Foliated Neighborhoods of Exceptional Submanifolds

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ABSTRACT. The present article deals with the classification of neighborhoods of negatively embedded submanifolds A of a complex manifold X. The main tools we use are one-dimensional foliations whose set of singularities is A and which are normally attracting at A. The linearization of these foliations is provided under general cohomological conditions. As a consequence, an extension of the classical embedding theorem of Grauert is obtained.

1. INTRODUCTION

In this paper, we consider a complex compact projective manifold A of dimension n negatively embedded in an $(n+m+1)$-dimensional complex manifold X. We denote by (X, A) the germ of the neighborhood of A in X. Our purpose is to establish a linearization theorem for one-dimensional foliations on (X, A) whose set of singularities is A and which are normally attracting at A. On the other hand, we will use these foliations as a tool for the classification of neighborhoods of negatively embedded compact submanifolds of a complex manifold, extending a classical theorem of Grauert [6]. More precisely, a one-dimensional foliation on (X, A) is defined by a collection of nontrivial local vector fields V_i defined on open subsets $U_i \subset X$, $i \in I$, which are part of a covering $(U_i)_{i\in I}$ of A, in such a way that, for each nonempty intersection $U_i \cap U_j \neq \varnothing$, we have $V_i = f_{ij}V_j$ with $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. The foliation is singular at A, if $V_i|_{A\cap U_i} = 0$, for each $i \in I$.

Let $\mathcal{F}_1$ be a complex one-dimensional foliation on (X, A), singular at A. We say that $\mathcal{F}_1$ is *normally attracting at* A if, for each $i \in I$, the linear part of V_i at each $p \in U_i \cap A$, $DV_i(p)$, is a linear operator whose action splits into two invariant subspaces $T_pX = T_pA + N_p$, and if $DV_i(p)|N_p$ has eigenvalues $\{\lambda_1, \dots, \lambda_{m+1}\} \subset \mathbb{C}$ whose convex hull does not contain $0 \in \mathbb{C}$. Clearly, this concept depends only on the foliation and not on the local vector fields. The linear part is defined by local expressions $DV_i(p), p \in A \cap U_i$, and $DV_i(p) = f_{ij}(p)DV_j(p)$ (whenever $p \in U_i \cap U_j$) on the normal bundle N of rank $m+1$ over A. The key question for the classification of these foliations is: Under which conditions is $\mathcal{F}_1$ holomorphically equivalent to its linear part? The case in which A is a point is classical. A *resonance* among the eigenvalues $\{\lambda_1, \dots, \lambda_{m+1}\} \subset \mathbb{C}$ is a relation of the kind $\lambda_i = \sum m_j \lambda_j$ where $m_j \geq 0$ and $\sum m_j \geq 2$. The theorem of Poincaré (see, e.g., [9] or [4]) states that if $0 \in \mathbb{C}^{m+1}$ is an attracting singularity of $\mathcal{F}_1$ and there are no resonances among the eigenvalues of the linear part of $\mathcal{F}_1$ at $0 \in \mathbb{C}^{m+1}$, then there is an analytic change of coordinates around $0 \in \mathbb{C}^{m+1}$ taking $\mathcal{F}_1$ to its linear part. This theorem can be extended in the presence of resonances to show the existence of a holomorphic change of coordinates taking $\mathcal{F}_1$ to a polynomial foliation in normal form, and involving only the terms in resonance (see, e.g., [9]). We consider the case in which A is exceptional in X, that is, in which there exist an analytic variety X' and a proper surjective holomorphic map $\Phi : X \to X'$ such that the following hold:

(1) $\varphi(A) = \{p\}$ is a single point;
(2) $\varphi : X - A \to X' - \{p\}$ is an analytic isomorphism;
(3) For small neighborhoods U' and U of p and A, respectively, $\mathcal{O}_{X'}(U') \to \mathcal{O}_X(U)$ is an isomorphism, where $O_X(U)$ is the ring of holomorphic functions in U.

We also say that A can be blown down to a point, or that it is *contractible* or *negatively embedded*. A vector bundle $V \to A$ over a complex manifold A is called negative (in the sense of Grauert) if its zero section is an exceptional variety in V. Naturally, $V \to A$ is called positive if its dual is negative. Let X be a smooth variety, and let A be a smooth subvariety. We say that the germ (X, A) is *strongly exceptional* if it is exceptional and the normal bundle of A in X is negative. The following gives us a generalization of Poincaré's theorem to the global situation, that is, when A is not a point.

Theorem 1.1. *Let $\mathcal{F}_1$ be a normally attracting one-dimensional foliation in a germ of strongly exceptional manifold (X, A). Assume that there are no resonances among the eigenvalues of the linear part of $\mathcal{F}_1$ along the normal direction of A. If*

$$H^1(A, N^{-\nu}) = 0, \ \nu = 1, 2, 3, \dots ,$$

then there is a biholomorphic map $(X, A) \to (N, A)$, where N is the normal bundle of A in X, which is a conjugacy between $\mathcal{F}_1$ and its linear part in (N, A).

For a vector bundle N on A, and $\mu \in \mathbb{N}$, we write N^μ to denote the symmetric μ-th power of N. Theorem 1.1 generalizes the linearization theorem proved in [2] where A is a one-dimensional compact curve embedded in a complex surface.

Of particular importance is the case where the germ of $\mathcal{F}_1$ at a point $p \in A$ is a *radial singularity* at p, that is, where all the normal eigenvalues of the linear part of $\mathcal{F}_1$ are equal—which means that, after a blow up normal to A, the lifted foliation of $\mathcal{F}_1$ becomes a transverse foliation to the blow-up divisor. We call $\mathcal{F}_1$ a *radial foliation*. In order to state our next results, we need the following cohomological conditions:

(I) Vanishing of cohomologies for arbitrary codimension of A on X:

$$H^1(A, N^{-\nu}) = 0 \quad \text{and} \quad H^1(A, TA \otimes N^{-\nu}) = 0, \ \nu = 1, 2, \ldots$$

(II) If the codimension of A in X is greater than one, then we have

$$H^2(A, \mathcal{O}_A) = 0, \quad \text{and} \quad H^1(A, N \otimes N^{-\nu}) = 0, \ \nu = 1, 2, \ldots .$$

The following theorem gives cohomological conditions for the existence of radial foliations.

Theorem 1.2. *Let (X, A) be a germ of strongly exceptional manifold satisfying the cohomological conditions (I) and (II). Then, there exists a germ of radial foliation in (X, A).*

The embedding theorem of Grauert [6] states that, under cohomological condition (I) on a codimension-one strongly exceptional embedding, there is a neighborhood of $A \subset X$ which is biholomorphically equivalent to a neighborhood of the zero section in the normal bundle N to A in X. Combining Theorem 1.1 and Theorem 1.2, we obtain the following generalization to any codimension of the embedding theorem of Grauert in [6].

Theorem 1.3. *Let (X, A) be a germ of strongly exceptional manifold satisfying cohomological conditions (I) and (II). Then, the germ of embedding of A in X is biholomorphic to the germ of embedding of A in N.*

As an example, let us restrict our focus to the case in which A is a Riemann surface and N is a direct sum of $m+1$ line bundles $N = L_1 \oplus L_2 \oplus \cdots \oplus L_{m+1}$. In this case, the Serre duality implies that cohomological condition (I) is equivalent to saying that $\Omega^1 \otimes N^\nu$ and $\Omega^1 \otimes \Omega^1 \otimes N^\nu$ have no global sections, where Ω^1 is the cotangent bundle of A. We have

$$N^\nu = \oplus_{i_1 + i_2 + \cdots + i_{m+1} = \nu,\ i_j \geq 0}\ L_1^{i_1} \otimes L_2^{i_2} \otimes \cdots \otimes L_{m+1}^{i_{m+1}},$$

and so (I) together with the strongly exceptional property follows from

$$c(L_i) < 0, \ c(L_i) < 4 - 4g, \ i = 1, 2, \ldots, m+1.$$

In a similar way, condition (II) is equivalent to saying that $A \cong \mathbb{P}^1$, and

$$|c(L_i) - c(L_j)| \leq 1, \ i, j = 1, 2, \ldots, m+1.$$

In this case, the decomposition of the normal bundle is automatic, and it is called the Birkhoff theorem. From this, we obtain as a corollary the following result of Laufer [8].

Corollary 1.4. *If $\mathbb{P}^1 \subset X$ is strongly exceptional, and if*

$$c(L_i) < 0, \ |c(L_i) - c(L_j)| \leq 1, \ i, j = 1, 2, \dots m + 1,$$

where the L_i are line bundles which appear in the decomposition of the normal bundle of A in X, then the germ $(X, \mathbb{P}^1)$ is biholomorphic to the germ $(N, \mathbb{P}^1)$.

In the case in which the codimension of A in X is greater than one, condition (II) seems to be necessary for our theorem. It imposes conditions on the submanifold A itself apart from negativity conditions on the normal bundle N. It would be of interest to show that, for instance, the Grauert theorem does not hold for Riemann surfaces of genus greater than zero and codimension greater than one. On the other hand, we may relax the negativity condition and ask for counter examples. Arnold in [1] constructs an elliptic curve embedded in a two-dimensional complex manifold and with zero self-intersection, such that Grauert's linearization theorem fails. Other counter examples to the linearization problem in the case of codimension-one embeddings can be found in [2].

The existence of a one-dimensional foliation $\mathcal{F}_1$, singular at A and normally attracting at A, implies, by the invariant manifold theorem [6], the existence of a regular foliation $\mathcal{F}_2$ in (X, A), transverse to A, whose leaves have dimension $m + 1$ and are invariant by $\mathcal{F}_1$. We call the pair $(\mathcal{F}_1, \mathcal{F}_2)$ a *bifoliation*. Reciprocally, we will establish in the proof of Theorem 4 cohomological conditions under which there exists a normally attracting foliation $\mathcal{F}_1$ tangent to a given $(m + 1)$-dimensional foliation transverse to A. This will give the following refinement of the theorem of Grauert.

Theorem 1.5. *Let $\mathcal{F}_2$ be a transverse regular foliation of dimension $m + 1$ in a germ of strongly exceptional manifold (X, A). Assume that (I) and (II) hold. Then, there is a biholomorphic map $(X, A) \to (N, A)$, where N is the normal bundle of A in X, which conjugates $\mathcal{F}_2$ with the foliation in (N, A) given by the fibers of N.*

The paper is organized as follows. In Section 2, we review some facts about exceptional varieties. In Section 3, we prove the key Proposition of the present text, which establishes cohomological conditions under which the restriction of line bundles from X to A is injective. The blow-up process along A reduces our problems in an arbitrary codimension to the codimension-one case. This is explained in Section 4. Then, Section 5 is dedicated to the proof of Theorem 1.1, while in Sections 6 and 7, we prove Theorem 1.2. Finally, in section 8, we prove Theorem 1.5.

2. GRAUERT'S VANISHING THEOREM

Let A be a complex compact manifold, and N be a negative line bundle on A. This is equivalent to saying that N^{-1} is a positive line bundle in the sense of Kodaira. The Kodaira vanishing theorem says that, for any coherent sheaf S on A, there is $\nu_0 \in \mathbb{N}$ such that

$$H^{\mu}(A, S \otimes N^{-\nu}) = 0, \ \nu \geq \nu_0, \ \mu = 1, 2, \dots . \tag{2.1}$$

Let us now be given a subvariety A of a variety X. Let $\mathcal{M}$ be the sheaf of holomorphic functions in (X, A) which vanish at A, and let S be a coherent sheaf in (X, A). For $\nu \in \mathbb{N}$, the sheaf $S(\nu) := S\mathcal{M}^{\nu}/S\mathcal{M}^{\nu+1}$ is a coherent sheaf with support A, and in fact,

$$S(\nu) \cong \tilde{S} \otimes N^{-\nu},$$

where $\tilde{S} = S(0)$ is the structural restriction of S to A. If there is no danger of confusion, we will also use S to denote $\tilde{S}$, since it is clear from the text which one we mean.

Theorem 2.1 (Grauert [6]***, Satz 2, p. 357).*** *Given a strongly exceptional submanifold A of a manifold X, there exists a positive integer ν_0 such that*

$$H^{\mu}(U, S\mathcal{M}^{\nu}) = 0, \;\; \mu \geq 1, \;\; \nu \geq \nu_0,$$

where U is a small, strongly pseudoconvex neighborhood of A in X. Moreover, ν_0 in the above theorem can be taken to be smaller than the same number ν_0 in (2.1).

3. Restriction of Line Bundles

First, we consider the case in which A is a hypersurface in X.

Proposition 3.1. *Let A be a strongly exceptional complex manifold of dimension n embedded in a manifold X of dimension $n+1$. Moreover, suppose that*

$$H^1(A, N^{-\nu}) = 0, \;\; \nu = 1, 2, 3, \dots ,$$

where N is the normal bundle of the embedding, and N^{-1} is the dual bundle. The restriction map

$$r : H^1(U, \mathcal{O}_U^*) \to H^1(A, \mathcal{O}_A^*)$$

is injective for some small neighborhood U of A in X.

Proof. The submanifold A is strongly exceptional in X, and so by Theorem 2.1 applied to $S = \mathcal{O}_X$, we have $H^1(U, \mathcal{M}) = 0$, where U is a strongly pseudoconvex neighborhood of A in X. The diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{M} & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O}_X & \to & \mathcal{O}_X^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O}_A & \to & \mathcal{O}_A^* & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{3.1}$$

gives us

$$
\begin{array}{ccccccc}
 & & H^1(U,\mathcal{M})=0 & & & & \\
 & & \downarrow & & & & \\
H^1(U,\mathbb{Z}) & \to & H^1(U,\mathcal{O}_X) & \to & H^1(U,\mathcal{O}_X^*) & \to & H^2(U,\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(A,\mathbb{Z}) & \to & H^1(A,\mathcal{O}_A) & \to & H^1(A,\mathcal{O}_A^*) & \to & H^2(A,\mathbb{Z})
\end{array}
\quad . \tag{3.2}
$$

By considering a smaller neighborhood U, if necessary, we can assume that A is a retraction of U, and so the maps induced in the homologies by the inclusion $A \hookrightarrow U$ are all isomorphisms. In particular, the first and fourth vertical morphisms in the above diagram are isomorphisms. In the argument we consider now, we do not mention the name of mappings, since it is clear from the above diagram which mapping we mean.

Let us consider $x_1 \in H^1(U,\mathcal{O}_X^*)$ which is mapped to the trivial bundle in $H^1(A,\mathcal{O}_A^*)$. Since the fourth vertical map is an isomorphism, x_1 maps to zero in $H^2(U,\mathbb{Z})$. This means that there is a $x_2 \in H^1(U,\mathcal{O}_X)$ which maps to x_1. Let x_3 be the image of x_2 in $H^1(A,\mathcal{O}_A)$. Since the above diagram is commutative, x_3 maps to the trivial bundle in $H^1(A,\mathcal{O}_A^*)$. Therefore, there exists an x_4 in $H^1(A,\mathbb{Z})$ which maps to x_3. Since the first vertical map is an isomorphism and the second vertical map is injective, we conclude that $x_4 \in H^1(U,\mathbb{Z}) \cong H^1(A,\mathbb{Z})$ maps to x_2, and so x_2 maps to $x_1 = 0$ in $H^1(U,\mathcal{O}_X^*)$. □

Now, we give some applications of Proposition 3.1. We assume that (X,A) has a transverse foliation, namely, $\mathcal{F}$. The normal bundle N of A in X has a meromorphic global section, namely, s. Let

$$\mathrm{div}(s) = \sum n_i D_i,\ n_i \in \mathbb{Z}.$$

We define the divisor D in X as follows:

$$D = A - \sum n_i \tilde{D}_i, \tag{3.3}$$

where $\tilde{D}_i$ is the saturation of D_i by $\mathcal{F}$. The line bundle L_D associated with D restricted to A is the trivial line bundle, because $N \cong \mathcal{O}_X(A)\,|_A$; thus, by Proposition 3.1, L_D is trivial. Equivalently, we have the following proposition.

Proposition 3.2. *Under the hypothesis of Proposition 3.1, there exists a meromorphic function g on (X,A) with $\mathrm{div}(g) = D$, where D is given by (3.3).*

We now give an application of Proposition 3.2.

Theorem 3.3. *Let A be a strongly exceptional codimension-one submanifold of X. Further, assume that*

$$H^1(A, N^{-\nu}) = 0,\ \ \forall \nu = 1,2,\ldots\ . \tag{3.4}$$

Any transverse holomorphic foliation in (X, A) is biholomorphic to the canonical transverse foliation of (N, A) by the fibers of N. In particular, the germs of any two holomorphic transverse foliations in (X, A) are equivalent.

For the case in which A is a Riemann surface, the theorem is proved in [2].

Proof. Let $\mathcal{F}$ be the germ of a transverse foliation in (X, A), and N the normal bundle of A in X. Let also $\mathcal{F}'$ be the canonical transverse foliation of (N, A), and let g (respectively, g') be the meromorphic function constructed in Proposition 3.2 for the pair (X, A) (respectively, (N, A)). We claim that at each point $a \in A$ there exists a unique biholomorphism

$$\psi_a : (X, A, a) \to (N, A, a)$$

with the following properties:

(1) ψ induces the identity map on A;
(2) ψ sends $\mathcal{F}$ to $\mathcal{F}'$;
(3) The pullback of g' by ψ is g.

The uniqueness property implies that these local biholomorphisms are restrictions of a global biholomorphism $\psi : (X, A) \to (N, A)$ which sends $\mathcal{F}$ to $\mathcal{F}'$.

We now prove our claim. Fix a coordinate system $x = (x_1, x_2, \cdots, x_n)$ in a neighborhood of a in A. We extend x to a coordinate system (x, x_{n+1}) of a neighborhood of a in X such that A (respectively, $\mathcal{F}$) in these coordinates is given by $x_{n+1} = 0$ (respectively, $dx_i = 0,\ i = 1, 2, \ldots, n$). We can write

$$g(x, x_{n+1}) = Q(x) x_{n+1} f(x, x_{n+1}),$$

where $Q(x)$ is a meromorphic function in a neighborhood of a in A. We can take Q and the coordinate system x independent of the choice of an embedding of A. Here, f is a holomorphic function in (X, a) without zeros. By changing the coordinates in x_{n+1}, we can assume that $f = 1$. Now, the coordinate system (x, x_{n+1}) such that $g = Q(x) x_{n+1}$ is unique, and it gives us the local biholomorphism ψ_a. □

4. Blow-up Along a Submanifold

Let N be a vector bundle of rank $m + 1$ over A, and let $\tilde{A} := \mathbb{P}(N)$ be the projectivization of the fibers of N. We have a canonical projection map $\pi : \tilde{A} \to A$ with fibers isomorphic to $\mathbb{P}^m$. The space $\tilde{A}$ carries a distinguished line bundle $\tilde{N}$ which is defined by

$$\tilde{N}_x = \text{ the line representing } x \text{ in the vector space } N_{\pi(x)}, \quad x \in \tilde{A}.$$

In some books, the notation $\mathcal{O}_{\tilde{A}}(-1)$ is used to denote the sheaf of sections of $\tilde{N}$, because the line bundle $\tilde{N}$ is the tautological bundle restricted to the fibers of π.

It has the following properties:

$$\pi_*(\mathcal{O}(\tilde{N}^{-\nu})) \cong \mathcal{O}(N^{-\nu}),\ \nu = 0, 1, 2, \dots$$

$$\pi_*(\mathcal{O}(\tilde{N}^{\nu})) = 0,\ \ \nu = 1, 2, \dots$$

$$H^q(\tilde{A}, \pi^*(S) \otimes \mathcal{O}(\tilde{N}^{-\nu})) \cong H^q(A, S \otimes \mathcal{O}(N^{-\nu})),\ \nu = 1, 2, \dots \tag{4.1}$$

for every locally free sheaf S on A (see [5], p. 178). Here, $\mathcal{O}$ of a bundle means the sheaf of its sections. When there is no ambiguity between a bundle and the sheaf of its sections, we do not write $\mathcal{O}$. We also use the following: if, for a sheaf of abelian groups S on $\tilde{A}$, we have $R^i\pi_*(S) = 0$ for all $i = 1, 2, \dots$, then

$$H^i(\tilde{A}, S) \cong H^i(A, \pi_* S),\ i = 0, 1, 2, \dots .$$

We will apply this for the sheaf of sections of $T\mathbb{P}^m \otimes \tilde{N}^{-\nu},\ \nu = 1, 2, \dots$, where $T\mathbb{P}^m$ is the subbundle of $T\tilde{A}$ corresponding to vectors tangent to the fibers of π.

By definition, $\tilde{N}$ is a subbundle of $\pi^* N$, and we have the short exact sequence

$$0 \to \tilde{N} \to \pi^* N \to T\mathbb{P}^m \to 0. \tag{4.2}$$

We then take $\mathcal{O}$ of the above sequence, make a tensor product with $\mathcal{O}(\tilde{N}^{-\nu}),\ \nu = 1, 2, \dots$, and apply π_*; we get

$$0 \to N^{-\nu+1} \to N \otimes N^{-\nu} \to \pi_*(T\mathbb{P}^m \otimes \tilde{N}^{-\nu}) \to 0 \tag{4.3}$$

(for simplicity we have not written $\mathcal{O}(\cdots)$). Note that $R^1\pi_*\mathcal{O}(\tilde{N}^{-\nu+1}) = 0$, for $\nu = 1, 2, \dots$. Note also that if N is not a line bundle, then $N \otimes N^{-1}$ may not be the trivial bundle.

The vector bundle $T\mathbb{P}^m$ appears also in the short exact sequence

$$0 \to \mathcal{O}(T\mathbb{P}^m) \to \mathcal{O}(T\tilde{A}) \to \pi^*\mathcal{O}(TA) \to 0, \tag{4.4}$$

where $\mathcal{O}(T\tilde{A}) \to \pi^*\mathcal{O}(TA)$ is the map obtained by derivation of $\tilde{A} \to A$, and by considering the pull-back of $\mathcal{O}(TA)$.

Let A be a compact submanifold of X with

$$n = \dim(A),\ m + 1 = \dim(X) - n,$$

and let $N = TX\mid_A / TA$ be the normal bundle of A in X. We make the blow-up of X along A:

$$\pi : \tilde{X} \to X,\ \tilde{A} := \pi^{-1}(A) = \mathbb{P}(N).$$

The normal bundle of $\tilde{A}$ in $\tilde{X}$ is, in fact,

$$\tilde{N} = N_{\tilde{X}/\tilde{A}} \cong \mathcal{O}_{\tilde{A}}(-1).$$

We will need all these facts, as well as the following proposition, in the next sections.

Proposition 4.1. *Let A be a strongly exceptional complex submanifold of X. Moreover, suppose that $H^1(A, N^{-\nu}) = 0$, $\nu = 1, 2, 3, \ldots$, where N is the normal bundle of the embedding, and N^{-1} is the dual bundle. The restriction map*

$$r : H^1(U, \mathcal{O}_U^*) \to H^1(A, \mathcal{O}_A^*)$$

is injective for a small strongly pseudoconvex neighborhood U of A in X.

Proof. Take U to be any strongly pseudoconvex neighborhood of A which can be contracted topologically to A. The proposition then follows from Proposition 3.1 and from the isomorphism (4.1). Note that the map $H^1(U, \mathcal{O}_U^*) \to H^1(\pi^{-1}(U), \mathcal{O}_{\pi^{-1}(U)}^*)$ is injective. Let L be a line bundle in U such that $\pi^* L$ is trivial in $\pi^{-1}(U)$ and so that there is a global section f of $\pi^* L$ which vanishes nowhere. For local nowhere vanishing sections f_i of L, we have the holomorphic functions $g_i := \frac{f}{\pi^* f_i}$, which are non-zero constants along the fibers of $\tilde{A} \to A$. Therefore, the g_i come from holomorphic nowhere-vanishing functions in U, namely, $\tilde{g}_i$. Multiplying these $\tilde{g}_i$ by the f_i, we get the trivialization of L. □

5. Proof of Theorem 1.1

First, we prove that there is a holomorphic vector field V on (X, A) tangent to the foliation $\mathcal{F}_1$ and singular at A. Indeed, by our hypothesis, such a vector field exists locally. Thus, there is a finite covering $(U_i)_{i \in I}$ of (X, A); moreover, for each $i \in I$, there is a vector field V_i on U_i such that, at any $p \in A \cap U_i$, $DV_i(p)$ has n eigenvalues equal to zero (along the direction of A) and eigenvalues $\{\lambda_1, \ldots, \lambda_{m+1}\}$ whose convex hull does not contain $0 \in \mathbf{C}$. On each nonempty intersection $U_i \cap U_j \neq \emptyset$, we have $V_i = f_{ij} V_j$, where the cocycle $L = \{f_{ij}\}$ is a line bundle. We write the linear part of $V_i = f_{ij} V_j$, and conclude that $f_{ij}|_A = 1$. This means that the restriction of L to A is the trivial bundle. The collection of vector fields V_i, $i \in I$, defines a global section of $TX \otimes L$, and by Proposition 3, L is a trivial bundle.

On the other hand, if V and $\tilde{V}$ are vector fields tangent to $\mathcal{F}_1$ on (X, A) and to its linear part $\tilde{\mathcal{F}}_1$ on (N, A), respectively, by the Poincaré linearization theorem (see, e.g., [3]), we know that locally there exists a unique biholomorphism $f_p : (X, A, p) \to (N, A, p)$ conjugating V to $\tilde{V}$. Since the f_p are unique, we conclude that they coincide in their common domains of definition; hence, they give us a biholomorphism $f : (X, A) \to (N, A)$ conjugating V to $\tilde{V}$.

6. PROOF OF THEOREM 1.2, CODIMENSION ONE

In this section, A is a codimension-one submanifold of X, N is the normal bundle of A in X, and TA is the tangent bundle of A.

Proposition 6.1. *Assume that*

$$H^1(A, N^{-1} \otimes TA) = 0. \tag{6.1}$$

Then, the pair $(TA \subset TX|_A)$ *is split, that is,* $TX|_A \cong N \oplus TA$.

Proof. It is enough to construct a vector bundle morphism $Y : N \to TX|_A$ with the image transverse to TA. First, we construct Y locally; that is, we find $Y_i : N|_{U_i} \to TX|_{U_i}$ with the desired property for an open covering U_i, $i \in I$ of A. Let $\tilde{Y}_i$ be the composition $N|_{U_i} \to TX|_{U_i} \to N|_{U_i}$. Then, $\tilde{Y}_i = a_{ij}\tilde{Y}_j$, where $\{a_{ij}\} \in H^1(A, \mathcal{O}_A^*)$ is a line bundle. Now, the $\tilde{Y}_i$ are sections of the trivial bundle $N^{-1} \otimes N$ with no zeros; thus, $\{a_{ij}\}$ is a trivial bundle, and so we can assume that $\tilde{Y}_i = \tilde{Y}_j$. Now,

$$\{Y_{ij}\} := \{Y_i - Y_j\} \in H^1(A, \mathrm{Hom}(N, TA)).$$

Since $\mathrm{Hom}(N, TA) \cong N^{-1} \otimes TA$, our assertion follows by the vanishing hypothesis (6.1). □

If A is a curve, we then can use the Serre duality, and thus the cohomological condition (6.1) follows from the fact that $A \cdot A < 4 - 4g$. Let $\mathcal{F}$ be a non-singular transverse foliation by curves in (X, A). We have the canonical embedding

$$T\mathcal{F}|_A \cong N \hookrightarrow TX|_A.$$

In Proposition 6.1 we constructed a transverse embedding $N \to TX|_A$, and it is natural to ask whether it comes from a holomorphic foliation as above.

Proposition 6.2. *Assume that A is a strongly exceptional codimension-one submanifold of X, and that*

$$H^1(A, N^{-\nu} \otimes TX\,|_A) = 0, \quad \nu = 2, 3, \dots . \tag{6.2}$$

Any transverse embedding $N \to TX|_A$ is associated with a non-singular transverse foliation $\mathcal{F}$ defined in a neighborhood of A.

Proof. We take local sections of N which trivialize N and have no zero point. The images of these sections under $N \subset TX|_A$ can be extended to vector fields X_i defined in $U_i, i \in I$, where $\{U_i\}_{i \in I}$ is a covering of (X, A). Therefore,

$$X_i|_A = f_{ij}X_j|_A, \; N^{-1} = \{f_{ij}\}.$$

The normal bundle N of A in X extends to a line bundle $\tilde{N}$ in (X, A) as follows. We take local holomorphic functions f_i in (X, A) such that $A = \{f_i = 0\}$. Now, $f_i = \tilde{f}_{ij} f_j$, and $\tilde{N} = \{\tilde{f}_{ij}\}$ is a line bundle in (X, A) which, being restricted to A, is the normal bundle. Now,

$$\{\Theta_{ij}\} = \{X_i - \tilde{f}_{ij} X_j\} \in H^1(X, \mathcal{M}_A \otimes TX \otimes N^{-1}).$$

By our hypothesis and Theorem 2.1, the cohomology group on the right-hand side is zero. □

Using the long exact sequence of

$$0 \to TA \otimes N^{-\nu} \to TX|_A \otimes N^{-\nu} \to N^{-\nu+1} \to 0,$$

one can see easily that the hypothesis (6.2) together with (6.1) follows from

$$H^1(A, N^{-\nu} \otimes TA) = 0, \;\; H^1(A, N^{-\nu}) = 0, \;\; \nu = 1, 2, \dots . \tag{6.3}$$

For the case in which A is a Riemann surface, we use Serre duality, and (6.3) follows from

$$A \cdot A < 4 - 4g \text{ for } g \geq 1 \text{ and } A \cdot A < 2 \text{ for } g = 0.$$

In this case, Propositions 6.1 and 6.2 and their generalization to foliations with tangencies were proved in [10].

7. Proof of Theorem 1.2, Codimension Greater than One

In this section, we perform blow-up along A. Recall the notation introduced in Section 4. We would like to construct a transverse holomorphic foliation in $(\tilde{X}, \tilde{A})$. This is already done in the previous section. We need the cohomological conditions

$$H^1(\tilde{A}, \tilde{N}^{-\nu} \otimes T\tilde{A}) = 0, \;\; H^1(\tilde{A}, \tilde{N}^{-\nu}) = 0, \;\; \nu = 1, 2, \dots . \tag{7.1}$$

Now, we would like to translate all these in terms of the data of the embedding $A \subset X$. First, note that

$$H^1(\tilde{A}, \tilde{N}^{-\nu}) \cong H^1(A, N^{-\nu}).$$

We make the tensor product of the sequence (4.4) with $\tilde{N}^{\nu}$, and write the long exact cohomology sequence. We conclude that if

$$H^1(\tilde{A}, T\mathbb{P}^m \otimes \tilde{N}^{-\nu}) = 0, \; H^1(A, TA \otimes N^{-\nu}) = 0, \;\; \nu = 1, 2, \dots ,$$

then

$$H^1(\tilde{A}, T\tilde{A} \otimes \tilde{N}^{-\nu}) = 0, \quad \nu = 1, 2, \dots .$$

Since $R^1\pi_*(T\mathbb{P}^m \otimes \tilde{N}^{-\nu}) = 0$, $\nu = 1, 2, \dots$, we have

$$H^1(\tilde{A}, T\mathbb{P}^m \otimes \tilde{N}^{-\nu}) = H^1(A, \pi_*(T\mathbb{P}^m \otimes \tilde{N}^{-\nu})).$$

We write the long exact sequence of (4.3), and conclude that if

$$H^1(A, N \otimes N^{-\nu}) = 0, \; H^2(A, N^{-\nu+1}) = 0, \quad \nu = 1, 2, \dots ,$$

then

$$H^1(\tilde{A}, T\mathbb{P}^m \otimes \tilde{N}^{-\nu}) = 0, \quad \nu = 1, 2, \dots .$$

Finally, we conclude that if

$$H^1(A, N \otimes N^{-\nu}) = 0, \; H^2(A, N^{-\nu+1}) = 0, \; H^1(A, TA \otimes N^{-\nu}) = 0, \; \nu = 1, 2, \dots ,$$

then

$$H^1(\tilde{A}, T\tilde{A} \otimes \tilde{N}^{-\nu}) = 0, \; \nu = 1, 2, \dots .$$

8. Proof of Theorem 1.5

Using Theorem 1.1, it is enough to construct a second foliation $\mathcal{F}_1$ such that $(\mathcal{F}_1, \mathcal{F}_2)$ is a germ of radial bifoliation. In codimension one, we have $\mathcal{F}_1 = \mathcal{F}_2$, and so we can assume that $m > 0$. After performing a blow-up along A, our problem is reduced to the following.

We let $\tilde{A}$ be a codimension-one submanifold of $\tilde{X}$, and we also let $\tilde{\mathcal{F}}_2$ be an $(m+1)$-dimensional regular foliation in X transverse to A. The transversality implies that $\tilde{\mathcal{F}}_2 \cap \tilde{A}$ is a regular foliation of dimension m in $\tilde{A}$. In fact, it is the foliation by the blow-up divisors $\mathbb{P}^m$, and its tangent bundle is denoted by $T\mathbb{P}^m$ in Section 4. We would like to construct a transverse to $\tilde{A}$ foliation $\tilde{\mathcal{F}}_1$ of dimension one such that its leaves are contained in the leaves of $\tilde{\mathcal{F}}_2$. The proof is a slight modification of Proposition 6.1 and Proposition 6.2. In both propositions, $TX|_A$ is replaced with $T\tilde{\mathcal{F}}_2|_{\tilde{A}}$, and TA is replaced with $T\mathbb{P}^m$. In Proposition 6.1, the cohomological condition is

$$H^1(\tilde{A}, \tilde{N}^{-1} \otimes T\mathbb{P}^m) = 0,$$

which follows from condition (II).

REFERENCES

[1] V. I. ARNOLD, *Bifurcations of invariant manifolds of differential equations, and normal forms of neighborhoods of elliptic curves*, Functional Anal. Appl. **10** (1976), no. 4, 1–12. MR0431285 (55 #4286)

[2] C. CAMACHO, H. MOVASTI, and P. SAD, *Fibered neighborhoods of curves in surfaces*, J. Geom. Anal. **13** (2003), no. 1, 55–66. http://dx.doi.org/10.1007/BF02930996. MR1967036 (2003m:32029)

[3] C. CAMACHO and P. SAD, *Pontos Singulares de Equações Diferenciais Analíticas*, Colóquios Brasileiros de Matemática, 1987. MR953780 (90a:58126)

[4] D. CERVEAU, *Singular points of differential equations: on a theorem of Poincaré. The scientific legacy of Poincaré*, Hist. Math. **36** (2010), 99–112. MR2647621 (2011k:37033)

[5] H. GRAUERT, Th. PETERNELL, and R. REMMERT (eds.), *Several complex variables. VII*, Encyclopaedia of Mathematical Sciences, vol. 74, Springer-Verlag, Berlin, 1994. Sheaf-theoretical methods in complex analysis, A reprint of *Current problems in mathematics. Fundamental directions. Vol. 74* (Russian), Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, http://dx.doi.org/10.1007/978-3-662-09873-8. MR1326617 (96k:32001)

[6] H. GRAUERT, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368. MR0137127 (25 #583)

[7] M. W. HIRSCH, C. C. PUGH, and M. SHUB, *Invariant manifolds*, Lecture Notes in Mathematics, vol. 583, Springer-Verlag, Berlin, 1977. MR0501173 (58 #18595)

[8] H. B. LAUFER, *On* $\mathbf{C}P^1$ *as an exceptional set*, Ann. of Math. Stud., vol. 100, Princeton Univ. Press, Princeton, N.J., 1981. MR627762 (82m:32012)

[9] Y. ILYASHENKO and S. YAKOVENKO, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, vol. 86, American Mathematical Society, Providence, RI, 2008. MR2363178 (2009b:34001)

[10] H. MOVASATI and P. SAD, *Embedded curves and foliations*, Publ. Mat. **55** (2011), 401–411. http://dx.doi.org/10.5565/PUBLMAT_55211_07. MR2839449 (2012g:32045)

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