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Automorphic forms for triangle groups: Integrality properties

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A B S T R A C T

We classify all primes appearing in the denominators of the Hauptmodul J and modular forms for non-arithmetic triangle groups with a cusp. These primes have a congruence condition in terms of the order of the generators of the group. As a corollary we show that for the Hecke group of type $(2, m, \infty)$, the prime p does not appear in the denominator of J if and only if $p \equiv \pm 1 \pmod{m}$.

1. Introduction

The theory of automorphic forms for Fuchsian groups was first developed by Poincaré. His construction is based on series carrying nowadays his name, analogous to the classical Eisenstein series (Fuchsian theta-series in his terminology). A disadvantage of this method is that explicit q-expansions which are fruitful part of the theory of modular forms for arithmetic groups are not available for these groups. An alternative approach with concentrating on explicit q-expansions for a special case, namely hyperbolic triangle groups is available. Here we briefly explain this method (for details see [3]).

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Let us consider the Halphen system

$$\begin{cases} \dot{t}_1 = (a-1)(t_1t_2 + t_1t_3 - t_2t_3) + (b+c-1)t_1^2 \\ \dot{t}_2 = (b-1)(t_2t_1 + t_2t_3 - t_1t_3) + (a+c-1)t_2^2 \\ \dot{t}_3 = (c-1)(t_3t_1 + t_3t_2 - t_1t_2) + (a+b-1)t_3^2 \end{cases} \tag{1}$$

with

$$1-a-b = \frac{1}{m_1}, \qquad 1-b-c = \frac{1}{m_2}, \qquad 1-a-c = \frac{1}{m_3} = 0, \tag{2}$$

and $m_1 \le m_2 \in \mathbb{N} \cup \{\infty\}$ with the hyperbolicity condition $\frac{1}{m_1} + \frac{1}{m_2} < 1$. Here $\dot{t} = q\frac{dt}{dq}$ and we consider $t_i \in \mathbb{C}[[q]]$ as formal power series in q with the initial condition:

$$t_1(0) = t_3(0) = 0,$$

$$t_2 = \begin{cases} -1 - (m_1+1)q + O(q^2) & \text{if } m_2 = \infty \\ -1 + (m_1^2m_2 + m_1^2 - m_1m_2^2 - m_2^2)q + O(q^2) & \text{otherwise.} \end{cases}$$

The recursion of Halphen system determines uniquely t_i's. If we set $q = \exp(\frac{2\pi i\tau}{h})$, where $h = 2\cos(\frac{\pi}{m_1}) + 2\cos(\frac{\pi}{m_2})$, then t_i's are meromorphic functions on $\mathrm{Im}(\tau) > \tau_0$ for some real positive τ_0. Now, rescaling q by multiplying a constant, t_i's become meromorphic on the whole upper half-plane with modular property with respect to the triangle group $\Gamma_{\mathfrak{t}} := \langle \gamma_1, \gamma_2, \gamma_3 \rangle \subset \mathrm{SL}(2,\mathbb{R})$ of type $\mathfrak{t} = (m_1, m_2, \infty)$, where

$$\gamma_1 = \begin{pmatrix} 2\cos(\frac{\pi}{m_1}) & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 2\cos(\frac{\pi}{m_2}) \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

$$\gamma_1\gamma_2\gamma_3 = \gamma_1^{m_1} = \gamma_2^{m_2} = -I_{2\times 2}. \tag{3}$$

The Hauptmodul for this triangle group is given by

$$J = \frac{t_3 - t_2}{t_3 - t_1}. \tag{4}$$

We define

$$E_{2k}^{(1)} := (t_1 - t_2)(t_3 - t_2)^{k-1} \in 1 + q\mathbb{Q}[[q]], \tag{5}$$

$$E_{2k}^{(2)} := (t_1 - t_2)^{k-1}(t_3 - t_2) \in 1 + q\mathbb{Q}[[q]]. \tag{6}$$

In [3] we showed that the algebra of automorphic forms for the group $\Gamma_{\mathfrak{t}}$ with $m_1 \le m_2 < \infty$ is generated by

$$E_{2k}^{(1)}, \quad 3 \le k \le m_1, \qquad E_{2k}^{(2)}, \quad 2 \le k \le m_2,$$

and when $m_1 < m_2 = \infty$, the algebra is generated by

$$E_{2k}^{(1)}, \quad 1 \le k \le m_1.$$

For the triangle group of type (∞, ∞, ∞) see [3]. The coefficients of J are rational numbers apart from the rescaling (transcendental) constant. This constant appears to fit the convergence of J in the whole upper half-plane (see [13] for a proof of transcendence of this constant). The rationality comes out from the recursion of the Halphen system for the coefficients of t_i. A natural question would be a classification of primes which appear in the denominators. In [3] we stated a conjecture concerning this problem. The aim of this article is to give a complete answer to this question. We recall that a power series f is called p-integral if, after multiplication of f by a constant, its coefficients are p-adic integers. We say an algebra of power series in $\mathbb{Q}[[q]]$ is p-integral if it has a basis with p-integral elements. We say an object (function or algebra) is 'almost' integral if it is p-integral for all but finitely many p.

Theorem 1. *Let $m_1 \le m_2 \in \mathbb{N}$ and p be a prime with $p > 2m_1m_2$. The Hauptmodul J, defined in (4), for the triangle group of type (m_1, m_2, ∞) is p-integral if and only if for some $\epsilon = \pm 1$ and $\epsilon' = \pm 1$ we have*

$$\left(p \overset{2m_1}{\equiv} \epsilon,\ p \overset{2m_2}{\equiv} \epsilon'\epsilon\right) \quad or \quad \left(p \overset{2m_1}{\equiv} m_1 + \epsilon,\ p \overset{2m_2}{\equiv} m_2 + \epsilon'\epsilon\right).$$

For the triangle group (m, ∞, ∞) and $p > 2m$ the Hauptmodul J is p-integral if and only if

$$p \overset{2m}{\equiv} \pm 1.$$

We need the condition $p > 2m_1m_2$ for the 'if' part of the theorem and the 'only if' part only requires only that p does not divide $2m_1m_2$. Some computations show that the theorem must be valid with this weak hypothesis on p. For example for $(2, 5, \infty)$, our experimental computations shows that the J function up to 183 terms is p-integral for $p = 11, 19$ (see below, Corollary 2).

Corollary 1. *The Hauptmodul J for a triangle group is almost integral if and only if*

$$\begin{aligned}(m_1, m_2, \infty) = {} & (2, 3, \infty), (2, 4, \infty), (2, 6, \infty), (2, \infty, \infty), (3, 3, \infty), (3, \infty, \infty), \\ & (4, 4, \infty), (6, 6, \infty), (\infty, \infty, \infty).\end{aligned}$$

This is the Takeuchi's classification in [12] of arithmetic triangle groups with a cusp and of type (m_1, m_2, ∞). For explicit uniformizations of modular curves attached to these 9 cases see [2].

Corollary 2. *Let $3 \leq n \in \mathbb{N}$. For a prime $p > 4n$ the Hauptmodul J of the Hecke group $\Gamma_{(2,n,\infty)}$ is p-integral if and only if $p \equiv \pm 1 \pmod{n}$.*

We remind that Corollary 2 was a conjecture made by Leo Garret in his PhD thesis [6]. He proved some partial results in this direction. Precisely, he showed that if $p \equiv 1 \bmod 4n$, then J is p-integral.

Corollary 3. *Let $p > 2m_1m_2$ be a prime number. The algebra of automorphic forms for the triangle group of type (m_1, m_2, ∞) is p-integral if and only if p satisfies the conditions of Theorem 1.*

Integrality problem for the coefficients of modular forms for noncongruence subgroups of $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})(= \Gamma_{(2,3,\infty)})$ was a task in [9]. There, Scholl proves that there exist positive integers d and N such that $d^n a_n \in \mathcal{O}_F[\frac{1}{N}]$, where a_n is the n-th Fourier coefficient of a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ for some subgroup of $\Gamma(1)$ and F a number field. A conjecture of Atkin and Swinnerton-Dyer predicts that $N = 1$ if and only if the subgroup contains a congruence subgroup (see [1]). The result of Scholl implies that at most finitely many distinct primes can appear in the denominators of modular forms for a noncongruence subgroup of $\Gamma(1)$. On the other hand, when the group is not commensurable with $\Gamma(1)$, one would expect infinitely many primes in the denominators. This prediction is compatible with our result in the case of hyperbolic triangle groups.

The paper is organized in the following way. In Section 2 we introduce the main technique for establishing the results of the paper. This is namely the Dwork method which is based on a Lemma due to Dieudonné and a Theorem due to Dwork, see [4,5]. In Section 3 we prove the corollaries and give the proof of the main theorem.

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2. Dwork method

The main idea in the proof of Theorem 1 is based on the Dwork method. Here we briefly review this method.

2.1. Dwork map

For the p-adic integers $\mathbb{Z}_p$, the Dwork map $\delta_p : \mathbb{Z}_p \to \mathbb{Z}_p$ is given by

$$x = \sum_{s=0}^{\infty} x_s p^s \longmapsto 1 + \sum_{s=0}^{\infty} x_{s+1} p^s, \quad 0 \leq x_s \leq p - 1.$$

In other words, for every x, with $x \equiv x_0 \pmod{p\mathbb{Z}_p}$, $\delta_p(x) := 1 + \frac{x - x_0}{p}$. Denote by $\mathbb{Z}_{(p)}$ the set of p-integral rational numbers. We have a natural embedding $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Z}_p$. The map δ_p leaves $\mathbb{Z}_{(p)}$ invariant because for $x \in \mathbb{Z}_{(p)}$, $\delta_p(x)$ is the unique number such that $p\delta_p(x) - x \in \mathbb{Z} \cap [0, p-1]$. For rational numbers there exists an alternative definition for the Dwork map as follows. Let $x = \frac{x_1}{x_2}$, with x_1 and $x_2 > 0$ integers and a prime p which does not divide x_2, we have

$$\delta_p(x) := \frac{p^{-1}x_1 \bmod x_2}{x_2}, \tag{7}$$

where p^{-1} is the inverse of p mod x_2 (note that x_1 and x_2 may have common factors). The denominators of x and $\delta_p(x)$ are the same and $\delta_p(1-x) = 1 - \delta_p(x)$. For any finite set of rational numbers, there is a finite decomposition of prime numbers such that in each class the function δ_p is independent of the prime p. Indeed for the set of primes $p \overset{x_2}{\equiv} r$, $\delta_p(x)$ only depends on x and r.

2.2. Gauss hypergeometric function

Let us consider the following hypergeometric differential operator

$$L : \theta^2 - z(\theta + a)(\theta + b), \tag{8}$$

with $\theta = z\frac{d}{dz}$ and

$$a = \frac{1}{2}\left(1 - \frac{1}{m_1} + \frac{1}{m_2}\right), \qquad b = \frac{1}{2}\left(1 - \frac{1}{m_1} - \frac{1}{m_2}\right), \tag{9}$$

where $2 \le m_1, m_2 \in \mathbb{N} \cup \{\infty\}$ and $\frac{1}{m_1} + \frac{1}{m_2} < 1$. Note that these a, b are slightly different from those in the introduction. From now on we will only use (9). The monodromy group of the corresponding differential equation is the triangle group of type (m_1, m_2, ∞), see for instance [3]. The Frobenius basis of (8) around $z = 0$ is given by $\{F(z), F(z)\log z + G(z)\}$, where

$$F(a,b|z) = 1 + \sum_{i=1}^{\infty} A_i(a,b)z^i = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{n!^2} z^n, \tag{10}$$

$$G(a,b|z) = \sum_{i=1}^{\infty} B_i(a,b)z^i = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{n!^2} \left(\sum_{i=0}^{n-1} \frac{1}{a+i} + \frac{1}{b+i} - \frac{2}{1+i}\right) z^n. \tag{11}$$

Let us define

$$D(a,b|z) := \frac{G(a,b|z)}{F(a,b|z)}, \qquad q(a,b|z) := z\exp\bigl(D(a,b|z)\bigr), \tag{12}$$

and D is called the Schwarz map. The Hauptmodul J introduced in the Introduction is given by

$$J = \frac{1}{z(\kappa \cdot q)}, \qquad \kappa := -2m_1^2 m_2^2,$$

where $z(q)$ is the inverse of q as a function in z, for more details see [3].

2.3. Dwork's theorem

The following lemma is the additive version of Dieudonné–Dwork lemma and frequently is used in the proof of p-integrality of power series.

Lemma 1. *Let $u(z) \in z\mathbb{Q}_p[[z]]$. Then $\exp(u(z)) \in 1 + z\mathbb{Z}_p[[z]]$, if and only if*

$$\exp\bigl(u\bigl(z^p\bigr) - pu(z)\bigr) \in 1 + p\mathbb{Z}_p[[z]].$$

For a more general statement and the proof see [5], p. 54. The following theorem is the main part of Dwork method.

Theorem 2. *Let D be the Schwarz map, defined in (12) and p a prime number coprime with $2m_1m_2$. We have*

$$D\bigl(\delta_p(a), \delta_p(b)|z^p\bigr) \equiv pD(a, b|z) \quad \bigl(\mathrm{mod}\ p\mathbb{Z}_p[[z]]\bigr).$$

As a remark we mention that the original Dwork's theorem is valid not only for arbitrary $a, b \in \mathbb{Z}_p$ but also for generalized hypergeometric series. For a proof see [4].

Corollary 4. *If*

$$\bigl\{\delta_p(a), \delta_p(b)\bigr\} = \{a, b\} \ \text{or} \ \{1 - a, 1 - b\} \tag{13}$$

holds then $q(a, b|z)$ is p-integral.

Proof. For $\{\delta_p(a), \delta_p(b)\} = \{a, b\}$, the statement is an immediate consequence of Dwork's theorem and Lemma 1 with $u(z) = D(a, b|z)$ and the fact that $\mathrm{ord}_p(n!) < n$. For the second case, thanks to the Euler identity

$$F(a, b|z) = (1 - z)^{1-a-b} F(1 - a, 1 - b|z),$$

one can easily check that the logarithmic solution of (8) and so G satisfy the same identity. Then the result follows from the first case. □

Corollary 4 gives a sufficient condition for p-integrality of $q(a, b|z)$. In order to proof Theorem 1 we need also a necessary condition. The following corollary is a step toward this goal.

Corollary 5. *Let p and $q(a,b|z)$ as before. If the function $q(a,b|z)$ is p-integral, then*

$$D\big(\delta_p(a),\delta_p(b)|z\big) \equiv D(a,b|z) \quad \big(\mathrm{mod}\ p\mathbb{Z}_p[[z]]\big), \tag{14}$$

and vice versa.

Proof. If $q(a,b|z)$ is p-integral, from Lemma 1 we have

$$D\big(a,b|z^p\big) - pD(a,b|z) = \log\big(1+ph(z)\big),$$

for some $h(z) \in z\mathbb{Z}_p[[z]]$. But

$$\log\big(1+ph(z)\big) = \sum_{n=1}^{\infty}(-1)^n\frac{p^n h(z)^n}{n} \in pz\mathbb{Z}_p[[z]].$$

Hence

$$D\big(a,b|z^p\big) \equiv pD(a,b|z) \quad \big(\mathrm{mod}\ p\mathbb{Z}_p[[z]]\big).$$

Combining with the congruence of Theorem 2 the result follows. The other side is similar. □

In the continuation we will determine complete conditions such that the congruency (14) holds. We will prove that it is equivalent to (13) in Corollary 4.

Lemma 2. *Let k be a field of characteristic $p \neq 2$ and $a_1,a_2,b_1,b_2 \in \mathsf{k}$. The coefficients of z^i, $i=1,2$, in*

$$D(a_2,b_2|z) \quad \textit{and} \quad D(a_1,b_1|z) \tag{15}$$

are equal if and only if

$$\{a_2,b_2\} = \{a_1,b_1\} \ \textit{or}\ \{1-a_1,1-b_1\}. \tag{16}$$

Note that in general $G(a_1,b_1|z)$, $a_1,b_1 \in \mathsf{k}$ (consequently $D(a_1,b_1|z)$), is not well-defined because in its expression we have division by primes. However, it makes sense to talk about the coefficients of z and z^2 in characteristic $p \neq 2$.

Proof. Let $\sigma = a+b$, $\tau = ab$. The coefficients of $D(a,b|z)$ can be written in terms of the symmetric polynomials σ,τ. Let $C_k(\sigma,\tau)$ be the k-th coefficient of $D(a,b|z)$. By definition we have $C_1(\sigma,\tau) = \sigma - 2\tau$, so the assumption implies that

$$\sigma_1 - 2\tau_1 \overset{p}{\equiv} \sigma_2 - 2\tau_2. \tag{17}$$

Now for $C_2(\sigma, \tau)$ we have

$$\begin{aligned}4C_2(\sigma_1,\tau_1) - 4C_2(\sigma_2,\tau_2) &= \sigma_1^2 - 5\sigma_1\tau_1 + 5\tau_1^2 - \sigma_2^2 + 5\sigma_2\tau_2 - 5\tau_2^2 + \sigma_1 - \tau_1 - \sigma_2 + \tau_2 \\ &= (\sigma_1 - 2\tau_1)^2 - (\sigma_2 - 2\tau_2)^2 + \tau_2(\sigma_2 - 2\tau_2) \\ &\quad - \tau_1(\sigma_1 - 2\tau_1) + \tau_2^2 - \tau_1^2 + \tau_1 - \tau_2 \\ &\equiv (\tau_2 - \tau_1)(\sigma_1 - 2\tau_1 + \tau_1 + \tau_2 - 1) \pmod p.\end{aligned}$$

In the above we have used the congruence (17) in the last line. Hence from the last congruence we conclude that either $\tau_1 \overset{p}{\equiv} \tau_2$ or $\sigma_1 \overset{p}{\equiv} \tau_1 - \tau_2 + 1$. The first (second) case together with the equation (17) gives the first (second) possibility mentioned in (16). □

3. Proofs

In this section we give a proof of Theorem 1 and its corollaries announced in the Introduction.

3.1. Proof of Theorem 1

First we show that for $p > 2m_1m_2$, the p-integrality of $q(a, b|z)$ is equivalent to condition (13). In fact if $q(a, b|z)$ is p-integral, then equation (14) holds. In particular we can apply Lemma 2 for $\{a_1, b_1\} = \{a, b\}$, $\{a_2, b_2\} = \{\delta_p(a), \delta_p(b)\}$ and the finite field $\mathsf{k} = \frac{\mathbb{Z}}{p\mathbb{Z}}$. It follows that $\{\delta_p(a), \delta_p(b)\}$ congruent to $\{a, b\}$ or $\{1 - a, 1 - b\}$ modulo k. But $p > 2m_1m_2$, in particular it is greater than the denominators of $a, b < 1$. Since the action of δ_p does not change the denominator, so the above congruence is indeed an equality in $\mathbb{Z}$. Hence the only thing to complete the proof is to show that the conditions of Theorem 1 are equivalent to equation (13). In order to do this we analyze the equality (13) case by case.

Recall that

$$a = \frac{a_1}{a_2} = \frac{m_1m_2 - m_1 + m_2}{2m_1m_2}, \qquad b = \frac{b_1}{b_2} = \frac{m_1m_2 - m_1 - m_2}{2m_1m_2}.$$

1. $\delta_p(a) = a$, $\delta_p(b) = b$ or $\delta_p(a) = 1 - a$, $\delta_p(b) = 1 - b$. By definition of the Dwork map in (7), in this case we have $p^{-1}a_1 \overset{a_2}{\equiv} \epsilon a_1$ and $p^{-1}b_1 \overset{b_2}{\equiv} \epsilon b_1$, where $\epsilon = 1$ corresponds to the first case and $\epsilon = -1$ corresponds to the second case. Since p is odd, the above congruences are equivalent to

$$\begin{aligned}p(m_1 + m_2) &\equiv \epsilon(m_1 + m_2) \pmod{2m_1m_2} \\ p(m_1 - m_2) &\equiv \epsilon(m_1 - m_2) \pmod{2m_1m_2}.\end{aligned} \tag{18}$$

Once adding and subtracting of congruences in (18) we find that $p \overset{m_i}{\equiv} \epsilon$ for both $i = 1, 2$. From this fact one can easily check that (18) is equivalent to

$$\big(p \overset{2m_1}{\equiv} \epsilon,\ p \overset{2m_2}{\equiv} \epsilon\big) \quad \text{or} \quad \big(p \overset{2m_1}{\equiv} m_1+\epsilon,\ p \overset{2m_2}{\equiv} m_2+\epsilon\big).$$

2. The case $\delta_p(a) = b$ and $\delta_p(b) = a$ or $\delta_p(a) = 1-b$, $\delta_p(b) = 1-a$. Again by definition of the Dwork map we have $p^{-1}a_1 \overset{a_2}{\equiv} \epsilon b_1$ and $p^{-1}b_1 \overset{b_2}{\equiv} \epsilon a_1$, where $\epsilon = 1$ corresponds to the first case and $\epsilon = -1$ corresponds to the second case. Like the previous case these congruences are equivalent to

$$\begin{aligned} p(m_1+m_2) &\equiv \epsilon(m_1-m_2) \pmod{2m_1m_2} \\ p(m_1-m_2) &\equiv \epsilon(m_1+m_2) \pmod{2m_1m_2}, \end{aligned} \tag{19}$$

and one can check that this is equivalent to

$$\big(p \overset{2m_1}{\equiv} \epsilon,\ p \overset{2m_2}{\equiv} -\epsilon\big) \quad \text{or} \quad \big(p \overset{2m_1}{\equiv} m_1+\epsilon,\ p \overset{2m_2}{\equiv} m_2-\epsilon\big).$$

Now for (m,∞,∞), from (9) we see that $a = b = \frac{m-1}{2m}$. Then Condition (13) is equivalent to $p(m-1) \overset{2m}{\equiv} \epsilon(m-1)$, canceling $m-1$ and the fact that p is odd proves the statement.

3.2. Proof of Corollary 1

We see that if one of m_i, $i = 1, 2$, does not belong to the set $\{2, 3, 4, 6, \infty\}$, then there is a residue like $r \neq \epsilon, m_i + \epsilon$ with $(r, 2m_i) = 1$. Then by Dirichlet theorem there are infinitely many primes $p \overset{2m_i}{\equiv} r$ and by Theorem 1, J is not p-integral for such primes, which is a contradiction. Now checking all possibilities we find the list given in the statement of the corollary.

3.3. Proof of Corollary 2

We note that the first condition in Theorem 1, namely p modulo $2m_1$ automatically holds for $m_1 = 2$ and every prime greater than 3. Hence J for Hecke group $\Gamma_{(2,n,\infty)}$ is p integral if and only if $p \overset{2n}{\equiv} \pm 1$ or $n \pm 1$. This is equivalent to $p \overset{n}{\equiv} \pm 1$.

3.4. Proof of Corollary 3

Let m_2 be finite (the case $m_2 = \infty$ resolves in a similar way). Let also $\mathfrak{m}$ be the algebra of automorphic forms for $\Gamma_{\mathfrak{t}}$. Using Theorem 1, it is enough to prove that J is p-integral if and only if $\mathfrak{m}$ is p-integral. Let $E_4 = E_4^{(2)}$ and $E_6 = E_6^{(2)}$, where $E_k^{(2)}$ are defined in (6). We have

$$J = \frac{E_4^3}{E_4^3 - E_6^2}.$$

If J is not p-integral then one of the functions E_4 or E_6 is not p-integral and hence $\mathfrak{m}$ is not p-integral (E_4 and E_6 are members of this algebra and we use the convention that the p-integrality property is defined up to multiplication by a constant). Now from the Halphen system one can check that

$$t_1 - t_2 = \frac{\dot{J}}{J}, \qquad t_3 - t_2 = \frac{\dot{J}}{J-1},$$

and so

$$E_{2k}^{(1)} = \frac{J-1}{J}\left(\frac{\dot{J}}{J-1}\right)^k, \qquad E_{2k}^{(2)} = \left(\frac{\dot{J}}{J}\right)^k \frac{J}{J-1}.$$

If J is p-integral then all its derivatives are p-integral, and so, all the above elements are p-integral. A subset of these functions form a basis for $\mathfrak{m}$. Note that if an algebra $\mathfrak{m}$ is p-integral and we have a basis A of $\mathfrak{m}$ then after multiplication of the elements of A by proper constants, A turns to be a basis of $\mathfrak{m}$ with p-integral elements.

3.5. Final remarks

We expect that Theorem 1 to be true for primes p less than and coprime to $2m_1m_2$. This is equivalent to say that Corollary 4 is "if and only if". If $q(a, b|z)$ is p-integral, then $q(\delta_p^n(a), \delta_p^n(b)|z)$ is p-integral for all $n \in \mathbb{N}$ and hence we can use Corollary 5 and Lemma 2 and conclude that

$$\left\{\delta_p^n(a), \delta_p^n(b)\right\} \overset{p}{\equiv} \{a, b\} \text{ or } \{1-a, 1-b\}. \tag{20}$$

This does not seem to be sufficient in order to conclude the true equality. In order to further investigate the p-integrality of $q(a, b|z)$ we need to use more data from the congruency (14).

Unfortunately, in the literature there are no applications for the q-expansion of automorphic forms for non-arithmetic triangle groups. The main reason is the lack of Hecke theory for such automorphic forms. The rationality of coefficients is one of the main obstacles. We saw that for a class of primes the integrality in the level of p-adic integers holds. One question here is whether this integrality has distinguished enumerative properties.

The hypergeometric functions $F, F\log(z) + G$ up to some Γ-factors are periods of the following family of curves

$$C_z^{a,b,c} : y = x^a(x-1)^b(x-z)^c,$$

where a, b, c are give in (2). Another interesting problem is to find a geometric description for the result of Theorem 1 using the above family of curves. This curve and its Jacobian, are extensively studied by Wolfart et al. in connection with the algebraic values of the Schwarz map, see [10,13] and references therein.

Another interesting problem which we would like to address here is the p-integrality for generalized hypergeometric equations of order n whose local exponents at $z = 0$ are all zero. As we mentioned before, Dwork's theorem is valid in this general case. The Gauss hypergeometric equation corresponds to $n = 2$. We obtain in a similar way p-integrality results for the mirror map (the analog of $q(a, b|z)$ for arbitrary n). For $n > 2$ in the absence of the Euler identity, the only sufficient condition for p-integrality of the mirror map is that δ_p acts as a permutation on the local exponents of the differential equation at ∞ (an analog of Corollary 4). Then an interesting question is the converse, as we did in this article for $n = 2$. An important situation with applications in algebraic geometry and mathematical physics is the case in which the mirror map is almost integral. Then a simple observation shows that in Lemma 5, the congruence (14) is indeed an equality. In [7], the author, using differential Galois theory, showed that, this equality holds if and only if δ_p acts as a permutation for almost all p This fact establishes the problem of classification of all hypergeometric equations with maximal unipotent monodromy and with integral mirror map. As a corollary for $n = 4$, which is important in mirror symmetry, the well-known 14 cases is obtained.

As a final remark we would like to mention the possible relationship between almost integrality and arithmeticity of the monodromy groups. The coincidence of the Takeuchi list and the list of Corollary 1 does not seem to be casual. However the intrinsic connection between these two different worlds is not yet clear. Despite the existence of Dwork's method for non-unipotent cases, this method does not determine the rest of the Takeuchi list, i.e., arithmetic triangle groups without cusp. The situation for $n > 2$ is more obscure. For example for $n = 4$ even in the case with maximal unipotent monodromy it has been shown that among the 14 cases some of them are arithmetic and some of them are not (see for instance [11]). For a nice discussion in this subject we refer the reader to [8].

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