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Automorphic forms for triangle groups

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For triangle groups, the (quasi-)automorphic forms are known just as explicitly as for the modular group $\mathrm{PSL}(2,\mathbb{Z})$. We collect these expressions here, and then interpret them using the Halphen differential equation. We study the arithmetic properties of their Fourier coefficients at cusps and Taylor coefficients at elliptic fixed-points — in both cases integrality is related to the arithmeticity of the triangle group. As an application of our formulas, we provide an explicit modular interpretation of periods of 14 families of Calabi–Yau three-folds over the thrice-punctured sphere.

1. Introduction

Although modular forms for congruence subgroups of the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \Gamma(1)$ go back to Euler, modular forms for more general Fuchsian groups (usually called *automorphic forms*) go back to Poincaré. He proved their existence by constructing functions (Fuchsian-theta series in his terminology) which nowadays are known as Poincaré series. Independently of Poincaré, Halphen in [20, 21] introduced a differential equation in three variables and three parameters, which nowadays bears his name. His motivation was a particular case studied by Darboux in [13] and he proved that in such a case the differential equation is satisfied by the logarithmic derivatives of theta functions. Despite the fact that Poincaré and Halphen were contemporaries and compatriots, the main relation between these works was not clearly understood, and Halphen's contribution was largely forgotten, only to be rediscovered several times.

The modular forms and functions for the modular group $\Gamma(1)$ have of course been well understood for many decades. What is less well known is that there is a natural infinite class of Fuchsian groups — the so-called *triangle groups* — where the automorphic forms and functions can be determined just as explicitly, even though all but a few are incommensurable with $\Gamma(1)$.

Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be any genus-0 finitely generated Fuchsian group of the first kind.[1] (See the following section for the definitions of these and other technical terms.) This means that $\Gamma\backslash\mathbb{H}_\Gamma$ is topologically a sphere, where $\mathbb{H}_\Gamma$ denotes the upper half-plane $\mathbb{H}$ extended by the cusps of Γ (if any). Let n_{cp} be the number of cusps and n_{el} be the number of elliptic fixed-points, and write $2 \leq n_i \leq \infty$ for the orders of their stabilizers. Then Gauss–Bonnet implies $2 < \sum_{j=1}^{n_{\mathrm{cp}}+n_{\mathrm{el}}}(1-1/n_j)$ (see, e.g., Theorem 2.4.3 of [32] for a generalization) and hence we have the inequality $n_{\mathrm{cp}}+n_{\mathrm{el}} \geq 3$. The field of automorphic functions of Γ is $\mathbb{C}(J_\Gamma)$ where the generator J_Γ maps $\Gamma\backslash\mathbb{H}_\Gamma$ bijectively onto the Riemann sphere $\mathbb{P}^1$. Knowing such a uniformizer J_Γ determines explicitly (in principle) all automorphic and quasi-automorphic forms. If Γ is commensurable with $\Gamma(1)$ (i.e., when $\Gamma\cap\Gamma(1)$ has finite index in both Γ and $\Gamma(1)$), then (in principle) a generator J_Γ can be determined from, e.g., the Hauptmodul $j(\tau) = q^{-1} + 196884q + \cdots$ of $\Gamma(1)$, where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, $\tau \in \mathbb{H}$. When Γ is not necessarily commensurable, it is useful to recall that J_Γ will satisfy a nonlinear third-order differential equation

$$-2\frac{J_\Gamma'''(\tau)}{J_\Gamma'(\tau)} + 3\frac{J_\Gamma''(\tau)^2}{J_\Gamma'(\tau)^2} = J_\Gamma'(\tau)^2 Q_\Gamma(J_\Gamma(\tau)) \tag{1.1}$$

coming from the Schwarzian derivative, where the prime here denotes $\frac{d}{d\tau}$ and Q_Γ is a rational function depending only on Γ (for triangle groups it is given in (2.16)).

The Schwarzian equation (1.1) is rather complicated. It can be replaced by a much simpler system of first-order differential equations in $n_{\mathrm{cp}}+n_{\mathrm{el}}$ variables, subject to $n_{\mathrm{cp}}+n_{\mathrm{el}}-3$ quadratic (nondifferential) constraints. In

[1] Since we only deal with subgroups of $\mathrm{PSL}(2,\mathbb{R})$ and not $\mathrm{SL}(2,\mathbb{R})$ all the corresponding automorphic forms will be of even weight.

this generality, the result is due to Ohyama [37], but the key ideas go back to the 19th century. In particular, Halphen [21] associated the system

$$\begin{cases} t_1' = (a-1)(t_1t_2 + t_1t_3 - t_2t_3) + (b+c-1)t_1^2, \\ t_2' = (b-1)(t_2t_1 + t_2t_3 - t_1t_3) + (a+c-1)t_2^2, \\ t_3' = (c-1)(t_3t_1 + t_3t_2 - t_1t_2) + (a+b-1)t_3^2, \end{cases} \tag{1.2}$$

where the prime denotes $d/d\tau$, to Gauss' hypergeometric equation

$$z(1-z)y'' + (a+c-(a+b+2c)z)y' - (a+b+c-1)cy = 0, \tag{1.3}$$

where now the prime denotes d/dz, and Brioschi [10] showed its equivalence to the corresponding version of (1.1) (namely (2.16) below). The Halphen system (1.2) has been rediscovered several times (including by one of the authors of this paper!), and over the past century has appeared in the study of monopoles, self-dual Einstein equations, WDVV equations, mirror maps, etc. In [23], the authors have used solutions of Halphen equation for many particular cases, including those with an arithmetic triangle group, to obtain replicable uniformizations of punctured Riemann surfaces of genus zero. Further particular cases of Halphen equation solved by classical theta series or modular forms are discussed in [1]. The idea to use Halphen equation and find new automorphic forms seems to be neglected in the literature.

Now, $Q_\Gamma(z)$ in (1.1) is a rational function depending on $n_{\mathrm{cp}} + n_{\mathrm{el}} - 2$ parameters. Unfortunately, these parameters depend on Γ in a very complicated nonalgebraic way and in general closed formulae for them cannot be found (see, e.g., [47] for an analysis of this question). However, when $n_{\mathrm{cp}} + n_{\mathrm{el}} = 3$ (the minimum value possible), this single parameter can be determined explicitly, using classical results on hypergeometric functions. In this case — where Γ is a triangle group — $J_\Gamma(\tau)$ and hence all quasi-automorphic forms for Γ can be explicitly determined.

One of the purposes of this paper is to write these explicit expressions down, both for arithmetic and nonarithmetic triangle groups. Special cases and partial results (mainly for arithmetic groups) are scattered throughout the literature, see for instance [5, 7, 52] and references therein, but to our knowledge these expressions have not appeared in the literature with this explicitness and in this generality, and certainly not all in one place and not including nonarithmetic triangle groups. Therefore, the intersection of our work with those in the literature is mainly limited to Takeuchi's 85 arithmetic triangle groups, see [45].

We do this in two ways. We begin with the classical approach, because of its familiarity: the multivalued ratio $\tau(z)$ of two solutions to the hypergeometric equation can in certain circumstances be regarded as the functional inverse of an automorphic function $z(\tau)$ for a triangle group. This determines $z(\tau)$ completely, but it is convenient to use (1.1) to recover its q-expansion. Differentiating $z(\tau)$ once yields all automorphic forms; differentiating it a second time yields all quasi-automorphic forms. For subgroups of $\mathrm{PSL}(2,\mathbb{R})$ there are no automorphic forms of odd weight, see Theorem 2. Although the basic ideas of this derivation are classical, going back to Fuchs and Poincaré, the details are unpleasant. *Our second approach, using the Halphen equation, is independent and turns this on its head,* even though the underlying mathematics is again that of the hypergeometric equation. We interpret solutions of Halphen's equation, when lifted to $\mathbb{H}$, as quasi-automorphic forms for a triangle group. Taking differences yields all automorphic forms, and ratios then yield all automorphic functions.

We suggest that in most respects, the (quasi-)automorphic forms of the triangle groups are close cousins of those of the modular group and can be studied analogously, even though these groups are (usually) not commensurable with $\Gamma(1)$ (and so, e.g., Hecke operators cannot be applied). In particular, everything is as explicit for arbitrary triangle groups as it is for the modular group.

Now, when the group contains a congruence subgroup $\Gamma(n)$ of $\Gamma(1)$, such modular forms have many arithmetic properties. It is natural to ask whether any such arithmeticity survives for general triangle groups. We explore the arithmeticity of both the local expansions at cusps and at elliptic fixed-points. The latter expansions are far less familiar, even though they were familiar to, e.g., Petersson in the 1930s [38], but they deserve more attention than they have received. For example, Rodriguez Villegas and Zagier [41] interpret the expansion coefficients of the Dedekind eta $\eta(\tau)$ at $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$ in terms of central values of Hecke L-functions.

The triangle groups are extremely special among the Fuchsian groups for a number of reasons, for instance:

(i) One is a consequence of Belyi's theorem. A Fuchsian group is a subgroup of finite index in a triangle group, iff for each weight $k \in 2\mathbb{Z}$, there is a basis of the $\mathbb{C}$-space of weight-k holomorphic automorphic forms whose expansion coefficients are all algebraic numbers (see, e.g., [42]). Of course, these coefficients are the primary reason for the importance of any automorphic forms.

(ii) The complement of a knot in S^3 has universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (the universal cover of $\mathrm{SL}(2,\mathbb{R})$), iff the knot is a torus knot [40]. In particular, the (p,q)-torus knot is diffeomorphic to $\widetilde{\mathrm{SL}}(2,\mathbb{R})/G$ for a certain lift of the (p,q,∞)-triangle group. For example, the complement of the trefoil is $\widetilde{\mathrm{SL}}(2,\mathbb{R})/\widetilde{\mathrm{SL}}(2,\mathbb{Z})$. The relevance to this here is that an automorphic form, of arbitrary weight, for Γ lifts to a function on $\widetilde{\mathrm{SL}}(2,\mathbb{R})/\widetilde{\Gamma}$. The relevance to torus knots of the automorphic forms of the (p,q,∞)-triangle group is developed in [46], following [31] and Section 2.4.3 of [16]. Now, recall that Gopakumar–Vafa duality would imply that the Chern–Simons knot invariants arise as Gromov–Witten invariants. This has been verified explicitly in [9] for the torus knots, by independently computing the two sets of invariants and showing they are equal. It seems very possible that reinterpreting [9] using automorphic forms for triangle groups would at least simplify their calculation, and could lead to a more conceptual explanation of the equality.

(iii) We see below that periods of some Calabi–Yau three-folds with one-dimensional (1D) moduli spaces can be interpreted as vector-valued automorphic forms (vvaf's) for certain triangle groups (e.g., $(5,\infty,\infty)$ for the dual of the quintic). Independently, all 26 sporadic finite simple groups are quotients of certain triangle groups [49], e.g., the Monster is a quotient of $(2,3,7)$ (and hence $\Gamma(1)$). This implies that, for each sporadic group G, there will exist vvaf's for some triangle group, whose multiplier ρ factors through to a faithful representation of G.

In [33], the author (HM) derived the Halphen differential equation using the inverse of a period map. One advantage of this point of view is the introduction of modular-type forms for finitely generated subgroups of $\mathrm{PSL}(2,\mathbb{C})$ which may not be even discrete, something which must sound dubious to most number theorists. Since Movasati [33] focusses on the differential and geometric aspects of such modular-type forms, we felt that we should now look at number theoretic aspects. The triangle groups provide interesting but nontrivial toy models, where the group is discrete but the automorphic forms are not so well-studied. This text is partly a result of this effort. We find it remarkable how naturally the (quasi-)automorphic forms for triangle groups arise in the Halphen system (1.2). We believe this observation is new (at least in this generality). In this case, the parameters a,b,c must be rational — in fact the combinations $1-a-b, 1-c-b, 1-a-c$ will equal the angular parameters $v_i = 1/m_i$, for $i = 1,2,3$, respectively, where $m_i \in \mathbb{Z}_{>0} \cup \{\infty\}$. However, some sort of modularity appears to persist though even when these angular parameters are complex.

Our main motivation for writing this paper is to establish the background needed to understand the modularity of the mirror map for examples such as the Calabi–Yau quintic, by relating the Halphen approach of one of the authors with that of vvaf's of another author. This required having completely explicit descriptions of the automorphic forms for the triangle group $(5, \infty, \infty)$, and as we could not find this adequately treated in the literature we did the calculations ourselves. The application to mirror maps will be forthcoming, although an initial step is provided in Section 6.

The outline of the paper is as follows. Section 2 provides the classical (i.e., hypergeometric) calculation of all data for the automorphic forms of the triangle groups. Section 3 recovers this data using solutions to Halphen's equation; we believe this approach is new. Section 4 specializes to the triangle groups commensurable with the modular group. Section 5 explores the arithmeticity of the Fourier and Taylor coefficients. Section 6 applies this material to periods of Calabi–Yau three-folds. Our proofs are collected in Section 7. Relevant facts on hypergeometric functions are collected in Appendix A.

The purpose of this paper is, firstly, to establish that the theory of automorphic forms for any triangle group with cusps is every bit as explicit as is that of $\mathrm{SL}_2(\mathbb{Z})$. We do this in two complementary ways: the classical argument from the hypergeometric equation, and a new approach using the Halphen equation. Some aspects of this lengthy calculation are scattered throughout the literature (see for instance [7,52] and references therein) and some seem missing (as we will explain in later sections), and we wanted to complete it and collect it all in one place. Moreover, we initiate a study of the arithmeticity of the coefficients of these automorphic forms; it appears that little work has been done on this, in particular for nonarithmetic groups and for Taylor expansion coefficients at elliptic points, but there are questions worth exploring. Finally, we explain the hidden modularity of Calabi–Yau periods.

Here is some notations used throughout the text.

- $\mathfrak{t} = (m_1, m_2, m_3)$: triangle group type;
- $\mathbb{H}$ resp. $\mathbb{H}_{\mathfrak{t}}$: the upper half-plane resp. extended upper half-plane;
- $\Gamma_{\mathfrak{t}} \subset \mathrm{PSL}(2, \mathbb{R})$: the realization of the triangle group of type $\mathfrak{t}$;
- γ_i, $i = 1, 2, 3$: matrix generators of $\Gamma_{\mathfrak{t}}$ (see (2.10));
- ζ_i, $i = 1, 2, 3$: fixed-points of γ_i (see (2.9));
- q_i resp. $\tilde{q}_i$: the local coordinate resp. normalized local coordinate, at ζ_i;
- $J_{\mathfrak{t}}$: the normalized Hauptmodul associated to the group $\Gamma_{\mathfrak{t}}$ (see (2.12));

- $v_i = \frac{1}{m_i}$, $i = 1, 2, 3$: the angular parameters;
- (a, b, c) resp. $(\tilde{a}, \tilde{b}, \tilde{c})$: parameters of the Halphen resp. hypergeometric systems; and
- (t_1, t_2, t_3): the solution of the Halphen system, defined in Section 3.

2. Classical computation of (quasi-)automorphic forms

In this section, we give the classical approach for computing automorphic forms through the Schwarzian and hypergeometric differential equations.

2.1. Background

See, e.g., [32] for the basics of Fuchsian groups and their automorphic forms. A *Fuchsian group* Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm 1\}$, the group of orientation-preserving isometries of the upper half-plane $\mathbb{H} := \{x + \mathrm{i}y \mid y > 0\}$. Γ is called *of first class* (the class of primary interest) if its fundamental domains in $\mathbb{H}$ have finite hyperbolic area. $\gamma \in \Gamma$ is called *parabolic* if γ has precisely one fixed-point on the boundary $\partial\mathbb{H} = \mathbb{RP}^1 = \mathbb{R} \cup \{\mathrm{i}\infty\}$; $x \in \mathbb{R} \cup \{\mathrm{i}\infty\}$ is called a *cusp* of Γ if it is fixed by some parabolic $\gamma \in \Gamma$. The *extended half-plane* together with all cusps; then for Γ of first class, the orbits $\Gamma\backslash\mathbb{H}_\Gamma$ naturally form a compact surface. The genus of this surface is called the genus of Γ.

If $\mathrm{i}\infty$ is a cusp of Γ, we call the smallest $h > 0$ with $\gamma_{\infty;h} := \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ the *cusp-width* h_∞. If $x \in \mathbb{R}$ is a cusp, its cusp-width h_x is the smallest $h > 0$ for which $\gamma_{x;h} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & -x \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -1 \\ 1 & -x \end{smallmatrix}\right) \in \Gamma$. The other special points in $\mathbb{H}_\Gamma$ are the *elliptic fixed-points*, which are $z \in \mathbb{H}$ stabilized by a nontrivial $\gamma \in \Gamma$. For each $z = x + \mathrm{i}y \in \mathbb{H}$, the stabilizer in Γ is finite cyclic, generated by

$$\gamma_{z;n} := \begin{pmatrix} y^{-1/2} & -y^{-1/2}x \\ 0 & y^{1/2} \end{pmatrix}^{-1} \begin{pmatrix} \cos(\pi/n) & \sin(\pi/n) \\ -\sin(\pi/n) & \cos(\pi/n) \end{pmatrix} \begin{pmatrix} y^{-1/2} & -y^{-1/2}x \\ 0 & y^{1/2} \end{pmatrix}$$

for a unique positive integer $n = n_z$ called the *order* of z. Write $n_x = \infty$ for a cusp x.

These numbers h_x, n_z are clearly constant along Γ-orbits. Let n_{el} denote the number of Γ-orbits of elliptic fixed-points and n_{cp} the number of Γ-orbits of cusps. Both n_{el} and n_{cp} must be finite, but can be zero; moreover, $n_{\mathrm{el}} + n_{\mathrm{cp}} \geq 3$.

For $z \in \mathbb{H}_\Gamma$, define Möbius transformations $\tau \mapsto \tau_z$, local coordinates q_z and automorphy factors $j_z(k;\tau)$ as follows. Choose $\tau_\infty = \tau$, $q_\infty = \mathrm{e}^{2\pi\mathrm{i}\tau/h_\infty}$ and $j_\infty(k;\tau) = 1$; for $x \in \mathbb{R}$ choose $\tau_x = -1/(\tau - x)$, $q_x = \mathrm{e}^{2\pi\mathrm{i}\tau_x/h_x}$ and $j_x(k;\tau) = \tau_x^k$; while for $z \in \mathbb{H}$ choose $\tau_z = (\tau - z)/(\tau - \overline{z})$, $q_z = \tau_z^{n_z}$ and $j_z(k;\tau) = (1-\tau_z)^k$. This factor j_z is, up to a constant, the standard weight-k automorphy factor associated to the transformation $\tau \mapsto \tau_z$.

The point is that any meromorphic function $f(\tau)$ invariant under the slash operator

$$(f|_k\gamma_{z;h})(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \tag{2.4}$$

for some $z \in \mathbb{H}_\Gamma$, where we write $\gamma_{z;h} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, will have a local expansion

$$f(\tau) = j_z(k;\tau) \sum_{n\in\mathbb{Z}} f\left[n + \frac{k}{n_z}\right]_z q_z^{n+\frac{k}{n_z}}. \tag{2.5}$$

The *order* $\mathrm{ord}_z(f)$ of an automorphic form f at a point $z \in \mathbb{H}_\Gamma$ is defined to be the smallest $r \in \mathbb{Q}$ such that $f[r]_z \neq 0$. Here, $f[r]_z$ is the coefficient of q_z^r in the Fourier expansion of f.

A *quasi-automorphic form* f of weight $k \in 2\mathbb{Z}$ and depth $\leq p$ for Γ can be defined [11] as a function meromorphic on $\mathbb{H}_\Gamma$ (meromorphicity at the cusps is defined shortly), satisfying the functional equation

$$(f|_k\gamma)(\tau) = \sum_{r=0}^{p} f_r(\tau)\left(\frac{c}{c\tau + d}\right)^r \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \tag{2.6}$$

for some functions f_r meromorphic in $\mathbb{H}_\Gamma$ and independent of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We say f is *meromorphic at the cusp* $z \in \{\mathrm{i}\infty\} \cup \mathbb{R}$ if all but finitely many coefficients $f[n]_z$ vanish for $n < 0$, and *holomorphic at* z if $f[n]_z = 0$ whenever the relevant power of q_z, namely $n + k/h_z$, is negative. When $p = 0$, f is called an *automorphic form*; when $p = k = 0$, it is called an *automorphic function.* When Γ is commensurable with $\Gamma(1)$, it is typical to replace "automorphic" with "modular."

This definition can be extended to any weight $k \in \mathbb{C}$ using the notion of automorphy factor, but we do not need it (though see the end of Section 2.4). It is elementary to verify that the orders $\mathrm{ord}_z(f)$ of an automorphic form f are constant on Γ-orbits Γz.

Suppose f is an automorphic function, not constant. Then $f' = \frac{d}{d\tau} f$ will be an automorphic form of weight 2 and $e_{2,f} = \frac{1}{f'} \frac{d^2}{d\tau^2} f$ will be quasi-automorphic of weight 2 and depth 1. In this case, the *Serre derivative* $D_k = \frac{d}{d\tau} - k\beta e_{2,f}(\tau)$, for some constant $\beta \in \mathbb{C}$ independent of f and k (computed for triangle groups in Theorem 2(ii) below), takes automorphic forms of weight k to those of weight $k+2$.

The automorphic functions form a field; when the genus of $\Gamma\backslash\mathbb{H}_\Gamma$ is zero, this field can be expressed as the rational functions $\mathbb{C}(f)$ in some generator f. By a *Hauptmodul* we mean any such generator. These Hauptmoduls f are mapped to each other by the Möbius transformations $\mathrm{PSL}(2,\mathbb{C})$, and therefore are determined by three complex parameters.

For example, for $\Gamma(1) = \mathrm{PSL}(2,\mathbb{Z})$, recall the classical Eisenstein series E_k given by

$$E_k(\tau) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1-q^n} \tag{2.7}$$

$k \in 2\mathbb{Z}_{>0}$, where $q = q_{\mathrm{i}\infty} = \exp(2\pi \mathrm{i}\tau)$. The holomorphic modular forms and quasi-modular forms yield the polynomial rings $\mathbb{C}[E_4, E_6]$ and $\mathbb{C}[E_2, E_4, E_6]$. The classical Hauptmodul is

$$j(\tau) = \frac{1728 E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} = q^{-1} + 744 + 196884q + \cdots . \tag{2.8}$$

Throughout this paper, by $E_k(\tau)$ and $j(\tau)$ we mean these modular forms for $\Gamma(1)$.

2.2. Triangle groups

In this paper, we focus on the *triangle groups*. These by definition are those genus-0 Fuchsian groups Γ of the first kind with $n_{\mathrm{el}} + n_{\mathrm{cp}} = 3$ (the minimal value possible). This means that there are exactly three Γ-orbits of cusps and elliptic fixed-points, in some combination. Let $2 \le m_1 \le m_2 \le m_3 \le \infty$ be the orders of the stabilizers of those three orbits. No Fuchsian group of the first kind can have types $(2,2,m)$ $\forall m \le \infty$, $(2,3,n)$ for $n \le 6$, (2,4,4) and (3,3,3); the remainder are called the hyperbolic types. We are primarily interested in the case where $m_3 = \infty$ — for $m_3 < \infty$ see Appendix B. As an abstract group, a triangle group has presentation $\langle g_1, g_2, g_3 \,|\, g_i^{m_i} = 1 = g_1 g_2 g_3 \rangle$; when $m_3 = \infty$ this is isomorphic to the free product $\mathbb{Z}_{m_1} * \mathbb{Z}_{m_2}$, where we write $\mathbb{Z}_k$ for the cyclic group with k elements.

Given one such triangle group, we can find another by conjugating by any $g \in \mathrm{PSL}(2,\mathbb{R})$. The triangle group of a given type $\mathfrak{t} = (m_1, m_2, \infty)$ is unique up to this conjugation [38], and so is determined by three real parameters. As the automorphic functions of Γ and $g\Gamma g^{-1}$ are related by $f(\tau) \leftrightarrow f(g^{-1}\tau)$, it is not so significant which realization is chosen. Of course, this conjugation will in general affect the integrality of Fourier coefficients, so in that sense some choices are better than others.

Write $v_i = 1/m_i$ for the angular parameters. A fundamental domain for a triangle group will be the double of a hyperbolic triangle in $\mathbb{H}_{\mathfrak{t}}$; we fix the triangle group by fixing the location of the corners of the triangle, which we take to be

$$\zeta_1 = -\mathrm{e}^{-\pi\mathrm{i}v_1},\ \zeta_2 = \mathrm{e}^{\pi\mathrm{i}v_2},\ \zeta_3 = \mathrm{i}\infty. \tag{2.9}$$

A fundamental domain for $\Gamma_{\mathfrak{t}}$ is the union of this triangle and its image under $\tau \mapsto \tau + \frac{h_3}{2}$, where $h_3 := 2\cos(\pi v_1) + 2\cos(\pi v_2)$. The Fuchsian group $\Gamma_{\mathfrak{t}}$ for this choice has generators

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 2\cos(\pi v_1) & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 2\cos(\pi v_2) \end{pmatrix}, \\ \gamma_3 &= \begin{pmatrix} 1 & 2\cos(\pi v_1) + 2\cos(\pi v_2) \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{2.10}$$

stabilizing the three corners $\zeta_1, \zeta_2, \zeta_3$, where

$$\gamma_1\gamma_2\gamma_3 = \gamma_1^{m_1} = \gamma_2^{m_2} = -I_{2\times 2}. \tag{2.11}$$

Thus, the cusp $\mathrm{i}\infty$ has cusp-width h_3; when $m_2 = \infty$, $\zeta_2 = 1$ is also a cusp, with cusp-width $h_2 = 1$. Of course the groups $\Gamma_{(m_{\pi 1}, m_{\pi 2}, m_{\pi 3})}$ are conjugate for any permutation $\pi \in \mathrm{Sym}(3)$.

The prototypical example is the modular group $\Gamma_{(2,3,\infty)} = \Gamma(1)$. More generally, the *Hecke groups* $\Gamma_{(2,m,\infty)}$, $m > 2$, have attracted a fair amount of attention.

2.3. Hauptmoduls for triangle groups

Given a type $\mathfrak{t} = (m_1, m_2, \infty)$, fix the triangle group $\Gamma_{\mathfrak{t}}$ as in (2.10). A Hauptmodul $J_{\mathfrak{t}}(\tau)$ for $\Gamma_{\mathfrak{t}}$ is determined by three independent *complex* parameters, which we fix by demanding

$$J_{\mathfrak{t}}(\zeta_1) = 1,\ J_{\mathfrak{t}}(\zeta_2) = 0,\ J_{\mathfrak{t}}(\mathrm{i}\infty) = \infty. \tag{2.12}$$

(We make this choice because $1728J_{(2,3,\infty)}$ then equals the classical choice (2.8) for $\Gamma(1)$.) We call the unique Hauptmodul satisfying (2.12) the *normalized Hauptmodul* for $\Gamma_{\mathfrak{t}}$. To find it, given any other Hauptmodul J, first note that $J(\zeta_i)$ must be distinct points in $\mathbb{CP}^1$ (since J is a Hauptmodul) so there will be a unique Möbius transformation mapping those three points to $1, 0, \infty$, respectively, and $J_{\mathfrak{t}}$ is the composition of that transformation with J. Note that $J_{(m_1,\infty,m_2)} = J^{-1}_{(m_1,m_2,\infty)}$, $J_{(m_2,\infty,m_1)} = (1 - J_{(m_1,m_2,\infty)})^{-1}$, etc. In the following theorem, we explicitly compute $J_{\mathfrak{t}}$, and in the following section do this in a different way.

Theorem 1. *Fix any hyperbolic type* $\mathfrak{t} = (m_1, m_2, \infty)$, $m_1 \leq m_2 \leq \infty$. *Let* q_i *be the local coordinates about the points* $\zeta_i \in \mathbb{H}_{\mathfrak{t}}$ *in* (2.9), *and write* $\widetilde{q}_i = \alpha_i q_i$ *for* α_i *defined by: if* $m_i = \infty$

$$\alpha_i = b'd' \prod_{k=1}^{b'-1} \left(2 - 2\cos\left(2\pi \frac{k}{b'}\right)\right)^{-\frac{1}{2}\cos\left(2\pi \frac{ka'}{b'}\right)} \times \prod_{l=1}^{d'-1} \left(2 - 2\cos\left(2\pi \frac{l}{d'}\right)\right)^{-\frac{1}{2}\cos\left(2\pi \frac{lc'}{d'}\right)}, \tag{2.13}$$

where we define positive integers a', b', c', d' *by* $a'/b' = (1 + v_1 - v_2)/2$, $c'/d' = (1 + v_1 + v_2)/2$, *g.c.d.*$(a', b') =$ *g.c.d.*$(c', d') = 1$; *if* $m_i < \infty$

$$\alpha_i = \frac{\cos(\pi(v_1 + v_2)/2)}{\cos(\pi(v_1 - v_2)/2)} \frac{\Gamma(1 + v_i)\Gamma((1 - v_i + v_{3-i})/2)^2}{\Gamma(1 - v_i)\Gamma((1 + v_1 + v_2)/2)^2}. \tag{2.14}$$

The normalized $J_{\mathfrak{t}}$ *in* (2.12) *has local expansions*

$$J_{\mathfrak{t}}(\tau) = 1 + \widetilde{q}_1 + \sum_{k=2}^{\infty} a_k \widetilde{q}_1^{\,k} = \widetilde{q}_2 + \sum_{k=2}^{\infty} b_k \widetilde{q}_2^{\,k} = \widetilde{q}_3^{\,-1} + \sum_{k=0}^{\infty} c_k \widetilde{q}_3^{\,k}. \tag{2.15}$$

These (normalized) coefficients a_k, b_k, c_k *are uniquely determined by*

$$-2\dddot{J}_{\mathfrak{t}}\dot{J}_{\mathfrak{t}} + 3\ddot{J}_{\mathfrak{t}}^2 - n_z^{-2}\dot{J}_{\mathfrak{t}}^2 = \dot{J}_{\mathfrak{t}}^4 \left(\frac{1 - v_2^2}{J_{\mathfrak{t}}^2} + \frac{1 - v_1^2}{(J_{\mathfrak{t}} - 1)^2} + \frac{v_1^2 + v_2^2 - 1}{J_{\mathfrak{t}}(J_{\mathfrak{t}} - 1)}\right) \tag{2.16}$$

[2] *for the choice $z = \zeta_1, \zeta_2, \zeta_3$, respectively, where each dot denotes $\widetilde{q}_j \frac{d}{d\widetilde{q}_j}$, and where n_z is the order of the stabilizer at z. The coefficients a_k, b_k, c_k are universal (i.e., type-independent) polynomials in $\mathbb{Q}[v_1, v_2]$, and are also unchanged if we replace $\Gamma_{\mathfrak{t}}$ by any conjugate.*

The key to this calculation, which we describe in Section 7.1, is the expression (using ratios of hypergeometric functions) of the uniformizing Schwarz map from the upper hemisphere in $\mathbb{CP}^1$ to a hyperbolic triangle in the Poincaré disc. Analytically continuing the (multivalued) hypergeometric functions amounts to reflecting in the sides of that triangle, resulting in a multivalued map from the thrice-punctured sphere to the disc. The (single-valued) functional inverse of this Schwarz map is a Hauptmodul; its automorphy traces back to the monodromy of the hypergeometric equation. The most convenient way to obtain (most of) the local expansion of that Hauptmodul is through the Schwarzian equation (2.16).

For instance we have

$$\begin{aligned}
c_0 &= (1-\gamma_-)/2,\ c_1 = (5 - 2\gamma_+ - 3\gamma_-^2)/64,\ c_2 = (-\gamma_-^3 - \gamma_+\gamma_- + 2\gamma_-)/54,\\
c_3 &= (-31 + 76\gamma_+ - 28\gamma_+^2 + 690\gamma_-^2 - 404\gamma_+\gamma_-^2 - 303\gamma_-^4)/32768,\\
c_4 &= (-274\gamma_- + 765\gamma_+\gamma_- - 314\gamma_+^2\gamma_- + 2807\gamma_-^3 - 1865\gamma_+\gamma_-^3 - 1119\gamma_-^5)/\\
&\quad 216000,\\
c_5 &= (19683 - 121770\gamma_+ + 199044\gamma_+^2 - 1909439\gamma_-^2 + 5990732\gamma_+\gamma_-^2\\
&\quad - 68472\gamma_+^3 + 12854105\gamma_-^4 - 2699804\gamma_+^2\gamma_-^2 - 9509386\gamma_+\gamma_-^4\\
&\quad - 4754693\gamma_-^6)/1528823808,\\
c_6 &= (341510\gamma_- - 2360379\gamma_+\gamma_- - 13805911\gamma_-^3 + 4269300\gamma_+^2\gamma_-\\
&\quad - 1587244\gamma_+^3\gamma_- + 48264782\gamma_+\gamma_-^3 + 70933968\gamma_-^5 - 23644656\gamma_+^2\gamma_-^3\\
&\quad - 57687959\gamma_+\gamma_-^5 - 24723411\gamma_-^7)/12644352000,
\end{aligned}$$

where $\gamma_\pm = v_1^2 \pm v_2^2$. To our knowledge, these formulas in this generality have not appeared in the literature, although Wolfart [51] computed (2.13) and (2.14). Replacing $\Gamma_{\mathfrak{t}}$ with any conjugate (using an element of $\mathrm{PSL}(2,\mathbb{R})$) affects $J_{\mathfrak{t}}$ by changing the value of α_3, the value of cusp-width h_3, and the choice of $\mathrm{i}\infty$ as a cusp. The only subtlety here is which α_3 corresponds to our choice (2.10) of $\Gamma_{\mathfrak{t}}$. We find that once one has chosen $\mathrm{i}\infty$ to be a cusp (it could have been anywhere in $\mathbb{R} \cup \{\mathrm{i}\infty\}$) and has fixed the cusp-width h_3 (it

[2] The left-hand side of (2.16) is derived from the left-hand side of (1.1) using classical properties of the Schwarzian derivative and the fact that derivation here is with respect to $\ln(\tau_z^{n_z})$ which is no more compatible with weights.

could have been any positive real number), then the modulus $|\alpha_3|$ is fixed for any conjugate; our choice (2.10) of generators then corresponds to α_3 being positive.

2.4. Automorphic forms for triangle groups

Knowing a Hauptmodul J for any genus-0 Fuchsian group — e.g., any triangle group — determines by definition all automorphic functions. It is less well known that from a Hauptmodul, all holomorphic (quasi-)automorphic forms can be quickly read off. We restrict here to triangle groups, although the argument works for any genus-0 group.

The following theorem constructs an automorphic form whose divisor is supported at the cusps, the analog here of the discriminant form $\Delta = \eta^{24}$ for $\Gamma(1)$. It constructs from this a "rational" basis for the space of automorphic forms (rational in a sense described after the theorem), and gives the analog here of E_2, and hence all quasi-automorphic forms. In Section 4, we compare this basis with more classical ones, for the nine triangle groups related to $\Gamma(1)$.

Theorem 2. (i) *For each $k \in \mathbb{Z}$, write $d_{2k} = k - \lceil k/m_1 \rceil - \lceil k/m_2 \rceil$ and let*

$$f_{2k} = (-1)^k \dot{J}_{\mathfrak{t}}^k J_{\mathfrak{t}}^{\lceil \frac{k}{m_2} \rceil - k} (J_{\mathfrak{t}} - 1)^{\lceil \frac{k}{m_1} \rceil - k} = \widetilde{q}_3^{d_{2k}} + O(\widetilde{q}_3^{d_{2k}+1}), \tag{2.17}$$

where the dot denotes $\widetilde{q}_3 d/d\widetilde{q}_3$. Then a basis for the $\mathbb{C}$-vector space $\mathfrak{m}_{2k}(\Gamma_{\mathfrak{t}})$ of holomorphic automorphic forms of weight $2k$ for $\Gamma_{\mathfrak{t}}$ is $f_{2k}(\tau)\, J_{\mathfrak{t}}(\tau)^l$ for each $0 \le l \le d_{2k}$. In particular

$$\dim(\mathfrak{m}_{2k}(\Gamma_{\mathfrak{t}})) = \begin{cases} d_{2k}+1 & \text{if } k \ge 0, \\ 0 & \text{if } k < 0. \end{cases} \tag{2.18}$$

The algebra $\mathfrak{m}(\Gamma_{\mathfrak{t}})$ of holomorphic automorphic forms of even weight has the following minimal set of generators:
when $\mathfrak{t} = (\infty, \infty, \infty)$, $\{f_2, J_{\mathfrak{t}} f_2\}$;
when $\mathfrak{t} = (m, \infty, \infty)$ for $m < \infty$, $\{f_2, \dots, f_{2m}\}$;
when $\mathfrak{t} = (m_1, m_2, \infty)$ for $m_1 \le m_2 < \infty$, $\{f_{2l}\}|_{2 \le l \le m_2} \cup \{J_{\mathfrak{t}}^{d_{2l}} f_{2l}\}|_{3 \le l \le m_1}$.

(ii) *Define L to be the least common multiple $lcm(m_1, m_2)$ where we write $lcm(m_1, \infty) = m_1$ and $lcm(\infty, \infty) = 1$. Then $\Delta_{\mathfrak{t}}(\tau) := f_{2L}(\tau)$ is a holomorphic automorphic form of weight $2L$, nonzero everywhere in $\mathbb{H}_{\mathfrak{t}}$*

except in the $\Gamma_{\mathfrak{t}}$*-orbit* $[\mathrm{i}\infty]$, *where* $\Delta_{\mathfrak{t}}$ *has a zero of order* $n_\Delta = L\,(1 - m_1^{-1} - m_2^{-1})$. *Define* $E_{2;\mathfrak{t}} = \frac{1}{2\pi\mathrm{i}}\Delta_{\mathfrak{t}}^{-1} d\Delta_{\mathfrak{t}}/d\tau$. *Then* $E_{2;\mathfrak{t}}$ *is holomorphic in* $\mathbb{H}_{\mathfrak{t}}$, $E_{2;\mathfrak{t}}$ *vanishes at any cusp* $\zeta_{m_j} \notin [\mathrm{i}\infty]$, *and* $E_{2;\mathfrak{t}}(\mathrm{i}\infty) = n_\Delta$. *Moreover,* $E_{2;\mathfrak{t}}$ *is quasi-automorphic of weight 2 and depth 1 for* $\Gamma_{\mathfrak{t}}$*: i.e., for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathfrak{t}}$

$$E_{2;\mathfrak{t}}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{n_\Delta c}{2\pi\mathrm{i}}(c\tau + d)E_{2;\mathfrak{t}}(\tau) + (c\tau + d)^2 E_{2;\mathfrak{t}}(\tau). \tag{2.19}$$

The derivation

$$D_k = \frac{1}{2\pi\mathrm{i}}\frac{d}{d\tau} - \frac{k}{L}E_{2;\mathfrak{t}}$$

sends weight k *automorphic forms to weight* $k+2$ *ones. The space of all holomorphic quasi-automorphic forms of* $\Gamma_{\mathfrak{t}}$ *is* $\mathfrak{m}(\Gamma_{\mathfrak{t}})[E_{2;\mathfrak{t}}]$.

The f_{2k} defined above is the unique holomorphic weight-$2k$ automorphic form with maximal order at the cusp $\mathrm{i}\infty$ and with the monic leading coefficient in the $\tilde{q}_3$-expansion. The weights of generators for $\mathfrak{m}(\Gamma)$ for any Fuchsian group of the first kind, are given in [48] and references therein; what we provide in Theorems 1 and 2 are explicit formulas and expansions for those generators, in the special case of triangle groups. Provided we expand in $\widetilde{q}_i = \alpha_i q_i$ instead of q_i, $J_{\mathfrak{t}}$ has rational coefficients; in this same sense, our bases for each $\mathfrak{m}_{2k}$ also has rational coefficients. Incidentally, according to Wolfart [51], α_3 is transcendental except for the types listed in table 1 below.

Although every triangle group shares many properties with $\Gamma(1)$, one difference is that $\mathfrak{m}(\Gamma_{\mathfrak{t}})$ will rarely be a polynomial algebra: in fact, $\mathfrak{m}(\Gamma_{\mathfrak{t}})$ is polynomial iff $\mathfrak{t} = (2, 3, \infty), (2, \infty, \infty)$ or (∞, ∞, ∞). On the other hand, Milnor [31] and Wolfart [50] consider the ring of holomorphic automorphic forms of $\Gamma_{\mathfrak{t}}$ for a root-of-unity-valued multiplier (which allows certain weights $k \notin 2\mathbb{Z}$), and find that larger ring always generated by three forms f_1, f_2, f_3 satisfying an identity of the form $f_1^{e_1} + f_2^{e_2} + f_3^{e_3} = 0$.

Incidentally, $\Delta_{\mathfrak{t}}$ can identify all automorphic forms with multiplier of arbitrary *complex* weight $k \in \mathbb{C}$. In particular, for any $w \in \mathbb{C}$ define $\Delta_{\mathfrak{t}}^{(w)}$ to be any nontrivial solution to

$$\frac{1}{2\pi\mathrm{i}}\frac{d}{d\tau}f = wE_{2;\mathfrak{t}}f. \tag{2.20}$$

Table 1: The triangle groups commensurable with $\Gamma(1)$.

(m_1, m_2, m_3)	$g\Gamma_t g^{-1}$	g	ζ_1	γ_1	ζ_2	γ_2	ζ_3	γ_3	α_3
$(2,3,\infty)$	$\Gamma(1)$	1	i	S	ω	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	∞	T	1728
$(2,4,\infty)$	$\Gamma_0^+(2)$	$\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\mathrm{i}/\sqrt{2}$	W_2	$(-1+\mathrm{i})/2$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$	∞	T	256
$(2,6,\infty)$	$\Gamma_0^+(3)$	$\left(\begin{smallmatrix} 3 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$\mathrm{i}/\sqrt{3}$	W_3	$(-3+\mathrm{i}\sqrt{3})/6$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}$	∞	T	108
$(2,\infty,\infty)$	$\Gamma_0(2)$	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$	$(1+\mathrm{i})/2$	$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$	0	U^2	∞	T	64
$(3,3,\infty)$	$\Gamma(1)^*$	1	ω^2	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	ω	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	∞	T^2	$48\sqrt{3}$
$(3,\infty,\infty)$	$\Gamma_0(3)$	$\left(\begin{smallmatrix} 1 & -1 \\ 0 & 3 \end{smallmatrix}\right)$	$(3+\mathrm{i}\sqrt{3})/6$	$\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	0	U^3	∞	T	27
$(4,4,\infty)$	$\Gamma_0^+(2)^*$	$\left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$(\mathrm{i}-1)/2$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$	$(1+\mathrm{i})/2$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$	∞	T^2	32
$(6,6,\infty)$	$\Gamma_0^+(3)^*$	$\left(\begin{smallmatrix} 3 & 0 \\ 1 & 1 \end{smallmatrix}\right)$	$(-3+\mathrm{i}\sqrt{3})/6$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}$	$(3+\mathrm{i}\sqrt{3})/6$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}$	∞	T^2	$12\sqrt{3}$
(∞,∞,∞)	$\Gamma(2)$	$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$	0	U^2	1	$\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$	∞	T^2	16

First note from the theory of ordinary differential equations (see, e.g., [25]), $\Delta_{\mathfrak{t}}^{(w)}$ exists and is holomorphic throughout $\mathbb{H}$. Locally, it corresponds to some branch of the power $\Delta_{\mathfrak{t}}^{w}$; that it transforms under $\Gamma_{\mathfrak{t}}$ like (and therefore is) a holomorphic automorphic form of weight $w \cdot \mathrm{lcm}\{m_1, m_2\}$ follows directly from (2.20). Then some f is a (meromorphic) automorphic *form* for $\Gamma_{\mathfrak{t}}$ with arbitrary weight $k \in \mathbb{C}$ automorphy factor, iff $f/\Delta_{\mathfrak{t}}^{(k/\mathrm{lcm}\{m_1,m_2\})}$ is an automorphic *function* for $\Gamma_{\mathfrak{t}}$ with the appropriate automorphy factor (namely some character of $\Gamma_{\mathfrak{t}}$).

3. Quasi-automorphic forms via Halphen's equation

In this section, we realize the (quasi-)automorphic forms of the triangle groups, using the Halphen differential equation. This material should be completely new; see [33] for some of the detailed calculations which are omitted here. For simplicity, we again require $m_3 = \infty$ — see Appendix B for some remarks on the generalization to finite m_3.

Fix any hyperbolic type $\mathfrak{t} = (m_1, m_2, \infty)$. Recall the angular parameters $v_i = 1/m_i$. Consider the Halphen differential equation (1.2), where a, b, c are the parameters

$$
\begin{aligned}
a &= \frac{1}{2}(1 + v_2 - v_1 - v_3),\\
b &= \frac{1}{2}(1 + v_3 - v_1 - v_2),\\
c &= \frac{1}{2}(1 + v_1 - v_2 - v_3).
\end{aligned}
\tag{3.21}
$$

In the original Halphen equation, the right-hand side of (1.2) is divided by $a + b + c - 2$.

Recall the normalized Hauptmodul $J_{\mathfrak{t}}$. We are interested in the particular solution of (1.2) given in Theorem 3(i) below. Because $v_3 = 0$ (i.e., $a + c = 1$), the Halphen vector field has the 1D singular locus $t_1 = t_3 = 0$; the solution of part (i) is a perturbation of this singular locus. The relation of the Halphen equation with hypergeometric functions goes back to Halphen, who is therefore ultimately responsible for parts (i) and (iii). Part (ii) follows from recursions coming from (1.2) (see Section 7.3 below), and is new. The automorphy of the Halphen solutions arises from the $\mathrm{SL}(2, \mathbb{C})$ action in part (iii), and can be also proved using generalizations of period maps, see Section 10 of [33].

Theorem 3. (i) *A solution to* (1.2) *is*

$$
\begin{aligned}
t_1(\tau) &= (a-1)z\,Q(z)\,F(1-a,b,1;z)\,F(2-a,b,2;z),\\
t_2(\tau) &= Q(z)\,F(1-a,b,1;z)^2 + t_1(\tau),\\
t_3(\tau) &= Q(z)\,z\,F(1-a,b,1;z)^2 + t_1(\tau),
\end{aligned}
$$

where $F = {}_2F_1$ *is the hypergeometric function and*

$$Q(z) = \frac{\pi \mathrm{i}\,(1-b)}{2\sin(\pi b)\,\sin(\pi a)}(1-z)^{b-a}\,, \quad z = (1-J_{\mathfrak{t}}(\tau))^{-1}.$$

(ii) *Write* $\hat{q} = \nu \mathrm{e}^{2\pi \mathrm{i}\tau/h_3}$ *where* $h_3 = 2\cos(\pi v_1) + 2\cos(\pi v_2)$ *and*

$$\nu = \begin{cases} \frac{1}{2}v_1^2 v_2^2 \alpha_3 & v_1 \neq 0,\ v_2 \neq 0,\\ \frac{1}{2}v_1^2\alpha_3 & v_2 = 0,\ v_1 \neq 0,\\ 8 & v_1 = 0,\ v_2 = 0. \end{cases} \tag{3.22}$$

Then the solution of (i) has the expansion

$$t_i = \frac{2\pi \mathrm{i}}{h_3} t_{i,0} + \kappa_i \sum_{j=1}^{\infty} \tilde{t}_{i,j}\hat{q}^j, \tag{3.23}$$

where $[t_{1,0}, t_{2,0}, t_{3,0}] = [0, -1, 0]$ *and*

$$
\begin{aligned}
[\kappa_1, \kappa_2, \kappa_3] = \frac{2\pi \mathrm{i}}{h_3}\big[&- m_1^2 m_2^2 - m_2^2 m_1 + m_2 m_1^2,\ \ m_2 m_1 + m_2 + m_1,\\
& m_1^2 m_2^2 - m_2^2 m_1 + m_2 m_1^2\big]\\
& \tilde{t}_{i,j} \in \mathbb{Q}[m_1, m_2].
\end{aligned} \tag{3.24}
$$

(iii) *If $t_i(\tau)$, $i = 1, 2, 3$, are the coordinates of any solution of the Halphen differential equation, then so are*

$$\frac{1}{(c'\tau+d')^2}\, t_i\left(\frac{a'\tau+b'}{c'\tau+d'}\right) - \frac{c'}{c'\tau+d'}, \quad \forall \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

For example, $\tilde{t}_{1,1} = \tilde{t}_{3,1} = 1$, $\tilde{t}_{2,1} = m_1 - m_2$,

$$\begin{aligned}
\tilde{t}_{1,2} &= \frac{1}{4}(2m_1m_2^2 - m_1^2m_2^2 - 7m_1^2 + 7m_2^2),\\
\tilde{t}_{3,2} &= \frac{1}{4}(m_1^2m_2^2 - 7m_1^2 + 7m_2^2 - 2m_1^2m_2),\\
\tilde{t}_{2,2} &= \frac{1}{8}(-m_1^3m_2^3 + 6m_1^2m_2^2 - 11m_1^3 + 11m_1^2m_2 - m_1^3m_2^2 - 3m_1^3m_2\\
&\quad - 11m_2^3 - m_1^2m_2^3 + 11m_1m_2^2 - 3m_1m_2^3),\\
\tilde{t}_{1,3} &= \frac{1}{48}(3m_1^4m_2^4 - 14m_1^2m_2^4 - 64m_1^3m_2^2 + 64m_1m_2^4 + 50m_1^4m_2^2\\
&\quad + 139m_1^4 + 139m_2^4 - 278m_1^2m_2^2),\\
\tilde{t}_{3,3} &= \frac{1}{48}(3m_1^4m_2^4 - 14m_1^4m_2^2 + 64m_1^4m_2 + 139m_1^4 - 64m_1^2m_2^3\\
&\quad + 139m_2^4 - 278m_1^2m_2^2 + 50m_1^2m_2^4).
\end{aligned}$$

Recall the triangle group $\Gamma_{\mathfrak{t}}$ of type $\mathfrak{t} = (m_1, m_2, \infty)$ generated by the matrices (2.10). We focus in this section on $\hat{q}$-expansions around the cusp $\mathrm{i}\infty$. The renormalization by ν of α_3 is natural from the point of view of the recursion coming from (1.2). For each $k \geq 2$, we set

$$\begin{aligned}
E^{(1)}_{2k,\mathfrak{t}} &:= \left(\frac{h_3}{2\pi\mathrm{i}}\right)^k (t_1 - t_2)\,(t_3 - t_2)^{k-1} \in 1 + \hat{q}\,\mathbb{Q}[\![\hat{q}]\!],\\
E^{(2)}_{2k,\mathfrak{t}} &:= \left(\frac{h_3}{2\pi\mathrm{i}}\right)^k (t_1 - t_2)^{k-1}\,(t_3 - t_2) \in 1 + \hat{q}\,\mathbb{Q}[\![\hat{q}]\!],\\
E_{4,\mathfrak{t}} &:= E^{(1)}_{4,\mathfrak{t}} = E^{(2)}_{4,\mathfrak{t}},\\
E_{6,\mathfrak{t}} &:= E^{(2)}_{6,\mathfrak{t}}.
\end{aligned}$$

Define $E_{2,\mathfrak{t}}$ using Theorem 4(iii). The notation and normalization is chosen so that when $\mathfrak{t} = (2, 3, \infty)$, $E_{k,\mathfrak{t}}$ for $k = 4, 6$ coincide with the classical series for $\Gamma(1)$. From now on we regard all t_i's as functions of τ. The convention throughout this paper is that the value of a polynomial $P(x)$ for $x = \infty$ is the coefficient of the monomial x^n of highest degree in $P(x)$.

Theorem 4. *Assume as usual that* $2 \leq m_1 \leq m_2 \leq \infty$ *and* $\mathfrak{t} = (m_1, m_2, \infty)$ *is hyperbolic. Then*

(i) *The* $t_i(\tau)$ *are quasi-automorphic. More precisely, they are meromorphic functions of* $\tau \in \mathbb{H}_{\mathfrak{t}}$, *and satisfy the following functional equation:*

$$(c'\tau + d')^{-2} t_i(\gamma(\tau)) - c'(c'\tau + d')^{-1} = t_i(\tau) \quad \forall \gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_{\mathfrak{t}}. \tag{3.25}$$

(ii) *The field generated by all meromorphic automorphic forms for* $\Gamma_{\mathfrak{t}}$ *consists of all rational functions in* $t_1 - t_2$ *and* $t_3 - t_2$.

(iii) *The relation with Theorems 1 and 2 is:* $t_1 - t_2 = \frac{2\pi\mathrm{i}}{h_3}\frac{J_{\mathfrak{t}}'}{J_{\mathfrak{t}}}$ *and* $t_3 - t_2 = \frac{2\pi\mathrm{i}}{h_3}\frac{J_{\mathfrak{t}}'}{(J_{\mathfrak{t}}-1)}$

$$\frac{1}{n_\Delta} E_{2;\mathfrak{t}} = \frac{b-a}{b} t_1 - t_2 + \frac{a+b-1}{b} t_3,$$
$$f_4 = E_{4,\mathfrak{t}}, \quad f_6 = \begin{cases} E_{6,\mathfrak{t}} & \text{if } m_1 = 2, \\ E_{6,\mathfrak{t}}/(J_{\mathfrak{t}} - 1) & \text{otherwise,} \end{cases}$$
$$J_{\mathfrak{t}} = \frac{t_3 - t_2}{t_3 - t_1} = \frac{E_{4,\mathfrak{t}}^3}{E_{4,\mathfrak{t}}^3 - E_{6,\mathfrak{t}}^2}.$$

Moreover, the function $j_{\mathfrak{t}} = 2m_2^2 m_1^2 J_{\mathfrak{t}} + (-m_2^2 m_1^2 + m_2^2 - m_1^2)$ *is the unique Hauptmodul for* $\Gamma_{\mathfrak{t}}$ *normalized so that* $j_{\mathfrak{t}}(\tau) = \frac{1}{\hat{q}} + O(\hat{q}^1)$.

(iv) *When* $m_2 \neq \infty$, *the algebra* $\mathfrak{m}(\Gamma_{\mathfrak{t}})$ *of holomorphic automorphic forms of even weight is generated by*

$$E_{2k,\mathfrak{t}}^{(2)},\ 2 \leq k \leq m_2, \quad E_{2k,\mathfrak{t}}^{(1)},\ 3 \leq k \leq m_1.$$

When $m_1 < \infty = m_2$, $\mathfrak{m}(\Gamma_{\mathfrak{t}})$ *is generated by*

$$E_{2k,\mathfrak{t}}^{(1)},\ 1 \leq k \leq m_1.$$

The case $m_1 = m_2 = m_3 = \infty$ corresponds to the classical Darboux–Halphen differential equation, see Section 4.2.

It should be emphasized that, although ultimately the approaches in Sections 2 and 3 both reduce to hypergeometric calculations, the approaches are independent in the sense that their outputs (a Hauptmodul in Section 2 compared with three quasi-automorphic forms in Section 3) are different. Both approaches are complete in the sense that all (quasi-)automorphic forms for the given triangle group $\Gamma_{\mathfrak{t}}$ can be obtained from their outputs by standard operations.

4. The modular triangle groups

By a *modular triangle group* Γ we mean a triangle group commensurable with $\Gamma(1)$ (i.e., $\Gamma \cap \Gamma(1)$ has finite index in both Γ and $\Gamma(1)$). There are precisely nine $\Gamma_{\mathfrak{t}}$ conjugate to a modular triangle group [45]. Such Fuchsian groups are called *arithmetic* (the definition of arithmetic Fuchsian groups can be extended to the case where there are no cusps, and Takeuchi [45] also identifies these). In this section, we show how our expressions for modular forms recover the classical ones in these nine cases.

In table 1, we list these nine types, together with one of the modular triangle groups which realizes it. We include the basic data for that conjugate $g\Gamma_{\mathfrak{t}}g^{-1}$. In the table and elsewhere, we write $\omega = \mathrm{e}^{2\pi\mathrm{i}/6}$, $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. The matrix $W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$ is called a *Fricke involution*. As usual, $\Gamma(N)$ consists of all $A \in \Gamma(1)$ with $A \equiv \pm I \pmod{N}$, $\Gamma_0(N)$ consists of all $A \in \Gamma(1)$ with entry $A_{2,1}$ divisible by N, and $\Gamma_0^+(N) := \langle \Gamma_0(N), W_N \rangle$. Given any triangle group Γ of type $(2, n, \infty)$, by Γ^* we mean the subgroup generated by the squares γ^2 of all elements $\gamma \in \Gamma$, together with any element in Γ of order n; then Γ^* has index 2 in Γ, and is a triangle group of type (n, n, ∞). Table 1 is largely taken from [7].

In this section, we recover explicitly the classical result that:

Proposition 1. *The algebra of holomorphic modular forms for each modular triangle group has a basis in $\mathbb{Z}[[Q]]$, where Q is some rescaling of q or $q^{1/2}$.*

Indeed, by Lemma 3 of [17], $1728J_{(2,3,\infty)}$, $256J_{(2,4,\infty)}$, $108J_{(2,6,\infty)}$, $16J_{(\infty,\infty,\infty)}$, $64J_{(2,\infty,\infty)}$ and $27J_{(3,\infty,\infty)}$ all have integer q- or $q^{1/2}$-coefficients (whichever is appropriate), and leading term $\pm q^{-1}$ or $\pm q^{-1/2}$. $144J_{(3,3,\infty)}$, $32J_{(4,4,\infty)}$ and $36J_{(6,6,\infty)}$ have $q^{1/2}$-coefficients in the Eisenstein $\mathbb{Z}[\omega]$ or Gaussian $\mathbb{Z}[\mathrm{i}]$ integers, but if Q is chosen to be $\mathrm{i}q^{1/2}/\sqrt{3}$, $\mathrm{i}q^{1/2}$ or $\mathrm{i}q^{1/2}/\sqrt{3}$, respectively, then these functions lie in $Q^{-1} + \mathbb{Z}[[Q]]$. This information is enough to verify that the basis given in Theorem 2 has integer coefficients. The exact rescaling of q or $q^{1/2}$ depends on the choice of realization of $\Gamma_{\mathfrak{t}}$.

4.1. Type $\mathfrak{t}_m = (2, m, \infty)$ for $m = 3, 4, 6$

For type $\mathfrak{t} = (2, 3, \infty)$, the triangle group $\Gamma_{\mathfrak{t}}$ is the full modular group $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$. Its algebra of holomorphic quasi-modular forms is generated by the classical Eisenstein series E_2, E_4, E_6 in (2.7). Their relation with the quasi-modular forms coming from the Halphen system are

$$E_{2;\mathfrak{t}} = E_2, \quad E_{4,\mathfrak{t}} = E_4, \quad E_{6,\mathfrak{t}} = E_6, \quad J_{\mathfrak{t}} = j/1728.$$

More generally, for any Hecke group $\Gamma_{(2,m,\infty)}$ (any $m \geq 3$), Eisenstein series $E_{k,\mathfrak{t}_m}(\tau)$ can be analogously defined (see, e.g., Section 4 of [28]). The spaces of holomorphic automorphic forms of weights 4 and 6 are both 1D, spanned by what we call $f_4(\tau) = E_{4,\mathfrak{t}_m}(\tau) = 1 + \cdots$ and $f_6(\tau) = E_{6,\mathfrak{t}_m}(\tau) = 1 + \cdots$, respectively. The normalized Hauptmodul is

$$J_{\mathfrak{t}}(\tau) = \frac{f_4(\tau)^3}{f_4(\tau)^3 - f_6(\tau)^2} \tag{4.26}$$

in perfect analogy with $\Gamma(1)$. In the special cases $m = 2p = 4, 6$ we are interested in here, we determine from Section 4.3.2 of [28] that for any $k \geq 2$

$$E_{2k,\mathfrak{t}_{2p}}(\tau) = (E_{2k}(\tau) + p^k E_{2k}(p\tau))/(p^k + 1) \tag{4.27}$$

and we find

$$\begin{aligned} J_{(2,4,\infty)} &= \frac{1}{256}q^{-1} + \frac{13}{32} + \frac{1093}{64}q + 376q^2 + \frac{620001}{128}q^3 + 41792q^4 + \cdots, \\ J_{(2,6,\infty)} &= \frac{1}{108}q^{-1} + \frac{1}{3} + \frac{371}{36}q + \frac{3643}{54}q^2 + \frac{20713}{36}q^3 - 34396q^4 + \cdots. \end{aligned}$$

4.2. Type (∞, ∞, ∞)

The most natural realization of $\mathfrak{t} = (\infty, \infty, \infty)$ is as $\Gamma(2)$, which has cusps at $\mathrm{i}\infty, 0, 1$. The local parameter at the infinite cusp is $q^{1/2} = \mathrm{e}^{\pi \mathrm{i} \tau}$ (the square-root of the parameter for $\Gamma(1)$). Recall the Jacobi theta functions

$$\begin{cases} \theta_2(\tau) := \displaystyle\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2}, \\ \theta_3(\tau) := \displaystyle\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2}, \\ \theta_4(\tau) := \displaystyle\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2}. \end{cases}$$

It is well known that θ_2^4, θ_3^4 and $\theta_4^4 = \theta_3^4 - \theta_2^4$ are modular forms for $\Gamma(2)$ of weight 2, and that they generate the ring of holomorphic modular forms. A Hauptmodul is

$$J_{(\infty,\infty,\infty)}(\tau) = \frac{\theta_3(\tau)^4}{\theta_2(\tau)^4} = \frac{1}{16}q^{-1/2} + \frac{1}{2} + \frac{5}{4}q^{1/2} - \frac{31}{8}q^{3/2} + \frac{27}{2}q^{5/2} + \cdots$$

which maps i∞ to ∞, cusp 0 to 1 and cusp 1 to 0. The normalized quasi-modular form is $e_2 = E_2/6$.

In 1878, G. Darboux studied the system of differential equations

$$\begin{cases} \dot{u}_1 + \dot{u}_2 = 2u_1u_2, \\ \dot{u}_2 + \dot{u}_3 = 2u_2u_3, \\ \dot{u}_1 + \dot{u}_3 = 2u_1u_3 \end{cases} \tag{4.28}$$

in connection with triply orthogonal surfaces in $\mathbb{R}^3$. Later Halphen in [21] found a solution of (4.28) in terms of theta series

$$u_1 = 2(\ln\theta_4(\tau))', \quad u_2 = 2(\ln\theta_2(\tau))', \quad u_3 = 2(\ln\theta_3(\tau))'.$$

The differential equation (4.28) after the change of variables $t_i := -2u_i$ turns to be (1.2). The relations between the series t_i in Section 3 and theta series are given by

$$\frac{-1}{4}t_i(8q^{\frac{1}{2}}) = 2q\frac{d}{dq}\ln\theta_{j_i},$$

where $(j_1, j_2, j_3) = (3, 2, 4)$.

4.3. Types $\mathbf{t}_m = (m, \infty, \infty)$, $m = 2, 3$

It is well known that a Hauptmodul for $\Gamma_0(N)$ when $N-1$ divides 24 is $J_{(N)}(\tau) = (\eta(\tau)/\eta(N\tau))^{24/(N-1)}$, which for $N = 2, 3$ rescales to the normalized Hauptmoduln

$$\begin{aligned} J_{(2,\infty,\infty)}(\tau) &= -\frac{1}{64}q^{-1} + \frac{3}{8} - \frac{69}{16}q + 32q^2 - \frac{5601}{32}q^3 \\ &\quad + 768q^4 - \frac{23003}{8}q^5 + \cdots, \\ J_{(3,\infty,\infty)}(\tau) &= -\frac{1}{27}q^{-1} + \frac{4}{9} - 2q + \frac{76}{27}q^2 + 9q^3 - 44q^4 + \frac{1384}{27}q^5 + \cdots. \end{aligned}$$

For any N (and in particular $N = 2, 3$)

$$q\frac{d}{dq}\log\left(\frac{\eta(\tau)}{\eta(N\tau)}\right) = E_2(\tau) - NE_2(N\tau)$$

is a holomorphic weight-2 modular form for $\Gamma_0(N)$. For $\Gamma_0(2)$, the algebra of holomorphic modular forms is generated by $E_2(\tau) - 2E_2(2\tau)$ and $E_4(\tau)$, while that for $\Gamma_0(3)$ is generated by $E_2(\tau) - 3E_2(3\tau)$, $E_4(\tau)$ and $E_6(\tau)$.

4.4. Type $\mathfrak{t}'_m = (m, m, \infty)$ for $m = 3, 4, 6$

Write $\mathfrak{t}_m = (2, m, \infty)$ as before. Recall from the beginning of this section that a Fuchsian group of type $\mathfrak{t}'_m$ (for any $m \geq 3$) can be chosen to be the index 2 subgroup $\Gamma^*_{\mathfrak{t}_m}$ of the Hecke group $\Gamma_{\mathfrak{t}_m}$. The normalized Hauptmodul for any $\mathfrak{t}'_m$ is

$$J_{(m,m,\infty)}(\tau) = \frac{1}{2}\left(\frac{E_{6,\mathfrak{t}_m}(\tau)}{\sqrt{E_{6,\mathfrak{t}_m}(\tau)^2 - E_{4,\mathfrak{t}_m}(\tau)^3}} + 1\right),$$

where $E_{k,\mathfrak{t}_m} = f_k$ here are the (normalized) Eisenstein series discussed in Section 4.1. The holomorphic modular forms are generated by $\sqrt{E^2_{6,\mathfrak{t}_m} - E^3_{4,\mathfrak{t}_m}}$ together with those for $\mathfrak{t}_m$ (since $\Gamma_{\mathfrak{t}'_m}$ is a subgroup of $\Gamma_{\mathfrak{t}_m}$). From this point of view, the only thing special about $m = 3, 4, 6$ is that we can easily express $E_{4,\mathfrak{t}_m}, E_{6,\mathfrak{t}_m}$ in terms of classical modular forms, as was done in (4.27) above. We find

$$\begin{aligned}
J_{(3,3,\infty)}(\tau) &= -\frac{\mathrm{i}\sqrt{3}}{144}q^{-1/2} + \frac{1}{2} + \frac{41\,\mathrm{i}\sqrt{3}}{12}q^{1/2} + \frac{1255\,\mathrm{i}\sqrt{3}}{8}q^{3/2} \\
&\quad + \frac{45925\,\mathrm{i}\sqrt{3}}{18}q^{5/2} + \cdots, \\
J_{(4,4,\infty)}(\tau) &= -\frac{\mathrm{i}}{32}q^{-1/2} + \frac{1}{2} + \frac{19\,\mathrm{i}}{8}q^{1/2} + \frac{351\,\mathrm{i}}{16}q^{3/2} + \frac{653\,\mathrm{i}}{4}q^{5/2} \\
&\quad + \frac{23425\,\mathrm{i}}{32}q^{7/2} + \cdots, \\
J_{(6,6,\infty)}(\tau) &= \frac{-\mathrm{i}\sqrt{3}}{36}q^{-1/2} + \frac{1}{2} + \frac{11\,\mathrm{i}\sqrt{3}}{12}q^{1/2} + \frac{17\,\mathrm{i}\sqrt{3}}{4}q^{3/2} \\
&\quad + \frac{713\,\mathrm{i}\sqrt{3}}{36}q^{5/2} + \cdots.
\end{aligned}$$

5. Observations and conjectures concerning coefficients

The raison d'être of modular forms is their q-expansions, i.e., the local (Fourier) expansions about the cusp i∞. Expansions about other cusps have the same familiar feel (although are usually ignored). The avoidance of considerations of (Taylor) expansions at points in $\mathbb{H}$, in particular at the elliptic fixed-points, is almost complete.

It is hard to justify this focus on the expansion at i∞, other than that it is exceedingly rich. However, a triangle group say has *three* special $\Gamma_{\mathfrak{t}}$-orbits, perhaps the other two may also prove interesting. For example, in the vvaf's of Section 6.3 below, it seems artificial to expand only about the large complex structure point (which corresponds to a cusp) but to refuse to expand about say the Landau–Ginzburg point (which corresponds to an elliptic fixed-point). For another example, consider the characters $\chi_M(\tau) = \sum_r a(M)_r q^r$ of irreducible modules M of rational vertex operator algebras. These χ_Ms are modular functions for some $\Gamma(N)$. A surprise happens at their expansions $\chi_M(\tau) = \sum_r a(M)_{x;r} q_x^r$ about certain cusps $x \in \mathbb{Q}$ (which x to choose depends only on N): there are signs $\epsilon_x(M)$ and *another* irreducible module M^x such that the coefficients at x of χ_M equal those at i∞ of $\epsilon_x(M)\,\chi_{M^x}$, that is, $a(M)_{x;r} = \epsilon_x(M)\,a(M^x)_r$. In other words, expanding one character about a different cusp can recover a different character at the usual cusp i∞. (This property of vertex operator algebra characters is implicit in Section 6.3.3 of [16].)

In any case, the Halphen or Schwarz differential equations can be used to compute arbitrarily many terms of Fourier or Taylor expansions of automorphic forms (on the third author's homepage one can find computer code written in singular [19] and the first few coefficients of $t_1, t_2, t_3, J_{\mathfrak{t}}$ at i∞). From these expansions, we are led to the conjectures (and results) gathered below.

We will find a deep connection to the arithmeticity (or otherwise) of $\Gamma_{\mathfrak{t}}$, and the integrality of those coefficients. This is hardly surprising. If a Fuchsian group has at least one cusp (as we have been assuming), then the definition of arithmeticity can be taken to be that it contains some conjugate of some congruence subgroup $\Gamma(N)$. By a theorem of Margulis [30], a Fuchsian group is arithmetic iff the commensurator

$$\mathrm{comm}(\Gamma) := \{\gamma \in \mathrm{PSL}(2,\mathbb{R}) \,:\, \gamma\Gamma\gamma^{-1} \text{ is commensurable with } \Gamma\}$$

is dense (recall that Γ_1, Γ_2 are *commensurable* iff $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_i). More precisely, when Γ is nonarithmetic, $\mathrm{comm}(\Gamma)$ is itself a

Fuchsian group of the first kind, in fact the largest containing Γ. On the other hand, if Γ contains some $\Gamma(N)$ then any $\gamma \in \mathrm{GL}^+(2,\mathbb{Q})$ (or rather its projection to $\mathrm{PSL}(2,\mathbb{R})$) will lie in $\mathrm{comm}(\Gamma)$. The relevance of the commensurator is that $\gamma \in \mathrm{comm}(\Gamma)$ directly yields Hecke operators for Γ. Given enough Hecke operators, the arithmeticity of coefficients will follow.

It is easy to see directly that, for the nonarithmetic triangle groups, something goes wrong with standard Hecke theory. Recall that the basis of Theorems 2 and 4 look like

$$f(\tau) = \sum_{n=0}^{\infty} a_n\, q_3^n, \quad a_n = r_n\, \alpha_3^n,$$

where $r_n \in \mathbb{Q}$ and $q_3 = \mathrm{e}^{\frac{2\pi \mathrm{i}\tau}{h}}$. Wolfart [51] proved that α_3 is transcendental, but that implies that $a_n a_m \neq a_{mn}$ whenever $m, n > 2$. Nor can we get multiplicativity if we absorb the α_3 into q_3. For weight k cusp forms for any Fuchsian group, we have the bound $a_n = O(n^{k/2})$ [32]. But this means that the r_n increase or decrease exponentially (depending on whether or not $|\alpha_3| < 1$), which is again incompatible with $r_n r_m = r_{nm}$ for sufficiently large m, n.

5.1. Coefficients at the cusps

Fix a hyperbolic type $\mathfrak{t} = (m_1, m_2, \infty)$. We do not require here that $m_1 \leq m_2$; the case where m_1 or m_2 is infinite is included in the formulas below using the aforementioned convention about the value of polynomials at ∞. Consider first the Fourier coefficients $c_n = c_{n;\mathfrak{t}}$ of (2.15). Note that the Euclidean types $(2,2,\infty)$ and (formally) $(1,\infty,\infty)$ correspond to polynomial solutions $\widetilde{q}_3^{\,-1} + \frac{1}{2} + \frac{1}{16}\widetilde{q}_3$ and $\widetilde{q}_3^{\,-1}$, respectively, of (2.16). This means that c_n vanishes when $m_1 = m_2 = 2\ \forall n \geq 2$, and also c_n vanishes at $m_1 = 1$, $m_2 = \infty\ \forall n \geq 0$, and hence

$$c_n = \frac{(m_1^2-4)P_{1;n}(m_1^2,m_2^2) + (m_2^2-4)P_{2;n}(m_1^2,m_2^2)}{(m_1^2 m_2^2)^{n+1}\, Q_n}, \quad n \geq 2, \tag{5.29}$$

$$c_n = \frac{(m_1^2-1)P'_{1;n}(m_1^2,m_2^2) + m_2^2 P'_{2;n}(m_1^2,m_2^2)}{(m_1^2 m_2^2)^{n+1}\, Q'_n}, \quad n \geq 1, \tag{5.30}$$

where $Q_n, Q'_n \in \mathbb{N}$ and $P_{i;n}, P'_{i;n}$ are type-independent polynomials with integral coefficients and total degree $\leq n-1$. The format (5.29) of c_n generalizes to any type (m_1, m_2, ∞) the observation of Akiyama [3] for Hecke groups described below, and (5.30) seems completely new. Note that it would be reasonable to absorb $(m_1^2 m_2^2)^n$ into $\tilde{q}_3$, at least when m_1, m_2 are both finite, and indeed this gives the $\hat{q}$ used in Section 3.

A more interesting symmetry is that for $n \geq 1$

$$c_{n;(m_1,m_2,\infty)} = (-1)^{n+1} c_{n;(m_2,m_1,\infty)}. \tag{5.31}$$

To prove this, first identify $\Gamma_{(m_2,m_1,\infty)}$ as a conjugate of $\Gamma_{(m_1,m_2,\infty)}$, and then use this to express $J_{(m_2,m_1,\infty)}$ in terms of $J_{(m_1,m_2,\infty)}$.

Some of this had already been worked out for the Hecke groups $\Gamma_{(2,m,\infty)}$. In particular, Lehner [27] and especially Raleigh [39] worked from the Schwarz equation, obtaining (2.13) in this special case as well as (5.29) without the $m^2 - 4$ factor. For $n \geq 2$ and again only for the Hecke groups, Akiyama [3] showed that c_n is a polynomial divisible by $m^2 - 4$. He also showed that the prime divisors of Q_n are not greater that $n + 1$. This follows immediately from the recursion given by the Halphen differential equation, where at the nth step of the recursion we divide by $n^2(n - 1)$, see Section 7.3. Leo [28] in his PhD thesis proved that c_n can be written as $\frac{C_n}{D_n(2^6 m^2)^{n+1}}$, where $C_n, D_n \in \mathbb{Z}$ are coprime and D_n has no prime factor of the form $p \equiv 1 \pmod{4m}$. He made also a precise conjecture about the prime factors of D_n. As with all these people, he focussed exclusively on the Hecke groups $\Gamma_{(2,m,\infty)}$.

A major conjecture, now attributed to Atkin and Swinnerton-Dyer [4], states that if f is a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ for some subgroup Γ of $\Gamma(1)$, and the Fourier coefficients are algebraic integers, then Γ (if it is chosen maximally) contains a congruence subgroup. See, e.g., [29] for a review. Scholl [42] has proved that when Γ is a subgroup of $\Gamma(1)$, there is an integer N and a scalar multiple $\widetilde{q}$ of $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ such that the space of modular forms for Γ of each weight $k \in \frac{1}{2}\mathbb{Z}$ has a basis with $\widetilde{q}$-expansion coefficients which are algebraic integers when multiplied by some power of N. We have $N = 1$ if (and conjecturally only if) Γ contains a congruence subgroup, i.e., is arithmetic. In other words, we know that at most finitely many distinct primes can appear in the denominators of modular forms for subgroups of $\Gamma(1)$. On the other hand, when Γ is not commensurable with $\Gamma(1)$, one would expect infinitely many distinct primes in the denominators.

Our observations are compatible with these conjectures. Recall from Section 4 the nine arithmetic triangle groups with at least one cusp: namely those of type

$$(\infty,\infty,\infty),(2,3,\infty),(3,3,\infty),(m,\infty,\infty),(2,2m,\infty),(2m,2m,\infty) \tag{5.32}$$

for $m = 2, 3$. This also coincides with the list of all triangle groups conjugate to a group commensurable with $\Gamma(1)$. All nine of those (up to conjugation) contain a congruence subgroup, as they must. In Section 4, we recovered the classical result that in these cases the algebra of modular forms for $\Gamma_{\mathfrak{t}}$ is defined over $\mathbb{Z}$. By that we mean that there is a rescaling Q of q_3, and some modular forms $f_i \in \mathbb{Z}[[Q]],\ i = 1, 2, \ldots$, such that the algebra of all holomorphic modular forms for $\Gamma_{\mathfrak{t}}$ is $\mathbb{C}[f_i,\ i = 1, 2, \ldots]$.

The algebra of automorphic forms for the hyperbolic triangle group $\Gamma_{(m_1,m_2,\infty)}$ is defined over $\mathbb{Z}$ if and only if the triangle group is arithmetic. The only if part of this affirmation is classical, and was reproved in Section 4. The other direction has been recently proved by the last two authors. For the nonarithmetic case, we are also able to prove that infinitely many primes do not appear in any denominators of the coefficients of $t_i,\ i = 1, 2, 3$ and $J_{\mathfrak{t}}$. We are led to the following conjecture experimentally:

Conjecture 1. *For any nonarithmetic hyperbolic triangle group of type (m_1, m_2, ∞), infinitely many primes appear in the denominators of the coefficients of $t_i,\ i = 1, 2, 3$ and $J_{\mathfrak{t}}$ at the infinite cusp.*

For nonarithmetic $\Gamma_{\mathfrak{t}}$ with $2 \le m_1 \le m_2 \le 30$ (and several other m_i chosen randomly), we looked at all denominators for terms up to q^{182}. The distribution of primes which appear, compared with those which do not, seem to be similar. We also observe that for each prime $p \neq 2$, $t_i(p\hat{q}),\ i = 1, 2, 3$ has no p in the denominators of its coefficients. This can be easily seen from the recursion given by the Halphen differential equation, see Section 7.3. More precisely, let p be a prime and f be an automorphic form for $\Gamma_{(m_1,m_2,\infty)}$. Define $m_{n,p}(f)$ to be the power of p in the denominator of a_n, where $f = \sum a_n \hat{q}^n$. Our data suggest the conjecture $\lim_{n\to\infty} \frac{m_{n,p}}{n} = 0$.

The main thing responsible for this nonintegrality is the coefficient Q_n in the denominator of (5.29). We suspect that each prime appears in the prime decomposition of some Q_n. The reason is that in the recursion for calculating the coefficients of $\tilde{q}^n$ we divide by $n^2(n-1)$. Although a priori a prime p could appear at $n = p$, we observe that it appears first at $n = p + 1$. Note that this observation does not imply Conjecture 1, since the denominator and numerator of c_n in (5.29) may have common factors.

The much simpler case of Hecke groups is extensively analyzed by Leo in [28]. For completeness we review his findings. Consider the triangle group of type $(2, m, \infty)$. Write

$$c_n = \frac{C_n}{D_n 2^{6n+6} m^{2n+2}},$$

where $C_n, D_n \in \mathbb{Z}$ and $\gcd(C_n, D_n) = 1$. Leo [28] conjectured that a prime p divides some D_n for $n \geq 1$, iff $p \neq 2$, p does not divide m, and $p \not\equiv \pm 1 \pmod{m}$. Moreover, he conjectures that the smallest n for which such a prime p divides D_n, is $n = p^k - 1$ for some k.

5.2. Integrality at elliptic fixed-points

Again, we propose studying these expansions because every triangle group has three special $\Gamma_{\mathfrak{t}}$-orbits, most of which are elliptic fixed-points. As already mentioned, Rodriguez Villegas and Zagier [41] have found some of these coefficients to be interesting.

Consider first $\Gamma_{(2,3,\infty)} = \Gamma(1)$. Recall the expansion (2.15). The coefficients at $\tau = \mathrm{i}$ are

$$a_2 = \frac{23}{54}, \quad a_3 = \frac{6227}{58320}, \quad a_4 = \frac{3319}{174960}, \quad a_5 = \frac{263489}{97977600}, \quad a_6 = \frac{1693777}{5290790400}, \ldots . \tag{5.33}$$

Not only are these nonintegral, but also the denominator seems to be growing without bound! But as we shall see shortly, there is a simple explanation for this.

The coefficients at elliptic fixed-points are more accessible than the coefficients at cusps. In particular, choose any point $z = x + \mathrm{i}y \in \mathbb{H}$ of order $m \geq 1$ and let $f(\tau) = j_z(k;\tau) q_z^{k/m} \sum c_n q_z^n$ be a weight-k automorphic form (recall (2.5)). Note that q_z is *not* rescaled here, so that series will have radius of convergence exactly 1 (provided f is holomorphic). Incidentally, Cauchy–Hadamard constrains the growth of these c_n: $\limsup_{n\to\infty} |c_n|^{1/n} = 1$, so they grow roughly like the usual (unscaled) Fourier coefficients.

These coefficients c_n are then computed by Bruinier *et al.* [11] and Rodriguez Villegas and Zagier [41]

$$c_n = \partial_k^{nm} f(z) \frac{(4\pi y)^{mn}}{(mn)!}, \tag{5.34}$$

where $z = x + \mathrm{i}y$ and $\partial_k^n = \partial_{k+n-2} \circ \cdots \circ \partial_{k+2} \circ \partial_k$, for the nonholomorphic modular derivative $\partial_k f = \frac{1}{2\pi \mathrm{i}} \frac{df}{d\tau} - \frac{kf(z)}{4\pi y}$. The mn arises because $q_z = \tau_z^m$ is a power. Hence in this sense we can think of these c_n as Taylor coefficients. The reason for the terrible denominators in (5.33) is the $n!$ in (5.34).

The important quantities should be the derivatives of f, in other words we should multiply the a_n by $n!$ (and rescale q_z). We find for $\Gamma(1)$ at $z = \mathrm{i}$ that $a_n(2n)!3^n$ are positive integers, with a single 3 in the denominators. The analogous calculation for the other elliptic fixed-point yields only positive integers. We expect:

Conjecture 2. *Consider any arithmetic triangle group $\Gamma_{(m_1,m_2,\infty)}$ and any elliptic fixed-point $z \in \mathbb{H}$. Then, the sequence $(m_1 n)!m_2^n a_n$ are strictly positive algebraic numbers with bounded denominators. There should exist a basis for the space of weight k holomorphic automorphic forms whose coefficients at z are algebraic integers when rescaled in this way.*

For $\mathfrak{t} = (3, 3, \infty)$, the denominator for $J_\mathfrak{t}$ is bounded by 8, while for $(4, 4, \infty)$ and $(6, 6, \infty)$ the denominators are all 1. For $(2, 4, \infty)$, the adjusted a_n have denominators bounded by 2, while the adjusted b_n have at most 3 in the denominators. For $(2, 6, \infty)$, the adjusted a_n have at most a 3 in the denominator, while the adjusted b_n is integral. The larger the order of the fixed-point, the greater the chance for integers, because the multipliers become so big. Note that for an arithmetic triangle group (m_1, m_2, ∞) it suffices to compute the values $\partial_k^{nm} f(z)$ for the generators f, as ∂_k is a derivation.

For nonarithmetic types, the situation is less clear. For example, for $\mathfrak{t} = (2, 5, \infty)$, the adjusted a_n has 5's appearing in the denominators to arbitrarily high powers, and the only other prime appearing in a denominator is 2, with power at most 3. In this case, $a_n(2n)!5^{2n}$ has bounded denominators. On the other hand, the adjusted b_n is integral. For $(m_1, m_2) = (2, 7), (2, 8), (3, 7)$, $a_n(m_1 n)!m_2^n$ has unbounded denominator but $a_n(m_1 n)!m_2^{2n}$ and $b_n(m_2 n)!m_1^{2n}$ both have bounded denominators. All of these were verified up to $n = 35$, but because of recursive formulas for these coefficients, it should not be difficult to prove this.

6. Periods and automorphic functions

The Gauss hypergeometric functions are periods up to some Γ-factors. This means that we can write them as integrals of algebraic differential forms over topological cycles. Looking in this way we can generalize automorphic functions beyond their classical context of Hermitian symmetric domains and action of groups, see for instance Section 6.2. In this section, we explain this idea.

6.1. Periods and Halphen

In [33], the third author has used integrals of the form $\int \frac{x^i dx}{(x-t_1)^a(x-t_2)^b(x-t_3)^c}$, in order to establish various properties of Halphen differential equations so that generalizations, for instance for arbitrary number of $x - t_i$ factors in the integrand, become realizable. We can view these integrals as periods in the following sense. We define a new variable y and consider the family of algebraic curves $C : y = (x - t_1)^a(x - t_2)^b(x - t_3)^c$ for rational numbers a, b, c. In this way hypergeometric functions up to some Γ factors can be written as periods $\int_\delta \omega$, where ω is a differential form on C with zero residues at poles and $\delta \in H_1(C, \mathbb{Z})$, see [43]. Now, one can use the algebraic geometry machinery in order to study the coefficients of q-expansions of automorphic functions, see for instance [26], or the arithmetic of hypergeometric functions, see [43]. In the next subsection, we describe a similar situation with Calabi–Yau periods.

6.2. Hypergeometric Calabi–Yau equations

Let $\widetilde{X}$ be a Calabi–Yau three-fold, and $\mathcal{M}$ its moduli space of complex structures. The (complex) dimension of $\mathcal{M}$ equals the Hodge number $h^{2,1}$. We are interested here in $h^{2,1} = 1$, in which we can, in the simplest cases, identify $\mathcal{M}$ with $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$, where the large complex structure point corresponds to $z = 0$, the conifold point to $z = 1$, and the Landau–Ginzburg point to $z = \infty$, see for instance [14]. The simplest example is the mirror family of the generic quintic hypersurface in $\mathbb{CP}^4$, which can be parametrized by $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5z^{-1/5}x_1x_2x_3x_4x_5 = 0$ for $z \in \mathcal{M}$.

A holomorphic family $\varpi(z)$ of holomorphic 3-forms will satisfy the *Picard–Fuchs equation*. This implies, for any topological 3-cycle $\gamma \in H_3(\widetilde{X}; \mathbb{Z})$, the period $\int_\gamma \varpi(z)$ will satisfy a generalized hypergeometric equation of order $2h^{2,1} + 2 = 4$, also called the Picard–Fuchs equation. Note that the topological cycle γ in $\widetilde{X}$ is fixed and the complex structure on $\widetilde{X}$, and hence $\varpi(z)$, is varying. Periods provide a (redundant) parametrization of $\mathcal{M}$. See, e.g., [35] for a systematic treatment of periods, Picard–Fuchs and related concepts.

There are precisely 23 integral variations of Hodge structure which can come from such $\widetilde{X}$ with $h^{2,1} = 1$, corresponding to 14 different Picard–Fuchs equations [14]. For simplicity we have selected in table 2 one representative

Table 2: Monodromy data of one-parameter models.

(a_1, a_2)	(n_1, n_2)	Type
$(\frac{1}{5}, \frac{2}{5})$	$(-4, -5)$	$(5, \infty, \infty)$
$(\frac{1}{6}, \frac{1}{3})$	$(-3, -3)$	$(6, \infty, \infty)$
$(\frac{1}{8}, \frac{3}{8})$	$(-3, -2)$	$(8, \infty, \infty)$
$(\frac{1}{10}, \frac{3}{10})$	$(-2, -1)$	$(10, \infty, \infty)$
$(\frac{1}{4}, \frac{1}{3})$	$(-4, -6)$	$(12, \infty, \infty)$
$(\frac{1}{6}, \frac{1}{4})$	$(-2, -2)$	$(12, \infty, \infty)$
$(\frac{1}{12}, \frac{5}{12})$	$(-3, -1)$	$(12, \infty, \infty)$
$(\frac{1}{4}, \frac{1}{2})$	$(-5, -8)$	(∞, ∞, ∞)
$(\frac{1}{3}, \frac{1}{2})$	$(-6, -12)$	(∞, ∞, ∞)
$(\frac{1}{6}, \frac{1}{2})$	$(-4, -4)$	(∞, ∞, ∞)
$(\frac{1}{3}, \frac{1}{3})$	$(-5, -9)$	(∞, ∞, ∞)
$(\frac{1}{4}, \frac{1}{4})$	$(-3, -4)$	(∞, ∞, ∞)
$(\frac{1}{6}, \frac{1}{6})$	$(-1, -1)$	(∞, ∞, ∞)
$(\frac{1}{2}, \frac{1}{2})$	$(-7, -16)$	(∞, ∞, ∞)

for each equation. The Picard–Fuchs equation satisfied by the periods is

$$\delta^4 - z \prod_{i=1}^{4} (\delta + a_i) = 0, \tag{6.35}$$

where we write $\delta = zd/dz$, $a_3 = 1 - a_2$ and $a_4 = 1 - a_1$. Periods are subject to monodromy as we circle the special points in $\mathcal{M}$, and these can be worked out explicitly.

In particular, fix an integral basis $\gamma_1, \ldots, \gamma_4$ of $H_3(\widetilde{X}; \mathbb{Z})$. This is done in [2, 18] using Meijer functions. Collect the periods into a column vector $\Pi(z) = [\int_{\gamma_1} \varpi(z), \ldots, \int_{\gamma_4} \varpi(z)]^{\mathrm{tr}}$. Then $\Pi(z)$ is a fundamental solution of (6.35). In terms of the Meijer basis, the monodromy matrices are

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} n_1 & 1-n_1 & n_2 & 1-n_2 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{6.36}$$

and $M_1 = M_0^{-1}M_\infty^{-1}$, using the parameters n_i of table 2, where M_0 is the monodromy picked up along a small counterclockwise circle going around $z = 0$, etc. In [12], the authors give a different presentation of the monodromy groups. The advantage of their approach is that the auxiliary parameters, which are related to (n_1, n_2), have explicit geometric interpretation.

Of course, these monodromy matrices together define a representation of $\pi_1(\mathcal{M}) \cong \Gamma_{(\infty,\infty,\infty)}$. In seven of the models, we can do better though. The orders of M_0 and M_1 will always be infinite, but those of M_∞ can sometimes be finite. If we let m be the order of M_∞, then this representation of $\Gamma_{(\infty,\infty,\infty)}$ factors through to a representation of $\Gamma_{(m,\infty,\infty)}$. This type (m, ∞, ∞) is collected in the final column of table 2. What we lose in going to a less familiar triangle group, we gain in getting a much tighter representation. Indeed, Brav and Thomas [8] show that for the first model in table 2, and a few others, the monodromy representation of $\Gamma_{(m,\infty,\infty)}$ is faithful; by contrast, the kernel of the natural surjection $\Gamma_{(\infty,\infty,\infty)} \to \Gamma_{(m,\infty,\infty)}$ is a free group of infinite rank for any $m < \infty$. After a conjugation of all the monodromy group in table 2, they become subgroups of $\mathrm{Sp}(4, \mathbb{Z})$. It is a remarkable fact that seven of the cases in table 2 are of infinite index and are triangle groups (see [8]) and three cases are of finite index, see [44].

6.3. Vector-valued automorphic forms

A solution to a Fuchsian differential equation over a compact surface, can be interpreted as a vvaf simply by lifting the surface minus singularities ($\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ here) to its universal cover $\mathbb{H}$. This is not a completely trivial statement — see [6] for the general argument — but in the special case of these models this will be made manifest shortly.

Definition. *Let $k \in 2\mathbb{Z}$, Γ be a Fuchsian group, and ρ a group homomorphism $\Gamma \to \mathrm{GL}(d, \mathbb{C})$. A* vvaf *$\mathbb{X}(\tau)$ of weight k and rank d on Γ with multiplier ρ is a meromorphic map $\mathbb{X} : \mathbb{H} \to \mathbb{C}^d$, meromorphic also at the cusps, obeying the functional equation*

$$(c\tau + d)^{-k}\, \mathbb{X}\left(\frac{a\tau + b}{c\tau + d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{X}(\tau), \quad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{6.37}$$

Choosing $\mathfrak{t} = (m, \infty, \infty)$ for either $m = \infty$ or any $m > 0$ with $\gamma_3^m = 1$, and the corresponding period vector $\Pi(z)$ in Section 6.2, $\mathbb{X}(\tau) := \Pi(J_\mathfrak{t}(\tau))$ is a vvaf of weight 0 for $\Gamma_\mathfrak{t}$, for multiplier which can be identified with the monodromy of the Picard–Fuchs differential equation. This gives a modular interpretation for periods.

Let us be more explicit. Perhaps, the simplest way to describe a vvaf $\mathbb{X}$ of weight k and rank d is to state a differential equation

$$D_k^d + f_{d-1}D_k^{d-1} + \cdots + f_0 = 0 \tag{6.38}$$

satisfied by all components of $\mathbb{X}$, together with enough information to identify which solution corresponds to each component. Here, f_j is an automorphic form for $\Gamma_{\mathfrak{t}}$ of weight $2j$, D_k is the differential operator of Theorem 2(ii), and $D_k^j = D_{k+2j-2} \circ \cdots \circ D_{k+2} \circ D_k$.

Recall the data for (∞, ∞, ∞) collected in Section 4.2. We have $D_2\theta_2^4 = (2\theta_3^4 - \theta_2^8)/3$, $D_2\theta_3^4 = (2\theta_3^4\theta_2^4 - \theta_3^8)/3$, $D_0 J_{\mathfrak{t}} = \theta_4^4 J_{\mathfrak{t}}$, $\Delta_{\mathfrak{t}} = \theta_2^4\theta_3^4\theta_4^4$. Recall the parameters a_1, a_2 collected in table 2. The vvaf $\mathbb{X}(\tau)$ has rank 4 and weight 0 and corresponds to the differential equation (6.38) with

$$\begin{aligned}
f_3 &= \frac{10B + 8C}{3}, \quad f_2 = \frac{20B^2}{9} + BC\left(a_1^2 + a_2^2 - a_2 - a_1 + \frac{41}{9}\right) + \frac{11C^2}{9},\\
f_1 &= -\frac{20B^3}{27} + B^2C\frac{-2 - 2a_2 - 2a_1 + 2a_1^2 + 2a_2^2}{3}\\
&\quad + BC^2\frac{1 + 12a_2^2 + 12a_1^2 - 12a_2 - 12a_1}{9} - \frac{C^3}{27},\\
f_0 &= C^3B(a_1^2a_2 - a_1^2a_2^2 - a_1a_2 + a_1a_2^2),
\end{aligned}$$

where we write $A = \theta_3^4, B = \theta_2^4, C = \theta_4^4 = A - B$. This looks more complicated because it is a uniform formula for all a_i.

The solutions all have an expansion $\sum_n c_n(\tau)q^{n/2}$ and each coefficient $c_n(\tau)$ is a polynomial of degree at most 3 in τ. We can identify which solution to call $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{X}_4$ — these form a basis of the solution space, and the components of a vvaf of weight 0 for $\Gamma(2)$. We know everything about these vvaf, e.g., their multiplier (i.e., to which representation of $\Gamma(2)$ they correspond), their local expansions at each of the three cusps 0,1,∞, etc. The components lie in $\mathbb{Q}[[\sqrt{q}]]$ but not $\mathbb{Z}[[\sqrt{q}]]$. So what we lose in the simplicity of the local expansions, we gain in the simplicity of the functional equations (which just involve the usual Möbius transformations defining $\Gamma(2)$).

The $\Gamma_{(m,\infty,\infty)}$ expressions should have some advantages, since that is *really* the group doing the acting — $\Gamma(2)$ is a bit of a formal trick. We will provide those expressions elsewhere. But the uniformity and familiarity of $\Gamma(2)$ of course has its advantages too. This gives an answer to the question: what is a modular interpretation for the Calabi–Yau three-fold periods? An alternate answer to this question generalizes the algebraic geometric definition of (quasi-)modular forms and the relation of the Halphen differential equation with the Gauss–Manin connection to the families of Calabi–Yau varieties, see [36]. The relation between these two approaches is discussed in the next subsection. In future work, we will reinterpret questions involving periods into the automorphic language and explore whether this sheds any new light on them.

6.4. Periods and modular-type functions

The most important modular object arising from the periods of Calabi–Yau varieties is the Yukawa coupling. Let $\psi_0 = 1 + O(z)$ and $\psi_1 := \psi_0 \ln(z) + O(z)$ be, respectively, the holomorphic and logarithmic solutions of the hypergeometric equation (6.35). The Yukawa coupling $Y := n_0 \frac{\psi_0^4}{(\psi_0\delta\psi_1 - \psi_1\delta\psi_0)^3(1-z)}$, where $\delta = z\frac{\partial}{\partial z}$, is holomorphic at $z = 0$ and so it can be written in the Calabi–Yau mirror map $q = \mathrm{e}^{\frac{\psi_1}{\psi_0}}$

$$Y = n_0 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1 - q^d}.$$

Here, $n_0 := \int_{\tilde{X}} \omega^3$, where $\tilde{X}$ is the A-model Calabi–Yau three-fold of mirror symmetry and ω is the Kähler 2-form of $\tilde{X}$ (the Picard–Fuchs equation (6.35) is satisfied by the periods of the B-model Calabi–Yau three-fold). The numbers n_d are supposed to count the number of rational curves of degree d in a generic $\tilde{X}$. For the first item in table 2 the first coefficients n_d are given by $n_d = 5, 2875, 609250, 317206375, \ldots$.

The field generated by $z, \delta^i\psi_0, \quad i = 0, 1, 2, 3, \delta\psi_1 - \psi_1\delta\psi_0, \psi_0\delta^2\psi_1 - \psi_1\delta^2\psi_0$ over $\mathbb{C}$ and written in the coordinate q, has many common features with the field of classical quasi-automorphic forms for triangle groups. This includes functional equations, the corresponding Halphen equation and so on. However, note that the former field is of transcendental degree 3, whereas this new field is of transcendental degree 7. This gives a second modular interpretation of the periods of Calabi–Yau varieties. For more details on this topic, see [36].

7. Proofs

This section contains the proof of the theorems announced earlier.

7.1. Proof of Theorem 1

Fix any hyperbolic $\mathfrak{t} = (m_1, m_2, \infty) \neq (\infty, \infty, \infty)$ (the extreme case (∞, ∞, ∞) can be verified using case ∞^4 in the appendix or by recalling familiar facts from the Fuchsian group $\Gamma(2)$). The hypergeometric parameters $\tilde{a}, \tilde{b}, \tilde{c}$ are related to the angular ones $v_i = 1/m_i$ via

$$\tilde{a} = \tilde{b} = (1 - v_1 - v_2)/2, \ \tilde{c} = 1 - v_1. \tag{7.39}$$

Let us begin with the derivation of the fundamental domain and generators of $\Gamma_{\mathfrak{t}}$. Define the Schwarz function

$$\phi(z) = \mu \frac{u_2(z)}{u_1(z)} = \mu \frac{z^{1-\tilde{c}} F(\tilde{a} - \tilde{c} + 1, \tilde{b} - \tilde{c} + 1, 2 - \tilde{c}; z)}{F(\tilde{a}, \tilde{b}, \tilde{c}; z)}, \tag{7.40}$$

where u_i are the independent solutions (A.49) to the hypergeometric equation given in (A.46) and the scale factor μ is [22]

$$\mu = \frac{\sin(\pi\,(\tilde{c} - \tilde{a}))}{\sin(\pi\,\tilde{a})} \frac{\Gamma(\tilde{a} - \tilde{c} + 1)^2\,\Gamma(\tilde{c})}{\Gamma(\tilde{a})^2\,\Gamma(2 - \tilde{c})} \tag{7.41}$$

and is chosen to fit the target into the unit disc. Then $\phi(z)$ maps the upper hemisphere of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ biholomorphically onto the (open) hyperbolic triangle in the Poincaré unit disc with vertices $\phi(0) = 0, \phi(1) = \xi_2, \phi(\infty) = \mathrm{e}^{\pi i v_1} = -\zeta_1^{-1}$, where $\xi_2 = \sin(\pi\tilde{a})/\sin(\pi(\tilde{c} - \tilde{a}))$. These values are calculated directly from the data in Appendix A. We can extend ϕ to all of $\mathbb{CP}^1$ by reflecting in the real axis (so the triangle becomes a quadrilateral), and we can make ϕ into a multivalued function onto the full Poincaré disc by reflecting in the sides of that quadrilateral. The local expansion of ϕ about $z = 0$ of course is obtained from (A.45), while those about $z = 1$ and ∞ are obtained from the formulas in cases $\infty^0, \infty^1, \infty^2$ of Appendix A.

We can map the unit disc to the upper half-plane via

$$\tau(z) = \frac{\phi(z) + \zeta_1}{\zeta_1 \phi(z) + 1}. \tag{7.42}$$

It is easy to verify that $\tau(z)$ maps the unit disc to $\mathbb{H}$, and sends $z = 0, 1, \infty$ to $\zeta_1, \zeta_2, \mathrm{i}\infty$. This means the normalized Hauptmodul $J_{\mathfrak{t}}(\tau)$ is related to the inverse map $z(\tau)$ by $J_{\mathfrak{t}} = 1 - z$. The monodromy of (1.3) with parameters computed from (7.39) and (3.21), directly yields the action $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right).\phi = (\delta\phi + \gamma)/(\beta\phi + \alpha)$, which up to conjugation reduces to the action of $\Gamma_{\mathfrak{t}}$ on τ. The values of α_i (and h_3) can be computed from the $z = 1, 0, \infty$ asymptotics given in Appendix A, but were already computed in [51]. Equation (2.16) is simply the Schwarz equation (1.1) expressed in local coordinates.

7.2. Proof of Theorem 2

Now turn to Theorem 2. Write $\mathfrak{m}_k$ for the space of holomorphic automorphic forms of weight k.

The divisor $\mathrm{div}\, f$ of a meromorphic automorphic form f (f not identically 0) is defined to be the formal (and finite) sum $\sum \mathrm{ord}_{[z]}(f)\,[z]$ where $[z]$ denotes the orbit $\Gamma_{\mathfrak{t}} z$. The degree of $\mathrm{div}\, f$ for any such f of weight k for a triangle group of type (m_1, m_2, m_3) is (see Theorem 2.3.3 of [32] for a generalization)

$$\deg(\mathrm{div}\, f) = \frac{k}{2}\left(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right). \tag{7.43}$$

By the classical argument, $\dot{J}_{\mathfrak{t}}$ is an automorphic form for $\Gamma_{\mathfrak{t}}$ of weight 2, since $J_{\mathfrak{t}}$ has weight 0. Clearly, the only poles of $\dot{J}_{\mathfrak{t}}$ are at the points in $[\mathrm{i}\infty]$, where we have a simple pole. Also, $\dot{J}_{\mathfrak{t}}$ has zeros at any other cusp (with order ≥ 1) and at elliptic points ζ_i (with order $\geq 1 - 1/m_i$). That these orders are equalities, and that $\dot{J}_{\mathfrak{t}}$ has no other zeros, follows from the formula for the degree of the divisor.

It is manifest from the formula for f_k that is an automorphic form of weight k, holomorphic everywhere in $\mathbb{H}_{\mathfrak{t}}$ except possibly at $[\mathrm{i}\infty]$. Note that for automorphic forms f, g of fixed weight, the orders of f and g at any point will differ by an integer, and thus the order of f_k at each point $\notin [\mathrm{i}\infty]$ is the minimum possible for $f \in \mathfrak{m}_k$.

The quantity d_k equals the order of f_k at $\mathrm{i}\infty$. If $d_k \geq 0$ then for each $0 \leq l \leq d_k$, $f_k J_{\mathfrak{t}}^l$ is holomorphic at $\mathrm{i}\infty$ (hence lies in $\mathfrak{m}_k$). In this case, for any $g \in \mathfrak{m}_k$, g/f_k will be an automorphic function holomorphic everywhere in $\mathbb{H}_{\mathfrak{t}}$ except possibly at $\mathrm{i}\infty$. This means g/f_k must equal some polynomial in $J_{\mathfrak{t}}$ of degree $\leq d_k$. Thus, the $f_k J_{\mathfrak{t}}^l$ span $\mathfrak{m}_k$. On the other hand, if $d_k < 0$, then $\mathfrak{m}_k = 0$ (again, look at g/f_k for any $g \in \mathfrak{m}_k$).

Consider now the generators of the algebra of holomorphic modular forms. Type (∞, ∞, ∞) can be obtained by recalling what is known for $\Gamma(2)$. Suppose first that $m_1 < m_2 = \infty$. Choose any $k \geq 0$ and write $k = k' + lm_1$ for $0 \leq k' < m_1$ and $l \in \mathbb{Z}$. Note that $f_{2k'}\, f_{2m_1}^l$ has weight $2k$ and has order $1 - k'/m_1$ (the smallest possible in $\mathfrak{m}_{2k}$) at ζ_1. Then, given $f \in \mathfrak{m}_{2k}$, a constant c can be found so that $f - c\, f_{2k'}\, f_{2m_1}^j$ will have order ≥ 1 at ζ_1. Since f_2 has order $1 - 1/m_1, 0, 0$ at $\zeta_1, \zeta_2, \zeta_3$, respectively, $(f - c\, f_{2k'}\, f_{2m_1}^j)/f_2 \in \mathfrak{m}_{2k-2}$. Thus by induction, $f_2, \ldots, f_{2m_1}$ generate all of $\mathfrak{m}_{2k}$, for any k.

The proof for $m_2 < \infty$ is similar. Define $f_{2l}^{(1)} := f_{2l} J_{\mathfrak{t}}^{d_{2l}}$ (minimal possible order at ζ_1 and $\mathrm{i}\infty$, maximal at ζ_2, in $\mathfrak{m}_{2l}$). Choose any $f \in \mathfrak{m}_{2k}$ for $k \geq 0$, and write $k = k_i + l_i m_i$ for $i = 1, 2$ where $0 \leq k_i < m_i$ and $l_i \in \mathbb{Z}$. Then it is possible to find constants c_i so that $g := f - c_2\, (f_{2m_2})^{l_2}\, f_{2k_2} - c_1\, (f_{2m_1})^{l_1}\, f_{2k_1}^{(1)}$ has order ≥ 1 at both ζ_1, ζ_2. This means $g/f_4 \in \mathfrak{m}_{k-4}$, so the result follows by induction on k.

As defined, $\Delta_{\mathfrak{t}}$ is manifestly a weight $2L$ automorphic form with no zeros or poles anywhere except possibly at $[\mathrm{i}\infty]$. In fact, since $J_{\mathfrak{t}}$ is a Hauptmodul, the order of $J_{\mathfrak{t}}(\tau) - J_{\mathfrak{t}}(\zeta_i)$ at ζ_i equals 1, which gives us the formula for n_Δ. That value is proportional to the area of a fundamental domain of $\Gamma_{\mathfrak{t}}$ (see, e.g., [32]), and so is strictly positive. Hence $\Delta_{\mathfrak{t}}$ vanishes at $\mathrm{i}\infty$.

The statement about holomorphicity of $E_{2;\mathfrak{t}}$ is immediate from the properties of $\Delta_{\mathfrak{t}}$. The functional equation for $E_{2;\mathfrak{t}}$ follows directly from that of $\Delta_{\mathfrak{t}}$, and the vanishing of $E_{2;\mathfrak{t}}(\zeta_j)$ at cusps ζ_j is a consequence of $\Delta_{\mathfrak{t}}(\zeta_j)$ being finite and nonzero there.

7.3. Proof of Theorem 3

The only new part of Theorem 3 is (ii). Write $h = h_3$. Many of their properties can be easily determined from those of the hypergeometric functions collected in Appendix A. In particular, they are meromorphic functions in $\mathbb{H}$ with possible poles only at the $\Gamma_{\mathfrak{t}}$-orbits of ζ_2 and ζ_1. Now, each t_i is a function of $\hat{q}$, because $J_{\mathfrak{t}}$ is. Write $t_i = \sum_{n=0}^{\infty} t_{i,n} \hat{q}^n$. We see directly from (i) that, in vector form

$$[t_{1,0}, t_{2,0}, t_{3,0}] = [0, -2\pi\mathrm{i}/h, 0]$$

(these are normalized differently in Theorem 3). Comparing $\hat{q}^n$ coefficients, for $n \geq 1$, we get a recursion

$$(M - nI_{3\times 3})[t_{1,n}, t_{2,n}, t_{3,n}]^{\mathsf{tr}} \in \mathbb{Q}^3[a, b, c][t_{i,m}]_{1 \leq i \leq 3, 0 \leq m < n}, \tag{7.44}$$

where tr denotes transpose and

$$M := \begin{pmatrix} \frac{m_1m_2+m_2-m_1}{2m_1m_2} & 0 & -\frac{m_1m_2+m_2-m_1}{2m_1m_2} \\ \frac{m_1m_2+m_1+m_2}{2m_1m_2} & 0 & \frac{m_1m_2+m_1+m_2}{2m_1m_2} \\ -\frac{m_1m_2-m_2+m_1}{2m_1m_2} & 0 & \frac{m_1m_2-m_2+m_1}{2m_1m_2} \end{pmatrix}.$$

Note that

$$(M - I_{3\times 3})[t_{1,1}, t_{2,1}, t_{3,1}]^{\mathsf{tr}} = 0$$

and so up to a constant ν'

$$\begin{aligned}[t_{1,1}, t_{2,1}, t_{3,1}] = \nu' \big[&- m_1^2m_2^2 - m_2^2m_1 + m_2m_1^2, \\ &- m_2^2m_1 - m_2^2 + m_2m_1^2 + m_1^2,\ \ m_1^2m_2^2 - m_2^2m_1 + m_2m_1^2 \big]\end{aligned}$$

when $m_2 < \infty$, while

$$[t_{1,1}, t_{2,1}, t_{3,1}] = \begin{cases} \nu'\big[-m_1^2 - m_1,\ \ -m_1 - 1,\ \ m_1^2 - m_1\big] & \text{if } m_1 < \infty = m_2, \\ \nu'\big[-1,\ \ 0,\ \ 1\big] & \text{if } m_1 = m_2 = \infty. \end{cases}$$

(The rule is that the value of a polynomial $P(x)$ for $x = \infty$ is the coefficient of the monomial x^n of highest degree in $P(x)$.) We chose the constant ν' here so that these expressions are polynomial in m_2 and m_1. That $\nu' = \nu$ follows by computing the leading term of t_1.

Note that $\det(M - nI_{3\times 3}) = -n^2(n-1)$ so the nth coefficients of t_i are well-defined polynomials in m_j for $n > 1$. The factor of $2\pi\mathrm{i}/h$ and power of ν in (3.23) follows from easy inductions. In order to see (3.24), we write (1.2) in the variables $x_1 = (m_1m_2)^{-2}(t_1 - t_3)$, $x_2 = \kappa_2^{-1}(t_2 + 1)$ and $x_3 = \kappa_3^{-1}t_3$, and we get a similar recursion as in (7.44) with different M such that $\det(M - nI_{3\times 3}) \neq 0$ even for $m_2 = 0$ and $m_1 = 0$.

7.4. Proof of Theorem 4

That the t_i obey (3.25) follows from its expression in terms of hypergeometric functions in Theorem 3 and the analytic continuation of such functions in

Appendix A. automorphy of $J_{\mathfrak{t}}$. We obtain from Appendix A that, when $m_2 \neq \infty$, the zero and pole orders of the t_i's at the ζ_j's are given in the table below:

	ζ_1	ζ_2	ζ_3
$t_2 - t_1$	$m_1 - 1$	-1	0
$t_3 - t_2$	-1	$m_2 - 1$	0
$t_1 - t_3$	-1	-1	1
t_1	-1	-1	1
t_2	-1	-1	0
t_3	-1	-1	1

while if $m_1 < \infty = m_2$, the table becomes

	ζ_1	ζ_2	ζ_3
$t_2 - t_1$	$m_1 - 1$	0	0
$t_3 - t_2$	-1	1	0
$t_1 - t_3$	-1	0	1
t_1	-1	0	1
t_2	-1	1	0
t_3	-1	1	1

(Note however that the orders of zeros for quasi-automorphic forms like t_i are not constant along orbits.) This table makes it easy to verify the automorphic form identities given in Theorem 4(iii). For the identity involving $E_{2;\mathfrak{t}}, t_1, t_2, t_3$ we must further calculate the residues of t_i's at elliptic points ζ_i's. Theorem 4(iv) follows by similar pole order arguments as in the proof of Theorem 2 and the above tables.

Appendix A. Hypergeometric functions: basic formulas

In this appendix, we review some classical facts about the *Gauss hypergeometric function* (or *series*)

(A.45)
$$F(\tilde{a}, \tilde{b}, \tilde{c}; z) = {}_2F_1(\tilde{a}, \tilde{b}, \tilde{c}; z) = \sum_{n=0}^{\infty} \frac{(\tilde{a})_n (\tilde{b})_n}{(\tilde{c})_n n!} z^n, \quad \tilde{c} \notin \{0, -1, -2, -3, \ldots\},$$

where $(x)_n := x(x+1)(x+2)\cdots(x+n-1)$, and its differential equation

$$z(1-z)y'' + (\tilde{c} - (\tilde{a} + \tilde{b} + 1)z)y' - \tilde{a}\tilde{b}y = 0 \tag{A.46}$$

which is called the *hypergeometric* or *Gauss equation.* A very complete reference is [15], though it has typos. In the following and throughout this paper, $\Gamma(z)$ denotes the gamma function and the digamma $\psi(z)$ denotes its logarithmic derivative. The values of ψ at rational z (the only ones we need) were calculated by Gauss to be

$$\begin{aligned}\psi(m/n) = &-\gamma - \ln n - \frac{\pi}{2}\cot(\pi m/n) \\ &+ \sum_{k=1}^{n/2}{}' \cos(2\pi mk/n)\ln(2 - 2\cos(2\pi k/m)),\end{aligned} \tag{A.47}$$

where γ is Euler's constant and the prime means that for even n the last term (namely, $k = n/2$) should be divided by 2. Another identity is useful

$$\psi(1-x) = \psi(x) + \pi \cot \pi x. \tag{A.48}$$

The values $\tilde{a}, \tilde{b}, \tilde{c}$ of interest here are given at the beginning of Section 7.1 and (more generally) Appendix B. As long as $\tilde{c} \notin \mathbb{Z}$ (i.e., except for case ∞^3 below), the solution space to (A.46) is spanned by

$$u_1(z) = F(\tilde{a}, \tilde{b}, \tilde{c}; z), \quad u_2(z) = z^{1-\tilde{c}}F(\tilde{a} - \tilde{c} + 1, \tilde{b} - \tilde{c} + 1, 2 - \tilde{c}; z). \tag{A.49}$$

We need to understand what $u_i(z)$ looks like about $z = 1$ and ∞, in order to understand the local expansions of the automorphic forms of $\Gamma_{(m_1,m_2,m_3)}$ about all cusps and elliptic fixed-points. Closely related to this, we need to understand the monodromy of (A.46) in order to explicitly identify the automorphic forms associated to $\Gamma_{(m_1,m_2,m_3)}$ (it is easy to identify them up to a conjugate of $\Gamma_{(m_1,m_2,m_3)}$, but we want to pin down that conjugate). These formulas only depend on the number of cusps, i.e., the number of m_i which equal ∞. We will require here (without loss of generality) that $m_1 \leq m_2 \leq m_3 \leq \infty$.

Case ∞^0: No cusps, i.e., $m_3 < \infty$.

This corresponds to all of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{c}-\tilde{a}-\tilde{b}, \tilde{a}-\tilde{b}$ being nonintegral. Analytic continuation for $|\arg(1-z)| < \pi$ resp. $|\arg(-z)| < \pi$ is

$$
\begin{aligned}
u_1(z) &= \frac{\Gamma(\tilde{c})\Gamma(\tilde{c}-\tilde{a}-\tilde{b})}{\Gamma(\tilde{c}-\tilde{a})\Gamma(\tilde{c}-\tilde{b})} F(\tilde{a}, \tilde{b}, \tilde{a}+\tilde{b}-\tilde{c}+1; 1-z) \\
&\quad + \frac{\Gamma(\tilde{c})\Gamma(\tilde{a}+\tilde{b}-\tilde{c})}{\Gamma(\tilde{a})\Gamma(\tilde{b})} (1-z)^{\tilde{c}-\tilde{a}-\tilde{b}} F(\tilde{c}-\tilde{a}, \tilde{c}-\tilde{b}, \tilde{c}-\tilde{a}-\tilde{b}+1; 1-z) \\
&= \frac{\Gamma(\tilde{c})\Gamma(\tilde{b}-\tilde{a})}{\Gamma(\tilde{b})\Gamma(\tilde{c}-\tilde{a})} (-z)^{-\tilde{a}} F(\tilde{a}, 1-\tilde{c}+\tilde{a}, 1-\tilde{b}+\tilde{a}; z^{-1}) \\
&\quad + \frac{\Gamma(\tilde{c})\Gamma(\tilde{a}-\tilde{b})}{\Gamma(\tilde{a})\Gamma(\tilde{c}-\tilde{b})} (-z)^{-\tilde{b}} F(\tilde{b}, 1-\tilde{c}+\tilde{b}, 1-\tilde{a}+\tilde{b}; z^{-1}), \\
u_2(z) &= \frac{\Gamma(2-\tilde{c})\Gamma(\tilde{c}-\tilde{a}-\tilde{b})}{\Gamma(1-\tilde{a})\Gamma(1-\tilde{b})} F(\tilde{a}, \tilde{b}, \tilde{a}+\tilde{b}-\tilde{c}+1; 1-z) \\
&\quad + \frac{\Gamma(2-\tilde{c})\Gamma(\tilde{a}+\tilde{b}-\tilde{c})}{\Gamma(\tilde{a}-\tilde{c}+1)\Gamma(\tilde{b}-\tilde{c}+1)} (1-z)^{\tilde{c}-\tilde{a}-\tilde{b}} \\
&\quad \times F(\tilde{c}-\tilde{a}, \tilde{c}-\tilde{b}, \tilde{c}-\tilde{a}-\tilde{b}+1; 1-z) \\
&= -\mathrm{e}^{-\pi \mathrm{i}\tilde{c}} \frac{\Gamma(2-\tilde{c})\Gamma(\tilde{b}-\tilde{a})}{\Gamma(\tilde{b}-\tilde{c}+1)\Gamma(1-\tilde{a})} (-z)^{-\tilde{a}} F(\tilde{a}-\tilde{c}+1, \tilde{a}, 1-\tilde{b}+\tilde{a}; z^{-1}) \\
&\quad - \mathrm{e}^{-\pi \mathrm{i}\tilde{c}} \frac{\Gamma(2-\tilde{c})\Gamma(\tilde{a}-\tilde{b})}{\Gamma(\tilde{a}-\tilde{c}+1)\Gamma(1-\tilde{b})} (-z)^{-\tilde{b}} F(\tilde{b}-\tilde{c}+1, \tilde{b}, 1-\tilde{a}+\tilde{b}; z^{-1}).
\end{aligned}
$$

From this, we obtain the monodromy matrices (in terms of the basis u_1, u_2) for small counterclockwise circles about $z = 0$, 1 and ∞

$$
M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-2\pi \mathrm{i}\tilde{c}} \end{pmatrix}, \tag{A.50}
$$

$$
M_1 = \begin{pmatrix} \frac{\xi s(\tilde{a})\, s(\tilde{b}) - s(\tilde{c}-\tilde{a})\, s(\tilde{c}-\tilde{b})}{s(\tilde{c})\, s(\tilde{c}-\tilde{a}-\tilde{b})} & \frac{\pi\, (\xi-1)\Gamma(1-\tilde{c})\Gamma(2-\tilde{c})}{s(\tilde{c}-\tilde{a}-\tilde{b})\Gamma(1-\tilde{a})\Gamma(1-\tilde{b})\Gamma(\tilde{a}-\tilde{c}+1)\Gamma(\tilde{b}-\tilde{c}+1)} \\ \frac{\pi\, (\xi-1)\Gamma(\tilde{c}-1)\, \Gamma(\tilde{c})}{s(\tilde{c}-\tilde{a}-\tilde{b})\, \Gamma(\tilde{c}-\tilde{a})\, \Gamma(\tilde{c}-\tilde{b})\, \Gamma(\tilde{a})\, \Gamma(\tilde{b})} & \frac{s(\tilde{a})\, s(\tilde{b}) - \xi s(\tilde{c}-\tilde{a})\, s(\tilde{c}-\tilde{b})}{s(\tilde{c})\, s(\tilde{c}-\tilde{a}-\tilde{b})} \end{pmatrix} \tag{A.51}
$$

and $M_\infty = M_1^{-1} M_0^{-1}$, where here $\xi = \mathrm{e}^{\pi \mathrm{i}(\tilde{c}-\tilde{a}-\tilde{b})}$ and $s(x) = \sin(\pi x)$.

Case ∞^1: Exactly one cusp, i.e., $m_2 < m_3 = \infty$.

This means $\tilde{a} = \tilde{b}$, and all of $\tilde{a}, \tilde{c}, \tilde{c} - 2\tilde{a}$ are nonintegral. Analytic continuation to $z = 1$ is as in case ∞^0, but to $z = \infty$ is given by

$$\begin{aligned} u_1(z) = {} & \frac{(-z)^{-\tilde{a}}\Gamma(\tilde{c})}{\Gamma(\tilde{a})\Gamma(\tilde{c}-\tilde{a})} \sum_{n=0}^{\infty} \frac{(\tilde{a})_n (1-\tilde{c}+\tilde{a})_n}{n!n!} (\ln(-z) + 2\psi(1+n) - \psi(\tilde{a}+n) \\ & - \psi(\tilde{c}-\tilde{a}-n))z^{-n}, \\ u_2(z) = {} & \frac{-\mathrm{e}^{-\pi \mathrm{i}\tilde{c}}(-z)^{-\tilde{a}}\Gamma(2-\tilde{c})}{\Gamma(1-\tilde{a})\Gamma(\tilde{a}-\tilde{c}+1)} \sum_{n=0}^{\infty} \frac{(\tilde{a})_n (1-\tilde{c}+\tilde{a})_n}{n!n!} (\ln(-z) + 2\psi(1+n) \\ & - \psi(\tilde{a}-\tilde{c}+n+1) - \psi(1-\tilde{a}-n))z^{-n}. \end{aligned}$$

Monodromy is given by the same matrices as in case ∞^0.

Case ∞^2: Exactly two cusps, i.e., $m_1 < m_2 = m_3 = \infty$.

This means $\tilde{a} = \tilde{b}$ and $\tilde{c} = 2\tilde{a}$, and both of $\tilde{c}, \tilde{a}$ are nonintegral. Analytic continuation to $z = \infty$ is as in case ∞^1, but to $z = 1$ is given by

$$\begin{aligned} u_1(z) = {} & \frac{\Gamma(2\tilde{a})}{\Gamma(\tilde{a})\Gamma(\tilde{c})} \sum_{n=0}^{\infty} \frac{(\tilde{a})_n (\tilde{a})_n}{n!n!} (2\psi(n+1) - 2\psi(\tilde{a}+n) - \ln(1-z))\,(1-z)^n, \\ u_2(z) = {} & \frac{z^{1-2\tilde{a}}\Gamma(2-2\tilde{a})}{\Gamma(1-\tilde{a})\Gamma(1-\tilde{a})} \sum_{n=0}^{\infty} \frac{(1-\tilde{a})_n (1-\tilde{a})_n}{n!n!} (2\psi(n+1) - 2\psi(1-\tilde{a}) \\ & - \ln(1-z))\,(1-z)^n. \end{aligned}$$

Monodromy is again given by the same matrices as in case ∞^0.

Case ∞^3: Three cusps, i.e., $m_1 = m_2 = m_3 = \infty$.

Then $\tilde{a} = \tilde{b} = 1/2, \tilde{c} = 1$. Take $u_1(z) = F(1/2, 1/2, 1; z)$ and

$$\begin{aligned} u_2(z) &= \mathrm{i}F(1/2, 1/2, 1; 1-z) \\ &= \frac{\mathrm{i}}{\pi} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_n}{n!n!} (2\psi(1+n) - 2\psi(1/2+n) - \ln(z))z^n, \end{aligned}$$

where the second equality is valid for $-\pi < \arg(z) < \pi$. Analytic continuation of u_1 is as in case ∞^2, but for u_2 is given by

$$\begin{aligned} u_2(z) &= \mathrm{i}F(1/2, 1/2, 1; 1-z) \\ &= \frac{\mathrm{i}}{\pi} z^{-1/2} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_n}{n!n!} (2\psi(1+n) - 2\psi(1/2+n) - \ln(z^{-1}))z^{-n}. \end{aligned}$$

The monodromy is

$$M_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}. \tag{A.52}$$

Appendix B. Triangular groups without cusps

In this paper (and the applications we have in mind), we are interested in triangle groups with cusps, but the same calculations work (though are messier) when there are no cusps, i.e., when all m_i are finite. In this appendix, we sketch the changes.

Equation (2.16) becomes

$$-2\dddot{J}_{\mathfrak{t}}\dot{J}_{\mathfrak{t}} + 3\ddot{J}_{\mathfrak{t}}^2 - n_z^{-2}\dot{J}_{\mathfrak{t}}^2 = \dot{J}_{\mathfrak{t}}^4\left(\frac{1-v_2^2}{J_{\mathfrak{t}}^2} + \frac{1-v_1^2}{(J_{\mathfrak{t}}-1)^2} + \frac{v_1^2+v_2^2-v_3^2-1}{J_{\mathfrak{t}}(J_{\mathfrak{t}}-1)}\right). \tag{B.53}$$

For example,

$$\begin{aligned}
c_0 &= \frac{-1+\gamma_- + v_3^2}{2(v_3^2-1)}, \quad c_1 = \frac{(5-2\gamma_+ - 3\gamma_-^2) + (-6+2\gamma_+)v_3^2 + v_3^4}{16(v_3^2-1)(v_3^2-4)},\\
c_2 &= \frac{(-2\gamma_- + \gamma_+\gamma_- + \gamma_-^3) + (2\gamma_- - \gamma_+\gamma_-)v_3^2}{6(v_3^2-9)(v_3^2-1)^2},\\
c_3 &= \frac{-31+76\gamma_+ + 690\gamma_-^2 - 28\gamma_+^2 - 404\gamma_-^2\gamma_+ - 303\gamma_-^4}{128(v_3^2-16)(v_3^2-4)^2(v_3^2-1)^3}\\
&\quad + \frac{100-244\gamma_+ + 88\gamma_+^2 - 1052\gamma_-^2 + 660\gamma_-^2\gamma_+ + 192\gamma_-^4}{128(v_3^2-16)(v_3^2-4)^2(v_3^2-1)^3}v_3^2\\
&\quad + \frac{-114+276\gamma_+ - 96\gamma_+^2 + 390\gamma_-^2 - 288\gamma_-^2\gamma_+ - 24\gamma_-^4}{128(v_3^2-16)(v_3^2-4)^2(v_3^2-1)^3}v_3^4\\
&\quad + \frac{\begin{array}{c}(52-124\gamma_+ + 40\gamma_+^2 - 24\gamma_-^2 + 32\gamma_-^2\gamma_+)v_3^6\\ +(-7+16\gamma_+ - 4\gamma_+^2 - 4\gamma_-^2)v_3^8\end{array}}{128(v_3^2-16)(v_3^2-4)^2(v_3^2-1)^3},
\end{aligned}$$

where $\gamma_\pm = v_1^2 \pm v_2^2$.

The table in Section 7.4, listing the orders of zeros and poles for the solutions of the Halphen system, generalizes to:

	ζ_3	ζ_2	ζ_1
$t_2 - t_1$	-1	-1	$m_1 - 1$
$t_3 - t_2$	-1	$m_2 - 1$	-1
$t_1 - t_3$	$m_3 - 1$	-1	-1
t_1	-1	-1	-1
t_2	-1	-1	-1
t_3	-1	-1	-1

As before, a basis for the ring of automorphic forms consists of the monomials of the form

$$(t_1 - t_2)^p (t_2 - t_3)^q (t_3 - t_1)^r$$

and the pole condition on the vertices implies that $p, q, r \geq 1$. The ring of holomorphic automorphic forms for the hyperbolic triangle group $\Gamma_{(m_1,m_2,m_3)}$ with the condition $m_1 \leq m_2 \leq m_3 < \infty$ is generated by holomorphic functions

$$\begin{aligned} E^{(3)}_{p,q;\mathtt{t}} &= (t_1 - t_2)^p (t_2 - t_3)^q (t_3 - t_1), \quad k = p + q, \\ E^{(1)}_{q,r;\mathtt{t}} &= (t_1 - t_2)(t_2 - t_3)^q (t_3 - t_1)^r, \quad k = q + r, \\ E^{(2)}_{p,r;\mathtt{t}} &= (t_1 - t_2)^p (t_2 - t_3)(t_3 - t_1)^r, \quad k = p + r. \end{aligned}$$

This list of generators is finite because for example holomorphicity at ζ_3 for $E^{(3)}_{p,q;\mathtt{t}}$ implies that $p + q \leq m_3 - 1$ and similarly for the rest. The space of automorphic forms of weight $2k$ is of dimension $k + 1 - \lceil \frac{k}{m_1} \rceil - \lceil \frac{k}{m_2} \rceil - \lceil \frac{k}{m_3} \rceil$.

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