkel method [CTV12, Remarque 5.10], which is an improvement of Kollár's construction. In the present article we prove the integral Hodge conjecture for a class of Fermat varieties.

Let $n \in \mathbb{N}$ be an even number and

$$(1) \qquad X = X_n^d \subset \mathbb{P}^{n+1} : \quad x_0^d + x_1^d + \cdots + x_{n+1}^d = 0,$$

be the Fermat variety of dimension n and degree d. It has the following well-known algebraic cycles of dimension $\frac{n}{2}$ which we call linear algebraic cycles:

$$(2) \qquad \mathbb{P}_{a,b}^{\frac{n}{2}} : \begin{cases} x_{b_0} - \zeta_{2d}^{1+2a_1} x_{b_1} = 0, \\ x_{b_2} - \zeta_{2d}^{1+2a_3} x_{b_3} = 0, \\ x_{b_4} - \zeta_{2d}^{1+2a_5} x_{b_5} = 0, \\ \vdots \\ x_{b_n} - \zeta_{2d}^{1+2a_{n+1}} x_{b_{n+1}} = 0, \end{cases}$$

where ζ_{2d} is a $2d$-primitive root of unity, b is a permutation of $\{0, 1, 2, \ldots, n+1\}$ and $0 \le a_i \le d-1$ are integers. In order to get distinct cycles we may further assume that $b_0 = 0$ and for i an even number, b_i is the smallest number in $\{0, 1, \ldots, n+1\} \backslash \{b_0, b_1, b_2, \ldots, b_{i-1}\}$. It is easy to check that the number of such cycles is

$$N := 1 \cdot 3 \cdots (n-1) \cdot (n+1) \cdot d^{\frac{n}{2}+1},$$

(for $d = 3, n = 2$ this is the famous 27 lines in a smooth cubic surface). In §3 we introduce a lattice $V_n^d \subset H_n(X, \mathbb{Z})$ such that if

$$(3) \qquad d \text{ is a prime number, or } d = 4, \text{ or } \gcd(d, (n+1)!) = 1$$

then by Poincaré duality $V_n^d \cong H^n(X, \mathbb{Z}) \cap H^{\frac{n}{2}, \frac{n}{2}}$. We prove the following theorem:

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Theorem 1. *For (n, d) in Table 1 we have:*

1. *The topological classes of $\mathbb{P}^{\frac{n}{2}}_{a,b}$ generate the lattice $V_n^d \subset H_n(X, \mathbb{Z})$ and so for those (n, d) with the property (3) the integral Hodge conjecture is valid (this includes the cases $(n, d) = (4, 4), (4, 5)$).*

2. *The elementary divisors and the discriminant of such a lattice are listed in the same table.*

(n, d)	Elementary divisors/discriminant	Rank
$(2, 3)$	$+1^7$	7
$(2, 4)$	$-1^{18} \cdot 8^2$	20
$(2, 5)$	$+1^{26} \cdot 5^{10} \cdot 25^1$	37
$(2, 6)$	$+1^{37} \cdot 3^{13} \cdot 12^8 \cdot 36^3 \cdot 108^1$	62
$(2, 7)$	$-1^{48} \cdot 7^{38} \cdot 49^5$	91
$(2, 8)$	$-1^{54} \cdot 2^{12} \cdot 8^{48} \cdot 16^2 \cdot 32^8 \cdot 64^4$	128
$(2, 9)$	$-1^{72} \cdot 3^7 \cdot 9^{72} \cdot 27^7 \cdot 81^{11}$	169
$(2, 10)$	$-1^{80} \cdot 5^{18} \cdot 10^{96} \cdot 20^{13} \cdot 100^{10} \cdot 500^1$	218
$(2, 11)$	$+1^{96} \cdot 11^{158} \cdot 121^{17}$	271
$(2, 12)$	$+1^{109} \cdot 3^5 \cdot 12^{192} \cdot 24^2 \cdot 48^2 \cdot 144^{21} \cdot 432^1$	332
$(2, 13)$	$+1^{120} \cdot 13^{254} \cdot 169^{23}$	397
$(2, 14)$	$-1^{126} \cdot 7^{20} \cdot 14^{288} \cdot 28^{15} \cdot 196^{20} \cdot 1372^1$	470
$(4, 3)$	$+1^{19} \cdot 3^1 \cdot 9^1$	21
$(4, 4)$	$+1^{100} \cdot 2^2 \cdot 4^4 \cdot 8^{30} \cdot 16^4 \cdot 32^2$	142
$(4, 5)$	$+1^{166} \cdot 5^{174} \cdot 25^{54} \cdot 125^7$	401
$(4, 6)$	$-1^{310} \cdot 3^{272} \cdot 6^{40} \cdot 12^{163} \cdot 36^{116} \cdot 108^{61} \cdot 216^{24} \cdot 648^{16}$	1002

Table 1: Elementary divisors of the lattice of linear Hodge cycles for X_n^d

The cases in the first part of Theorem 1 for Fermat surfaces ($n = 2$) are particular instances of a general theorem proved in [SSvL10] and [Deg15] for $d = 4$ or $\gcd(d, 6) = 1$. The integral Hodge conjecture for cubic fourfolds is proved in [Voi13], see Theorem 2.11 and the comments thereafter. Nevertheless, note that Theorem 1 in this case does not follow from this. The rational Hodge conjecture in all the cases covered in our theorem is well-known, see [Shi79]. The limitation for (n, d) in our main theorem is due to the fact that its proof uses the computation of Smith normal form of integer matrices, for further discussion on this see §4.

After the first draft of this article was written, A. Degtyarev informed us about his joint paper [DS16] with I. Shimada in which the authors give a criterion for primitivity of the sub-lattice of $H_n(X, \mathbb{Z})$ generated by linear algebraic cycles. They also use a computer to check the primitivity in many cases of (n, d) in our table, see [DS16, Section 5]. This together with Shioda's result implies the first statement in Theorem 1. The verification of this mathematical statement by computer, using two different methods and different people, bring more trust on such a statement even without a theoretical proof. Their method is based on computation of the intersection number of a linear algebraic cycle with a vanishing cycle ([DS16, Theorem 2.2]), whereas our method is based on the computation of both intersection matrices of vanishing cycles, see (16), and linear algebraic cycles, see (6). The advantage of our method is the computation of elementary divisors and the discriminant of the lattice of Hodge cycles.

In Table 1, $a^b \cdot c^d \cdots$ means a, b times, followed by c, d times etc. Interpreting this as usual multiplication we get the discriminant of the lattice of Hodge cycles. The discriminant is an invariant of a lattice and it is easy to see that the list of elementary divisors are also, see [Coh93, Theorem 2.4.12]. We have computed such elementary divisors from two completely different

bases of V_n^d. The first one is computed using the intersection matrix of linear algebraic cycles and the other from the intersection matrix of Hodge cycles written in terms of vanishing cycles. We can give an interpretation of the power of 1 in the list of elementary divisors, see §2.3. For instance, the lattice V_4^6 modulo 3 (resp. 2) is of discriminant zero and its largest sub lattice of non-zero discriminant has rank 310 (resp. $310 + 272$). In all the cases in Table 1, for primes p such that p does not divide d, the lattice V_n^d modulo p has non-zero discriminant. Finally, it is worth mentioning that I. Shimada in [Shi01, Theorem 1.5 and 1.6] for the case $(n, d) = (4, 3)$ and in characteristic 2 proves that the lattice spanned by linear algebraic cycles is isomorphic to the laminated lattice of rank 22 . He also computes the discriminant of such a lattice in the case $(n, d) = (6, 3)$. For an explicit basis of $V_2^4 \cong H^2(X_2^4, \mathbb{Z}) \cap H^{1,1}$ using linear algebraic cycles and the corresponding intersection matrix see [SS17, Table 3.1].

2 Preliminaries on lattices

2.1 Smith normal form:

Let $\mathbf{A}$ be a $\mu \times \check{\mu}$ matrix with integer coefficients. There exist $\mu \times \mu$ and $\check{\mu} \times \check{\mu}$ integer-valued matrices $\mathbf{U}$ and $\mathbf{T}$, respectively, such that $\det(\mathbf{U}) = \pm 1$, $\det(\mathbf{T}) = \pm 1$ and

$$(4) \qquad \mathbf{U} \cdot \mathbf{A} \cdot \mathbf{T} = \mathbf{S}$$

where $\mathbf{S}$ is the diagonal matrix $\text{diag}(a_1, a_2, \ldots, a_m, 0, \cdots, 0)$. Here, m is the rank of the matrix $\mathbf{A}$ and the natural numbers $a_i \in \mathbb{N}$ satisfy $a_i \mid a_{i+1}$ for all $i < m$ and are called the elementary divisors. The matrix $\mathbf{S}$ is called the Smith normal form of $\mathbf{A}$. The equality (4) is also called the Smith decomposition of $\mathbf{A}$.

2.2 Elementary divisors of a Lattice:

A lattice V is a free finitely generated $\mathbb{Z}$-module equipped with a symmetric bilinear form $V \times V \to \mathbb{Z}$ which we also call the intersection bilinear map. We usually fix a basis $v_1, v_2, \ldots, v_\mu$ of V. Its discriminant (resp. elementary divisors) is the determinant (resp. elementary divisors) of the intersection matrix $[v_i \cdot v_j]$. The discriminant does not depend on the choice of the basis v_i, neither do the elementary divisors. Let V be a lattice, possibly of discriminant zero. Let also $V^\perp$ be the sublattice of V containing all elements $v \in V$ such that $v \cdot V = 0$. The quotient $\mathbb{Z}$-module $V/V^\perp$ is free and it inherits a bilinear form from V, and so it is a lattice. Therefore, by definition $\text{rank}(V/V^\perp) \leq \text{rank}(V)$ and $\text{disc}(V/V^\perp) \neq 0$. Let V be a lattice and its intersection matrix $\mathbf{A} = [v_i \cdot v_j]$ in a basis v_i of V. In order to find a basis, discriminant, and elementary divisors of $V/V^\perp$ we use the Smith normal form of $\mathbf{A}$. We make the change of basis $[w_i]_{\mu \times 1} := \mathbf{U}[v_i]_{\mu \times 1}$ and the intersection matrix is now $[w_i \cdot w_j] = \mathbf{U} \cdot \mathbf{A} \cdot \mathbf{U}^{\text{tr}} = \mathbf{S} \cdot \mathbf{T}^{-1} \cdot \mathbf{U}^{\text{tr}}$. Therefore, using Smith normal form of matrices we can always find a basis w_i of V with

$$(5) \qquad [w_i \cdot w_j] = \mathbf{S} \cdot \mathbf{R},$$

where $\mathbf{R}$ is $\mu \times \mu$ integer-valued matrix with $\det(\mathbf{R}) = \pm 1$ and $\mathbf{S}$ is a diagonal matrix as in (4). Since $[w_i \cdot w_j]$ is a symmetric matrix, it follows that $\mathbf{R}$ is a block diagonal matrix with two blocks of size $m \times m$ and $(\mu - m) \times (\mu - m)$ in the diagonal. Both blocks have determinant ± 1 because $\det(\mathbf{R}) = \pm 1$. A basis of $V^\perp$ (resp. $V/V^\perp$) is given by w_i, $i = m + 1, \cdots, \mu$ (resp. $i = 1, 2, \cdots, m$). It follows that the elementary divisors of $V/V^\perp$ in this basis are $a_1, a_2, \cdots, a_m$.

2.3 Lattices mod primes

For a lattice V let $V_p := V/pV$ which is a $\mathbb{F}_p$-vector space with the induced $\mathbb{F}_p$-bilinear map $V_p \times V_p \to \mathbb{F}_p$. In a similar way as before we define $V_p^\perp$.

Proposition 1. *Let $a_1, a_2, \ldots, a_m$ be the elementary divisors of the lattice V. For a prime number such that $p \mid a_{s+1}$ but $p \nmid a_j$ for all $j \leq s$, the number s is the rank of $V_p/V_p^\perp$.*

Proof. This follows from the intersection matrix of V with respect to the basis w_i of V in (5). $\square$

3 Fermat variety

Let $X \subseteq \mathbb{P}^{n+1}$ be the Fermat variety of degree d and even dimension n. In $H_n(X, \mathbb{Z})$ we have the algebraic cycle $[Z_\infty]$, where Z_∞ is the intersection of a linear projective space $\mathbb{P}^{\frac{n}{2}+1}$ with X. Every algebraic subvariety $Z \subseteq X$ of codimension $\frac{n}{2}$ induces an algebraic cycle $[Z] \in H_n(X, \mathbb{Z})$. By abuse of notation, we will also denote the algebraic cycle by Z instead of $[Z]$, letting it be understood in the context which one we are referring to. A cycle $\delta \in H_n(X, \mathbb{Z})$ is called primitive if its intersection with Z_∞ is zero. For any subset $V \subset H_n(X, \mathbb{Z})$ we denote by $V_0 \subset V$ the set of primitive elements of V. Let $v_1, v_2, \cdots, v_\mu$ be any collection of primitive cycles (possibly linearly dependent in $H_n(X, \mathbb{Z})$) and let V be the free $\mathbb{Z}$-module generated by v_i's. It has a natural lattice structure induced by the intersection of v_i's in $H_n(X, \mathbb{Z})$. From the Hodge index theorem it follows that $V^\perp$ maps to zero in $H_n(X, \mathbb{Z})$ and $V/V^\perp$ is a sublattice of $H_n(X, \mathbb{Z})$. We will use this fact twice, for linear algebraic cycles and for Hodge cycles written in terms of vanishing cycles.

3.1 The lattice of linear algebraic cycles

Let us index arbitrarily the linear subspaces defined in (2) as $P_i, \quad i = 1, 2, \ldots, N$. A linear algebraic cycle P_i intersects Z_∞ transversally in a point, and so $P_i \cdot Z_\infty = 1$. Moreover, one can use the adjunction formula and show

$$(6) \qquad\qquad P_i \cdot P_j = \frac{1 - (-d+1)^{m+1}}{d}$$

where $\dim(P_i \cap P_j) = m$, see [Mov18, Section 17.6]. The primitive part of the sublattice of $H_n(X, \mathbb{Z})$ generated by P_i, is generated by

$$(7) \qquad\qquad \check{P}_i := P_i - P_1, \quad i = 2, \ldots, N.$$

The linear subspace $\mathbb{P}^{\frac{n}{2}+1}$ can be chosen as $\{x_0 - \zeta_{2d}x_1 = \cdots = x_n - \zeta_{2d}x_{n+1} = 0\}$, so that the polarization cycle Z_∞ splits as a sum of d linear algebraic cycles P_i which intersect each other in a linear $\mathbb{P}^{\frac{n}{2}-1} \subset \mathbb{P}^{n+1}$. Therefore, the primitive cycle $dP_i - Z_\infty$ is a $\mathbb{Z}$-linear combination of (7). For the proof of Theorem 1 we first write the Smith normal form of the $(N-1) \times (N-1)$ matrix

$$(8) \qquad\qquad \mathtt{A}_1 := [\check{P}_i \cdot \check{P}_j].$$

3.2 The lattice of primitive Hodge cycles

For all missing proofs and references in this paragraph see [Mov18, Chapter 15,16]. Let L be the affine chart of the Fermat variety X given by $x_0 \neq 0$. A topological cycle $\delta \in H_n(X, \mathbb{Z})$ is primitive if and only if it is supported in L. We also know that the homology group $H_n(L, \mathbb{Z})$ is free of rank $\mu := (d-1)^{n+1}$ and we can choose a basis

$$\delta_\beta, \ \beta \in I := \{0, 1, 2, \ldots, d-2\}^{n+1}$$

of $H_n(L, \mathbb{Z})$ by homological classes homeomorphic to n-dimensional spheres, see [Mov18, §15.2]. These are Lefschetz' famous vanishing cycles, which are obtained by the shifts of Pham polyhedron by deck translations, see [Pha65]. On the other hand, we can also choose a basis of the

image of the restriction map $F^{\frac{n}{2}+1}H^n_{\mathrm{dR}}(X) \to H^n_{\mathrm{dR}}(L)$ (where $F^{\frac{n}{2}+1}$ is the $(\frac{n}{2}+1)$-th piece of the Hodge filtration of $H^n_{\mathrm{dR}}(X)$). In fact, take

$$(9) \qquad \omega_\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}} \left(\sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1} \right), \beta \in I_1,$$

where

$$(10) \qquad I_1 := \left\{ (\beta_1, \cdots, \beta_{n+1}) \in \mathbb{Z}^{n+1} \,\middle|\, 0 \le \beta_i \le d-2, \ \sum_{i=1}^{n+1} \frac{\beta_i + 1}{d} \notin \mathbb{Z} \text{ and } < \frac{n}{2} \right\}.$$

Therefore, the lattice of Hodge cycles is given by

$$(11) \qquad \mathrm{Hodge}_n(X, \mathbb{Z}) := \left\{ \delta \in H_n(X, \mathbb{Z}) \,\middle|\, \int_\delta \omega_\beta = 0, \ \ \forall \beta \in I_1 \right\}.$$

Let I_2 be the set of $\beta \in I$ such that

1. There is an integer β_0 satisfying the conditions $0 \le \beta_0 \le d-2$ and $\sum_{i=0}^{n+1} \frac{\beta_i + 1}{d} := \frac{n}{2} + 1$.

2. There is no permutation σ of $0, 1, \ldots, n+1$ without fixed point and with σ^2 being the identity such that $\beta_i + \beta_{\sigma(i)} = d - 2$ for all i.

The space of linear Hodge cycles is defined in the following way:

$$(12) \qquad V_n^d := \left\{ \delta \in H_n(X, \mathbb{Z}) \,\middle|\, \int_\delta \omega_\beta = 0, \ \ \forall \beta \in I_1 \cup I_2 \right\}.$$

We know that the topological classes of the primitive linear algebraic cycles $\check{P}_i$ are in $(V_n^d)_0$ (the subindex 0 was introduced at the beginning of §3) and over rational numbers they generate it, see for instance [Aok83, Theorem A], (the first statement also follows from the integration formula in the main theorem of [MV18]). Moreover, if d is prime or $d = 4$ or d is coprime with $(n+1)!$ then I_2 is the empty set and so $\mathrm{Hodge}_n(X, \mathbb{Z}) = V_n^d$. One can easily compute the integrals $\int_{\delta_\beta} \omega_{\beta'}$, $\beta \in I_1 \cup I_2$ explicitly, see [DMOS82, Lemma 7.12] or [Mov18, Proposition 15.1], and it turns out that $\delta := \sum_{\beta \in I} c_\beta \delta_\beta \in (V_n^d)_0$, $c_\beta \in \mathbb{Z}$ if and only if the coefficient matrix $\mathsf{C} = [c_\beta]_{1 \times \mu}$ satisfies $\mathsf{C} \cdot Q = 0$, where Q is the $\mu \times \check{\mu}$ matrix given by:

$$(13) \qquad Q := \left[\prod_{i=1}^{n+1} \left(\zeta_d^{(\beta_i+1)(\beta_i'+1)} - \zeta_d^{\beta_i(\beta_i'+1)} \right) \right]_{\beta \in I, \beta' \in I_1 \cup I_2}$$

$\zeta_d := e^{\frac{2\pi i}{d}}$ is the d-th primitive root of unity and $\check{\mu} := \#(I_1 \cup I_2)$. Since the minimal polynomial of ζ_d is of degree $\varphi(d)$ (Euler's totient function) we write $Q = \sum_{i=0}^{\varphi(d)-1} Q_i \zeta_d^i$ and define

$$(14) \qquad \mathsf{A}_2 := [Q_0 | Q_1 | \cdots | Q_{\varphi(d)-1}]$$

which is a $\mu \times (\check{\mu} \cdot \varphi(d))$ matrix obtained by concatenation of Q_i's. Now, the coefficients matrix C of a primitive Hodge cycle satisfies $\mathsf{C} \cdot \mathsf{A}_2 = 0$. We write A_2 in the Smith normal form. Let Y be the $(\mu - m) \times \mu$ matrix which is obtained by joining the $(\mu - m) \times m$ zero matrix with the $(\mu - m) \times (\mu - m)$ identity matrix in a row. The rows of

$$(15) \qquad \mathsf{X} := \mathsf{Y} \cdot \mathsf{U}_2$$

form a basis of the integer-valued $1 \times \mu$ matrices $\mathtt{C}$ such that $\mathtt{C} \cdot \mathtt{A}_2 = \mathtt{0}$. The intersection numbers of topological cycles δ_β are given by the the following rules:

(16)
$$\begin{aligned}
\langle \delta_\beta, \delta_{\beta'} \rangle &= (-1)^n \langle \delta_{\beta'}, \delta_\beta \rangle, \quad \forall \beta, \beta' \in I, \\
\langle \delta_\beta, \delta_\beta \rangle &= (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^n), \quad \forall \beta \in I \\
\langle \delta_\beta, \delta_{\beta'} \rangle &= (-1)^{\frac{n(n+1)}{2}} (-1)^{\Sigma_{k=1}^{n+1} \beta'_k - \beta_k}
\end{aligned}$$

for those $\beta, \beta' \in I$ such that for all $k = 1, 2, \ldots, n+1$ we have $\beta_k \leq \beta'_k \leq \beta_k + 1$ and $\beta \neq \beta'$. In the remaining cases, except those arising from the previous ones by a permutation, we have $\langle \delta_\beta, \delta_{\beta'} \rangle = 0$, see [Mov18, Proposition 7.7]. This computation of intersection number of δ_β's is essentially due to F. Pham, see [Pha65] and [AGZV88, Page 66]. Let $\Psi := [\langle \delta_\beta, \delta_{\beta'} \rangle]$ be the $\mu \times \mu$ intersection matrix. The intersection matrix for primitive Hodge cycles is:

$$\mathtt{A}_3 := \mathtt{X} \cdot \Psi \cdot \mathtt{X}^{\mathsf{tr}}.$$

3.3 Proof of Theorem 1

The proof is reduced to computation of elementary divisors of $\mathtt{A}_1$ and $\mathtt{A}_3$ (counting with repetitions) and observing that they are the same. In other words, one has to check if the output of the algorithms (1) and (2) are the same. Such elementary divisors might be obtained from those listed in Table 1 in the following way: decrease the power of 1 by two and add d to the list of elementary divisors. This proves that the sublattice of $H_n(X, \mathbb{Z})_0$ generated by (7) coincides with $(V_n^d)_0$ (the lattice of primitive Hodge cycles). Since $H_n(X, \mathbb{Z}) = H_n(X, \mathbb{Z})_0 + \mathbb{Z} \cdot P_1$ we get Theorem 1. We had to compute elementary divisors of V_n^d in Table 1 separately, as we did not find any straightforward argument relating this with the elementary divisors of its primitive part. For the sake of completeness, we also computed the sign of each discriminant in Table 1. This is $\det(\mathtt{U}_1) \cdot \det(\mathtt{T}_1)$ for the Smith normal form of $\mathtt{A}_1$ (one could also use $\mathtt{A}_3$ for this purpose). The direct computation of the determinant in general does not work, and we had to compute the Smith normal form of $\mathtt{U}_1, \mathtt{T}_1$ and count the number of -1 in the corresponding two matrices $\mathtt{S}$.

Algorithm 1: Computation of elementary divisors of the intersection matrix of linear algebraic cycles for a Fermat variety $X_n^d \subseteq \mathbb{P}^{n+1}$.

Data: Pair (n, d), where n is an even positive integer and d a natural number, representing the dimension and the degree, respectively, of a Fermat variety $X_n^d \subseteq \mathbb{P}^{n+1}$.

Result: Elementary divisors of the intersection matrix of linear algebraic cycles of the Fermat variety $X_n^d \subseteq \mathbb{P}^{n+1}$.

begin

    $\mathcal{I} \leftarrow$ Family of ideals of the subvarieties $\mathbb{P}^{\frac{n}{2}}_{a,b}$ described in (2);

    for $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{I} \times \mathcal{I}$ **do**

        $\mathfrak{c} \leftarrow$ Ideal generated by $\mathfrak{a}$ and $\mathfrak{b}$;

        $m \leftarrow \dim(\mathfrak{c})$;

        $M_{\mathfrak{a}, \mathfrak{b}} \leftarrow \dfrac{1 - (-d+1)^{m+1}}{d}$;

    $L \leftarrow$ List of elementary divisors obtained from the Smith decomposition of M;

    return L;

end

Algorithm 2: Computation of elementary divisors of the intersection matrix of primitive Hodge cycles for a Fermat variety $X_n^d \subseteq \mathbb{P}^{n+1}$.

Data: Pair (n, d), where n is an even positive integer and d a natural number, representing the dimension and the degree, respectively, of a Fermat variety $X_n^d \subseteq \mathbb{P}^{n+1}$.

Result: Elementary divisors of the intersection matrix of primitive Hodge cycles of the Fermat variety $X_n^d \subseteq \mathbb{P}^{n+1}$.

begin

$\quad I, I_1, I_2 \leftarrow$ index sets as defined in subsection 3.2;

$\quad Q \leftarrow \left[\prod_{i=1}^{n+1} \left(\zeta_d^{(\beta_i+1)(\beta_i'+1)} - \zeta_d^{\beta_i(\beta_i'+1)} \right) \right]_{\beta \in I, \beta' \in I_1 \cup I_2}$;

$\quad \mathtt{A_2} \leftarrow [Q_0 | Q_1 | \cdots | Q_{\varphi(d)-1}]$ where Q_i satisfy $Q = \sum_{i=0}^{\varphi(d)-1} Q_i \zeta_d^i$;

$\quad \mathtt{U_2} \leftarrow$ matrix arising in Smith normal form of $\mathtt{A_2}$ as in (4);

$\quad \mathtt{X} \leftarrow$ defined as in (15);

$\quad \Psi \leftarrow \left[\langle \delta_\beta, \delta_{\beta'} \rangle \right]$, see (16);

$\quad \mathtt{A_3} \leftarrow \mathtt{X} \cdot \Psi \cdot \mathtt{X}^{\mathsf{tr}}$;

$\quad L \leftarrow$ List of elementary divisors obtained from the Smith decomposition of $\mathtt{A_3}$;

$\quad$ **return** L;

end

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4 Our computational experience

The matrix $\mathtt{A_1}$ in (8) is implemented in `Matlab`, $\mathtt{A_2}$ in `Singular`, and $\mathtt{A_3}$ in `Mathematica`. All the Smith normal forms are performed in `Mathematica`. For all data produced in our computations see the first author's web page. We started to prepare Table 1 with a computer with processor `Intel Core i7-7700`, 16 GB Memory plus 16 GB swap memory and the operating system `Ubuntu 16.04`. It turned out that for the case $(n, d) = (4, 6)$ we get the 'Memory Full' error. This was during the computation of the Smith normal form of the matrix $\mathtt{A_1}$. Therefore, we had to increase the swap memory up to 170 GB. Despite the low speed of the swap which slowed down the computation, the computer was able to use the data and give us the desired output. For this reason the computation in this case took more than two days. We only know that at least 56 GB of the swap were used. We were able to compute the elementary divisors of the lattice of primitive Hodge cycles in Table 2 but were not able to compute the elementary divisors of the lattice of linear algebraic cycles.

(n, d)	Elementary divisors
$(6, 3)$	$+1^{54} \cdot 3^8 \cdot 9^7 \cdot 27^1$
$(6, 4)$	$-1^{576} \cdot 2^{68} \cdot 4^{49} \cdot 8^{296} \cdot 16^{48} \cdot 32^{69} \cdot 512^2$
$(8, 3)$	$-1^{172} \cdot 3^{35} \cdot 9^{34} \cdot 27^{11}$
$(10, 3)$	$+1^{559} \cdot 3^{144} \cdot 9^{144} \cdot 27^{76} \cdot 81^1$

Table 2: Elementary divisors of the lattice of primitive Hodge cycles of X_n^d

For the case $(n, d) = (10, 3)$, 36 GB of swap were used and the number of linear algebraic cycles is $N = 7577955$ and we were even not able to produce the matrix $\mathtt{A_1}$ which is the intersection matrix of the lattice of linear algebraic cycles.

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