Quasi-modular forms attached to elliptic curves: Hecke operators

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Abstract

In this article we introduce Hecke operators on the differential algebra of geometric quasi-modular forms. As an application for each natural number \(d\) we construct a vector field in six dimensions which determines uniquely the polynomial relations between the Eisenstein series of weight 2, 4 and 6 and their transformation under multiplication of the argument by \(d\), and in particular, it determines uniquely the modular curve of degree \(d\) isogenies between elliptic curves.

1 Introduction

The theory of quasi-modular forms was first introduced by Kaneko and Zagier in [4] due to its applications in mathematical physics. In [6] and [7] we have described how one can introduce quasi-modular forms in the framework of the algebraic geometry of elliptic curves, and in particular, how the Ramanujan differential equation between Eisenstein series can be derived from the Gauss-Manin connection of families of elliptic curves. In the present article we proceed further our investigation and we introduce Hecke operators for quasi-modular forms and give some applications. However, the main motivation behind this work is to prepare the ground for similar topics in the case of Calabi-Yau varieties, see [8].

In [3] the authors describe a differential equation in the \(j\)-invariant of two elliptic curves which is tangent to all modular curves of degree \(d\) isogenies of elliptic curves. This differential equation can be derived from the Schwarzian differential equation of the \(j\)-function and the later can be calculated from the Ramanujan differential equation between Eisenstein series. This suggests that there must be a relation between Ramanujan differential equation and modular curves. In this article we also establish this relation.

Consider the Ramanujan ordinary differential equation

\[
\begin{align*}
\dot{s}_1 &= \frac{1}{12}(s_1^2 - s_2) \\
\dot{s}_2 &= \frac{1}{3}(s_1s_2 - s_3) \\
\dot{s}_3 &= \frac{1}{2}(s_1s_3 - s_2^2) \\
\dot{s}_k &= \frac{\partial s_k}{\partial \tau}
\end{align*}
\]

which is satisfied by Eisenstein series:

\[
s_i(\tau) = a_i E_{2i}(q) := a_i \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right),
\]

\(i = 1, 2, 3, \quad q = e^{2\pi i \tau}, \quad \text{Im}(\tau) > 0\)
Theorem 2. The vector field \( V \) is tangent to the affine variety \( V_d \).

I do not know the complete classification of all \( R_d \)-invariant algebraic subvarieties of \( \mathbb{A}^6_k \). We consider \( R_d \) as a differential operator:

\[
\begin{align*}
&k[t, s] \to k[t, s], \\
&f \mapsto R_d(f) := df(R_d).
\end{align*}
\]
From Theorem 2 and the fact that \(V_d\) is irreducible (see §11), it follows that:

\[ R_j^d(I_{d,i}) \in \text{Radical}(I_{d,1}, I_{d,2}, I_{d,3}), \quad i = 1, 2, 3, \quad j \in \mathbb{N} \cup \{0\}. \]

Note that the ideal \((I_{d,1}, I_{d,2}, I_{d,3}) \subset k[t, s]\) may not be radical. We can compute \(I_{d,i}\)'s using the \(q\)-expansion of Eisenstein series, see §11. This method works only for small degrees \(d\). However, for an arbitrary \(d\) we can introduce some elements in the radical of the ideal generated by \(I_{d,i}\), \(i = 1, 2, 3\). Let

\[
J_{d,i} = \det \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{m_d,i} \\
R_d(\alpha_1) & R_d(\alpha_2) & \cdots & R_d(\alpha_{m_d,i}) \\
\vdots & \vdots & \cdots & \vdots \\
R_d^{m_d,i-1}(\alpha_1) & R_d^{m_d,i-1}(\alpha_2) & \cdots & R_d^{m_d,i-1}(\alpha_{m_d,i})
\end{pmatrix},
\]

where for \(i = 1\), \(\alpha_j\)'s are the monomials:

\[
ti^a_1 s_1^a s_2^a s_3^a, \quad i \cdot \psi(d) = ia_0 + a_1 + 2a_2 + 3a_3, \quad a_0, a_1, a_2, a_3 \in \mathbb{N}_0
\]

and for \(i = 2, 3\), \(\alpha_j\)'s are the above monomials with \(a_1 = 0\). The polynomial \(J_{d,i}\) is weighted homogeneous of degree

\[
i \cdot \psi(d) + i \cdot \psi(d) + 1 + i \cdot \psi(d) + 2 + \cdots + i \cdot \psi(d) + m_{d,i} - 1 = m_{d,i} \cdot \frac{i \cdot \psi(d) + \frac{m_{d,i} \cdot (m_{d,i} - 1)}{2}}{2}.
\]

**Theorem 3.** We have

\[ J_{d,i} \in \text{Radical}(I_{d,1}, I_{d,2}, I_{d,3}), \quad i = 1, 2, 3. \]

and so

\[ J_{d,i}(d^{2i} \cdot s_i(d \cdot \tau), s_1(\tau), s_2(\tau), s_3(\tau)) = 0, \quad i = 1, 2, 3. \]

Throughout the text we will state our results over a field \(k\) of characteristic zero and not necessarily algebraically closed. Such results are valid if and only if the same results are valid over the algebraic closure \(\overline{k}\) of \(k\). By Lefschetz principle, see for instance [10] p.164, it is enough to prove such results over the complex numbers.

The article is organized in the following way. In §2 and §3 we recall the definition of full quasi-modular forms in the framework of both algebraic geometry and complex analysis. In §4 we describe some facts relating isogenies and algebraic de Rham cohomology of elliptic curves. Using isogeny of elliptic curves we introduce Hecke operators in §5. Theorem 1, Theorem 2 and Theorem 3 are respectively proved in §8, §9 and §10. Finally, in §11 we give some examples.

The main idea behind the proof of Theorem 3 is due to J. V. Pereira in [9]. Here, I would like to thank him for teaching me such an elegant and simple argument.

## 2 Geometric quasi-modular forms

In this section we recall some definitions and theorems in [5, 6]. The reader is also referred to [7] for a complete account of quasi-modular forms in a geometric context. Note that in [7] the \(t\) parameter is in fact \((\frac{1}{12} t_1, 12 \frac{1}{12} t_2, 8 \frac{1}{12} t_3)\). Let \(k\) be any field of characteristic zero and let \(E\) be an elliptic curve over \(k\). The first algebraic de Rham cohomology of \(E\), namely \(H^{1}_{dR}(E)\), is a \(k\)-vector space of dimension two and it has a one dimensional space \(F^1\) consisting of elements represented by regular differential 1-forms on \(E\).
Theorem 4. ([7], §5.5) The set $T(k) = \mathbb{A}^3_k \setminus \Delta$, where $\Delta := \{(t_1, t_2, t_3) \in \mathbb{A}^3_k \mid t_1^3 - t_2^3 = 0\}$, is the moduli of the pairs $(E, \omega)$, where $E$ is an elliptic curve and $\omega \in H^1_{\text{dR}}(E) \setminus F^1$. For $(t_1, t_2, t_3) \in T(k)$, the corresponding pair $(E, \omega)$ is given by

$$E : 3y^2 = (x - t_1)^3 - 3t_2(x - t_1) - 2t_3, \quad \omega = \frac{1}{12} \frac{xdx}{y}.$$  

From now on an element of $T(k)$ is denoted either by $(t_1, t_2, t_3)$ or $(E, \omega)$. We can regard $t_i$ as a rule which for any pair $(E, \omega)$ as above it associates an element $t_i = t_i(E, \omega) \in k$. We will also use $t_i$ as an indeterminate variable or an element in $k$, being clear from the text which we mean. A full quasi-modular form $f$ of weight $m$ and differential order $n$ is a homogeneous element in the $k$-algebra

$$M := k[t_1, t_2, t_3], \quad \text{weight}(t_i) = 2i, \ i = 1, 2, 3,$$

with $\text{deg}(f) = m$ and $\text{deg}_{t_i} f \leq n$. The set of such quasi-modular forms is denoted by $M_m^n$.

For a pair $(E, \omega) \in T(k)$ we have also a unique element $\alpha \in F^1$ satisfying $\langle \alpha, \omega \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the intersection form in the de Rham cohomology, see for instance [7] §2.10. For this reason we sometimes use $(E, \{\alpha, \omega\})$ instead of $(E, \omega)$. The algebraic groups $G_a := (k,+)$ and $G_m := (k - \{0\}, \cdot)$ act from the right on $T(k)$:

$$(E, \omega) \cdot k := (E, k\omega), \quad k \in G_m, \quad (E, \omega) \cdot k := (E, \omega + k\alpha), \quad k \in G_a$$

and so they act from the left on $M$. It can be shown that $M^n_m$ is invariant under these actions and the functions $t_i : T \to k, \ i = 1, 2, 3$ satisfy

$$k \cdot t_i = t_i + k, \quad k \cdot t_i = t_i, \quad k \in G_a, \quad k \cdot t_i = k^{-2i}t_i, \quad k \in G_m.$$

Let $R$ be the the Ramanujan vector field in $T$. It is the unique vector field in $T$ which satisfies $\nabla_R \alpha = -\omega$, $\nabla_R \omega = 0$, where $\nabla$ is the Gauss-Manin connection of the universal family of elliptic curves over $T$, see for instance [7] §2. The $k$-algebra of full quasi-modular forms has a differential structure which is given by:

$$M^n_m \to M^{n+1}_{m+2}, \quad t \mapsto R(t) := \sum_{i=1}^{3} \frac{\partial}{\partial t_i} R_{t_i},$$

where $R = \sum_{i=1}^{3} R_i \frac{\partial}{\partial t_i}$ is the Ramanujan vector field.

3 Holomorphic quasi-modular forms

Now, let us assume that $k = \mathbb{C}$. The period domain is defined to be

$$\mathcal{P} := \left\{ \left( \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right) \mid x_i \in \mathbb{C}, \ x_1x_4 - x_2x_3 = 1, \ \text{Im}(x_1x_3) > 0 \right\}.$$  

We let the group $\text{SL}(2, \mathbb{Z})$ act from the left on $\mathcal{P}$ by usual multiplication of matrices. In $\mathcal{P}$ we consider the vector field

$$X = -x_2 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_3}$$
which is invariant under the action of \( \text{SL}(2, \mathbb{Z}) \) and so it induces a vector field in the complex manifold \( \text{SL}(2, \mathbb{Z}) \setminus \mathcal{P} \). For simplicity we denote it again by \( X \). The Poincaré upper half plane \( \mathbb{H} \) is embedded in \( \mathcal{P} \) in the following way:

\[
\tau \mapsto \left( \begin{array}{cc} \tau & -1 \\ 1 & 0 \end{array} \right)
\]

and so we have a canonical map \( \mathbb{H} \to \text{SL}(2, \mathbb{Z}) \setminus \mathcal{P} \).

**Theorem 5.** ([7] §8.4 and §8.8) The period map

\[
\text{pm} : T(\mathbb{C}) \to \text{SL}(2, \mathbb{Z}) \setminus \mathcal{P}
\]

\[
t \mapsto \left[ \frac{1}{\sqrt{-2\pi i}} \left( \int_{\delta} \alpha \int_{\gamma} \omega \right) \right]
\]

is a biholomorphism, where \( \{\delta, \gamma\} \) is a basis of \( H_1(E, \mathbb{Z}) \) with \( \langle \delta, \gamma \rangle = -1 \). Under this biholomorphism the Ramanujan vector field is mapped to \( X \). The pull-back of \( t_i \) by the composition

\[
(7) \quad \mathbb{H} \to \text{SL}(2, \mathbb{Z}) \setminus \mathcal{P} \xrightarrow{\text{pm}^{-1}} T(\mathbb{C}) \hookrightarrow \mathbb{A}_\mathbb{C}^3,
\]

is the Eisenstein series \( a_i E_{2i}(e^{2\pi i \tau}) \) in (2).

The algebra of full holomorphic quasi-modular forms is the pull-back of \( k[t_1, t_2, t_3] \) under the composition (7). We can also introduce it in a classical way using functional equations plus growth conditions: a holomorphic function \( f \) on \( \mathbb{H} \) is called a (holomorphic) quasi-modular form of weight \( m \) and differential order \( n \) if the following two conditions are satisfied:

1. There are holomorphic functions \( f_i, i = 0, 1, \ldots, n \) on \( \mathbb{H} \) such that

\[
(8) \quad (cz + d)^{-m} f(Az) = \sum_{i=0}^{n} \binom{n}{i} c^i (cz + d)^{-i} f_i, \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).
\]

2. \( f_i, i = 0, 1, 2, \ldots, n \) have finite growths when \( \text{Im}(z) \) tends to +∞, i.e.

\[
\lim_{\text{Im}(z) \to +\infty} f_i(z) = a_i, \infty < \infty, \quad a_{i, \infty} \in \mathbb{C}.
\]

For the proof of the equivalence of both notions of quasi-modular forms see [7] §8.11. We have \( f_0 = f \) and the associated functions \( f_i \) are unique. In fact, \( f_i \) is a quasi-modular form of weight \( m - 2i \) and differential order \( n - i \) and with associated functions \( f_{ij} := f_{i+j} \). It is useful to define

\[
(9) \quad f|_m A := (\det A)^{m-1} \sum_{i=0}^{n} \binom{n}{i} \left( \frac{-c}{\det(A)} \right)^i (cz + d)^{-i} f_i(Az),
\]

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}), \quad f \in M^n_m.
\]
In this way, the equality (8) is written in the form

\[ \langle f^* \omega, f^* \alpha \rangle = d \cdot \langle \omega, \alpha \rangle. \]

Here, \( \langle \cdot, \cdot \rangle : H^1_{\text{dR}}(E) \times H^1_{\text{dR}}(E) \rightarrow k \) is the intersection form in the de Rham cohomology, see [7] §2.10.

\textbf{Proof.} It is enough to prove the proposition over algebraically closed field. Since the above formula is \( k \)-linear in both \( \omega \) and \( \alpha \), it is enough to prove it in the case \( \omega = \frac{dx}{y}, \alpha = \frac{xdx}{y} \), where \( x, y \) are the Weierstrass coordinates of \( E \). Since \( \langle \frac{dx}{y}, \frac{xdx}{y} \rangle = 1 \), we have to prove that \( \langle f^* \left( \frac{dx}{y} \right), f^* \left( \frac{xdx}{y} \right) \rangle = d \). Let \( f^{-1}(0_2) = \{p_1, p_2, \ldots, p_d \} \). The differential form \( f^* \left( \frac{dx}{y} \right) \) is again a regular differential form and \( f^* \left( \frac{xdx}{y} \right) \) has poles of order two at each \( p_i \). Consider the covering \( U = \{U_0, U_1 \} \) of \( E_1 \), where \( U_0 = E_1 \setminus f^{-1}(0_E) \) and \( U_1 \) is any other open set which contains \( f^{-1}(0_E) \). The differential forms \( f^* \left( \frac{dx}{y} \right), i = 0, 1 \) as elements in \( H^1_{\text{dR}}(E_1) \) are represented by the pairs

\[
\left( \frac{d\tilde{x}}{\tilde{y}}, \frac{\tilde{d}\tilde{x}}{\tilde{y}} \right), \left( \frac{\tilde{x}d\tilde{x}}{\tilde{y}}, \frac{\tilde{d}\tilde{x}}{\tilde{y}} - \frac{1}{2} \frac{d}{\tilde{x}}(\tilde{y}) \right),
\]

where \( \tilde{x} = f^* x, \tilde{y} = f^* y \). We have \( \frac{dx}{y} \cup \frac{\tilde{d}\tilde{x}}{\tilde{y}} = \{\omega_{01} \} \), where \( \omega_{01} = \frac{-1}{2} \frac{d\tilde{x}}{x} \) and so

\[
\langle f^* \left( \frac{dx}{y} \right), f^* \left( \frac{xdx}{y} \right) \rangle = \left( \frac{d\tilde{x}}{\tilde{y}}, \frac{\tilde{d}\tilde{x}}{\tilde{y}} \right) = \sum_{i=1}^{d} \Residue(\frac{-1}{2} \frac{d\tilde{x}}{x}, p_i) = \sum_{i=1}^{d} 1 = d.
\]

\( \square \)

\textbf{Proposition 2.} We have:
1. Let \( f : E_1 \to E_2 \) be an isogeny defined over \( k \). The induced map \( f^* : H^1_{\text{dR}}(E_2) \to H^1_{\text{dR}}(E_1) \) is an isomorphism.

2. Let \( [d]_E : E \to E \) be the multiplication by \( d \in \mathbb{N} \) map. We have \( [d]_E^* : H^1_{\text{dR}}(E) \to H^1_{\text{dR}}(E), \omega \mapsto d \cdot \omega \).

Proof. In the complex context, \( E = \mathbb{C}/(\tau,1) \) and \([d]_E \) is induced by \( \mathbb{C} \to \mathbb{C}, z \mapsto d \cdot z \). Moreover, a basis of the \( C^\infty \) de Rham cohomology is given by \( dz, d\bar{z} \). This proves the second part of the proposition. For the first part we take the dual isogeny and use the first part.

For \( E \) an elliptic curve over an algebraically closed field \( k \) of characteristic zero, the number of isogenies \( f : E_1 \to E \) of degree \( d \) is equal to \( \sigma_1(d) := \sum_{c|d} c \). To prove this we may work in the complex context and assume that \( E = \mathbb{C}/(\tau,1) \). The number of such isogenies is the number of subgroups of order \( d \) of \( (\mathbb{Z}/d\mathbb{Z})^2 \).

5 Geometric Hecke operators

In this section all the algebraic objects are defined over \( k \) unless it is mentioned explicitly. Let \( d \) be a positive integer. The Hecke operator \( T_d \) acts on the space of full quasi-modular forms as follow:

\[
T_d : M^n_m \to M^n_m,
\]

\[
T_d(t)(E,\omega) = \frac{1}{d} \sum_{f:E_1 \to E, \deg(f) = d} t(E_1, f^*\omega), \quad t \in M^n_m
\]

where the sum runs through all isogenies \( f : E_1 \to E \) of degree \( d \) defined over \( \overline{k} \). Since \((E,\omega)\) and \(t\) are defined over \( k\), \( T_d(t)(E,\omega) \) is invariant under \( \text{Gal}(\overline{k}/k) \) and so it is in the field \( k \). This implies that \( T_d(t) \) is defined over \( k \). The statement \( T_d \in k[t_1,t_2,t_3] \) is not at all clear. In order to prove this, we assume that \( k = \mathbb{C} \) and we prove the same statement for holomorphic quasi-modular forms, see §6. The functional equation of \( T_d \) with respect to the action of the algebraic groups \( \mathbb{G}_m \) and \( \mathbb{G}_a \) can be proved in the algebraic context as follow:

**Proposition 3.** The action of \( \mathbb{G}_m \) commutes with Hecke operators, that is,

\[
(11) \quad k \bullet T_d(t) = T_d(k \bullet t), \quad t \in M, \quad k \in \mathbb{G}_m
\]

and the action of \( \mathbb{G}_a \) satisfies:

\[
k \bullet T_d(t) = T_d((d \cdot k) \bullet t), \quad t \in M, \quad k \in \mathbb{G}_a.
\]

Proof. The first equality is trivial:

\[
(k \bullet T_d(t))(E,\omega) = T_d(t)(E,k\omega) = \frac{1}{d} \sum t(E_1,f^*(k\omega)) = \frac{1}{d} \sum (k \bullet t)(E_1,f^*(\omega)) = T_d(k \bullet t)(E,\omega)
\]

For the second equality we use Proposition 1:

\[
(k \bullet T_d(t))(E,\omega) = T_d(t)(E,\omega + k\alpha) = \frac{1}{d} \sum t(E_1,f^*(\omega + k\alpha)) = \frac{1}{d} \sum ((d \cdot k) \bullet t)(E_1,f^*(\omega)) = T_d(d \cdot k \bullet t)(E,\omega).
\]
We can also define the Hecke operators in the following way:

\[ T_d(t)(E, \omega) = d^{m-1} \sum_{g: E \to E_1, \deg(g) = d} t(E_1, g_* \omega), \quad t \in M^n_m \]

where the sum runs through all isogenies \( g: E \to E_1 \) of degree \( d \) defined over \( \overline{k} \). Both definitions of \( T_d(t) \) are equivalent: for an isogeny \( f: E_1 \to E \) of degree \( d \) defined over \( \overline{k} \) we have a unique dual isogeny \( g: E \to E_1 \) such that

\[ f \circ g = [d]_E, \quad g \circ f = [d]_{E_1}. \]

Therefore by Proposition 2 we have \( d \cdot g_* \omega = [d]_{E_1}^*(g_* \omega) = f^* \omega \) and so

\[ t(E_1, f^* \omega) = t(E_1, d \cdot g_* \omega) = d^m t(E_1, g_* \omega). \]

It can be shown that the geometric Eisenstein modular form \( G_k \) (see \([7] \S 6.5\)) is an eigenform with eigenvalue

\[ \sigma_{k-1}(d) := \sum_{c|d} c^{k-1} \]

for the Hecke operator \( T_d \), that is

\[ T_d G_k = \sigma_{k-1}(d) G_k, \quad d \in \mathbb{N}, \quad k \in 2\mathbb{N} \]

see for instance \([5]\).

The differential operator \( R: M \to M \) and the Hecke operator \( T_d \) commute, that is

\[ R \circ T_d = T_d \circ R, \quad \forall d \in \mathbb{N}. \]

For the proof we may assume that \( k = \mathbb{C} \). In this way using Theorem 5 it is enough to prove the same statement for holomorphic quasi-modular forms, see for instance \([6]\) Proposition 4.

## 6 Holomorphic Hecke operators

In this section we want to use the biholomorphism in Theorem 5 and describe the Hecke operators on holomorphic quasi-modular forms. Let us take \( k = \mathbb{C} \).

**Proposition 4.** The \( d \)-th Hecke operator on the vector space of quasi-modular forms of weight \( m \) and differential order \( n \) is given by

\[ T_d: M^n_m \to M^n_{m}, \quad T_d f = \sum_A f||_m A, \]

where \( A \) runs through the the set \( \text{SL}(2, \mathbb{Z}) \setminus \text{Mat}_d(2, \mathbb{Z}) \) and \( || \) is the double slash operator (9) for quasi-modular forms.
Proof. Let us consider two points \((E_i, \{\alpha_i, \omega_i\}), \ i = 1, 2\) in the moduli space \(T\). Let us also consider a \(d\)-isogeny \(f : E_1 \to E_2\) with
\[ f^*\omega_2 = \omega_1, \ f^*\alpha_2 = d \cdot \alpha_1. \]
We can take a symplectic basis \(\delta_1, \gamma_1\) of \(H_1(E_1, \mathbb{Z})\) and \(\delta_2, \gamma_2\) of \(H_1(E_2, \mathbb{Z})\) such that
\[ f^*[\delta_1, \gamma_1] = A[\delta_2, \gamma_2], \]
where \(A\) is an element in the set \(\text{Mat}_d(2, \mathbb{Z})\) of \(2 \times 2\) matrices with coefficients in \(\mathbb{Z}\) and with determinant \(d\). From another side we have
\[ [\alpha_1, \omega_1]B = f^*[\alpha_2, \omega_2], \ \text{where} \ B = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \]
Therefore, if the period matrix associated to \((E_i, \{\alpha_i, \omega_i\}, \{\delta_i, \gamma_i\}), i = 1, 2\) is denoted respectively by \(x'\) and \(x\) then
\[ x'B = Ax. \]
Using Theorem 5, the \(d\)-th Hecke operator acts on the space of \(\text{SL}(2, \mathbb{Z})\)-invariant holomorphic functions on \(\mathcal{P}\) by:
\[ T_d F(x) = \frac{1}{d} \sum_A F(AxB^{-1}), \ x \in \mathcal{P} \]
where \(A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\) runs through \(\text{SL}(2, \mathbb{Z})\setminus\text{Mat}_d(2, \mathbb{Z})\). Let \(f \in M^n_m\) be a holomorphic quasi-modular form defined on the upper half plane. By definition there is a geometric modular form \(\tilde{f} \in \mathbb{C}[t_1, t_2, t_3]\) such that \(f\) is the pull-back of \(\tilde{f}\) by the composition (7). Let \(F\) be the holomorphic function on \(\mathcal{P}\) obtained by the push-forward of \(\tilde{f}\) by the period map and then its pull-back by \(\mathcal{P} \to \text{SL}(2, \mathbb{Z})\setminus\mathcal{P}\). We have
\[
T_d f(\tau) = T_d F\left(\begin{array}{cc}
\tau & -1 \\
1 & 0
\end{array}\right)
\]
\[ = \frac{1}{d} \sum_A F(A \begin{pmatrix} \tau & -1 \\ 1 & 0 \end{pmatrix} B^{-1})
\]
\[ = \frac{1}{d} \sum_A F(\begin{pmatrix} A\tau & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d^{-1}(c_1\tau + d_1) & -c_1 \\ 0 & (c_1\tau + d_1)^{-1}d \end{pmatrix})
\]
\[ = \frac{1}{d} \sum_A d^n(c_1\tau + d_1)^{-m}(n_i)^i(-c_1)^i(c_1\tau + d_1)^i d^{-i} f_i(A(\tau))
\]
\[ = \sum_A f||mA
\]
One can take the representatives
\[
\{A_i\} := \left\{ \begin{pmatrix} d & b \\ 0 & c \end{pmatrix} \right| c \mid c, \ 0 \leq b < c \right\}
\]
for the quotient \( SL(2, \mathbb{Z}) \backslash \text{Mat}_d(2, \mathbb{Z}) \) and so

\[
T_d f(\tau) = \frac{1}{d} \sum_{ac=1, c \geq b < c} \sum_{0 \leq b < c} c^m f\left( \frac{c' \tau + b}{c} \right).
\]

In a similar way to the case of modular forms (see [1] §6) one can check that

\[
T_p \circ T_q = \sum_{d(p,q)} d^{m-1} T_{\frac{a}{d^2}}.
\]

If we write \( f = \sum_{n=0}^{\infty} f_n q^n \) then we have:

\[
(T_d f)_n = \sum_{c|(d,n)} c^{m-1} f_{\frac{ad}{c^2}}.
\]

In particular if we set \( n = 0 \) then the constant term of \( T_d(f) \) is \( f_0 \sigma_m(d-1) \). If \( f \) is the eigenvector of \( T_d \) and the constant term of \( f \) is non-zero then the corresponding eigenvalue is \( \sigma_m(d-1) \).

7 Refined Hecke operators

Let \( W_d \) be the set of subgroups of \( \frac{\mathbb{Z}}{d_1 \mathbb{Z}} \times \frac{\mathbb{Z}}{d_2 \mathbb{Z}} \) of order \( d \) and \( S_d \) be the set (up to isomorphism) of finite groups of order \( d \) and generated by at most two elements. We have a canonical surjective map \( W_d \to S_d \).

**Proposition 5.** We have bijections

\[
\begin{align*}
(13) \quad & \quad SL(2, \mathbb{Z}) \backslash \text{Mat}_d(2, \mathbb{Z}) \cong W_d, \\
(14) \quad & \quad SL(2, \mathbb{Z}) \backslash \text{Mat}_d(2, \mathbb{Z})/SL(2, \mathbb{Z}) \cong S_d,
\end{align*}
\]

both given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{\mathbb{Z}^2}{\mathbb{Z}(a, b) + \mathbb{Z}(c, d)}.
\]

If \( d \) is square-free then the both side of the second bijection are single points.

**Proof.** First note that the induced maps are both well-defined. The first bijection is already proved and used in §6. The action of \( SL(2, \mathbb{Z}) \) from the left on \( \text{Mat}_d(2, \mathbb{Z}) \) corresponds to the base change in the lattice \( \mathbb{Z}(a, b) + \mathbb{Z}(c, d) \) in the right hand side of the bijection. The action of \( SL(2, \mathbb{Z}) \) from the right on \( \text{Mat}_d(2, \mathbb{Z}) \) corresponds to the isomorphism of finite groups in the right hand side of the bijection.

Any element in \( S_d \) is isomorphic to the group \( \frac{\mathbb{Z}}{d_1 \mathbb{Z}} \times \frac{\mathbb{Z}}{d_2 \mathbb{Z}} \) for some \( d_1, d_2 \in \mathbb{N} \) with \( d = d_1^2 d_2 \). In the right hand side of (14) the corresponding element is represented by the matrix \( \begin{pmatrix} d_1 d_2 & 0 \\ 0 & d_2 \end{pmatrix} \). In the geometric context, this means that any isogeny of elliptic curves \( E_1 \to E_2 \) over an algebraically closed field can be decomposed into \( E_1 \overset{\alpha}{\to} E_1 \overset{\beta}{\to} E_2 \),
where \( \alpha \) is the multiplication by \( d_2 \) and \( \beta \) is a degree \( d_1 \) isogeny with cyclic center. Note that

\[
\sigma(d) := \sum_{d=d_2^2d_1} \psi(d_1).
\]

We conclude that we have a natural decomposition of the both geometric and holomorphic Hecke operators:

\[
T_d t = \sum_{d=d_2^2d_1} d_2^{-m-2} \cdot T_d^0 t, \quad t \in M^m
\]

where in the geometric context

\[
T_d^0 : M^m_m \to M^m_m, \quad T_d^0(t)(E, \omega) = \frac{1}{d} \sum_{f:E_1 \to E, \deg(f)=d, \ker(f) \text{ is cyclic}} t(E_1, f^*\omega),
\]

and in the holomorphic context

\[
T_d^0 : M^m_m \to M^m_m, \quad T_d^0 f = \sum_{A \in (\text{SL}(2,\mathbb{Z}) \setminus \text{Mat}_d(2,\mathbb{Z}))^0} f||_{mA}.
\]

Here, \((\text{SL}(2,\mathbb{Z}) \setminus \text{Mat}_d(2,\mathbb{Z}))^0\) is the fiber of the map

\[
\text{SL}(2,\mathbb{Z}) \setminus \text{Mat}_d(2,\mathbb{Z}) \to \text{SL}(2,\mathbb{Z}) \setminus \text{Mat}_d(2,\mathbb{Z})/\text{SL}(2,\mathbb{Z})
\]

over the matrix \( \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \). Note that in the geometric context using the Galois action \( \text{Gal} (\bar{k}/k) \), we can see that \( T_d^0 \) is defined over the field \( k \) and not its algebraic closure. We call \( T_d^0 \) the refined Hecke operator. The refined Hecke operator \( T_d^0 \) will be used in the next sections. Note that if \( d \) is square free then \( T_d^0 = T_d \).

8 Proof of Theorem 1

For a holomorphic quasi-modular form of weight \( m \) we associate the polynomial

\[
P^0_f(x) := \prod_{A \in (\text{SL}(2,\mathbb{Z}) \setminus \text{Mat}_d(2,\mathbb{Z}))^0} (x - d \cdot f||_{mA}) = \sum_{j=0}^{\psi(d)} P^0_{f,j} x^j.
\]

**Proposition 6.** \( P^0_{f,j} \) is a full quasi-modular form of weight \((\psi(d) - j) \cdot m\).

**Proof.** The coefficient \( P^0_{f,j} \) of \( x^j \) is a homogeneous polynomial with rational coefficients and of degree \( \psi(d) - j \) in

\[
T_d^0(f^k), \quad k = 1, 2, \ldots, \psi(d) - j, \quad \text{weight}(T_d^0(f^k)) = k,
\]

where \( T_d^0 : M \to M \) is the refined \( d \)-th Hecke operator defined in §7. For instance, the coefficient of \( x^{\psi(d) - 1} \) is \( -d \cdot T_d^0 f \) and the coefficient of \( x^{\psi(d) - 2} \) is \( \frac{d}{2}((T_d^0 f)^2 - T_d^0 f^2) \). Now the assertion follows from the fact that the Hecke operator \( T_d^0 \) sends a quasi-modular form of weight \( m \) to a quasi-modular form of weight \( m \). \( \square \)
Using the fact that the algebra of quasi-modular forms over $\mathbb{Q}$ is isomorphic to $\mathbb{Q}[E_2, E_4, E_6]$ we conclude that $P_f^0(x)$ is a homogeneous polynomial of degree $\psi(d) \cdot m$ in the ring

$$\mathbb{Q}[x, E_2, E_4, E_6], \text{ weight}(E_{2k}) = 2k, \ k = 1, 2, 3, \ \text{ weight}(x) = m.$$ 

The geometric definition of the polynomial $P_f^0(x)$ is:

$$P_f^0(x)(E, \omega) = \prod (x - f(E_1, g^* \omega))$$

where the product is taken over all degree $d$ isogenies $g : E_1 \to E$ with cyclic kernel.

**Proof of Theorem 1.** Let us regard $s_i$'s as holomorphic functions on the upper half plane and $t_i$'s as variables. We define

$$I_{d,i} = P_{s_i}^0(t_i), \ i = 1, 2, 3.$$ 

For $i = 2, 3$, $T^2_i s_i^{k}$ is a homogeneous polynomial of degree $ki$ in $\mathbb{Q}[s_2, s_3]$, weight($s_i$) = $2i$ and for $i = 1$, $T^2_i s_i^{k}$ is a homogeneous polynomial of degree $ki$ in $\mathbb{Q}[s_1, s_2, s_3]$, weight($s_1$) = $2i$. We conclude that $I_{d,i}(t_1, s_1, s_2, s_3)$ is a homogeneous polynomial of degree $i \cdot \psi(d)$ in the weighted ring (3) and for $i = 2, 3$ it does not depend on $t_1$. By definition it is monic in $t_i$. We have

$$s_i ||_{2i} \left( \begin{array}{cc} d & 0 \\ 0 & 1 \end{array} \right) = d^{2i-1} t_i(d \tau)$$

and so $I_{d,i}(d^{2i} \cdot t_i(d \cdot \tau), s_1(\tau), s_2(\tau), s_3(\tau)) = 0.$

9 **Proof of Theorem 2**

Since the period map $\text{pm}$ is a biholomorphism, it is enough to prove the same statement for the push-forward of $R$ and $V_d$ under the product of two period maps:

$$\text{pm} \times \text{pm} : T \times T \to \text{SL}(2, \mathbb{Z}) \backslash \mathcal{P} \times \text{SL}(2, \mathbb{Z}) \backslash \mathcal{P}.$$ 

First we describe the push-forward of $V_d$. Using Theorem 4 and the comparison of Hecke operators in both algebraic and complex context, we have:

$$V_d \cap (T \times T) = \{(E_1, \omega_1), (E_2, \omega_2) \in T \times T \mid \exists f : E_1 \to E_2, \ f^* \omega_2 = \omega_1, \ \ker(f) \text{ is cyclic of order } d\}.$$ 

Let us now consider the elliptic curves $E_i, \ i = 1, 2$ as complex curves. For a $d$-isogeny $f : E_1 \to E_2$ such that $\ker(f)$ is cyclic, we can take a symplectic basis $\delta_1, \gamma_1$ of $H_1(E_1, \mathbb{Z})$ and $\delta_2, \gamma_2$ of $H_1(E_2, \mathbb{Z})$ such that

$$f_* \delta_1 = d \cdot \delta_2, \ f_* \gamma_1 = \gamma_2.$$ 

Therefore, if the period matrix associated to $(E_i, \{\alpha_i, \omega_i\}, \{\delta_i, \gamma_i\})$, $i = 1, 2$ is denoted respectively by $x$ and $y$ then

$$x = \pi_d(y) := \left( \begin{array}{cc} y_1 & dy_2 \\ d^{-1}y_3 & y_4 \end{array} \right).$$
Therefore, the push-forward of $V_d$ under $\text{pm} \times \text{pm}$ and then its pull-back to $\mathcal{P} \times \mathcal{P}$ is given by:

$$V_d^* = \{(\pi_d(y), y) \mid y \in \mathcal{P}\}.$$ 

The push-forward of the vector field $R_d$ by $\text{pm} \times \text{pm}$ and then its pull-back in $\mathcal{P} \times \mathcal{P}$ is given by the vector field

$$R^* = d(y_2 \frac{\partial}{\partial y_1} + y_4 \frac{\partial}{\partial y_3}) - (x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3})$$

where we have used the coordinates $(x, y)$ for $\mathcal{P} \times \mathcal{P}$. Now, it can be easily shown that the above vector field $R^*$ is tangent to $V_d^*$.

**Remark 1.** The locus $V_d^*$ contains the one dimensional locus:

$$(16) \tilde{H} := \{(\left(\frac{\tau}{d}, -\frac{1}{d}\right), \left(\frac{\tau}{1}, -1\right)) \mid \tau \in \mathbb{H}\}.$$ 

Note also that the push-forward of the Ramanujan vector field $R$ is tangent to the image of $\mathbb{H} \to \mathcal{P}$ and restricted to this locus it is $\frac{\partial}{\partial \tau}$. Therefore, $R^*$ is tangent to the locus $\tilde{H}$ and restricted to there is again $\frac{\partial}{\partial \tau}$.

### 10 Proof of Theorem 3

From Theorem 1 it follows that $I_{d,1}$ is a linear combination of the monomials (4) and $I_{d,i}, i = 2, 3$ is a linear combination of the same monomials with $a_1 = 0$. The proof is a slight modification of an argument in holomorphic foliations, see [9]. We prove that $J_{d,i}$’s restricted to $V_d$ are identically zero. We know that $I_{d,i}$ is a linear combination of $\alpha_j$’s with $\mathbb{C}$ (in fact $\mathbb{Q}$) coefficients:

$$I_{d,i} = \sum c_j \alpha_j.$$ 

Since $R_d$ is tangent to the variety $V_d$, we conclude that $R_d^*(\sum c_j \alpha_j) = \sum c_j R_d^* \alpha_j$ restricted to $V_d$ is zero. This in turn implies that the matrix used in the definition of $J_{d,i}$ restricted to $V_d$ has non-zero kernel and so its determinant restricted to $V_d$ is zero.

### 11 Examples and final remarks

In order to calculate $I_{d,i}$’s using Theorem 3 we can proceed as follow: We use the Gröbner basis algorithm and find the irreducible components of the affine variety given by the ideal $\langle J_{d,1}, J_{d,2}, J_{d,3} \rangle$ and among them identify the variety $V_d$. In practice this algorithm fails even for the simplest case $d = 2$. In this case we have $\deg(J_{2,1}) = 42$, $\deg(J_{2,2}) = 40$, $\deg(J_{2,3}) = 69$ and calculating the Gröbner basis of the ideal $\langle J_{d,1}, J_{d,2}, J_{d,3} \rangle$ is a huge amount of computations. We use the $q$-expansion of $t_i$’s and we calculate $I_{d,i}, i = 1, 2, 3, d = 2, 3$. We have written powers of $t_i$ in the first row and the corresponding coefficients in the second row. For more examples see the author’s web-page.

$$I_{2,1} : \begin{pmatrix} t_1^3 & t_2^2 & t_1 & 1 \\ 1 & -6s_1 & 12s_1^2 - 3s_2 & -8s_1^3 + 6s_1s_2 - 2s_3 \end{pmatrix}$$

$$I_{2,2} : \begin{pmatrix} t_1^2 & t_2^2 & t_2 & 1 \\ 1 & -18s_2 & 33s_2^2 & 484s_3^2 - 500s_3 \end{pmatrix}$$

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We consider the polynomial \( I_{d,i} : i = 2, 3 \) is monic in \( t_i \) and it has integral coefficients. This follows from the formula (15) and the fact that geometric modular forms can be defined over a field of characteristic \( p \). Our computations shows that \( I_{d,1} \) is also monic in \( t_1 \) and it is with integral coefficients. However, the notion of a geometric quasi-modular form is only elaborated over field of characteristic zero (see [7]). The proof of such a statement for \( I_{d,1} \) would need a reformulation of the definition of geometric quasi-modular forms.

For \( j = 0, 1, \ldots, \psi(d) \), the coefficients of \( t_j \) in \( I_{d,i}, i = 2, 3 \) is homogeneous of degree \( i \cdot \psi(d) - i \cdot j \) in \( s_2 \) and \( s_3 \) with weight\( s_2 = 2 \) and weight\( s_3 = 3 \). From this it follows that \( I_{d,i} \) can be written in a unique way as a polynomial in \( t_i, s_i \) and \( s_3^3 - s_2^2 \), say it \( I_{d,i} = P_{d,i}(t_i, s_i, s_3^3 - s_2^2) \). It follows that

\[
\langle I_{d,1}, I_{d,2}, I_{d,3}, s_2^3 - s_3^2 \rangle = \langle I_{d,1}, P_{d,2}(t_2, s_2, 0), P_{d,3}(t_2, s_2, 0), s_2^3 - s_3^2 \rangle
\]

and so the irreducible components of the variety \( I_{d,1} = I_{d,2} = I_{d,3} = s_2^3 - s_3^2 = 0 \) are given by

\[
(17) \quad t_2 - a_2 s_2 = t_3 - a_3 s_3 = s_2^3 - s_3^2 = I_{d,1} = 0,
\]

where \( a_i, i = 2, 3 \) is a root of the polynomial \( P_{d,i}(t_i, 1, 0) \). We have also

\[
\langle I_{d,1}, I_{d,2}, I_{d,3}, t_2^3 - t_3^2 \rangle = \langle I_{d,1}, I_{d,2}, I_{d,3}, t_2^3 - t_3^2, B(s_2, s_3) \rangle,
\]

where \( B(s_2, s_3) \) is the resultant of the polynomials \( I_{d,2}(t^3, s_2, s_3) \) and \( I_{d,3}(t^2, s_2, s_3) \) with respect to the variable \( t \). From the definition of the resultant it follows that \( B \) is in the ideal generated by \( I_{d,i}, i = 2, 3 \) and \( t_2^3 - t_3^2 \). Finally, we conclude that the variety \( V_d \cap \{ s_2^3 - s_3^2 = 0 \} \) has many irreducible components of dimension 2 and given by (17) and \( V_d \cap \{ t_2^3 - t_3^2 = 0 \} \) is an irreducible of dimension 2. By the arguments in §9 we know that \( V_d \cap (T \times T) \) is irreducible and so \( V_d \) is also irreducible.

References


