Some aspects of abelian integrals

Hossein Movasati
IMPA, Instituto de Matemática Pura e Aplicada, Brazil
www.impa.br/~hossein/

My research in mathematics turns around abelian integrals which I am going to explain with a simple example. Let us take the polynomial

\[ f = y^2 - x^3 + 3x \]

in two variables \( x \) and \( y \) and the family of elliptic curves \( E_t : \{ f - t = 0 \} \), \( t \in \mathbb{C} \). Only for \( t = -2, 2 \) the curve \( E_t \) is singular. For \( t \) a real number between 2 and \(-2\) the level surface \( f^{-1}(t) \subset \mathbb{R}^2 \) contains an oval \( \delta_t \). The polynomial \( f \) is a first integral of the differential equation

\[
\mathcal{F}_\epsilon : \begin{cases}
\dot{x} = 2y + \epsilon \frac{x^2}{2} \\
\dot{y} = 3x^2 - 3 + \epsilon sy
\end{cases}
\]

with \( \epsilon = 0 \). If the abelian integral \( \int_{\delta_t} \left( \frac{x^2}{2} dy - sy dx \right) = 0 \) is zero for \( t = 0 \) or equivalently if \( s \approx 0.9025 \) then for \( \epsilon \) near to 0, \( \mathcal{F}_\epsilon \) has a limit cycle near \( \delta_0 \). In fact for \( \epsilon = 1 \) such a limit cycle still exists and it is depicted in Figure (1). The origin of the above discussion comes form the second part of the Hilbert sixteen problem on limit cycles ([1]).

Every abelian integral satisfies a linear differential equation which is called the Picard-Fuchs equation (coming from geometry). For instance \( \int_{\delta_t} \frac{dx}{y} \) satisfies

(1)

\[
\frac{5}{36} I + 2t I' + (t^2 - 4)I'' = 0
\]

I general one has the linear system

\[
Y'' = \frac{1}{t^2 - 4} \left( \frac{-1}{6} I + \frac{1}{3} \right) Y
\]

which is called the Gauss-Manin connection of the family \( E_t, \ t \in \mathbb{C} \). The main point behind the calculation of Picard-Fuchs equations and Gauss-Manin connections is the techniques of derivation of an integral with respect to a parameter and simplifying the result (see [4] for the implementation of algorithms for tame polynomials in \textsc{Singular}).

We may transfer the singularities \(-2, 2\) of (1) to 0 and 1 and obtain a recursive formula for the coefficients of the Taylor series around 0 of its solutions. Since the integral \( \int_{\delta_t} \frac{dx}{y} \) is holomorphic around \( t = -2 \), we get

\[
\int_{\delta_t} \frac{dx}{y} = \frac{-2\pi}{\sqrt{3}} F\left(1, 5, 1; \frac{t + 2}{4}\right),
\]

where

\[
F(a, b, c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,
\]

Figure 1: A limit cycle crossing \((x, y) \sim (-1.79, 0)\)
is the Gauss hypergeometric function and $\alpha, \beta := \alpha(a+1)(a+2)\cdots(a+n-1)$. An elegant way to prove the statement

$$(2) \quad F\left(\frac{5}{6}, \frac{1}{6}, 1 \mid \pm \frac{\sqrt{54001}}{120} + \frac{1}{2} \right) \frac{\pi^2}{\Gamma(\frac{1}{3})^3} \in \overline{\mathbb{Q}},$$

is as follows: The elliptic curve $L_t$ has the $j$ invariant $\frac{3}{t^2-4}$. For the values of $t$ such that $j = 2^4 \cdot 3^2 \cdot 5^3$, $L_t$ admits a complex multiplication by the field $\mathbb{Q}(\sqrt{-3})$. Now one uses the Chowla-Selberg Theorem on the periods of differential forms of the first kind on elliptic curves with complex multiplication. In the next paragraph we give another interpretation of (2) in terms of a Hodge cycle of a four dimensional cubic hypersurface.

Let us consider the affine hypersurface

$$U_c : x_1^3 + x_2^3 + \cdots + x_5^3 - x_1 - x_2 - c = 0, \quad c \in \mathbb{C} - \{\pm \frac{4}{3\sqrt{3}}, 0\}$$

in $\mathbb{C}^5$ and its compactification $M_c$ in the projective space of dimension 5. The Hodge decomposition of the 4-th primitive cohomology of $M_c$ has the Hodge numbers 0, 1, 20, 1, 0 and a generator of $H^{3,1}$ piece restricted to $U_c$ is represented by the differential 4-form

$$\alpha := \left(972a^2 - 192\right)x_1x_2 + \left(-405c^3 - 48c\right)x_2 + \left(-405c^3 - 48c\right)x_1 + \left(243c^4 - 36c^3 + 64\right)$$

$$\cdot \sum_{i=1}^{5} (-1)^{i-1} x_i dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_5$$

(see [3,5]). Therefore, a cycle $\delta \in H_4(M_c, \mathbb{Z})$ with support in $U_c$ is a Hodge cycle if and only if $\int_{U_c} \alpha = 0$. It turns out that the $\mathbb{Q}$-vector space of the periods of $\alpha$ is spanned over $\mathbb{Q}$ by $\Gamma(\frac{1}{3})^3Q(\zeta_3)$ times the periods of $\frac{d\alpha}{\alpha}$ over the elliptic curve $L_t : y^2 - x^3 + 3x - t$, $t = 2 - \frac{4}{\sqrt{3}}c^2$. For $j = \frac{3}{t^2-4} = 2^4 \cdot 3^2 \cdot 5^3$, $L_t$ has a complex multiplication by $\mathbb{Q}(\zeta_3)$ and this gives us a Hodge cycle $\delta$ in $H_4(M_c, \mathbb{Q})$. One of the consequences of the Hodge conjecture is that for $c \in \overline{\mathbb{Q}}$ the integration over $\delta$ of any 4-differential form in $\mathbb{C}^5$, which is defined over $\overline{\mathbb{Q}}$ and is without residue at infinity, belongs to $\pi^2\overline{\mathbb{Q}}$. Since the Hodge conjecture is proved for cubic hypersurfaces of dimension 4, we get another interpretation of (2). For more details see [2].

Finally one can take families of elliptic curves depending on many parameters and investigate certain differential equations in parameter spaces. For instance, for the family of elliptic curves

$$y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3,$$

the abelian integral $\int \frac{dx}{y}$ is constant along the solutions of the Ramanujan ordinary differential equation

$$\begin{cases} 
\dot{t}_1 = t_1^2 - \frac{4}{t_2} t_2 \\
\dot{t}_2 = 4t_1 t_2 - 6t_3 \\
\dot{t}_3 = 6t_1 t_3 - \frac{3}{4} t_2^2 
\end{cases}$$

(3)

Using this, one can prove that every transcendental leaf of (3) intersects points with algebraic coordinates at most once. For more details see [4].