A course in Hodge Theory: with emphasis on multiple integrals

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Chapter 1

Introduction

The study of multiple integrals of dimension $n$, that is the integration of differential $n$-forms over topological cycles of dimension $n$ and lying in algebraic varieties, goes back to 19th century. Abel, Riemann, Poincaré were among many mathematicians who studied the one dimensional integrals. Picard was the first person who studied systematically the two dimensional integrals and, jointly with Simart, wrote a two-volume treatise on this subject$^1$. Between 1911 and 1924 Lefschetz, motivated by such an study and with *Analysis Situs* of Poincaré in hand, started a complete investigation of the topology of algebraic varieties$^2$, in his own words, he wanted to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry. It is a remarkable fact that at the time Lefschetz’s work was being done, while the study of algebraic topology was getting under way, the topology tools available were still primitive. It took some decades for the precise formulations and proofs of Lefschetz ideas using harmonic integrals and sheaf theory. However, these methods only obtain the homology groups with complex or real coefficients, whereas the direct method of Lefschetz enables us to use integer coefficient. Nowadays, few mathematics students and university professors know that many achievements in algebraic geometry and algebraic topology were originated by Lefschetz study of the topology of algebraic varieties, which in turn, originated by the works of Picard for understanding double integrals.

In his mathematical autobiography$^3$, Lefschetz writes: From the $\rho_0$-formula of Picard, applied to a hyperelliptic surface $\Phi$ (topologically the product of four circles) I had come to believe that the second Betti number $R_2(\Phi) = 5$, where as clearly $R_2(\Phi) = 6$. What was wrong? After considerable time it dawned upon me that Picard only dealt with finite 2-cycles, the only useful cycles for calculating periods of certain double integrals. Missing link? The cycle at infinity, that is the plane section of the surface at infinity. This drew my attention to cycles carried by an algebraic curve, that is to algebraic cycles, and ... the harpoon was in. Therefore, the desire of classifying cycles carried by algebraic varieties goes back to the early state of both Algebraic geometry and Algebraic Topology. Lefschetz himself formulated a criterion for cycles carried by an algebraic curve which can be generalized to real codimension two cycles on algebraic varieties and nowadays it is known as Lefschetz (1, 1) theorem. He states his result in the following form: On an algebraic surface $V$ a 2-dimensional homology class contains the carrier cycle of a virtual

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algebraic curve if and only if all the algebraic double integrals of the first kind have zero periods with respect to it. I was surprised when I found that no modern book in Hodge theory states the Lefschetz (1,1) theorem in its original form, namely, using integrals. Complain in this direction is expressed by V. Arnold: 


... students who have sat through courses on differential and algebraic geometry (read by respected mathematicians) turned out to be acquainted neither with the Riemann surface of an elliptic curve \( y^2 = x^3 + ax + b \) nor, in fact, with the topological classification of surfaces (not even mentioning elliptic integrals of first kind and the group property of an elliptic curve, that is, the Euler-Abel addition theorem). They were only taught Hodge structures and Jacobi varieties!

A criterion, called nowadays the Hodge conjecture, to distinguish topological cycles carried by algebraic varieties was formulated after the discovery of the Hodge decomposition for Kähler manifolds\(^5\). The objective of the present text is to collect recent and old developments on the Hodge conjecture, with emphasis on multiple integrals. In the same time we want to rebuild the higher dimensional integral theory of Picard.

### 1.1 Elliptic and abelian integrals

In order to trace back the origins of the Hodge conjecture, one must goes back to the study of elliptic integrals of the form

\[
\int_{a}^{b} \frac{dy}{\sqrt{f(x)}}
\]

where \( f(x) \) is a polynomial of degree 3 and with three distinct real roots \( t_1, t_2, t_3 \) and \( a, b \) are chosen from the roots and \( \pm \infty \). It is an easy exercise to show that all the above integrals can be calculated in terms of only two of them. One way to see this is to consider the integration in the complex domain \( x \in \mathbb{C} \), in which we may discard the assumption that \( f \) has only real roots. We may go even further and reduce our integrals to integrals of the form

\[
\int_{\delta} \frac{dx}{y}, \quad \delta \in H_1(E, \mathbb{Z})
\]

where

\[
E : y^2 = f(x)
\]

We may define \( H_1(E, \mathbb{Z}) \) as the abelization of the fundamental group of \( E \). We know the topology of \( E \); it is a punctured torus and so \( H_1(E, \mathbb{Z}) \) has only two linearly independent generators.

The above discussion can be made for \( f \) an arbitrary polynomial with distinct roots. In this case the curve \( E \) is a punctured Riemann surface of genus \( g = \left\lfloor \frac{d-1}{2} \right\rfloor \) and so \( H_1(E, \mathbb{Z}) \) is of rank \( 2g \) and we have \( 2g \) linearly independent integrals.

### 1.2 Multiple integrals

As we saw, the study of linear independent one dimensional integrals naturally leads to the study of the topology of the curves. Therefore, for the study of higher dimensional integrals we need a better understanding of the topology of algebraic varieties.

It took more than half a decade to conclude the precise definition of the (co)homologies:

\[ X \text{ a smooth variety of dimension } n \rightarrow H_q(X, \mathbb{Z}), H^q(X, \mathbb{Z}), \ q = 0, 1, \ldots, n \]

The fundamentals of algebraic topology were constructed by Poincaré in his paper "Analysis Situs" a series of addenda. What really was studied in these works were the rank and torsion elements of a homology and not the homology itself. One of the varieties for which Poincaré applies his theory is the affine variety

\[ y^2 = f(x, y), \text{ where } f(x, y) \text{ is a smooth curve} \]

Poincaré studied the topology of this variety by cutting it with the hyperplanes \( y = c \), where \( c \) is a constant.\(^6\) This idea was later used by Lefschetz. A generic pencil of hyperplanes nowadays is called a Lefschetz pencil. Poincaré needed this in order to study the double integral

\[ \int \int \frac{dxdy}{\sqrt{f(x, y)}} \]

in the article *Sur les residus des int\égrales doubles*. In this articles he even associates to Cauchy the interest to such an study *C’est à Cauchy que revient la gloire d’avoir fondé la theorie des int\égrales prises entre des limites imaginaire ....*

E. Picard, together with Simart, in 1900 and 1904 published two books on multiple integrals with emphasis on double integrals. The main tools in his books are the algebraic geometry of Noether, Severi, Castelnuovo and others, and the fundamentals of algebraic topology after Betti and Poincaré. Lefschetz after reading these two books felt the need for a systematic study of the topology of algebraic varieties and after eleven years of labor and isolation he published his treatise in 1924. After Lefschetz the study of multiple integrals were forgotten.

1.3 An example

Let us take the polynomial \( f(x, y) = x^n + y^n - 1 \) and the surface

\[ V : z^2 = f(x, y), \ V \in \mathbb{C}^3. \]

The compactification \( \mathbb{C}^3 \subset \mathbb{P}^3 \) with the coordinates \((x, y, z, w)\), or in other words \( \mathbb{P}^3 = \mathbb{C}^3 \cup \mathbb{P}^2_{\infty} \), gives us the projective variety \( \tilde{V} \) such that \( V_\infty := \tilde{V} \setminus V \) is given by \( \{ x^n + y^n = 0 \} \subset \mathbb{P}^2 = \mathbb{C}^3 \subset \mathbb{P}^3 \). This is a union of \( n \) curves isomorphic to \( \mathbb{P}^1 \). Let \( C \) be any curve in \( \tilde{V} \). We have the topological classes

\[ [V_\infty], [C] \in H_2(\tilde{V}, \mathbb{Z}) \]

which are called algebraic cycles. This is in the same way that we associate to any curve in a Riemann surface a one dimensional homological class. The curve \( C \) intersects \( V_\infty \) in a finite number of points, namely \( n \) and counting with multiplicity, and so \( \langle [C], [V_\infty] \rangle = n \). We set \( m = \langle [V_\infty], [V_\infty] \rangle \) and we have

\[ \langle [\delta], [V_\infty] \rangle = 0, \ \delta := m[C] - n[V_\infty]. \]

The conclusion is that the support of $\delta$ is in $V$. For simplicity we write $\delta \in H_2(V,\mathbb{Z})$. The cycle $\delta$ is a very special cycle. For instance, from Lefschetz $(1,1)$ theorem it follows that
\[ \int_{\delta} \frac{dxdy}{\sqrt{f(x,y)}} = 0. \]

Further, a theorem of Deligne implies that
\[ \int_{\delta} p(x,y)dxdy \sqrt{f(x,y)} \in \bar{\mathbb{Q}} \cdot \pi \]
for any polynomial $p$ with algebraic coefficients. These two properties shows the importance of algebraic cycles in the study of multiple integrals.

In general, for a smooth projective variety $X$ of even dimension $n$ and any subvariety, not necessarily smooth, $Y$ of $X$ of dimension $\frac{n}{2}$ we have
\[ [Y] \in H_n(X,\mathbb{Z}) \]
and any $\mathbb{Z}$-linear combination of such classes is called an algebraic cycle. The Hodge conjecture claims to give a criterion to distinguish algebraic cycles from other cycles. The case $n = 2$ is already the Lefschetz $(1,1)$-theorem. In our example it turns out to be the following statement: If for a cycle $\delta \in H_2(V,\mathbb{Z})$ we have
\[ \int_{\delta} x^iy^j dxdy = 0, \forall i, j \in \mathbb{N} \cup \{0\}, \frac{1}{2} + \frac{i+1}{n} + \frac{j+1}{n} < 1 \]
then $\delta$ is an algebraic cycle. In particular, for $n = 2, 3, 4$ all the cycles are algebraic.

### 1.4 Lefschetz’s Puzzle

As a part of the present introduction we state the $\rho_0 = 5$ puzzle of Lefschetz and we explain what dawned upon him. Our history begins from page 445 of the second volume of Picard’s treatise: A hyperelliptic curve of genus two is given by the equation
\[ S : y^2 = f(x) \]
in $\mathbb{C}^2$, where $f$ is a degree 5 polynomial and it has not repeated roots. All integrals on this curve reduces to the integrals $\int \frac{x^i dx}{y}, y = 0, 1, 2, 3$. Let us set $\Phi = S \times S/ \sim$, where $\sim$ is defined by $((x_1, y_1), (x_2, y_2)) \sim ((x_2, y_2), (x_1, y_1))$. We call $\Phi$ the hyperelliptic surface. The image of $S \times S$ under the map given by
\[ (x_1, y_1, x_2, y_2) \mapsto (x, y, z), \quad x := x_1 + x_2, \quad y = x_1x_2, \quad z = y_1 + y_2 \]
gives $\Phi$ in the affine coordinate $(x, y, z)$. The double integrals
\[ \int \int (x_1^p x_2^q - x_1^q x_2^p) \frac{dx_1 dx_2}{y_1 y_2}, \quad p, q = 0, 1, 2, 3, p < q \]
gives us essentially six double integrals on $\Phi$ of the form
\[ \int R(x, y, z)dxdy. \]

In this example the second Betti number of $\Phi$ is 6 but its affine part in $(x, y, z)$-coordinates has the Betti number 5.

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Chapter 2

Homology theory

In mathematics an object may be constructed during the decades and by many mathematicians and finally one finds that such an object satisfies an enumerable number of axioms, previously stated as theorems, and these axioms determine the object uniquely. An outstanding example to this is the homology theory of topological spaces. It was founded by Henri Poincaré under the name *Analysis Situs*\(^1\), further developed by Solomon Lefschetz and many others and finally, it was axiomatized by Samuel Eilenberg and Norman Steenrod around 1950 (see [15]). A fascinating fact is the study of the topology of algebraic varieties by Lefschetz right at the beginning of homology theory. This goes back even further, to the study of integrals started by Abel and pursued by Picard.

In this chapter we present the axiomatic approach to Homology theory introduced by Eilenberg and Steenrod in [15]. After many years of using homology theory in my own research my impression is that instead of spending time and effort to construct the homology theory, one has to get the feeling that one uses it correctly, even without knowing its precise definition.

2.1 Eilenberg-Steenrod axioms of homology

An admissible category \( A \) of pairs \((X, A)\) of topological spaces \(X\) and \(A\) with \(A \subset X\) and the maps between them \(f : (X, A) \to (Y, B)\) with \(f : X \to Y\) and \(f(A) \subset B\) satisfies the following conditions:

1. If \((X, A)\) is in \(A\) then all pairs and inclusion maps in the lattice of \((X, A)\)

\[
\begin{array}{ccc}
(X, A) & \to & (X, X) \\
(\emptyset, \emptyset) & \to & (A, \emptyset) \\
(A, A) & \leftarrow & (X, \emptyset)
\end{array}
\]

are in \(A\).

2. If \(f : (X, A) \to (Y, B)\) is in \(A\) then \((X, A), (Y, B)\) are in \(A\) together with all maps that \(f\) defines of members of the lattice of \((X, A)\) into corresponding members of the lattice of \((Y, B)\).

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\(^1\)Henri Poincaré, Analysis Situs, Journal de l’École Polytechnique ser 2, 1 (1895) pages 1-123. See also a series of addenda after this paper

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3. If \( f_1 \) and \( f_2 \) are in \( \mathcal{A} \) and their composition is defined then \( f_1 \circ f_2 \in \mathcal{A} \).

4. If \( I = [0, 1] \) is the closed unit interval and \( (X, A) \in \mathcal{A} \) then the cartesian product

\[
(X, A) \times I := (X \times I, A \times I)
\]

is in \( \mathcal{A} \) and the maps given by

\[
g_0, g_1 : (X, A) \to (X, A) \times I
\]

\[
g_0(x) = (x, 0), \ g_1(x) = (x, 1)
\]

are in \( \mathcal{A} \).

5. There is in \( \mathcal{A} \) a space consisting of a single point. If \( X, P \in \mathcal{A} \) and \( P \) is a single point space and \( f : P \to X \) is any map then \( f \in \mathcal{A} \).

The category of all topological pairs and the category of polyhedra are admissible. In this text we will not need these general categories. The reader is referred to [15] for examples of admissible categories. What we need in this text is the category of real differentiable manifolds, possibly with boundaries, which is admissible and it is a sub category of the category of polyhedra. A polyhedra is also called a triangulabel space.

An axiomatic homology theory is a collections of functions

\[
(X, A) \mapsto H_q(X, A), \ q = 0, 1, 2, 3, \ldots
\]

from a admissible category of pairs \((X, A)\) of topological spaces \(X\) and \(A\) with \(A \subset X\) to the category of abelian groups such that to each continuous map \(f : (X, A) \to (Y, B)\) in the category it is associated a homomorphism:

\[
f_q : H_q(X, A) \to H_q(Y, B), \ q = 0, 1, 2, \ldots
\]

In addition we have the following axioms of Eilenberg and Steenrod.

1. If \( f \) is the identity map then \( f^* \) is also the identity map.

2. For \((X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)\) we have \((g \circ f)_* = g_* \circ f_*\).

3. There are connecting homomorphisms

\[
\partial : H_q(X, A) \to H_{q-1}(A)
\]

such that such that for any \((X, A) \to (Y, B)\) in \( \mathcal{A} \) the following diagram commutes:

\[
\begin{array}{ccc}
H_q(X, A) & \to & H_q(Y, B) \\
\downarrow & & \downarrow \\
H_{q-1}(A) & \to & H_{q-1}(B)
\end{array}
\]

Here \( A \) means \((A, \emptyset)\).

4. The exactness axiom: The homology sequence

\[
\cdots \to H_{q+1}(X, A) \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \to \cdots \to H_0(X, A)
\]

where \( i_* \) and \( j_* \) are induced by inclusions, is exact. This means that the kernel of every homomorphism coincides with the image of the previous one.
5. The homotopy axiom: for homotopic maps \( f, g : (X, A) \to (Y, B) \) we have \( f_* = g_* \).

6. The excision axiom: If the closure of a subset \( U^2 \) in \( X \) is contained in the interior of \( A \) and the inclusion
\[
(X \setminus U, A \setminus U) \hookrightarrow (X, A)
\]
belongs to the category, then \( i_* \) is an isomorphism.

7. The dimension axiom: For a single point set \( X = \{p\} \) we have \( H_q(X) = 0 \) for \( q > 0 \).

The coefficient group of a homology theory is defined to be \( H_0(X) \) for a single point set \( X \). From the first and second axioms it follows that for any two single point sets \( X_1 \) and \( X_2 \) we have an isomorphism \( H_0(X_1) \cong H_0(X_2) \).

Axiomatic cohomology theories are dually defined, i.e for \( (X, A) \xrightarrow{f} (Y, B) \) we have \( H^q(Y, B) \xrightarrow{f^*} H^q(X, A) \) and the coboundary maps \( \delta : H^{q-1}(A) \to H^q(X, A) \) with similar axioms as listed above. We just change the direction of arrows and insted of subscript \( q \) we use superscript \( q \).

In [31] we find the Milnor’s additivity axiom which does not follow from the previous ones if the admissible category of topological spaces has topological sets which are disjoint union of infinite number of other topological sets:

8. Milnor additivity Axiom. If \( X \) is the disjoint union of open subsets \( X_\alpha \) with inclusion maps \( i_\alpha : X_\alpha \hookrightarrow X \), all belonging to the category, the the homomorphisms
\[
(i_\alpha)_q : H_q(X_\alpha) \to H_q(X)
\]
must provide an injective representation of \( H_q(X) \) as a direct sum.

The amazing point of the above axims is the following:

**Theorem 2.1.** In the category of polyhedra the homology (cohomology) theory exists and it is unique for a given coefficient group.

The singular homology (cohomology) is the first explicit example of the homology (cohomology) theory. Its precise construction took more than sixty years in the history of mathematics. For more details, the reader is referred to [30]. The uniqueness is a fascinating observation of Eilenberg and Steenrod. For a proof of uniqueness the reader is referred to [15] page 100 and [31].

In order to stress the role of the coefficient group \( G \) we sometimes write:
\[
H_q(X, A, G) = H_q(X, A)
\]
and so on.

Using the abobe axiom we can show that
\[
H_0(S^1, \mathbb{Z}) \cong H_1(S^1, \mathbb{Z}) \cong \mathbb{Z}.
\]

Later, we will need this in order to calculate the homology of a torus.

\(^2\)In [15] it is asumed that \( U \) is open. However, in the page 200 of the same book, the authors show that for singular homology or cohomology the openness condition is not necessary.
2.2 Singular homology

As we mentioned before, one of the reasons for the development of algebraic topology was a systematic study of multiple integrals. For this reason, the first example of homology theory is the singular homology constructed from simplicial complexes, where our integrations take place.

Fix an abelian group $G$. Let $X$ be a $C^\infty$ manifold and $\Delta^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1 \text{ and } x_i \geq 0, i = 0, 1, 2, \ldots, n\}$ be the standard $n$-simplex. A $C^\infty$ map $f : \Delta^n \to M$ is called a singular $n$-simplex. The map $f$ need not be neither surjective nor injective. Therefore, its image may not be so nice as $\Delta^n$. Let $C_n(X)$ the set of all formal and finite sums $\sum_i n_if_i, n_i \in G, f_i$ a singular $n$-complex. $C_n(X)$ has a natural structure of an abelian group. For the simplex $\Delta^n$ we denote by $I_k : \Delta^{n-1} \to \Delta^n, I_k(x_1, x_2, \ldots, x_n) := (x_1, x_2, \ldots, x_k, 0, x_{k+1}, \cdots, x_n)$ the canonical inclusion for which the image is the $k$-th face of $\Delta^n$. For $f$ a singular $n$-complex we define $\partial_n f = \sum_{k=0}^n (-1)^k f \circ I_k \in C_{n-1}(X)$ and by linearity we extend it to the homomorphism of abelian groups: $\partial = \partial_n : C_n(X) \to C_{n-1}(X)$, which we call it the boundary map. The kernel of the boundary map is $Z_n(X) = \ker(\partial_n)$ and is called the group of singular $n$-cycles. The image of the boundary map is $B_n(X) = \text{Im}(\partial_{n+1})$ and is called the group of singular $n$-boundaries. It is an easy exercise to show that $\partial_n \circ \partial_{n+1} = 0$ and so $B_n(X) \subset C_n(X)$. The $n$-th homology group of $X$ with coefficients in $G$ is defined to be $H_n(X, G) := H_n(C_\bullet(X), \partial) = \frac{Z_n(X)}{B_n(X)}$. The elements of $H_n(X, G)$ are called homology classes with coefficients in $G$.

In order to construct relative homologies we proceed as follows: For the pair $(X, A)$ of topological spaces with $A \subset X$ we define $H_n(X, A, G) := H_n(\frac{C_\bullet(X)}{C_\bullet(A)}, \partial)$.

The boundary map $\delta : H_n(X, A, G) \to H_{n-1}(A, G)$ is given by the boundary map $\partial_n$ (prove that it is well-defined).
2.3 Some consequences of the axioms

From the axioms it is possible to prove the following classical theorems in singular homology theory. For proofs see [15].

1. Universal coefficient theorem for homology: For a polyhedra $X$ there is a natural short exact sequence

$$0 \to H_q(X, \mathbb{Z}) \otimes G \to H_q(X, G) \to \text{Tor}(H_{q-1}(X, \mathbb{Z}), G) \to 0$$

For two abelian group $A$ and $B$, $\text{Tor}(A, B) := \text{Tor}_G^\mathbb{Z}(A, B)$ is the Tor functor and we will define it later. For the moment, it is sufficient to know the following:

- Let $0 \to F_1 \xrightarrow{h} F_0 \xrightarrow{k} A \to 0$ be a short exact sequence with $F_0$ a free abelian group (it follows that $F_1$ is free too). Then there is an exact sequence as follows:

$$0 \to \text{Tor}(A, B) \to F_1 \otimes B \xrightarrow{h \otimes 1} F_0 \otimes B \xrightarrow{k \otimes 1} A \otimes B \to 0$$

One can use this property to define or calculate $\text{Tor}(A, B)$. It is recommended to an student to prove the bellow properties using only this one.

- $\text{Tor}(A, B)$ and $\text{Tor}(B, A)$ are isomorphic.
- If either $A$ or $B$ is torsion free then $\text{Tor}(A, B) = 0$.
- For $n \in \mathbb{N}$ we have

$$\text{Tor}\left(\frac{\mathbb{Z}}{n\mathbb{Z}}, A\right) \cong \{ x \in A \mid nx = 0 \}$$

and so $\text{Tor}\left(\frac{\mathbb{Z}}{n\mathbb{Z}}, \frac{\mathbb{Z}}{m\mathbb{Z}}\right) = \frac{\mathbb{Z}}{\gcd(n,m)\mathbb{Z}}$.

For further properties of Tor see [30], p. 270.

2. Universal coefficient theorem for cohomology: For a polyhedra $X$ there is a natural short exact sequence

$$0 \to \text{Ext}(H_{q-1}(X, \mathbb{Z}), G) \to H^q(X, G) \to \text{Hom}(H_q(X, \mathbb{Z}), G) \to 0$$

For two abelian group $A$ and $B$, $\text{Ext}(A, B)$ is the Ext functor and we will define it later. For the moment, it is sufficient to know the following:

- Let $0 \to F_1 \xrightarrow{h} F_0 \xrightarrow{k} A \to 0$ be a short exact sequence with $F_0$ a free abelian group (it follows that $F_1$ is free too). Then there is an exact sequence as follows:

$$0 \leftarrow \text{Ext}(A, B) \leftarrow \text{Hom}(F_1, B) \xrightarrow{\text{Hom}(h, 1)} \text{Hom}(F_0, B) \xrightarrow{\text{Hom}(k, 1)} \text{Hom}(A, B) \leftarrow 0$$

One can use this property to define or calculate $\text{Ext}(A, B)$. It is recommended to an student to prove the bellow properties using only this one.

- If $A$ is a free abelian group then $\text{Ext}(A, B) = 0$ for any abelian group $B$.
- If $B$ is a divisible group then $\text{Ext}(A, B) = 0$ for any abelian group $A$.
- For $n \in \mathbb{N}$ we have

$$\text{Ext}\left(\frac{\mathbb{Z}}{n\mathbb{Z}}, B\right) \cong \frac{B}{nB}.$$
For further properties of Ext see [30], p. 313.

Note that $H^q(X, G) \to \text{Hom}(H_q(X, \mathbb{Z}), G)$ gives us a natural pairing

$$(2.1) \quad H^q(X, G) \times H_q(X, \mathbb{Z}) \to G, \quad (\alpha, \beta) \mapsto \int_\beta \alpha.$$ 

3. For a triple $Z \subset Y \subset X$ of polyhedras we have the long exact sequence:

$$\cdots \to H_q(Y, Z; \mathbb{Z}) \to H_q(X, Z; \mathbb{Z}) \to H_q(X, Y; \mathbb{Z}) \to H_{q-1}(Y, Z; \mathbb{Z}) \to \cdots$$

4. For $X$ and $Y$ two polyhedra we have a cross product maps

$$H^p(X, G_1) \times H^q(Y, G_2) \to H^{p+q}(X \times Y, G_1 \otimes G_2), \quad (\omega_1, \omega_2) \mapsto \omega_1 \times \omega_2$$

For a list of its properties see [30], 174.

5. Künneth theorem for homology: Let $X$ and $Y$ be two polyhedra. Then we have a natural exact sequence

$$0 \to \bigoplus_{i+j=q} H_i(X, \mathbb{Z}) \otimes H_j(Y, \mathbb{Z}) \to H_q(X \times Y, \mathbb{Z}) \to \bigoplus_{i+j=q-1} \text{Tor}(H_i(X, \mathbb{Z}), H_j(Y, \mathbb{Z})) \to 0.$$ 

6. Künneth theorem for cohomology: Let $X$ and $Y$ be two polyhedra. Let us assume that all the cohomologies of $X$ with coefficients in $\mathbb{Z}$ are finitely generated and at least one of the two spaces $X$ and $Y$ has all cohomology groups torsion free. Then we have a canonical isomorphism

$$\bigoplus_{i+j=q} H^i(X, \mathbb{Z}) \otimes H^j(Y, \mathbb{Z}) \cong H^q(X \times Y, \mathbb{Z})$$

given by the cross product. see [30], p. 196.

7. For $X, Y$ as in the previous item and $\delta_1 \in H_i(X, \mathbb{Z}), \delta_2 \in H_j(Y, \mathbb{Z}), \omega_1 \in H^i(X, \mathbb{Z}), \omega_2 \in H^j(X, \mathbb{Z})$ we have

$$\int_{\delta_1 \otimes \delta_2} \omega_1 \otimes \omega_2 = \int_{\delta_1} \omega_1 \int_{\delta_2} \omega_2.$$ 

8. There are natural cup and cap products:

$$H^p(X, G_1) \times H^q(X, G_2) \to H^{p+q}(X, G_1 \otimes G_2), \quad (\alpha, \beta) \mapsto \alpha \cup \beta$$

$$H^p(X, G_1) \times H^q(X, G_2) \to H_{q-p}(X, G_1 \otimes G_2), \quad (\alpha, \beta) \mapsto \alpha \cap \beta$$

For some properties which $\cup$ and $\cap$ satisfy see [30], p. 329, for instance we have

$$\alpha \cap (\beta \cap \gamma) = (\alpha \cup \beta) \cap \gamma, \quad \alpha \in H^p(X, G_1), \quad \beta \in H^q(X, G_2), \quad \gamma \in H_r(X, G_3).$$

The cup product is defined using the cross product. Let $d : X \to X \times X, \ d(x) = (x, x)$ be the diagonal map. We define

$$\omega_1 \cup \omega_2 = d^*(\omega_1 \times \omega_2).$$
9. The cap product for \( p = q, G_1 = G, G_2 = \mathbb{Z} \) and \( X \) a connected space generalizes the integration map (2.1):

\[
H^q(X, G_1) \times H_q(X, G_2) \to G_1 \otimes G_2, \quad (\alpha, \beta) \mapsto \int_\beta \alpha := \alpha \cap \beta.
\]

10. Top (co)homology: Let \( X \) be a compact connected oriented manifold of dimension \( n \). We have \( H^n(X, \mathbb{Z}) \cong \mathbb{Z}, H_n(X, \mathbb{Z}) \cong \mathbb{Z} \). The choice of a generator of \( H_n(X, \mathbb{Z}) \) or \( H^n(X, \mathbb{Z}) \) corresponds to the choice of an orientation and so we sometimes refer to it as a choice of an orientation for \( X \). We denote by \([X]\) a generator of \( H_n(X, \mathbb{Z})\) and write

\[
\int_X \alpha := \int_{[X]} \alpha, \quad \alpha \in H^n(X, \mathbb{Z}).
\]

Let \( Y \) be a compact connected oriented manifold of dimension \( m \). For a \( C^\infty \) map \( f : X \to Y \) we have the map \( f_* : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z}) \). If there is no danger of confusion then the image of \([X]\) in \( H_n(Y; \mathbb{Z}) \) is denoted again by \([X]\). In many cases \( X \) is a submanifold of \( Y \) and \( f \) is the inclusion.

11. Intersection map: Let \( X \) be an oriented manifold. There is a natural intersection map

\[
H_p(X, \mathbb{Z}) \times H_q(X, \mathbb{Z}) \to H_{p+q-n}(X, \mathbb{Z}), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta
\]

If \( X \) is connected for \( q = n - p \) this gives us

\[
H_p(X, \mathbb{Z}) \times H_{n-p}(X, \mathbb{Z}) \to \mathbb{Z}
\]

12. Poincaré duality theorem: Let \( X \) be a compact oriented manifold of dimension \( n \). The intersection map (2.3) is unimodular. This means that any linear function \( H_{n-q}(X, \mathbb{Z}) \to \mathbb{Z} \) is expressible as intersection with some element in \( H_q(X, \mathbb{Z}) \) and any class in \( H_q(X, \mathbb{Z}) \) having intersection number zero with all classes in \( H_{n-q}(X, \mathbb{Z}) \) is a torsion class. This is equivalent to say that

\[
P : H^q(X, \mathbb{Z}) \to H_{n-q}(X, \mathbb{Z}), \quad \alpha \mapsto \alpha \cap [X]
\]

is an isomorphism. For \( \alpha \in H^q(X, \mathbb{Z}) \) we say that \( \alpha \) and \( P(\alpha) \) are Poincaré duals. We have the equality

\[
\int_{P(\alpha)} \omega = \int_X \omega \cup \alpha, \quad \omega \in H^{n-q}(X, \mathbb{Z}), \quad \alpha \in H^q(X, \mathbb{Z})
\]

By Poincaré duality the intersection map in homology is dual to cup product map and in particular (2.3) is dual to

\[
H^{n-q}(X, \mathbb{Z}) \times H^q(X, \mathbb{Z}) \to \mathbb{Z}, \quad (\omega_1, \omega_2) \mapsto \int_X \omega_1 \land \omega_2.
\]

The first and main example of homology theory with \( \mathbb{Z} \)-coefficients is the singular homology

\[
H^k_{sing}(X, \mathbb{Z}), \quad k = 0, 1, 2, \ldots
\]

see for instance [30]. For the interpretation of integration we will need this interpretation of homology theory. For cohomology theory there are in fact different constructions. First, it was constructed singular cohomology and then it appeared Čech cohomology of constant sheaves (see [5]). The introduction of the de Rham cohomology was a significant step toward the formulation of Hodge theory.
2.4 Leray-Thom-Gysin isomorphism

Let us be given a closed oriented submanifold $N$ of real codimension $c$ in an oriented manifold $M$. One can define a map

\begin{equation}
(2.4) \quad \tau : H_{q-c}(N, \mathbb{Z}) \to H_q(M, M \setminus N, \mathbb{Z})
\end{equation}

for any $q$, with the convention that $H_q(N) = 0$ for $q < 0$, in the following way: Let us be given a cycle $\delta$ in $H_{q-c}(N)$. Its image by this $\tau$ is obtained by thickening a cycle representing $\delta$, each point of it growing into a closed $c$-disk transverse to $N$ in $M$ (see for instance [8] p. 392).

**Theorem 2.2** (Leray-Thom-Gysin isomorphism). The map $(2.4)$ is an isomorphism.

**Proof.** Recall that a tubular neighborhood of $N$ in $M$ is a $C^\infty$ embedding $f : E \to M$, where $E$ is the normal bundle of $N$ in $M$, such that

1. $f$ restricted to the zero section of $E$ induces the identity map in $N$.
2. $f(E)$ is an open neighborhood of $N$ in $M$.

We know that a tubular neighborhood of $N$ in $M$ exists ([23], Theorem 5.2). Now using Excision property, it is enough to prove the theorem for a $M$ a vector bundle and $N$ its zero section.

Let $M$ and $N$ as above. Writing the long exact sequence of the pair $(M, M \setminus N)$ and using $(2.4)$ we obtain:

\begin{equation}
(2.5) \quad \cdots \to H_q(M, \mathbb{Z}) \xrightarrow{\tau} H_{q-c}(N, \mathbb{Z}) \xrightarrow{\sigma} H_{q-1}(M \setminus N, \mathbb{Z}) \xrightarrow{i} H_{q-1}(M, \mathbb{Z}) \to \cdots
\end{equation}

The map $\tau$ is the intersection with $N$. The map $\sigma$ is the composition of the boundary operator with $(2.4)$.

2.5 Intersection map

Let us be given a closed submanifold $N$ of real codimension $c$ in a manifold $M$. Using Leray-Thom-Gysin isomorphism we defined the intersection map

$H_q(M) \to H_{q-c}(N)$.

If $\tilde{M}$ is another submanifold of $M$ which intersects $N$ transversely then it is left to the reader the construction of the relative intersection map

\begin{equation}
(2.6) \quad H_q(M, N) \to H_{q-c}(\tilde{M}, N \cap \tilde{M}).
\end{equation}

**Exercises**

1. Using the Eilenberg-Steenrod axioms, calculate the homology and cohomology groups of
   - the $n$-dimensional sphere $S^n$,
   - the projective spaces $\mathbb{R}P^n(n), \mathbb{C}P(n)$.
2. List all the axioms of a cohomology theory.
3. Show that the Milnor additivity axiom for finite disjoint union of topological space follows from the axioms 1 till 7.

4. Let us be given a homology theory $H_q(X, \mathbb{Z})$ with $\mathbb{Z}$-coefficients. Does $\text{Hom}(H_q(X, \mathbb{Z}), \mathbb{Z})$ is a cohomology theory? If no, which axiom fails?

5. Try to prove some of the consequences 1 till 7 of homology theory by yourself. For this purpose you can consult [15].

6. Show that
   \[ \partial_n \circ \partial_{n+1} = 0. \]
   and that $\delta : H_n(X, A, G) \rightarrow H_{n-1}(A, G)$ is well-defined.

7. Prove all the properties of Tor mentioned in this text using just the first property in the list.

8. Construct the relative intersection map (2.6).

9. Show that any point in $\mathbb{R}^n$ is a deformation retract of $\mathbb{R}^n$. 
Chapter 3

Lefschetz theorems

Here’s to Lefschetz
Who’s as argumentative as hell,
When he’s at last beneath the sod
Then he’ll start to heckle God\(^1\)

As I see it at last it was my lot to plant the harpoon of algebraic topology into
the body of the whale of algebraic geometry, Solomon Lefschetz in [29].

In 1924 Lefschetz published his treatise on the topology of algebraic varieties. When it was
written knowledge of topology was still primitive and Lefschetz “made use most uncritically
of early topology à la Poincaré and even of his own later developments” (see [29]). Later,
Lefschetz theorems were proved using harmonic forms or Morse theory or sheaf theory and
spectral sequences. But none of these very elegant methods yields Lefschetz’s full geometric
insight. Two temptations to give precise proofs for Lefschetz theorems are due to A. Wallace
1958 and K. Lamotke 1981, see [27]. In this chapter we use the later source and we present
Lefschetz’s theorems on hyperplane sections. Unfortunately, up to the time of writing the
present text there is no topological proof for the so called ”hard Lefschetz theorem”. In
this chapter if the coefficient ring of the homology or cohomology is not mentioned then
it is supposed to be the ring of integers \(\mathbb{Z}\).

3.1 Main Theorem

Let \(X\) be a smooth projective variety of dimension \(n\). By definition \(X\) is embedded in some
projective space \(\mathbb{P}^N\) and it is the zero set of a finite collection of homogeneous polynomials.
Let also \(Y, Z\) be two smooth codimension one hyperplane sections of \(X\). \(^2\) We assume
that \(Y\) and \(Z\) intersect each other transversely at \(X' := Y \cap Z\). We do not assume that \(Y\)
and \(Z\) are hyperplane sections associated to the same embedding \(X \subset \mathbb{P}^N\). However, we
assume that for some \(k \in \mathbb{N}\) the divisor \(Y - kZ\) is principal, i.e. for some meromorphic
function \(f\) on \(X\), \(Y\) is the zero divisor of order one of \(f\) and \(Z\) is the pole divisor of order
\(k\) of \(f\). We will need the following Theorem:

**Theorem 3.1.** We have

\[
H_q(X \setminus Z, Y \setminus X') = \begin{cases} 
0 & \text{if } q \neq n \\
\text{Free } \mathbb{Z}\text{-module of finite rank} & \text{if } q = n 
\end{cases}, \quad n := \dim(X).
\]

\(^2\) A modern terminology is to say that \(Y, Z\) are two smooth very ample divisors of \(X\).
where \( n := \dim(X) \).

Later we will see how to calculate the rank of \( H_q(X \setminus Z, Y \setminus X') \) by means of algebraic methods. A proof and further generalizations of Theorem 3.1 will be presented in Chapter 4 in which we develop the Picard-Lefschetz theory. Let us state some consequence of this theorem. Let

\[
U := X \setminus Z, \ V := Y \setminus X'.
\]

\( U \) is an affine variety and \( V \) is an affine subvariety of codimension one.

**Corollary 3.1.** Let \( X \) be smooth projective space and \( Z \) be a smooth hyperplane section of \( X \). We have

\[
H_q(U; \mathbb{Z}) = 0, \text{ for } q > \dim U, \ U := X \setminus Z.
\]

The homology group \( H_n(U; \mathbb{Z}) \) is free of finite rank.

**Proof.** We write the long exact sequence of the pair \( V \subset U \) and we get a five term exact sequence and the isomorphisms

\[
H_q(U) \cong H_q(V), \ q \neq n, n - 1.
\]

Now our result follows by induction on \( n \). For \( n = 1 \) it is trivial because \( U \) is not compact. Let us assume that it is true for \( n \). We know that \( X' \) is also a smooth hyperplane section of \( Y \) and \( \dim(Y) = n - 1 \). Therefore, the corollary in dimension \( n - 1 \) implies the corollary in dimension \( n \).

Applying the first part of the corollary to \( V \) we conclude that \( H_n(V) = 0 \). The five term exact sequence mentioned above reduces to the four term exact sequence:

\[
(3.1) \quad 0 \rightarrow H_n(U) \rightarrow H_n(U,V) \rightarrow H_{n-1}(V) \rightarrow H_{n-1}(U) \rightarrow 0.
\]

This implies that \( H_n(U) \) is a subset of \( H_n(U,V) \) and so by theorem 3.1 it is free.

**Remark 3.1.** In the exact sequence (3.1), \( H_n(U), H_n(U,V) \) and \( H_{n-1}(V) \) are free \( \mathbb{Z} \)-modules and \( H_{n-1}(U) \) may have torsions. Later we will see that for complete intersection affine varieties \( H_{n-1}(U) = 0 \) and so (3.1) reduces to three terms which are all free \( \mathbb{Z} \)-modules.

**Corollary 3.2.** The intersection mappings

\[
H_{n+q}(X) \rightarrow H_{n+q-2}(Y), \ x \mapsto [Y] \cdot x, \ q = 2, 3, \ldots
\]

are isomorphism
Proof. We write the long exact sequence of the pair $X \setminus Y \subset X$ and use the Leray-Thom-Gysin isomorphism and obtain

\begin{equation}
\cdots \to H_{n+q}(X \setminus Y) \to H_{n+q}(X) \to H_{n+q-2}(Y) \to H_{n+q-1}(X \setminus Y) \to \cdots
\end{equation}

Now, our statement follows from Corollary (3.1).

\begin{remark}
The dual of the intersection mapping in Corollary 3.2

\[ H^{n+q-2}(Y) \to H^{n+q}(X) \]

is an isomorphism of weight $(1, 1)$ of Hodge structures (it sends $(p, q)$-forms to $(p+1, q+1)$-forms). Let us assume that $n + q$ is even. If the Hodge conjecture is true then we have: Any algebraic cycle of dimension $\frac{n+q}{2} - 1$ in $Y$ is obtained by intersecting an algebraic cycle of dimension $\frac{n+q}{2}$ in $X$ with $Y$.

\section{3.2 Lefschetz theorem on hyperplane sections}

In this section we are going to prove the following theorem:

\begin{theorem}
Let $X$ be a smooth projective variety of dimension $n$ and $Y \subset X$ be a smooth hyperplane section. Then

\[ H_q(X, Y; \mathbb{Z}) = 0, \ 0 \leq q \leq n - 1, \]

In other words, the inclusion $Y \hookrightarrow X$ induces isomorphisms of the homology groups in all dimensions strictly less than $n - 1$ and a surjective map in $H_{n-1}$.

\end{theorem}

Proof. The proof is essentially based on the long exact sequence of triples

\[ V \subset U \subset X, \]

\[ V \subset Y \subset X, \]

the Leray-Thom-Gysin isomorphisms for the pairs $(V, Y)$ and $(U, X)$ and Theorem 3.1.

The long exact sequence of the first triple together with Theorem 3.1 gives us isomorphisms

\[ H_q(X, V) \cong H_q(X, U), \ q \neq n, n + 1 \]

induced by inclusions, and the five term exact sequence:

\[ 0 \to H_{n+1}(X, V) \to H_{n+1}(X, U) \to H_n(U, V) \to H_n(X, V) \to H_n(X, U) \to 0 \]

Now, we use Thom-Leray-Gysin isomorphism for the pair $(U, X)$ and obtain $H_q(X, U) \cong H_{q-2}(Z)$. Combining these two isomorphisms we get

\begin{equation}
H_q(X, V) \cong H_{q-2}(Z), q \neq n, n + 1
\end{equation}

We write the long exact sequence of the second triple and $(X', Z)$ in the following way:

\[ \cdots \to H_q(Y, V) \to H_q(X, V) \to H_q(X, Y) \to H_{q-1}(Y, V) \to \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \cdots \to H_{q-2}(X') \to H_{q-2}(Z) \to H_{q-2}(Z, X') \to H_{q-3}(X') \to \cdots \]
Some words must be said about the down arrows: The first and fourth down arrows are the Leray-Thom-Gysin isomorphism for the pair $(V,Y)$. The second down arrow is the isomorphism (3.3). The third morphism is obtained by intersecting the cycles with $Z$. It is left to the reader to show that the above diagram commutes and so by five lemma, there is an isomorphism

$$H_q(X,Y) \cong H_{q-2}(Z,X'), \ q \neq n, n+1, n+2.$$  

Now the theorem is proved by induction on $n$. \hfill \Box

**Remark 3.3.** Let $q$ be an even natural number. The Hodge conjecture implies that for any algebraic cycle $\delta \in H_q(X,Z)$, $q < n-1$ there are algebraic suvareties $Z_i \subset Y$, $i = 1, 2, \ldots, r$, $\dim(Z_i) = \frac{q}{2}$ and $n_i \in \mathbb{Z}$ such that $\delta = \sum_{i=1}^{r} n_i[Z_i]$. This seems to be a nontrivial statement.

### 3.3 Topology of complete intersections

We start this section by studying the topology of projective spaces:

**Proposition 3.1.** For $i \in \mathbb{N}_0$ we have

$$H_i(\mathbb{P}^n) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ (\mathbb{P}^1) & \text{if } i \text{ is even} \end{cases}$$

where $\mathbb{P}^1$ is any linear projective subspace of $\mathbb{P}^n$.

**Proof.** We take a linear suspace $\mathbb{P}^{n-1} \subset \mathbb{P}^n$, write the long exact sequence of $\mathbb{P}^n \setminus \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and use the equalities

$$H_i(\mathbb{P}^n \setminus \mathbb{P}^{n-1}) = 0, \ i \neq 0.$$  

Note that $\mathbb{P}^n \setminus \mathbb{P}^{n-1} \cong \mathbb{C}^n$ can be retracted to a point. We conclude that the intersection with $\mathbb{P}^{n-1}$ mappings

$$H_i(\mathbb{P}^n) \to H_{i-2}(\mathbb{P}^{n-2})$$

are isomorphism and $H_1(\mathbb{P}^n) = 0$. Now the proposition follows by induction on $n$. \hfill \Box

Iterate the sequence $X \supset Y \supset X'$ to

$$X = X_0 \supset X_1 = Y \supset X_2 = X' \supset X_3 \supset \cdots \supset X_n \supset X_{n+1} = \emptyset$$

so that $X_q$ is a smooth hyperplane section of $X_{q-1}$ and hence $\dim X_q = n - q$.

**Proposition 3.2.** We have

$$H_q(X, X_i) = 0, \ q \leq \dim(X_i) = n - i$$

**Proof.** From theorem 3.2 it follows that

$$H_q(X_i, X_{i+1}) = 0, \ i \leq \dim(X_{i+1}) = n - (i + 1)$$

Now, we prove the proposition by induction on $i$. For $i = 1$ it is Theorem 3.2. Assume that $H_q(X, X_i) = 0, \ q \leq \dim(X_i) = n - i$. We write the long exact sequence of the triple $X_{i+1} \subset X_i \subset X$ and use the induction hypothesis and (3.5) to obtain the proposition for $i + 1$. \hfill \Box
Proposition 3.3. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension $n$. Then

$$H_q(\mathbb{P}^{n+1}, X) = 0, \ q \leq n$$

In particular, we have

$$H_q(X) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ \mathbb{Z} & \text{if } q \text{ is even} \end{cases} \quad \text{for } q \leq n - 1$$

Proof. We can use the Veronese embedding of $\mathbb{P}^{n+1}$ such that $X$ becomes a smooth hyperplane section of $\mathbb{P}^{n+1}$. 

We have seen that a hypersurface in $\mathbb{P}^N$ is a smooth hyperplane section. A projective variety $X \subset \mathbb{P}^N$ of dimension $n$ is called a complete intersection if it is given by $N - n$ homogeneous polynomials $f_1, f_2, \ldots, f_{N-n} \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ such that the matrix

$$\left[ \frac{\partial f_i}{\partial x_j} \right]_{i=1,2,\ldots,N-n, j=0,1,\ldots,N}$$

has the maximum rank $N - n$.

Proposition 3.4. If $X \subset \mathbb{P}^N$ is a complete intersection of dimension $n$. We have

$$H_q(\mathbb{P}^N, X) = 0, \ q \leq n$$

In particular,

$$H_q(X) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ \mathbb{Z} & \text{if } q \text{ is even} \end{cases} \quad \text{for } q \leq n - 1$$

Proof. There is a sequence of projective varieties

$$\mathbb{P}^N = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_{N-n} = X$$

such that $X_i$ is a hyperplane section of $X_{i-1}$ for $i = 1, 2, \ldots, N - n$. In fact for $i = 1, 2, \ldots, N - n$, $X_i$ is induced by the zero set of $f_1, f_2, \ldots, f_i$. Now, our assertion follows from Proposition 3.2. 

Let $X$ be a complete intersection in $\mathbb{P}^N$ and $Z$ be a smooth hyperplane section corresponding to $X \subset \mathbb{P}^N$. We call $U := X \setminus Z$ an affine complete intersection.

Proposition 3.5. Let $U$ be an affine variety as above. We have

1. For $q \leq n - 1$, $H_q(U) = 0$ and so by Corollary (3.1) all $H_q(U)$ is zero except for $q = n$.

2. The intersection mappings

$$H_{n-q}(X) \to H_{n-q-2}(Y), \ x \mapsto [Y] \cdot x, \ q = 1, 2, \ldots$$

are isomorphism.

3. The four term exact sequence (3.1) reduces to three term exact sequence of free finitely generated $\mathbb{Z}$-modules.

Proof. We have a sequence $U = U_{N-n} \subset U_{N-n-1} \subset \cdots \subset U_1 \subset U_0 = \mathbb{C}^N$ such that each consecutive inclusion induces an isomorphism in $H_q, \ q \leq n - 1$.

The second and third part follows from the long exact sequence of $(X, U)$ and the Leray-Thom-Gysin isomorphism. 

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### 3.4 Some remarks on hard Lefschetz theorem

Let us consider the sequence (3.4).

**Theorem 3.3** (Hard Lefschetz theorem). For every $q = 1, 2, \ldots, n$ the intersection with $X_q$

$$H_{n+q}(X, \mathbb{Q}) \rightarrow H_{n-q}(X, \mathbb{Q}), \quad x \mapsto x \cdot [X_q]$$

is an isomorphism.

First of all note that the Hard Lefschetz theorem is stated with rational coefficients and hence it is valid for any field of characteristic zero. It is false with $\mathbb{Z}$-coefficients for trivial reasons. Take for instance $X$ a Riemann surface and $Y$ a hyperplane section which consists of $m$ points with $m > 1$. We use canonical isomorphisms $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$, $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and obtain the map $\mathbb{Z} \rightarrow \mathbb{Z}$, $x \mapsto mx$ which is not surjective. For the moment I do not have any counterexample with the non injective map in Theorem 3.3 with $\mathbb{Z}$ coefficients.

Since Theorem 3.3 is valid with $\mathbb{Q}$ coefficients, it is natural to look a counterexample which is a two dimensional projective variety $X$ with $\dim(X) = 2$ and a torsion $\alpha \in H_3(X; \mathbb{Z})$ with zero intersection with $Y$.

There is no topological proof of hard Lefschetz theorem in the literature. This is in some sense natural because it is a theorem with coefficients in $\mathbb{Q}$. The only precise proof available in the literature is due to Hodge by means of harmonic integrals. There is also an arithmetic version due to Deligne. We will give a precise proof of Theorem 3.3 after introducing the de Rham cohomology of algebraic varieties.

First we remark that it is enough to prove Theorem 3.3 for $q = 1$. For an arbitrary $q$ the theorem follows from the case $q = 1$, the diagram

$$
\begin{array}{ccc}
H_{n+q}(X; \mathbb{Q}) & \overset{[X_q]}{\rightarrow} & H_{n-q}(X; \mathbb{Q}) \\
\downarrow & & \uparrow \\
H_{(n-1)+q-1}(X_1; \mathbb{Q}) & \overset{[X_q+1]}{\rightarrow} & H_{(n-1)-(q-1)}(X_1; \mathbb{Q})
\end{array}
$$

and induction on $q$, where the up arrow is induced by the inclusion.

We write the long exact sequence of the pairs $X \setminus Y \subset X$ and $Y \subset X$ and we have

$$
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
H_n(X) & \rightarrow & H_n(X, Y) & \rightarrow & H_n(X \setminus Y) & \rightarrow H_n(X) \rightarrow H_{n-2}(Y) \rightarrow 0 \\
& \downarrow & & & \downarrow & \\
& & & & & \\
H_{n-1}(X) & \rightarrow & H_{n-1}(X, Y) & \rightarrow & H_{n-1}(X \setminus Y) & \rightarrow H_{n-1}(X) \rightarrow H_{n-3}(Y) \rightarrow 0 \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & & & & & \\
\end{array}
$$

Hard-Lefschetz theorem and following statements are equivalent:

1. $H_{n-1}(Y; \mathbb{Q}) = \text{Im}(H_{n+1}(X; \mathbb{Q}) \rightarrow H_{n-1}(Y; \mathbb{Q})) \oplus \text{ker}(H_{n-1}(Y; \mathbb{Q}) \rightarrow H_{n-1}(X; \mathbb{Q}))$. 

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2.
\[ \text{Im}(H_{n+1}(X; \mathbb{Q}) \to H_{n-1}(Y; \mathbb{Q})) \cap \ker(H_{n-1}(Y; \mathbb{Q}) \to H_{n-1}(X; \mathbb{Q})) = \{0\}. \]

Note that by Poincaré duality \( H_{n+1}(X; \mathbb{Q}) \) and \( H_{n-1}(X; \mathbb{Q}) \) have the same dimension. For more equivalent versions of hard Lefschetz theorem see [27].

**Proposition 3.6.** Let \( Y \) be a smooth hyperplane section of a projective variety \( X \). The intersection with \([Y]\) induces an isomorphism
\[ H_{n+1}(X)_{\text{tors}} \cong H_{n-1}(Y)_{\text{tors}}. \]

In particular if \( \dim(X) = 2 \) then \( H_3(X) \) is torsion free.

**Proof.** This follows from the horizontal line of the diagram (3.6) and the fact that \( H_n(X\setminus Y) \) has no torsion. \( \square \)

**Remark 3.4.** Let \( n + q \) be even. If the Hodge conjecture is true for cycles in \( H_{n+q}(X, \mathbb{Q}) \) then it is true for cycles in \( H_{n-q}(X, \mathbb{Q}) \) but not necessarily vice versa. It also implies that any algebraic cycle in \( H_{n-q}(X, \mathbb{Q}) \) is an intersection of an algebraic cycle in \( H_{n+q}(X, \mathbb{Q}) \) with \( Y \).

### 3.5 Lefschetz decomposition

Let us consider the sequence (3.4). We assume that all hyperplane sections in this sequence are associated to the same embedding \( X \subset \mathbb{P}^N \). Each \( X_q \) gives us a homology class
\[ [X_q] \in H_{2n-2q}(X; \mathbb{Z}) \]

It is left to the reader to check the equalities:

\[ [X_q] \cdots [X_q] = [X_{q+q}]. \]

An element \( x \in H_{n+q}(X), \ q = 0, 1, 2, \ldots, n \) is called primitive if
\[ [X_{q+1}] \cdot x = 0 \]

Recall that by hard Lefschetz theorem if \( [X_q] \cdot x = 0 \) then \( x = 0 \). Let us define
\[ H_{n+q}(X; \mathbb{Q})_{\text{prim}} := \{ x \in H_{n+q}(X; \mathbb{Q}) \mid [X_{q+1}] \cdot x = 0 \}. \]

**Theorem 3.4.** Every element \( x \in H_{n+q}(X) \) can be written uniquely as
\[ x = x_0 + [X_1] \cdot x_1 + [X_2] \cdot x_2 + \cdots \]

and every element \( x \in H_{n-q}(X) \) as
\[ x = [X_q] \cdot x_0 + [X_{q+1}] \cdot x_1 + [X_{q+2}] \cdot x_2 + \cdots \]

where \( x_i \in H_{n+q+2i}(X) \) are primitive and \( q \geq 0 \).
Proof. We use the fact that intersection with \([X_q]\) induces an isomorphism \(H_{n+q}(X) \to H_{n-q}(X)\) which transforms (3.8) into (3.9). Therefore, it is enough to prove (3.8). For this we use decreasing induction on \(q\) starting from \(q = n, n - 1\), where every element is primitive. For the induction step from \(n + q + 2\) to \(n + q\) it suffices to show that every \(x \in H_{n+q}(X)\) can be written uniquely as

\[(3.10) \quad x = x_0 + [X_1] \cdot y, \quad x_0 \text{ primitive}\]

because the induction hypothesis applied to \(y\) then yields the decomposition (3.8). In order to prove (3.10) consider \([X_{q+1}] \cdot x\) according to Theorem 3.3 there is exactly one \(y \in H_{n+q+2}(X)\) with \([X_{q+2}] \cdot y = [X_{q+1}] \cdot x\) and thus

\[x_0 := x - [X_1] \cdot y\]

is primitive. In order to show the uniqueness assume that \(0 = x_0 + [X_1] \cdot y\) with \(x_0\) primitive. Then \([X_{q+1}] \cdot x_0 = 0\), hence \([X_{q+2}] \cdot y = 0\) and Theorem 3.3 implies \(y = 0\), and hence \(x_0 = 0\).

\[\square\]

3.6 Lefschetz theorems in cohomology

Let \(u \in H^2(X, \mathbb{Z})\) denote the Poincaré dual of of the algebraic cycle \([Y] \in H_{2n-2}(X, \mathbb{Z})\), i.e.

\[u \cap [X] = [Y].\]

Let also

\[u^q = u \cup u \cup \cdots \cup u \in H^{2q}(X, \mathbb{Z}).\]

which is the Poincaré dual of \(X_q\). We have

\[u^q \cap x = [X_q] \cdot x\]

and so Theorem 3.3 says that for every \(q = 1, 2, \ldots, n\) the cap product with the \(q\)-th power \(u^q\)

\[H_{n+q}(X, \mathbb{Q}) \to H_{n-q}(X, \mathbb{Q}), \quad \alpha \mapsto u^q \cap \alpha\]

is an isomorphism. Poincaré dual to the Hard Lefschetz theorem is the following: For every \(q = 1, 2, \ldots, n\) the cup product with the \(q\)-th power of \(u \in H^2(X, \mathbb{Z})\) is an isomorphism

\[L_q : H^{n-q}(X, \mathbb{Q}) \to H^{n+q}(X, \mathbb{Q}), \quad \alpha \mapsto u^q \cup \alpha\]

Define the primitive cohomology in the following way:

\[H^{n-q}(X, \mathbb{Q})_{\text{prim}} := \ker (L^{q+1} : H^{n-q}(X, \mathbb{Q}) \to H^{n+q+2}(X, \mathbb{Q}))\]

The Poincaré dual to the decomposition theorem 3.4 is:

**Theorem 3.5 (Lefschetz decomposition).** The natural map

\[\oplus_q L^q : \oplus_q H^{m-2q}(X, \mathbb{Q})_{\text{prim}} \to H^m(X, \mathbb{Q})\]

is an isomorphism.
Exercises

1. Prove the equality (3.7).

2. Let $Y$ be a hypersurface of dimension $n - 1$. Does $H_{n-1}(Y)$ has torsion? If the answer is yes then by diagram 3.6 and Proposition 3.6 we have a hypersurface $X$ of dimension $n$ such that $H_{n+1}(X)$ has torsion and so the surjectivity of hard Lefschetz theorem with $\mathbb{Z}$ coefficients fails.
Chapter 4

Picard-Lefschetz Theory

In 1924 Solomon Lefschetz published his treatise [28] on the topology of algebraic varieties. He considered a pencil of hyperplanes in general position with respect to that variety in order to study its topology. In this way he founded the so called Picard-Lefschetz theory. In fact, the idea of taking a pencil goes back further to Poincaré and Picard. In the this chapter we introduce basic concepts of Picard-Lefschetz theory and we prove Theorem 3.1 in Chapter 3.

4.1 Ehresmann’s fibration theorem

Many of the Lefschetz intuitive arguments are made precise by appearance of a critical fiber bundle which is the basic stone of the so called Picard-Lefschetz theory. Despite the fact that the theorem below is proved two decades after Lefschetz treatise, it is the starting point of Picard-Lefschetz theory.

**Theorem 4.1.** (Ehresmann’s Fibration Theorem [14]). Let \( f : Y \to B \) be a proper submersion between the \( C^\infty \) manifolds \( Y \) and \( B \). Then \( f \) fibers \( Y \) locally trivially i.e., for every point \( b \in B \) there is a neighborhood \( U \) of \( b \) and a \( C^\infty \)-diffeomorphism \( \phi : U \times f^{-1}(b) \to f^{-1}(U) \) such that

\[
f \circ \phi = \pi_1 = \text{the first projection}.
\]

Moreover if \( N \subset Y \) is a closed submanifold such that \( f \mid_N \) is still a submersion then \( f \) fibers \( Y \) locally trivially over \( N \) i.e., the diffeomorphism \( \phi \) above can be chosen to carry \( U \times (f^{-1}(b) \cap N) \) onto \( f^{-1}(U) \cap N \).

The map \( \phi \) is called the fiber bundle trivialization map. Ehresmann’s theorem can be rewritten for manifolds with boundary\(^1\) and also for stratified analytic sets. In the last case the result is known as the Thom-Mather theorem.

In the above theorem assume that \( f \) is not be submersion. Let \( C' \) be the union of critical values of \( f \) and critical values of \( f \mid_N \), and \( C \) be the closure of \( C' \) in \( B \). By a critical point of the map \( f \) we mean the point in which \( f \) is not submersion. Now, we can apply the theorem to the function

\[
f : Y \setminus f^{-1}(C) \to B \setminus C = B'
\]

\(^1\)In fact one needs this version of Ehresmann’s fibration theorem in the local Picard-Lefschetz theory developed for singularities, see [2] and §4.4.
For any set $K \subset B$, we use the following notations

$$Y_K = f^{-1}(K), \quad Y'_K = Y_K \cap N, \quad L_K = Y_K \setminus Y'_K$$

and for any point $c \in B$, by $Y_c$ we mean the set $Y_{\{c\}}$. By $f : (Y, N) \to B$ we mean the mentioned map. It is called the critical fiber bundle map.

**Definition 4.1.** Let $A \subset R \subset S$ be topological spaces. $R$ is called a strong deformation retract of $S$ over $A$ if there is a continuous map $r : [0, 1] \times S \to S$ such that

1. $r(0, \cdot) = \text{Id}$,
2. $\forall x \in S, \ y \in R, \ r(1, x) \in R, \ r(1, y) = y$,
3. $\forall t \in [0, 1], \ x \in A, \ r(t, x) = x$.

Here $r$ is called the contraction map. In a similar way we can do this definition for the pairs of spaces $(R_1, R_2) \subset (S_1, S_2)$, where $R_2 \subset R_1$ and $R_1, S_2 \subset S_1$.

We use the following theorem to define generalized vanishing cycle and also to find relations between the homology groups of $Y \setminus N$ and the generic fiber $L_c$ of $f$.

**Theorem 4.2.** Let $f : Y \to B$ and $C'$ as before, $A \subset R \subset S \subset B$ and $S \cap C$ be a subset of the interior of $A$ in $S$, then every retraction from $S$ to $R$ over $A$ can be lifted to a retraction from $L_S$ to $L_R$ over $L_A$.

**Proof.** According to Ehresmann’s fibration theorem $f : L_{S \setminus C} \to S \setminus C$ is a $C^\infty$ locally trivial fiber bundle. The homotopy covering theorem, [35], §11.3, implies that the contraction of $S \setminus C$ to $R \setminus C$ over $A \setminus C$ can be lifted so that $L_{R \setminus C}$ becomes a strong deformation retract of $L_{S \setminus C}$ over $L_{A \setminus C}$. Since $C \cap S$ is a subset of the interior of $A$ in $S$, the singular fibers can be filled in such a way that $L_R$ is a deformation retract of $L_S$ over $L_A$. \hfill $\square$

### 4.2 Monodromy

Let $\lambda$ be a path in $B' = B \setminus C$ with the initial and end points $b_0$ and $b_1$. In the sequel by $\lambda$ we will mean both the path $\lambda : [0, 1] \to B$ and the image of $\lambda$; the meaning being clear from the text.

**Proposition 4.1.** There is an isotopy

$$H : L_{b_0} \times [0, 1] \to L_\lambda$$

such that for all $x \in L_{b_0}$, $t \in [0, 1]$ and $y \in N$

$$H(x, 0) = x, \ H(x, t) \in L_{\lambda(t)}, \ H(y, t) \in N$$

For every $t \in [0, 1]$ the map $h_\lambda = H(\cdot, t)$ is a homeomorphism between $L_{b_0}$ and $L_{\lambda(t)}$. The different choices of $H$ and paths homotopic to $\lambda$ would give the class of homotopic maps

$$\{h_\lambda : L_{b_0} \to L_{b_1}\}$$

where $h_\lambda = H(\cdot, 1)$. 29
Proof. The interval $[0, 1]$ is compact and the local trivializations of $L_\lambda$ can be fitted together along $\gamma$ to yield an isotopy $H$. \qed

The class $\{h_\lambda : L_{b_0} \to L_{b_1}\}$ defines the maps

$$h_\lambda : \pi_*(L_{b_0}) \to \pi_*(L_{b_1})$$

$$h_\lambda : H_*(L_{b_0}) \to H_*(L_{b_1})$$

In what follows we will consider the homology class of cycles, but many of the arguments can be rewritten for their homotopy class.

Definition 4.2. For any regular value $b$ of $f$, we can define

$$h : \pi_1(B', b) \times H_*(L_b) \to H_*(L_b)$$

$$h(\lambda, \cdot) = h_\lambda(\cdot)$$

The image of $\pi_1(B', b)$ in $\text{Aut}(H_*(L_b))$ is called the monodromy group and its action $h$ on $H_*(L_b)$ is called the action of monodromy on the homology groups of $L_b$.

Following the article [7], we give the generalized definition of vanishing cycles.

Definition 4.3. Let $K$ be a subset of $B$ and $b$ be a point in $K \setminus C$. Any relative $k$-cycle of $L_K$ modulo $L_b$ is called a $k$-thimble above $(K, b)$ and its boundary in $L_b$ is called a vanishing $(k - 1)$-cycle above $K$.

4.3 Vanishing cycles

What we studied in the previous section is developed first in the complex context. Let $Y$ be a complex compact manifold, $N$ be a submanifold of $Y$ of codimension one, $B = \mathbb{P}^1$ and $f$ be a holomorphic function. The set $C$ of critical values of $f$ is finite and so each point in $C$ is isolated in $\mathbb{P}^1$. We write

$$C := \{c_1, c_2, \ldots, c_s\}.$$  

Let $c_i \in C$, $D_i$ be an small disk around $c_i$ and $\tilde{\lambda}_i$ be a path in $B'$ which connects $b \in B'$ to $b_i \in \partial D_i$. Put $\lambda_i$ the path $\tilde{\lambda}_i$ plus the path which connects $b_i$ to $c_i$ in $D_i$ (see Figure 4.1). Define the set $K$ in the three ways as follows:

$$K^s = \begin{cases} 
\lambda_i & s = 1 \\
\lambda_i \cup D_i & s = 2 \\
\tilde{\lambda}_i \cup \partial D_i & s = 3
\end{cases}$$

In each case we can define the vanishing cycle in $L_b$ above $K^s$. $K^1$ and $K^3$ are subsets of $K^2$ and so the vanishing cycle above $K^1$ or $K^3$ is also vanishing above $K^2$.

In case $K^1$ is the intuitional concept of vanishing cycle. If $c_i$ is a critical point of $f \mid_N$, we can see that the vanishing cycle above $K^2$ may not be vanishing above $K^1$. 

\[ \]
The case $K^3$ gives us the vanishing cycles obtained by a monodromy around $c_i$. In this case we have the Wang isomorphism

$$v : H_{k-1}(L_b) \xrightarrow{\sim} H_k(L_K, L_b)$$

see [8]. Roughly speaking, the image of the cycle $\alpha$ by $v$ is the footprint of $\alpha$, taking the monodromy around $c_i$. Let $\gamma_i$ be the closed path which parametrize $K_3$ i.e., $\gamma_i$ starts from $b$, goes along $\lambda_i$ until $b_i$, turn around $c_i$ on $\partial D_i$ and finally comes back to $b$ along $\lambda_i$. Let also $h_{\gamma_i} : H_k(L_{b_i}) \rightarrow H_k(L_{b_i})$ be the monodromy around the critical value $c_i$. We have

$$\partial \circ v = h_{\lambda_i} - \text{Id}$$

where $\partial$ is the boundary operator. Therefore the cycle $\alpha$ is a vanishing cycle above $K^3$ if and only if it is in the image of $h_{\lambda_i} - \text{Id}$. For more information about the generalized vanishing cycle the reader is referred to [7].

### 4.4 The case of isolated singularities

First, let us recall some definitions from local theory of vanishing cycles. For all missing proofs the reader is referred to [2]. Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at $0 \in \mathbb{C}^n$. We can take an small closed disc $D$ with center $0 \in \mathbb{C}$ and a closed ball $B$ with center $0 \in \mathbb{C}^n$ such that

$$f : (f^{-1}(D) \cap B, f^{-1}(D) \cap \partial B) \rightarrow D$$

is a $C^\infty$ fiber bundle over $D \setminus \{0\}$. Here fibers are real manifolds with boundaries which lie in $\partial B$. In what follows we consider the mentioned domain and image for $f$. The Milnor number of $f$ is defined to be

$$\dim \left( \frac{\mathcal{O}_{\mathbb{C}^n, 0}}{(\frac{df}{dx_1} : \frac{df}{dx_2} : \cdots : \frac{df}{dx_n})} \right)$$

where $(x_1, x_2, \cdots, x_n)$ is a local coordinate system around 0.

**Proposition 4.2.** For $b \in \partial D$ the relative homology group $H_k(f^{-1}(D), f^{-1}(b))$ is zero for $k \neq n$ and it is a free $\mathbb{Z}$-module of rank $\mu$, where $\mu$ is the Milnor number of $f$. A basis of $H_n(f^{-1}(D), f^{-1}(b))$ is given by hemispherical homology classes.

By definition a hemispherical homology class is the image of a generator of infinite cyclic group $H_n(B^n, S^{n-1}) \cong \mathbb{Z}$, where

$$S^{n-1}_t := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum x_j^2 = t, \ 0 \leq t \leq 1, \ S^{n-1} := S^{n-1}_1$$

and

$$B^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum x_j^2 \leq 1 \} = \cup_{t \in [0,1]} S^{n-1}_t$$

under the homeomorphism induced by a continuous mapping of the closed $n$-ball $B^n$ into $f^{-1}(\lambda)$ which sends the $(n - 1)$-sphere $S^{n-1}_t$, to $L_{\lambda(t)}$. Here $\lambda : [0,1] \rightarrow D$ is a straight path which connects $0$ to $b$. We denote the image of $B^n$ in $B$ by $\Delta$ and call it a Lefschetz thimble. Its boundary is denoted by $\delta$ and it is called the Lefschetz vanishing cycle.
Let us consider the simplest case, i.e.

\[ f(x_1, x_2, \ldots, x_n) = x_2^2 + x_2^2 + \cdots + x_n^2. \]

and assume that \( b = 1 \). The Milnor number of \( f \) is one and \( H_n(f^{-1}(D), f^{-1}(b)) \) is generated by the image of the generator of \( H_n(\mathbb{R}^n, S^{n-1}) \) under the inclusion \( \mathbb{R}^n \subset \mathbb{C}^n \).

Let us come back to the global context of the previous section.

**Proposition 4.3.** Assume that \( c \in \mathbb{P}^1 \) is not a critical point of \( f \) restricted to \( N \) and \( f \) has only isolated critical points \( p_1, p_2, \ldots, p_k \) in \( L_c \) and these are all the critical points of \( f : (Y, N) \to \mathbb{P}^1 \) within \( Y_c \). Let also \( K = K^2 \) The following statements are true:

1. For all \( k \neq n \) we have \( H_k(L_K, L_b) = 0 \). This means that there is no \((k-1)\)-vanishing cycle along \( \lambda_i \) for \( k \neq n \);

2. \( H_n(L_K, L_b) \) is freely generated by hemispherical homology classes. It is a free \( \mathbb{Z} \)-module of rank

\[ \mu_c := \sum_{i=1}^{k} \mu_{p_i}. \]

**Proof.** We reconstruct the proof from [27], paragraph 5.4.1.

**Remark 4.1.** The monodromy \( h_i \) around the critical value \( c_i \) is given by the Picard-Lefschetz formula

\[ h(\delta) = \delta + ( -1)^{\frac{n(n+1)}{2}} \langle \delta, \delta_i \rangle \delta_i, \quad \delta \in H_{n-1}(L_b) \]

where \( \langle \cdot, \cdot \rangle \) denotes the intersection number of two cycles in \( L_b \).

**Remark 4.2.** In the above example vanishing above \( K^1 \) and \( K^2 \) are the same. Also by the Picard-Lefschetz formula the reader can verify that three types of the definition of a vanishing cycle coincide. In what follows by vanishing along the path \( \lambda_i \) we will mean vanishing above \( K^2 \).

### 4.5 Vanishing Cycles as Generators

Let \( \{c_1, c_2, \ldots, c_s\} \) be a subset of the set of critical values of \( f : (Y, N) \to \mathbb{P}^1 \), and \( b \in \mathbb{P}^1 \). Consider a system of \( s \) paths \( \lambda_1, \ldots, \lambda_s \) starting from \( b \) and ending at \( c_1, c_2, \ldots, c_s \), respectively, and such that:

1. each path \( \lambda_i \) has no self intersection points,

2. two distinct path \( \lambda_i \) and \( \lambda_j \) meet only at their common origin \( \lambda_i(0) = \lambda_j(0) = b \) (see Figure 4.2).

This system of paths is called a distinguished system of paths. The set of vanishing cycles along the paths \( \lambda_i, \ i = 1, \ldots, s \) is called a distinguished set of vanishing cycles related to the critical points \( c_1, c_2, \ldots, c_s \).

Fix a point \( b_{\infty} \in \mathbb{P}^1 \) which may be the critical value of \( f \). Assume that \( Y \) is a compact complex manifold, \( f : (Y, N) \to \mathbb{P}^1 \) restricted to \( N \) has no critical values, except probably \( b_{\infty} \), and \( f \) has only isolated critical points in \( Y \setminus Y_{b_{\infty}} \).
Theorem 4.3. The relative homology group $H_k(L_{P^1 \setminus \{b_\infty\}}, L_b)$ is zero for $k \neq n$ and it is a freely generated $\mathbb{Z}$-module of rank $r$ for $k = n$.

Proof. Our proof is a slight modification of arguments in [27] Section 5. We consider our system of distinguished paths inside a large disk $D_+$ so that $b_\infty \in P^1 \setminus D_+$, the point $b$ is in the boundary of $D_+$ and all critical values $c_i$’s in $C\setminus \{b_\infty\}$ are interior points of $D_+$.

Small disks $D_i$ with centers $c_i = 1, \ldots, r$ are chosen so that they are mutually disjoint and contained in $D_+$. Put $K_i = \lambda_i \cup D_i$, $K = \bigcup_{i=1}^r K_i$.

The pair $(K, b)$ is a strong deformation retract of $(D_+, b)$ and $(D_+, b)$ is a strong deformation retract of $(L_{P^1 \setminus \{b_\infty\}}, L_b)$. The set $\lambda = \bigcup \lambda_i$ can be retract within itself to the point $b$ and so $(L_K, L_b)$ and $(L_{D_i}, L_b)$ have the same homotopy type.

By the excision theorem we conclude that

$$H_k(L_{D_+}, L_b) \simeq \sum_{i=1}^r H_k(L_{K_i}, L_b) \simeq \sum_{i=1}^r H_k(L_{D_i}, L_{b_i})$$

and the so Proposition 4.3 finishes the proof.

Corollary 4.1. Suppose that $H_{n-1}(L_{P^1 \setminus \{b_\infty\}}) = 0$ for some $b_\infty \in P^1$, which may be a critical value. Then a distinguished set of vanishing $(n-1)$-cycles related to the critical points in the set $C\setminus \{b_\infty\} = \{c_1, c_2, \ldots, c_r\}$ generates $H_{n-1}(L_b)$.

Proof. Write the long exact sequence of the pair $(L_{P^1 \setminus \{b_\infty\}}, L_b)$:

$$(4.3) \quad \ldots \to H_n(L_{P^1 \setminus \{b_\infty\}}) \to H_n(L_{P^1 \setminus \{b_\infty\}}, L_b) \xrightarrow{\sigma} H_{n-1}(L_b) \to H_{n-1}(L_{P^1 \setminus \{b_\infty\}}) \to \ldots$$

Knowing this long exact sequence, the assertion follows from the hypothesis.

4.6 Lefschetz pencil

Let $P^N$ be the projective space of dimension $n$. The hyperplanes of $P^N$ are the points of the dual projective space $\check{P}^N$ and we use the notation

$$H_y \subset P^N, \quad y \in \check{P}^N$$

Let $X \subset P^N$ be a smooth projective variety of dimension $n$. By definition $X$ is a connected complex manifold. The dual variety of $X$ is defined to be:

$$\check{X} := \{y \in \check{P}^N \mid H_y \text{ is not transverse to } X\}$$

One can show that $\check{X}$ is an irreducible variety of dimension at most $N - 1$. Any line $G$ in $P^N$ gives us in $P^N$ a pencil of hyperplanes $\{H_t\}_{t \in G}$ which is the collection of all hyperplanes containing a projective subspace $A$ of dimension $N - 2$ of $P^N$. $A$ is called the axis of the pencil.
Let $G$ be a line in $\mathbb{P}^N$ which intersects $\bar{X}$ transversely. If $\dim(\bar{X}) < N - 1$ this means that $G$ does not intersects $\bar{X}$ and if $\dim(\bar{X}) = N - 1$ this means that $G$ intersects $\bar{X}$ transversely in smooth points of $\bar{X}$. We define

$$X_t := X \cap H_t, \ t \in G, \ X' := X \cap A$$

One can prove that

1. $A$ intersects $X$ transversely and so $X'$ is a smooth codimension two subvariety of $X$.
2. For $t \in G \cap \bar{X}$ the hyperplane section $X_t$ has a unique singularity which lies in $X_t \setminus X'$ and in a local holomorphic coordinates is given by

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0.$$  

We fix an isomorphism $G \to \mathbb{P}^1$ such that $t = 0, \infty \notin G \cap \bar{X}$. Let $L_0$ and $L_\infty$ be the linear polynomials such that $\{L_0 = 0\} = H_0$ and $\{L_0 = 0\} = H_\infty$. Our pencil of hyperplanes is now given by

$$bL_0 + aL_\infty = 0, \ [a; b] \in \mathbb{P}^1.$$  

We define

$$f = \frac{L_0}{L_\infty} |_X$$

which is a meromorphic function on $X$ with indeterminacy set $X'$.

Let us now see how Ehresmann’s theorem applies to a Lefschetz pencil $\{X_t\}_{t \in G}$. Let

$$C := G \cap \bar{X}.$$  

**Proposition 4.4.** The map $f : X \setminus X' \to G$ is a $C^\infty$ fiber bundle over $G \setminus C$.

**Proof.** We consider the blow-up of $X$ along the indeterminacy points of $f$, i.e.

$$Y := \{(x, t) \in X \times G \mid x \in X_t\}.$$  

There are two projections

$$X \xleftarrow{p} Y \xrightarrow{g} G$$

We have

$$Y' := p^{-1}(X') = X' \times G$$

and $p$ maps $Y \setminus Y'$ isomorphically to $X \setminus X'$. Under this map $f$ is identified with $g$. We use Ehresmann’s theorem and we conclude that $g$ is a fiber bundle over $G \setminus C$. Since it is regular restricted to $Y'$, we conclude that $g$ restricted to $Y \setminus Y'$ is a $C^\infty$ fiber bundle over $G \setminus C$. \hfill $\square$

### 4.7 Proof of Theorem 3.1

Theorem 3.1 follows from Theorem 4.3. By our hypothesis there is a meromorphic function $f$ on $X$ such that $Y$ is the zero divisor of order one of $f$ and $Z$ is the pole divisor of order $k$ of $f$. Since $Y$ intersects $Z$ transversely, a similar blow up argument as in Proposition 4.4 implies that $f$ is a $C^\infty$ fiber bundle map over $C \setminus C$, where $C$ is the set of critical values of $f$ restricted to $X \setminus Z$. Now, we have to prove that $f$ in $X \setminus Z$ has isolated singularities. If
this is not the case then we take an irreducible component $S$ of the locus of singularities of $f$ restricted to $X \setminus Z$ which is of dimension bigger than one. The variety $S$ necessarily intersects $Y$ in some point $p$. The point $p$ does not lie in $Y \setminus Z$ because $Y \setminus Z$ is smooth. It does not lie in $X'$ because $Y$ intersects $Z$ transversely and hence in some coordinate system $(x, y, \ldots)$ around $p \in X'$, the meromorphic function $f$ can be written as $\frac{x}{y^k}$.

**Exercises**

1. In local context prove that three different definition of a vanishing cycle and thimble coincide.

2. $\hat{X}$ is an irreducible variety of dimension at most $N - 1$ and the set of lines in $\mathbb{P}^N$ transverse to a variety form an non-empty Zariski open subset of lines in $\mathbb{P}^N$.

3. Prove the properties 1 and 2 of the Lefschetz pencil.
Chapter 5

Hodge conjecture

In this chapter we present the Hodge conjecture in a form which does not need the Hodge decomposition.

5.1 De Rham cohomology

Let $M$ be a $C^\infty$ manifold and $\Omega^i_{M^\infty}$, $i = 0, 1, 2, \ldots$ be the sheaf of $C^\infty$ differentiable $i$-forms on $M$. By definition $\mathcal{O}_{M^\infty} = \Omega^0_{M^\infty}$ is the sheaf of $C^\infty$-functions on $M$. We have the de Rham complex

$$
\Omega^0_{M^\infty} \xrightarrow{d} \Omega^1_{M^\infty} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^i_{M^\infty-1} \xrightarrow{d} \Omega^i_{M^\infty} \xrightarrow{d} \cdots
$$

and the de Rham cohomology of $X$ is defined to be

$$
H^i_{dR}(M) = H^n(\Gamma(M, \Omega^i_{M^\infty}), d) := \frac{\text{global closed } i\text{-forms on } M}{\text{global exact } i\text{-forms on } M}.
$$

Remark 5.1. By Poincaré Lemma we know that if $M$ is a unit ball then

$$
H^i_{dR}(M) = \begin{cases} 
\mathbb{R} & \text{if } i = 0 \\
0 & \text{if } i = 0
\end{cases}
$$

It follows that $\mathbb{R} \rightarrow \Omega^\bullet_{M^\infty}$ is the resolution of the constant sheaf $\mathbb{R}$ on a $C^\infty$ manifold $M$. Since the sheaves $\Omega^i_{M^\infty}$, $i = 0, 1, 2, \ldots$ are fine we conclude that

$$
H^i(M, \mathbb{R}) \cong H^i_{dR}(M), \ i = 0, 1, 2, \ldots
$$

where $H^i(M, \mathbb{R})$ can be interpreted as the Cech cohomology of the constant sheaf $\mathbb{R}$ on $M$. All these will be explained in details in Appendix 6.

5.2 Integration

Let $H_i(M, \mathbb{Z})$ be the $i$-th singular homology of of $M$. We have the integration map

$$
H_i(M, \mathbb{Z}) \times H^i_{dR}(M) \rightarrow \mathbb{R}, \ (\delta, \omega) \mapsto \int_{\delta} \omega
$$

which is defined as follows: let $\delta \in H_i(M, \mathbb{Z})$ be a homology class which is represented by a piecewise smooth $p$-chain

$$
\sum a_i f_i, \ a_i \in \mathbb{Z}
$$
and \( f_i \) is a \( C^\infty \) map from a neighborhood of the standard \( p \)-simplex \( \Delta \subset \mathbb{R}^p \) to \( M \). Let also \( \omega \) a \( C^\infty \) global differential form on \( M \). Then

\[
\int_\delta \omega := \sum a_i \int_\Delta f_i^* \omega
\]

By Stokes theorem this definition is well-defined and does not depend on the class of both \( \delta \) and \( \omega \) in \( H_1(M, \mathbb{Z}) \), respectively \( H^1_{dR}(M) \).

### 5.3 Hodge decomposition

Now, let \( M \) be a complex manifold. All the discussion in the previous chapter is valid replacing the \( \mathbb{R} \) coefficients with \( \mathbb{C} \)-coefficients. Let \( \Omega^{p,q}_M \) (resp. \( Z^{p,q}_M \)) be the sheaf of \( C^\infty \) differential \((p,q)\)-forms (resp. closed \((p,q)\)-forms) on \( M \). We define

\[
H^{p,q} = \frac{\Gamma(M, Z^{p,q}_M)}{d \Gamma(\Omega^{p+q-1}_M) \cap \Gamma(M, Z^{p,q}_M)}
\]

We have the canonical inclusion:

\[
H^{p,q} \rightarrow H^m_{dR}(M)
\]

**Theorem 5.1.** Let \( M \) be a projective smooth variety . We have

\[
H^m_{dR}(M) = H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{1,m-1} \oplus H^{0,m},
\]

which is called the Hodge decomposition.

One can prove the above theorem using harmonic forms, see for instance M. Green’s lectures [17], p. 14. The Hodge theory, as we learn it from the literature, starts from the above theorem. Surprisingly, we will not need the above theorem and so we did not prove it.

We have the conjugation mapping

\[
H^m_{dR}(M) \rightarrow H^m_{dR}(M), \ \omega \mapsto \bar{\omega}
\]

which leaves \( H^m(M, \mathbb{R}) \) invariant and maps \( H^{p,q} \) isomorphically to \( H^{q,p} \).

**Remark 5.2.** In order to prove the Hodge decomposition it is enough to prove that:

\[
H^m_{dR}(M) = H^{m,0} + H^{m-1,1} + \cdots + H^{1,m-1} + H^{0,m}.
\]

Let \( \alpha_{p,m-p} \in H^{p,m-p}, \ p = 0, 1, 2, \ldots, m \) and \( \sum_{p=0}^m \alpha_{p,m-p} = 0 \). This equality implies that the wedge product of \( \alpha_{p,m-p} \) with every element in \( H^m_{dR}(M) \) is zero and hence \( \alpha_{p,m-p} = 0 \).

### 5.4 Hodge conjecture

One of the central conjectures in Hodge theory is the so called Hodge conjecture. Let \( m \) be an even natural number and \( M \) a fixed complex compact manifold. Consider a holomorphic map \( f : Z \rightarrow M \) from a complex compact manifold \( Z \) of dimension \( \frac{m}{2} \) to \( M \). We have then the homology class

\[
[Z] \in H_m(M, \mathbb{Z})
\]

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which is the image of the generator of $H_m(Z, \mathbb{Z})$ (corresponding to the canonical orientation of $Z$) in $H_m(X, \mathbb{Z})$. The image $\tilde{Z}$ of $f$ in $X$ is a subvariety (probably singular) of $X$ and if $\dim(\tilde{Z}) < \frac{m}{2}$ then using resolution of singularities we can show that $[\tilde{Z}] = 0$. Let us now

$$Z = \sum_{i=1}^{s} r_i Z_i$$

where $Z_i$, $i = 1, 2, \ldots, s$ is a complex compact manifold of dimension $\frac{m}{2}$, $r_i \in \mathbb{Z}$ and the sum is just a formal way of writing. Let also $f_i : Z \to X$, $i = 1, 2, \ldots, s$ be holomorphic maps. We have then the homology class

$$\sum_{i=1}^{s} r_i [Z_i] \in H_m(M, \mathbb{Z})$$

which is called an algebraic cycle with \mathbb{Z}-coefficients (see [4]). We denote by $F_{alg} H_m(M, \mathbb{Z})$ the \mathbb{Z}-module of algebraic cycles of set $H_m(M, \mathbb{Z})$.

**Proposition 5.1.** For an algebraic cycle $\delta \in H_m(M, \mathbb{Z})$ we have

$$\int_{\delta} \omega = 0,$$

for all $C^\infty (p, q)$-form $\omega$ on $M$ with $p + q = m$, $p \neq \frac{m}{2}$.

**Proof.** The pull-back of a $(p, q)$-form with $p + q = m$ and $p \neq \frac{m}{2}$ by $f_i$ is identically zero because at least one of $p$ or $q$ is bigger than $\frac{m}{2}$. \hfill \Box

Any torsion element $\delta \in H_m(M, \mathbb{Z})$ satisfies the property (5.2) and so sometimes it is convenient to consider $H_m(M, \mathbb{Z})$ up to torsions.

**Definition 5.1.** A cycle $\delta \in H_m(M, \mathbb{Z})$ with the property (5.2) is called a Hodge cycle. We denote by $F_{Hodge} H_m(M, \mathbb{Z})$ the \mathbb{Z}-module of Hodge cycles in $H_m(M, \mathbb{Z})$. By definition it contains all the torsion elements of $H_m(M, \mathbb{Z})$.

**Conjecture 1** (Hodge conjecture). For any Hodge cycle $\delta \in H_m(M, \mathbb{Z})$ there is an integer $a \in \mathbb{N}$ such that $a \cdot \delta$ is an algebraic cycle.

Note that if $\delta$ is a torsion then there is $a \in \mathbb{N}$ such that $a \delta = 0$ and in the way that we have introduced the Hodge conjecture, torsions do not violate the Hodge conjecture.

**Remark 5.3.** Let $\delta \in H_m(M, \mathbb{Z})$ be a Hodge cycle and let $P^{-1}(\delta)$ be its Poincaré dual. We have

$$\int_{\delta} \omega = \int_{X} \omega \wedge P^{-1}\delta, \ \omega \in H^m(X, \mathbb{C})$$

and so $\omega \wedge P^{-1}(\delta) = 0$ for all $(p, q)$-form $\omega$ with $p + q = m$, $p \neq q$. By Hodge decomposition and its relation with the intersection form (5.3) we see that $P^{-1}(\delta) \in H^{n-\frac{m}{2}, n-\frac{m}{2}}$ and so

$$P^{-1}(\delta) \in H^{n-\frac{m}{2}, n-\frac{m}{2}} \cap H^{2n-m}(M, \mathbb{Z}).$$

where we have identified $H^{2n-m}(M, \mathbb{Z})$ with its image in $H^{2n-m}(M, \mathbb{C})$ and hence we have killed the torsions. An element of the above set is called a Hodge class. The Poincaré duality gives a bijection between the \mathbb{Q}-vector space of Hodge cycles and the \mathbb{Q}-vector space of Hodge classes. In the literature one usually call an element of $H^{n-\frac{m}{2}, n-\frac{m}{2}} \cap H^{2n-m}(M, \mathbb{Q})$ to be a Hodge class. Note that

$$H^{n-\frac{m}{2}, n-\frac{m}{2}} \cap H^{2n-m}(M, \mathbb{Z}) = F^{n-\frac{m}{2}} (H^{2n-m}(M, \mathbb{C})) \cap H^{2n-m}(M, \mathbb{Z}).$$

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5.5 Real Hodge cycles

Let \( k \) be a field with \( \mathbb{Q} \subset k \subset \mathbb{C} \). We can define the set \( F_{\text{Hodge}} H_m(M, k) \) of Hodge cycles defined over \( k \).

**Proposition 5.2.** We have

\[
\dim_{\mathbb{R}} F_{\text{Hodge}} H_m(M, \mathbb{R}) = h^{\frac{m}{2}} \cdot \frac{m}{2}
\]
\[
\dim_{\mathbb{C}} F_{\text{Hodge}} H_m(M, \mathbb{C}) = h^{\frac{m}{2}} \cdot \frac{m}{2}
\]

where \( h^{\frac{m}{2}} \cdot \frac{m}{2} = \dim H^{\frac{m}{2}} \cdot \frac{m}{2} \).

**Proof.** The proof is an easy linear algebra. Note that if \( k \subset \mathbb{R} \) then in (5.2) we can assume that \( p < \frac{m}{2} \).

Therefore, in order to find counterexamples for the Hodge conjecture over real numbers it is sufficient to give a variety such that

\[
\dim_{\mathbb{Q}} F_{\text{Alg}} H_m(M, \mathbb{Q}) < h^{\frac{m}{2}} \cdot \frac{m}{2}.
\]

For instance, the product of two elliptic curves have this property.

5.6 Counterexamples

The Hodge conjecture is false when it formulated with \( a = 1 \). This means that a Hodge cycle \( \delta \in H_m(M, \mathbb{Z}) \) may not be an algebraic cycle. It is natural to look for a counterexample \( \delta \) which is a torsion. The first example of torsion non-algebraic homology elements was obtained by Atiyah and Hirzebruch in [3]. A new point of view is presented by Totaro in [36]. In [1] Kollár and van Geemen shows that if \( X \subset \mathbb{P}^4 \) is a very general threefold of degree \( d \) and \( p \geq 5 \) is a prime number such that \( p^3 \) divides \( d \), the degree of every curve \( C \) contained in \( X \) is divisible by \( p \). This provides a counterexample to the Hodge conjecture over \( \mathbb{Z} \) not involving torsion classes, since it implies that the generator \( \alpha \) of \( H^4(X, \mathbb{Z}) \) is not algebraic whereas \( d\alpha \) is algebraic. In [33] Voisin and Soulé remarks that the methods of Atiyah-Hirzebruch and Totaro cannot produce non-algebraic \( p \)-torsion classes for prime numbers \( p > \dim_{\mathbb{C}}(X) \). They then show that for every prime number \( p \geq 3 \) there exist a fivefold \( Y \) and a non-algebraic \( p \)-torsion class in \( H^5(Y, \mathbb{Z}) \).

The Hodge decomposition is also valid for Kähler manifolds, however, the Hodge conjecture is not valid in this case, see [40, 37]

5.7 Polarization

Let \( Y \) be a hyperplane section of \( X \) and let \( u \in H^2(X, \mathbb{Z}) \) denote the Poincaré dual of of the algebraic cycle \( [Y] \in H_{2n-2}(X, \mathbb{Z}), \) i.e.

\[
u \cap [X] = [Y].
\]

In the de Rham cohomology \( H^2_{\text{dR}}(M) \), \( u \) is of type \((1, 1)\) and so the Lefschetz decomposition

\[
H^m(X, \mathbb{Q}) \cong \oplus_q H^{m-2q}(X, \mathbb{Q})_{\text{prim}}
\]

is compatible with the Hodge decomposition. We have the wedge product map:

$$\psi : H^m_{\text{dR}}(M) \times H^m_{\text{dR}}(M) \to \mathbb{C}, \psi(\omega_1, \omega_2) = \frac{1}{(2\pi i)^m} \int_M u^{n-m} \wedge \omega_1 \wedge \omega_2,$$

where $n = \dim_{\mathbb{C}} M$. This form is symmetric for $m$ even and alternating otherwise.

**Proposition 5.3.** We have

(5.3) $$\psi(H^{i,m-i}, H^{m-j,j}) = 0 \text{ unless } i = j,$$

(5.4) $$(-1)^{\frac{m(m-1)}{2} + p} \psi(\omega, \bar{\omega}) > 0, \forall \omega \in H^{p,m-p} \cap H^m(M, \mathbb{C})_{\text{prim}}, \omega \neq 0.$$

**Proof.** The first part follows from the fact for $\omega_k$, $k = 1, 2$ of type $(i, m-i)$ and $(j, m-j)$, respectively, the $2n$-form $u^{n-m} \wedge \omega_1 \wedge \omega_2$ is identically zero for $i \neq j$.

The proof of the second part uses Harmonic forms and can be found in [38]. The proof for an $\omega$ which is locally of the form $f dz^p \wedge d\bar{z}^q$, where $p + q = m$ and $f$ is a local $C^\infty$ function and $dz^p$ (resp. $d\bar{z}^q$) is a wedge product of $p$ (resp. $q$) $dz_i$'s (resp. $d\bar{z}_i$) (this is the case for instance for $p = n = m$). Up to positive numbers we have:

$$\omega \wedge \bar{\omega} = |f|^2 dz^p \wedge d\bar{z}^q \wedge d\bar{z}^p \wedge dz^q$$

$$= (-1)^{p+q} p! dz^p \wedge d\bar{z}^q \wedge d\bar{z}^p \wedge dz^q$$

$$= (-1)^q (-1)^{\frac{m(m-1)}{2}} dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots$$

$$= (-1)^q (-1)^{\frac{m(m-1)}{2}} (-2i)^m d\text{Re}(z_{i_1}) \wedge d\text{Im}(z_{i_1}) \wedge \cdots$$

\[\square\]

**Exercises**

1. Show that the wedge product in the de Rham cohomology corresponds to the cup product in the singular cohomology.

2. Show that the top de Rham cohomology of an oriented compact manifold is one dimensional.
Chapter 6

Cech cohomology

In chapter 2 we discussed the axiomatic approach to homology and cohomology theories and we saw that the singular homology and cohomologies are examples of such theories. In this Appendix we discuss another construction of cohomology theory, namely Cech cohomology of constant sheaves. It has the advantage that it is easy to calculate and it generalizes to the cohomology of sheaves. We assume that the reader is familiar with sheaves of abelian groups on manifolds.

6.1 Cech cohomology

A sheaf \( S \) of abelian groups on a topological space \( X \) is a collection of abelian groups

\[ S(U), \ U \subset X \text{ open} \]

with restriction maps which satisfy certain properties, for instance see [21]. In particular \( S(X) \) is called the set of global sections of \( S \) and the following equivalent notations

\[ S(X) = \Gamma(X, S) = H^0(X, S). \]

is used.

It is not difficult to see that for an exact sequence of sheaves of abelian groups

\[ 0 \to S_1 \to S_2 \to S_3 \to 0. \]

we have

\[ 0 \to S_1(X) \to S_2(X) \to S_3(X) \]

and the last map is not necessarily surjective. In this section we want to construct abelian groups \( H^i(X, S), \ i = 0, 1, 2 \ldots, \ H^0(X, S) = S(X) \) such that we have the long exact sequence

\[
0 \to H^0(X, S_1) \to H^0(X, S_2) \to H^0(X, S_3) \to H^1(X, S_1) \to H^1(X, S_2) \to H^1(X, S_3) \to H^2(X, S_1) \to \cdots
\]

Let \( X \) be a topological space, \( S \) a sheaf of abelian groups on \( X \) and \( \mathcal{U} = \{U_i, \ i \in I\} \) a covering of \( X \) by open sets. In this paragraph we want to define the Cech cohomology of the covering \( \mathcal{U} \) with coefficients in the sheaf \( S \). Let \( \mathcal{U}^p \) denotes the set of \( p \)-tuples
\[ \sigma = (U_{i_0}, \ldots, U_{i_p}), \ i_0, \ldots, i_p \in I \text{ and for } \sigma \in \mathcal{U}^p \text{ define } |\sigma| = \cap_{j=0}^{p} U_{i_j}. \] A \( p \)-cochain \( f = (f_\sigma)_{\sigma \in \mathcal{U}^p} \) is an element in

\[ C^p(\mathcal{U}, \mathcal{S}) := \prod_{\sigma \in \mathcal{U}^p} H^0(|\sigma|, \mathcal{S}) \]

Let \( \pi \) be the permutation group of the set \( \{0, 1, 2, \ldots, p\} \). It acts on \( \mathcal{U}^p \) in a canonical way and we say that \( f \in C^p(\mathcal{U}, \mathcal{S}) \) is skew-symmetric if \( f_{\pi \sigma} = \text{sign}(\pi) f_\sigma \). The set of skew-symmetric cochains form an abelian subgroup \( C_s(\mathcal{U}, \mathcal{S}) \).

For \( \sigma \in \mathcal{U}^p \) and \( j = 0, 1, \ldots, p \) denote by \( \sigma_j \) the element in \( \mathcal{U}^{p-1} \) obtained by removing the \( j \)-th entry of \( \sigma \). We have \( |\sigma| < |\sigma_j| \) and so the restriction maps from \( H^0(|\sigma_j|, \mathcal{S}) \) to \( H^0(|\sigma|, \mathcal{S}) \) is well-defined. We define the boundary mapping

\[ \delta : C^p_s(\mathcal{U}, \mathcal{S}) \to C^{p+1}_s(\mathcal{U}, \mathcal{S}), \ (\delta f)_\sigma = \sum_{j=0}^{p+1} (-1)^j f_{\sigma_j} |\sigma| \]

It is left to the reader to check that \( \delta \) is well-defined, \( \delta \circ \delta = 0 \) and so

\[ 0 \to C^0_s(\mathcal{U}, \mathcal{S}) \to C^1_s(\mathcal{U}, \mathcal{S}) \to C^2_s(\mathcal{U}, \mathcal{S}) \to \cdots \]

can be viewed as cochain complexes, i.e. the image of a map in the complex is inside the kernel of the next map.

**Definition 6.1.** The Čech cohomology of the covering \( \mathcal{U} \) with coefficients in the sheaf \( \mathcal{S} \) is the cohomology groups

\[ H^p(\mathcal{U}, \mathcal{S}) := \frac{\text{Kernel}(C^p_s(\mathcal{U}, \mathcal{S}) \to C^{p+1}_s(\mathcal{U}, \mathcal{S}))}{\text{Image}(C^{p-1}_s(\mathcal{U}, \mathcal{S}) \to C^p_s(\mathcal{U}, \mathcal{S}))}. \]

The above definition depends on the covering and we wish to obtain cohomologies \( H^p(\mathcal{X}, \mathcal{S}) \) which depends only on \( \mathcal{X} \) and \( \mathcal{S} \). We recall that the set of all coverings \( \mathcal{U} \) of \( \mathcal{X} \) is directed: \( \mathcal{U}_1 \leq \mathcal{U}_2 \) if \( \mathcal{U}_1 \) is a refinement of \( \mathcal{U}_2 \), i.e. each open subset of \( \mathcal{U}_1 \) is contained in some open subset of \( \mathcal{U}_2 \). For two covering \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) there is another covering \( \mathcal{U}_3 \) such that \( \mathcal{U}_3 \leq \mathcal{U}_1 \) and \( \mathcal{U}_3 \leq \mathcal{U}_2 \). It is not difficult to show that for \( \mathcal{U}_1 \leq \mathcal{U}_2 \) we have a well-defined map

\[ H^p(\mathcal{U}_2, \mathcal{S}) \to H^p(\mathcal{U}_1, \mathcal{S}). \]

Now the Čech cohomology of \( \mathcal{X} \) with coefficients in \( \mathcal{S} \) is defined in the following way:

\[ H^p(\mathcal{X}, \mathcal{S}) := \text{dir lim}_\mathcal{U} H^p(\mathcal{U}, \mathcal{S}). \]

### 6.2 How to compute Čech cohomologies

The covering \( \mathcal{U} \) is called acyclic with respect to \( \mathcal{S} \) if \( \mathcal{U} \) is locally finite, i.e. each point of \( \mathcal{X} \) has an open neighborhood which intersects a finite number of open sets in \( \mathcal{U} \), and \( H^p(U_{i_1} \cap \cdots \cap U_{i_k}, \mathcal{S}) = 0 \) for all \( U_{i_1}, \ldots, U_{i_k} \in \mathcal{U} \) and \( p \geq 1 \).

**Theorem 6.1.** (Leray lemma) Let \( \mathcal{U} \) be an acyclic covering of a variety \( \mathcal{X} \). There is a natural isomorphism

\[ H^p(\mathcal{U}, \mathcal{S}) \cong H^p(\mathcal{X}, \mathcal{S}). \]
For a sheaf of abelian groups $S$ over a topological space $X$, we will mainly use $H^1(X, S)$. Recall that for an acyclic covering $U$ of $X$ an element of $H^1(X, S)$ is represented by

$$f_{ij} \in S(U_i \cap U_j), \ i, j \in I$$

$$f_{ij} + f_{jk} + f_{ki} = 0, \ f_{ij} = f_{ji}, \ i, j, k \in I$$

It is zero in $H^1(X, S)$ if and only if there are $f_i \in S(U_i), \ i \in I$ such that $f_{ij} = f_j - f_i$.

6.3 Acyclic sheaves

A sheaf $S$ of abelian groups on a topological space $X$ is called acyclic if

$$H^k(X, S) = 0, \ k = 1, 2, \ldots$$

The main examples of fine sheaves that we have in mind are the following: Let $M$ be a $C^\infty$ manifold and $\Omega^i_{M^\infty}$ be the sheaf of $C^\infty$ differential $i$-forms on $M$.

**Proposition 6.1.** The sheaves $\Omega^i_{M^\infty}, \ i = 0, 1, 2, \ldots$ are acyclic.

*Proof.* The proof is based on the partition of unity and is left to the reader. \qed

A sheaf $S$ is said to be flasque or fine if for every pair of open sets $V \subset U$, the restriction map $S(U) \rightarrow S(V)$ is surjective.

**Proposition 6.2.** Flasque sheaves are acyclic.

See [38], p.103, Proposition 4.34.

6.4 Resolution of sheaves

A complex of abelian sheaves is the following data:

$$S^\bullet : S^0 \xrightarrow{d_0} S^1 \xrightarrow{d_1} \ldots \xrightarrow{d_{k-1}} S^k \xrightarrow{d_k} \ldots$$

where $S^i$'s are sheaves of abelian groups and $S^k \rightarrow S^{k+1}$ are morphisms of sheaves of abelian groups such that the composition of two consecutive morphism is zero, i.e

$$d_{k-1} \circ d_k = 0, \ k = 1, 2, \ldots$$

A complex $S^k, k \in \mathbb{N}_0$ is called a resolution of $S$ if

$$\text{Im}(d^k) = \ker(d^{k+1}), \ k = 0, 1, 2, \ldots$$

and there exists an injective morphism $i : S \rightarrow S^0$ such that $\text{Im}(i) = \ker(d^0)$. We write this simply in the form

$$S \rightarrow S^\bullet$$

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Theorem 6.2. Let $S$ be a sheaf of abelian groups on a topological space $X$ and $S \to S^\bullet$ be an acyclic resolution of $S$, i.e. a resolution for which each $S^k$ is acyclic, then

$$H^k(X, S) \cong H^k(\Gamma(X, S^\bullet), d), \; k = 0, 1, 2, \ldots$$

where

$$\Gamma(X, S^\bullet) : \Gamma(S^0) \xrightarrow{d_0} \Gamma(S^1) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} \Gamma(S^k) \xrightarrow{d_k} \cdots$$

and

$$H^k(\Gamma(X, S^\bullet), d) := \frac{\ker(d_k)}{\text{Im}(d_{k-1})}.$$

Let us come back to the sheaf of differential forms. Let $M$ be a $C^\infty$ manifold. The de Rham cohomology of $M$ is defined to be

$$H^i_{\text{dR}}(M) = H^n(\Gamma(M, \Omega^i_{M, \infty}), d) := \text{global closed } i\text{-forms on } M \\overline{\text{global exact } i\text{-forms on } M}.$$

Theorem 6.3. (Poincaré Lemma) If $M$ is a unit ball then

$$H^i_{\text{dR}}(M) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ 0 & \text{if } i = 0 \end{cases}$$

The Poincaré lemma and Proposition 6.1 imply that

$$\mathbb{R} \to \Omega^i_M$$

is the resolution of the constant sheaf $\mathbb{R}$ on the $C^\infty$ manifold $M$. By Proposition 6.2 we conclude that

$$H^i(M, \mathbb{R}) \cong H^i_{\text{dR}}(M), \; i = 0, 1, 2, \ldots$$

where $H^i(M, \mathbb{R})$ is the Cech cohomology of the constant sheaf $\mathbb{R}$ on $M$.

6.5 Cech cohomology and Eilenberg-Steenrod axioms

Let $G$ be an abelian group and $M$ be a polyhedra. We can consider $G$ as the sheaf of constants on $M$ and hence we have the Cech cohomologies $H^k(X, G), \; k = 0, 1, 2, \ldots$. This notation is already used in Chapter 2 to denote a cohomology theory with coefficients group $G$ which satisfies the Eilenberg-Steenrod axioms. The following theorem justifies the usage of the same notation.

Theorem 6.4. In the category of polyhedra the Cech cohomology of the sheaf of constants in $G$ satisfies the Eilenberg-Steenrod axioms.

Therefore, by uniqueness theorem the Cech cohomology of the sheaf of constants in $G$ is isomorphic to the singular cohomology with coefficients in $G$. We present this isomorphism in the case $G = \mathbb{R}$ or $\mathbb{C}$.

Recall the definition of integration from Chapter ??

$$H^i_{\text{sing}}(M, \mathbb{Z}) \times H^i_{\text{dR}}(M) \to \mathbb{R}, \; (\delta, \omega) \mapsto \int_\delta \omega$$

This gives us

$$H^i_{\text{dR}}(M) \to H^i_{\text{sing}}(M, \mathbb{R}) \cong H^i_{\text{sing}}(M, \mathbb{R})$$
Theorem 6.5. The integration map gives us an isomorphism

\[ H^i_{\text{dR}}(M) \cong H^i_{\text{sing}}(M, \mathbb{R}) \]

Under this isomorphism the cup product corresponds to

\[ H^i_{\text{dR}}(M) \times H^j_{\text{dR}}(M) \to H^{i+j}_{\text{dR}}(M), \ (\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2, \ i, j = 0, 1, 2, \ldots \]

where \( \wedge \) is the wedge product of differential forms.

If \( M \) is an oriented manifold of dimension \( n \) then we have the following bilinear map

\[ H^i_{\text{dR}}(M) \times H^{n-i}_{\text{dR}}(M) \to \mathbb{R}, \ (\omega_1, \omega_2) \mapsto \int_M \omega_1 \wedge \omega_2, \ i = 0, 1, 2, \ldots \]

6.6 Dolbeault cohomology

Let \( M \) be a complex manifold and \( \Omega_{M, \infty}^{p,q} \) be the sheaf of \( C^\infty \) \((p,q)\)-forms on \( M \). We have the complex

\[ \Omega_{M, \infty}^{p,0} \to \Omega_{M, \infty}^{p,1} \to \cdots \to \Omega_{M, \infty}^{p,q} \to \cdots \]

and the Dolbeault cohomology of \( M \) is defined to be

\[ H^{p,q}_{\bar{\partial}}(M) := H^q(\Gamma(M, \Omega_{M, \infty}^{p,\bullet}), \bar{\partial}) = \text{global } \bar{\partial}\text{-closed } (p,q)\text{-forms on } M \]

\[ = \text{global } \bar{\partial}\text{-exact } (p,q)\text{-forms on } M \]

Theorem 6.6 (Dolbeault Lemma). If \( M \) is a unit disk or a product of one dimensional disks then \( H^{p,q}_{\bar{\partial}}(M) = 0 \)

Let \( \Omega^p \) be the sheaf of holomorphic \( p \)-forms on \( M \). In a similar way as in Proposition 6.1 one can prove that \( \Omega_{M, \infty}^{p,q} \)'s are fine sheaves and so we have the resolution of \( \Omega^p \):

\[ \Omega^p \to \Omega_{M, \infty}^{p,\bullet} \]

By Proposition 6.2 we conclude that:

Theorem 6.7 (Dolbeault theorem). For \( M \) a complex manifold

\[ H^q(M, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(M) \]
Chapter 7

Category theory

The category theory is a language which unifies many similar ideas which had been appearing in mathematics since the invention of singular homology and cohomology. It tells us how similar arguments in topology, geometry and algebraic geometry can be unified in the context of a unique language. In this unique language the morphisms are not more functions from a set to another set. We have used mainly the book [25] of M. Kashiwara and P. Schapira, [16] of S. Gelfand and Y. Manin, [38] of C. Voisin and [12] of A. Dimca.

7.1 Objects and functions

A category $\mathcal{A}$ consists of

1. The set of objects $\text{Ob}(\mathcal{A})$.

2. For $X,Y \in \text{Ob}(\mathcal{A})$ a set $\text{Hom}(X,Y) = \text{Hom}_\mathcal{A}(X,Y)$, called the set of morphisms. Instead of $f \in \text{Hom}(X,Y)$ we usually write:

   $$f : X \rightarrow Y \text{ or } X \xrightarrow{f} Y.$$ 

   Note that this is just a way of writing and it does not mean that $X$ and $Y$ are sets and $f$ is a map between them. Of course, our main examples of the category theory have this interpretation.

3. For $X,Y,Z \in \text{Ob}(\mathcal{A})$ a map

   $$\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \rightarrow \text{Hom}(X,Z), \ (f,g) \mapsto g \circ f$$

   called the composition map.

It satisfies the following axioms:

1. The composition of morphisms is associative, i.e

   $$f \circ (g \circ h) = (f \circ g) \circ h.$$ 

   for $X \xrightarrow{h} Y \xrightarrow{g} Z \xrightarrow{f} W$. 

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2. For all objects $X$, there is $\text{Id}_X : X \to X$ such that
\[ f \circ \text{Id}_X = f, \quad \text{Id}_X \circ g = g \]
for $f, g$ that the equality is defined. It is easy to see that $\text{Id}_X$ with this property is unique. We define the isomorphism in the category $\mathcal{A}$ in the following way: For $X, Y \in \text{Ob}(\mathcal{A})$ we write $X \cong Y$ if there are $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ such that
\[ g \circ f = \text{Id}_X, \quad f \circ g = \text{Id}_Y. \]
(7.1)

We say that the morphism $f$ is an isomorphism and its inverse is $g$. The inverse of $f$ is unique and is denoted by $f^{-1}$.

In some texts one has also the following axiom in the definition of a category:

If $X \cong Y$ then $X = Y$.

We have not included this property in the definition of a category. However, when we say that an object $X$ is unique we mean that it is unique up to the above isomorphism.

### 7.2 Some definitions

For two categories $\mathcal{A}_1, \mathcal{A}_2$ we say that $\mathcal{A}_1$ is a subcategory of $\mathcal{A}_2$ if $\text{Ob}(\mathcal{A}_1) \subseteq \text{Ob}(\mathcal{A}_2)$ and for $X, Y \in \text{Ob}(\mathcal{A}_1)$ we have $\text{Hom}_{\mathcal{A}_1}(X,Y) \subseteq \text{Hom}_{\mathcal{A}_2}(X,Y)$. We say that $\mathcal{A}_1$ is a full subcategory of $\mathcal{A}_2$ if for $X, Y \in \text{Ob}(\mathcal{A}_1)$ we have $\text{Hom}_{\mathcal{A}_1}(X,Y) = \text{Hom}_{\mathcal{A}_2}(X,Y)$.

For a category $\mathcal{A}$ we can associate the opposite category $\mathcal{A}^\circ$ which is defined by: $\text{Ob}(\mathcal{A}^\circ) = \text{Ob}(\mathcal{A})$ and for all $X, Y \in \text{Ob}(\mathcal{A})$ we have $\text{Hom}_{\mathcal{A}^\circ}(X,Y) = \text{Hom}_{\mathcal{A}}(Y,X)$. In the opposite category we have just changed the direction of arrows and it is easy to see that $\mathcal{A}^\circ$ is in fact a category.

A morphism $f : X \to Y$ is called a monomorphism (or an injective morphism) if for any $Z \in \text{Ob}(\mathcal{A})$ and $g, g' \in \text{Hom}(Z,X)$ such that $f \circ g = f \circ g'$, one has $g = g'$. We usually write
\[ X \hookrightarrow Y \]
to denote a monomorphism. $f$ is called an epimorphism (or surjective morphism) if it is a monomorphism in the opposite category.

An object $P$ in the category $\mathcal{A}$ is called initial if $\text{Hom}(P,X)$ has exactly one element for all $Y \in \text{Ob}(\mathcal{A})$. It is called final if it is initial in the opposite category. Up to isomorphism, there is only one initial or final object.

### 7.3 Functors

A (covariant) functor $F$ between two categories $\mathcal{A}_1$ and $\mathcal{A}_2$ is a collection of maps
\[ F : \text{Ob}(\mathcal{A}_1) \to \text{Ob}(\mathcal{A}_2), \]
\[ F : \text{Hom}(X,Y) \to \text{Hom}(F(X),F(Y)), \quad \forall X, Y \in \mathcal{A}_1 \]
such that

1. $F(\text{Id}_X) = \text{Id}_{F(x)}$ for all $x \in \mathcal{A}_1$,
2. \( F(f \circ g) = F(f) \circ F(g) \) for all \( X \xrightarrow{g} Y \xrightarrow{f} Z \) in \( \mathcal{A}_1 \).

A contravariant functor \( F : \mathcal{A}_1 \to \mathcal{A}_2 \) is a covariant functor \( \mathcal{A}_1^\circ \to \mathcal{A}_2 \). For instance, for \( X \in \text{Ob}(\mathcal{A}) \)
\[
\text{Hom}(X, \cdot) : \mathcal{A} \to \text{Set}, \ Z \mapsto \text{Hom}(X, Z)
\]
is a covariant functor. In a similar way \( \text{Hom}(\cdot, X) \) is a contravariant functor. Here \( \text{Set} \) is
the category of sets and functions between them.

A morphism between two functors \( F_1, F_2 : \mathcal{A}_1 \to \mathcal{A}_2 \) contains the following data. For all \( X \in \text{Ob}(\mathcal{A}_1) \) we have \( \theta(X) \in \text{Hom}(F_1(X), F_2(X)) \) such that for all \( f \in \text{Hom}_{\mathcal{A}_1}(X, Y) \)
the following diagram commutes:
\[
\begin{array}{ccc}
F_1(X) & \xrightarrow{\theta(X)} & F_2(X) \\
\downarrow F_1(f) & & \downarrow F_2(f) \\
F_1(Y) & \xrightarrow{\theta(Y)} & F_2(Y)
\end{array}
\]

For any two categories \( \mathcal{A}_1, \mathcal{A}_2 \) we get a new category called the category of functors for
which the objects are the functors from \( \mathcal{A}_1 \) to \( \mathcal{A}_2 \) and the morphisms are as above. It is
left to the reader to show that this is a category.

Now, we can talk about an isomorphism between two functors. It is simply an isomor-
phism in the category of functors. Let us describe this in more details. We say that two
functors \( F_1, F_2 : \mathcal{A}_1 \to \mathcal{A}_2 \) are isomorphic and write \( F_1 \cong F_2 \) if for \( X \in \text{Ob}(\mathcal{A}_1) \) we have
\( \theta_1(X) \in \text{Hom}(F_1(X), F_2(X)), \ \theta_2(X) \in \text{Hom}(F_2(X), F_1(X)) \) such that
\[
\theta_2(X) \circ \theta_1(X) = \text{Id}_{F_1(X)}, \ \theta_1(X) \circ \theta_2(X) = \text{Id}_{F_2(X)}
\]
and the following diagrams commute:
\[
\begin{array}{ccc}
F_1(X) & \xrightarrow{\theta_1(X), \theta_2(X)} & F_2(X) \\
\downarrow F_1(f) & & \downarrow F_2(f) \\
F_1(Y) & \xrightarrow{\theta_1(Y), \theta_2(Y)} & F_2(Y)
\end{array}
\]
In another words \( F_1(X) \cong F_2(X) \) and \( F_1(Y) \cong F_2(Y) \) and under these isomorphisms
\( F_1(f) \) is identified with \( F_2(f) \).

A functor \( F : \mathcal{A} \to \text{Set} \) is called representable if there exists \( X \in \text{Ob}(\mathcal{A}) \) such that it
is isomorphic to the functor \( \text{Hom}(X, \cdot) \). It is not difficult to see that up to isomorphism
\( X \) is unique.

### 7.4 Additive categories

A category \( \mathcal{A} \) is called additive if

1. The set \( \text{Hom}(X, Y) \) has a structure of an abelian group such that the composition
   map is bilinear.

2. There is 0 \( \in \text{Ob}(\mathcal{A}) \) such that \( \text{Hom}(0, 0) \) is the zero abelian group \( \{0\} \). It can be
   proved that \( \text{Hom}(X, 0) \) and \( \text{Hom}(0, X) \) are zero groups.
3. For any two objects \( X_1, X_2 \in \text{Ob}(A) \) there exist an object \( Y \in \text{Ob}(A) \) and morphisms

\begin{align*}
Y & \xrightarrow{i_2} X_1 \xrightarrow{\pi_1} \xrightarrow{\pi_2} X_2 \xrightarrow{i_1} Y
\end{align*}

such that

\begin{align*}
\pi_1 \circ i_1 &= \text{Id}_{X_1}, \\
\pi_2 \circ i_2 &= \text{Id}_{X_2}, \\
i_1 \circ \pi_1 + i_2 \circ \pi_2 &= \text{Id}_Y, \\
\pi_2 \circ i_1 &= 0, \\
\pi_1 \circ i_2 &= 0.
\end{align*}

One can show that \( Y \) with this definition is unique. We shall denote such \( Y \) by \( X_1 \oplus X_2 \) and we shall call it the direct sum of \( X_1 \) and \( X_2 \). For any two morphisms \( f : X_1 \to Z \) and \( g : X_2 \to Z \) we define \( h : f \circ \pi_1 + g \circ \pi_2 \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f} & Z \\
\downarrow & & \nearrow \\
X_1 \oplus X_2 & \xrightarrow{h} & Z \\
\uparrow & & \searrow \\
X_2 & & \\
\end{array}
\]

### 7.5 Kernel and cokernel

The kernel of a morphism \( X \xrightarrow{f} Y \) in an additive category \( A \) is an object \( C \) in \( A \) with the following properties: It is equipped with a morphism \( C \xrightarrow{i} X \) such that for any other object \( M \) in the category \( A \) the mapping:

\[
\text{Hom}(M, C) \to \{ \psi \in \text{Hom}(M, X) \mid f \circ \psi = 0 \}, \ g \mapsto i \circ g
\]

is well-defined and it is a bijective map. The kernel of \( f \), if it exists, is usually denoted by \( \ker(f) \). One can check that the map \( i : \ker(f) \to X \) is a monomorphism. A better way to define \( \ker(f) \) is to say that the contravariant functor

\[
F : A \to \text{Set}, \ F(M) = \text{Hom}(M, X) \mid f \circ \psi = 0
\]

is isomorphic to a representable functor.

The cokernel of \( f \), denoted by \( \text{coker}(f) \), in the category \( A \) is defined to be the kernel of \( f \) in the opposite category \( A^\circ \). In explicit words, the cokernel of \( f \) is an object \( C \) with the morphism \( j : Y \to C \) such that for any other object \( M \) in the category \( A \) the mapping:

\[
\text{Hom}(C, M) \to \{ \psi \in \text{Hom}(Y, M) \mid \psi \circ f = 0 \}, \ g \mapsto g \circ j.
\]

is well-defined and it is a bijective map. The image and coimage of a morphism \( X \xrightarrow{f} Y \) are defined by

\[
\text{Im}(f) =: \ker(j), \ \text{Coim}(f) =: \text{coker}(i).
\]

### 7.6 Abelian categories

An additive category \( A \) is called an abelian category if it satisfies

1. For any \( f : X \to Y \) in \( A \) the kernel and cokernel of \( f \) exist.
2. The canonical morphism \( \text{Coim}(f) \to \text{Im}(f) \) is an isomorphism.
Some words must be said about the canonical map $\text{Coim}(f) \to \text{Im}(f)$. Let $r : X \to \text{Coim}(f)$ be the map in the definition of $\text{Coim}(f) := \text{coker}(i)$. Therefore, we have a one to one map:

$$\text{Hom}(\text{Coim}(f), M) \to \{ \psi \in \text{Hom}(X, M) \mid \psi \circ i = 0 \}, \ g \mapsto g \circ r.$$ 

which is well-defined and it is an isomorphism. Put $M = Y$ and $\psi = f$. We have $f \circ i = 0$ and by the above universal property for $\text{coker}(i)$ we have a map $o : \text{Coim}(f) \to Y$ such that $f = o \circ r$. Now, we use the universal property of $\text{ker}(j)$: We have a map $s : \text{Im}(f) \to Y$ and a one to one map

$$\text{Hom}(M, \text{Im}(f)) \to \{ \psi \in \text{Hom}(M, Y) \mid j \circ \psi = 0 \}, \ g \mapsto s \circ g.$$ 

We set $M = \text{Coim}(f)$ and $\psi = o$. We have $j \circ o = 0$ because

$$j \circ f = (j \circ o) \circ r = 0 \quad \text{and} \quad r \text{ is surjective}$$

We conclude that there is a morphism $O : \text{Coim}(f) \to \text{Im}(f)$ such that $s \circ O = o$.

### 7.7 Additive functors

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two abelian categories. A functor from $\mathcal{A}_1$ to $\mathcal{A}_2$ is called additive if for all $X, Y \in \text{Ob}(\mathcal{A}_1)$ the map $\text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y))$ is a morphism of additive groups. An additive functor $F$ from $\mathcal{A}_1$ to $\mathcal{A}_2$ is called left (resp. right) exact if for any exact sequence in $\mathcal{A}_1$:

$$0 \to X_1 \to X_2 \to X_3 \to 0 \quad \text{(resp.} \quad X_1 \to X_2 \to X_3 \to 0)$$

the sequence

$$0 \to F(X_1) \to F(X_2) \to F(X_3) \quad \text{(resp.} \quad F(X_1) \to F(X_2) \to F(X_3) \to 0)$$

is exact.

### 7.8 Injective and projective objects

Let $\mathcal{A}$ be an abelian category. An object $A \in \text{Ob}(\mathcal{A})$ is injective if for any diagram

$$
\begin{array}{ccc}
0 & \to & X \\
\downarrow & & \downarrow \\
& & A \\
& \nearrow & \\
Y & \to & 
\end{array}
$$

in $\mathcal{A}$ there is $Y \to A$ which makes the diagram commutative. The object $A$ is called projective if it is injective in the opposite category, i.e. for any diagram

$$
\begin{array}{ccc}
0 & \leftarrow & X \\
\uparrow & & \uparrow \\
& & A \\
& \searrow & \\
Y & \leftarrow & 
\end{array}
$$

in $\mathcal{A}$ there is $Y \leftarrow A$ which makes the diagram commutative. As an exercise, it is left to the reader to show that any short exact sequence $0 \to X \to Y \to Z \to 0$ with $X$ an injective object is split. In the category of abelian groups the injective objects are exactly divisible abelian groups, i.e. a group $G$ such that for all $g \in G$ and $n \in \mathbb{N}$ there is $g' \in G$ such that $ng' = g$. 

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### 7.9 Complexes

Let \( \mathcal{A} \) be an abelian category. A complex in this category is a sequence of morphisms

\[
\cdots \to X^{n-1} \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \to \cdots ,
\]

with

\[
d^n \circ d^{n-1} = 0.
\]

We usually do not write the name of the morphisms \( d^n \). We write, for short, \( X^\bullet \) or \( X^k, k \in \mathbb{Z} \) to denote a complex. When we write \( X^k, k \in \mathbb{N}_0 \) we mean a complex for which \( X^k = 0, k = -1, -2, \ldots \).

The category of complexes \( C(\mathcal{A}) \) is a category whose objects are the set of complexes in \( \mathcal{A} \). A morphism between two complexes \( X^\bullet \) and \( Y^\bullet \) is the following commutative diagram:

\[
\cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \cdots \\
\cdots \to Y^{n-1} \to Y^n \to Y^{n+1} \to \cdots
\]

We have the following natural functor

\[
H^k : C(\mathcal{A}) \to \mathcal{A}, \quad H^k(X) := \frac{\ker(d^k)}{\text{Im}(d^{k-1})}, \quad k \in \mathbb{Z}
\]

For a morphism \( f : X \to Y \) of complexes we have a canonical morphism \( H^k(f) : H^k(X) \to H^k(Y) \). A morphism of complexes \( f \) is called a quasi-isomorphism if the induced morphisms \( H^k(f), k \in \mathbb{Z} \) are isomorphisms.

We have three full subcategories of \( C(\mathcal{A}) \). The category of left complexes \( C^+(\mathcal{A}) \) contains the complexes

\[
\cdots \to 0 \to 0 \to X_n \to X_{n+1} \to \cdots
\]

and the category of right complexes \( C^-(\mathcal{A}) \) contains the complexes

\[
\cdots \to X_{n-1} \to X_n \to 0 \to 0 \to \cdots
\]

the category of bounded complexes is \( C^b(\mathcal{A}) \) for which the objects are

\[
\text{Ob}(C^+(\mathcal{A})) \cap \text{Ob}(C^-(\mathcal{A}))
\]

### 7.10 Homotopy

Let \( \mathcal{A} \) be an abelian category. The shift functor of degree \( k \)

\[
[k] : C(\mathcal{A}) \to C(\mathcal{A}), \quad k \in \mathbb{Z}.
\]

is defined in the following way. The complex \( [k](X) \) is denoted by \( X[k] \) and it is defined by

\[
X[k]^n := X^{n+k}, \quad d_X^m := (-1)^k d_X^{m+k}.
\]

A homomorphism \( f : X \to Y \) in \( C(\mathcal{A}) \) is called homotopic to zero if there exists morphisms \( s^n : X^n \to Y^{n-1} \) in \( \mathcal{A} \) such that

\[
f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n
\]

We say that \( f : X \to Y \) is homotopic to \( g : X \to Y \) if \( f - g \) is homotopic to zero. The following easy proposition is left to the reader
Proposition 7.1. Let $X$ and $Y$ be two complexes in $\mathcal{A}$ and $f, g : X \to Y$ be two homotopic maps. Then the induced maps

$$H^n(f), H^n(g) : H^n(X) \to H^n(Y), \ n \in \mathbb{Z}.$$ 

are equal.

7.11 Resolution

Let $\mathcal{A}$ be an abelian category. A complex $X^k, k \in \mathbb{N}_0$ is called a resolution of $X \in \text{Ob}(\mathcal{A})$ if

$$\text{Im}(d^k) = \ker(d^{k+1}), \ k = 0, 1, 2, \ldots$$

and there exists an injective morphism $i : X \to X^0$ such that $\text{Im}(i) = \ker(d^0)$. We write this simply in the form

$$X \to X^\bullet$$

We have a natural injective map $\text{Im}(d^k) \to \ker(d^{k+1})$ and by the equality above we mean that this injective map is an isomorphism.

An object $I$ of the category $\mathcal{A}$ is called injective if for any injective morphism $A \hookrightarrow B$ and a morphism $A \to I$ there exists a morphism $B \to I$ such that the following diagram is commutative:

$$A \hookrightarrow B \quad \downarrow \quad \check{\leftarrow}$$

$$I$$

We say that the category $\mathcal{A}$ has enough injectives if for all object $A$ in $\mathcal{A}$ there are an injective element $I$ and an injective morphism $A \to I$. For instance the category of sheaves of abelian groups over a topological space has enough injectives. For a given sheaf $A$, $I$ is the sheaf of all not necessarily continuous sections of $A$.

Proposition 7.2. If an abelian category contains enough injectives then any object $X$ of $\mathcal{A}$ admits a resolution.

For a proof see [38], Lemma 4.26.

Proposition 7.3. Let $A \overset{i}{\to} I^\bullet$ and $B \overset{j}{\to} J^\bullet$ be resolutions of $A$ and $B$ respectively and $\phi : A \to B$ be a morphism. Then if the second resolution is injective there exists a morphism of complexes $\phi^\bullet : I^\bullet \to J^\bullet$ satisfying $\phi^0 \circ i = j \circ \phi$. Moreover, if we have two such morphisms $\phi^\bullet$ and $\psi^\bullet$, there exists a homotopy between $\phi^\bullet$ and $\psi^\bullet$.

See [38] Proposition 4.27.

If both $I^\bullet$ and $J^\bullet$ are injective resolutions of $A$ then we apply Proposition 7.3 to the case where $A = B$ and $\phi$ is the identity map. We obtain $\phi^\bullet : I^\bullet \to J^\bullet$ and $\psi^\bullet : J^\bullet \to I^\bullet$ such that $\phi^\bullet \circ \psi^\bullet$ and $\psi^\bullet \circ \phi^\bullet$ are both homotopic to identity. Therefore, injective resolutions are unique up to homotopy.

7.12 Derived functors

Now, we are in position to announce the first important theorem of this chapter.
**Theorem 7.1.** Let $\mathcal{A}_1, \mathcal{A}_2$ be two abelian categories and $F$ be a left exact functor from $\mathcal{A}_1$ to $\mathcal{A}_2$. Assume that $\mathcal{A}_1$ has enough injectives. For any object $X$ of $\mathcal{A}_1$ there are objects $R^kF(X), k = 0, 1, 2, \ldots$ with the following properties:

1. $R^0F(X) = F(X)$.

2. For all exact sequence

   \[ 0 \to A \to B \to C \to 0 \]

   in $\mathcal{A}_1$ one can construct a long exact sequence

   \[ 0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to R^2F(B) \to \cdots \]

3. For any injective object object $I$ of $\mathcal{A}_1$ we have $R^kF(I) = 0, k = 1, 2, \ldots$

The objects $R^kF(X)$ with the above properties are unique up to isomorphisms.

For a proof see [38] Theorem 4.28. The construction of $R^iF(A)$ is as follows. Let $A \to I^\bullet$ be an injective resolution of $A$. we define

\[ R^iF(A) = H^iF(I^\bullet). \]

For a morphism $\phi : A \to B$ in $\mathcal{A}$ we have canonical morphisms

\[ R^iF(\phi) : R^iF(A) \to R^iF(B). \]

### 7.13 Acyclic objects

Let $\mathcal{A}_1, \mathcal{A}_2$ be two abelian categories and $F$ be a left exact functor from $\mathcal{A}_1$ to $\mathcal{A}_2$. An object $X$ of $\mathcal{A}_1$ is called $F$-acyclic, or simply acyclic if one understand $F$ from the context, if

\[ R^kF(X) = 0, k = 1, 2, 3 \ldots \]

By definition, injective objects are acyclic but not vice verse.

**Proposition 7.4.** Let $X$ be an object of $\mathcal{A}_1$ and

\[ X \to X^\bullet \]

be an $F$-acyclic resolution of $X$, i.e. a resolution for which each $X^i$ is acyclic, then

\[ R^kF(X) \cong H^k(F(X^\bullet)). \]

See [38], Proposition 4.32.

### 7.14 Cohomology of sheaves

We consider the category of sheaves of abelian groups on topological spaces. From this category to the category of abelian groups we have the global section functor $\Gamma$ which associates to each sheaf $\mathcal{S}$ the set of its global sections. For a sheaf $\mathcal{S}$ of abelian groups on a topological space $X$ we define:

\[ H^k(X, \mathcal{S}) := R^k\Gamma(\mathcal{S}) \]

Cech cohomology of sheaves gives a nice interpretation of $H^k(X, \mathcal{S})$ in terms of covering of $X$ with open sets (see Chapter 6 and [21, 5]).
Proposition 7.5. Let $X$ be a topological space and $S$ a sheaf of abelian groups on $X$. Then we have a canonical isomorphism between $H^k(X, S)$ and the Cech cohomology of $X$ with coefficients in $S$.

Proof. We take a covering $\mathcal{U} = \{U_i\}_{i \in I}$ such that the intersection $V$ of any finite number of $U_i$’s satisfies

$$H^k(V, S^i) = 0, \ k = 1, 2, \ldots, i = 0, 1, 2, \ldots$$

For any open set $V$ in $X$ let $S|_V$ be the sheaf for which the stalk over $x \in V$ coincides with the stalk of $S$ over $x$ and outside of $V$ it is zero. Let $I^{k-1}$ be the direct product of all the sheaves $S|_{U_1 \cap U_2 \cap \cdots U_k}$. We have now the acyclic resolution of $S$

$$S \to I^0 \xrightarrow{\delta} I^1 \xrightarrow{\delta} I^2 \cdots$$

where $\delta$ is defined in a similar way as the $\delta$ in the Cech cohomology. Now our assertion follows from Proposition 7.4. 

7.15 Derived functors for complexes

We follow [38] page 186-194

Let $\mathcal{A}_1$ be an abelian category which has enough injective objects.

Proposition 7.6. For a left-bounded complex $M^\bullet$ in $\mathcal{A}_1$, there exists a left-bounded complex $I^\bullet$ in $\mathcal{A}_1$ such that each $I^k$, $k \in \mathbb{Z}$ is injective object of $\mathcal{A}_1$, and a morphism $\phi^\bullet : M^\bullet \to I^\bullet$ of complexes which is a quasi-isomorphism and for all $k \in \mathbb{Z}$, $\phi^k : M^k \to I^k$ is injective.

See [38], Proposition 8.4.

It is a natural question to ask which objects of the category $C^+(\mathcal{A}_1)$ are injective. Note that a left-bounded complex $I^\bullet$ with $I^k$ injective objects is not necessarily an injective object of $C^+(\mathcal{A}_1)$.

Let $\mathcal{A}_1, \mathcal{A}_2$ be two abelian categories, where $\mathcal{A}_1$ has enough injective objects. Let also $F$ be a left exact functor from $\mathcal{A}_1$ to $\mathcal{A}_2$. Let $M^\bullet$ be a left-bounded complex in $\mathcal{A}_1$ and $\phi^\bullet : M^\bullet \to I^\bullet$ as in Proposition 7.6. We define the derived functors

$$R^kF(M^\bullet) := H^k(F(I^\bullet)), \ k \in \mathbb{Z}.$$ 

Note that for an object $A$ in $\mathcal{A}_1$ we have

$$R^kF(\cdots \to 0 \to A \to 0 \to \cdots) = R^kF(A),$$

where $A$ is positioned in the 0-th place.

Proposition 7.7. If we have two quasi-isomorphisms $M^\bullet \to I^\bullet$ and $M^\bullet \to J^\bullet$ as in Proposition 7.6, then we have canonical isomorphisms

$$H^k(F(I^\bullet)) \cong H^k(F(J^\bullet)).$$

See [38] Proposition 8.6.

Proposition 7.8. Let $\phi^\bullet : M^\bullet \to I^\bullet$ be a quasi-isomorphism of left-bounded complexes in the category $\mathcal{A}_1$ with $I^k$ injective for all $k \in \mathbb{Z}$. Then we have an isomorphism

$$R^kF(M^\bullet) \cong H^k(F(I^\bullet))$$
See [38] Proposition 8.8. The only difference with Proposition 7.6 is that we do not assume that each \( \phi^k \) is injective.

**Proposition 7.9.** Let \( \phi^* : M^* \to N^* \) be a quasi-isomorphism of left bounded complexes in \( A_1 \). Then we have a canonical isomorphisms
\[
R^k F(\phi^*) : R^k F(M^*) \cong R^k F(N^*)
\]

See [38], Corollary 8.9.

**Proposition 7.10.** Let \( \phi^* : M^* \to N^* \) be a quasi-isomorphism of left bounded complexes in \( A_1 \). Assume that all \( N^k \)'s are acyclic for the functor \( F \). Then we have a canonical isomorphisms
\[
R^k F(M^*) \cong H^k(F(N^*)�).
\]

See [38], Proposition 8.12.

### 7.16 Hypercohomology

In the case where \( A_1 \) is the category of sheaves of abelian groups over a topological space \( X \) and \( F \) is the global section functor
\[
\mathcal{S} \to \Gamma(X, \mathcal{S})
\]
we write
\[
\mathbb{H}^k(X, \mathcal{S}^*) := R^k \Gamma(\mathcal{S}^*)�.
\]
The group \( \mathbb{H}^k(X, \mathcal{S}^*) \) is called the \( k \)-the hypercohomology of the complex \( \mathcal{S}^* \).

### 7.17 De Rham cohomology

Let \( M \) be a complex manifold and \( \Omega_M^* \) and \( \Omega_{M,\infty}^* \) be the complex of the sheaves of holomorphic, respectively \( C^\infty \), differential forms on \( M \). In the category of sheaves of abelian groups on \( M \) we have a quasi-isomorphism
\[
\Omega_M^* \to \Omega_{M,\infty}^*
\]
induced by inclusion. We also know that the sheaves \( \Omega_{M,\infty}^k \) are \( \Gamma \)-acyclic. By Proposition 7.10 we conclude that
\[
\mathbb{H}^k(M, \Omega_M^*) \cong H^k_{dR}(M)�.
\]
We also know that
\[
\mathbb{C} \to \Omega_M^*
\]
is the resolution of the constant sheaf \( \mathbb{C} \). This is the same to say that the complex
\[
\cdots \to 0 \to \mathbb{C} \to 0 \to \cdots
\]
with \( \mathbb{C} \) in the 0-th place, is quasi-isomorphic to the complex \( \Omega_M^* \) and so by Proposition 7.9 we have
\[
H^k(M, \mathbb{C}) \cong \mathbb{H}^k(M, \Omega_M^*)�.
\]
7.18 How to calculate hypercohomology

Let us be given a complex of sheaves of abelian groups on a topological space $X$.

\begin{equation}
S : \quad S^0 \xrightarrow{d} S^1 \xrightarrow{d} S^2 \xrightarrow{d} \cdots \xrightarrow{d} S^n \xrightarrow{d} \cdots, \quad d \circ d = 0.
\end{equation}

Let $U = \{U_i\}_{i \in I}$ be a covering of $X$. Consider the double complex

\begin{equation}
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & \uparrow & & \uparrow & \uparrow \\
S_0^n & \rightarrow & S_1^n & \rightarrow & S_2^n & \rightarrow \cdots \rightarrow S_n^n \\
& & \uparrow & & \uparrow & \uparrow \\
S_0^{n-1} & \rightarrow & S_1^{n-1} & \rightarrow & S_2^{n-1} & \rightarrow \cdots \rightarrow S_{n-1}^{n-1} \\
& & \vdots & & \vdots & \vdots \\
& \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
S_2^0 & \rightarrow & S_1^0 & \rightarrow & S_1^0 & \rightarrow \cdots \rightarrow S_1^0 \\
& \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
S_1^0 & \rightarrow & S_0^1 & \rightarrow & S_0^1 & \rightarrow \cdots \rightarrow S_0^1 \\
& \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
S_0^0 & \rightarrow & S_0^0 & \rightarrow & S_0^0 & \rightarrow \cdots \rightarrow S_0^0 \\
\end{array}
\end{equation}

Here $S^i_j$ is the disjoint union of global sections of $S^i$ in the open sets $\cap_{i \in I_1} U_i, I_1 \subset I, \# I_1 = j$. The horizontal arrows are usual differential operator $d$ of $S^i$'s and vertical arrows are differential operators $\delta$ in the sense of Cech cohomology (see Chapter 6). The $k$-th piece of the total chain of (7.3) is

$$L_k := \bigoplus_{i=0}^k S^i_{k-i}$$

with the the differential operator

$$d' = d + (-1)^k \delta : L^k \rightarrow L^{k+1}.$$

**Proposition 7.11.** The hypercohomology $H^m(M, S^\bullet)$ is canonically isomorphic to the direct limit of the total cohomology of the double complex (7.3), i.e.

$$H^m(M, S^\bullet) \cong \text{dirlim}_U H^m(L^\bullet, d').$$

If

\begin{equation}
H^k(U_{i_1} \cap U_{i_2} \cap \cdots U_{i_r}, S^i) = 0, \quad k, r = 1, 2, \ldots, \quad i = 0, 1, 2, \ldots
\end{equation}

then

\begin{equation}
H^m(M, S^\bullet) \cong H^m(L^\bullet, d').
\end{equation}

**Proof.** The fact that the direct limit dirlim$_U H^m(L^\bullet, d')$ is canonically isomorphic to $H^m(L^\bullet, d')$ with $U$ satisfying (7.4) is classic and is left to the reader. Therefore, we just prove the second part of the proposition. The proof is similar to the proof of Proposition 7.5. Let us redefine $I^k_j$ to be the direct product of

$$S^i_{\cap_{i \in I_1} U_i, I_1 \subset I, \# I_1 = j}$$

and

$$I^k := \bigoplus_{i=0}^k I^i_{k-i}$$

We have a canonical quasi-isomorphism $S^\bullet \rightarrow I^\bullet$ with all $I^k$ acyclic. Now our assertion follows from Proposition 7.10. \qed
If one takes (7.5) as a definition of the hypercohomology then it can be shown that this definition is independent of the choice of \( \mathcal{U} \).

Let us assume that \( M \) is a complex manifold and \( \mathcal{U} \) is a Stein covering, i.e. each \( U_i \) is Stein. Further assume that all \( S^k \)'s are coherent. It can be shown that the intersection of finitely many of them is also Stein, see for instance [6] Proposition 1.5. We conclude that for a Stein covering of \( M \) we have (7.5).

### 7.19 Filtrations

For a complex \( S \) and \( k \in \mathbb{Z} \) we define the truncated complexes

\[
S^{\leq k} : \cdots \to S^{k-1} \to S^k \to 0 \to 0 \to \cdots
\]

and

\[
S^{\geq k} : \cdots \to 0 \to 0 \to S^k \to S^{k+1} \to \cdots
\]

We have canonical morphisms of complexes:

\[
S^{\leq k} \to S, \quad S^{\geq k} \to S
\]

Assume that \( S \) is a left-bounded complex,

\[
S : \cdots \to 0 \to S^k \to S^{k+1} \to \cdots
\]

The morphism \( S^{\geq i} \to S \) induces a map in hypercohomologies and we define

\[
F^i := \text{Im}(\mathbb{H}^m(X, S^{\geq i}) \to \mathbb{H}^m(X, S))
\]

This gives us the filtration

\[
\cdots \subset F^i \subset F^{i-1} \subset \cdots \subset F^{k-1} \subset F^k := \mathbb{H}^m(M, S^\bullet).
\]

**Proposition 7.12.** If the maps

\[
H^a(M, S^i) \to H^a(M, S^{i+1}), \quad a \in \mathbb{N}_0, \quad i \in \mathbb{Z}
\]

induced by \( S^i \to S^{i+1} \) are zero then we have canonical isomorphisms

\[
F^i/F^{i+1} \cong H^{m-i}(M, S^i)
\]

**Proof.** We use the second part of Proposition 7.11 (the details are done in the class).

**Remark 7.1.** The hypothesis of Proposition 7.12 is satisfied for the complex of holomorphic differential forms on a complex manifold, see [19], II. The corresponding filtration in this case is called the Hodge filtration.

**Proposition 7.13.** If \( H^a(M, S^i) = 0, \quad a > 0, \quad i \in \mathbb{Z} \) then

\[
\mathbb{H}^m(M, S^\bullet) \cong H^m(\Gamma S^\bullet, d).
\]

**Proof.** We use Proposition 7.11. The hypothesis implies that the vertical arrows in 7.3 are exact. Every element in \( L^k \) is reduced to an element in \( S^k_0 \) whose \( \delta \) is zero and so corresponds to a global section of \( S^k \).

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Exercises

1. Prove that in a category 1. the identity morphism $\text{Id}_X$ is unique 2. the morphism $i : \ker(f) \to X$ is a monomorphism and $j : Y \to \coker(f)$ is an epimorphism.

2. In a category $\mathcal{A}$ prove that up to isomorphism, there is a unique initial or final object.

3. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two categories. Show that the set of functors from $\mathcal{A}_1$ to $\mathcal{A}_2$ and morphisms between them is a category. A functor $F : \mathcal{A} \to \text{Set}$ is called representable if there exists $X \in \text{Ob}(\mathcal{A})$ such that it is isomorphic to the functor $\text{Hom}(X, \cdot)$. Prove that up to isomorphism $X$ is unique.

4. Prove that for an additive category $\text{Hom}(X, 0)$ and $\text{Hom}(0, X)$ are zero group.

5. Use the definition of $X_1 \oplus X_2$ and show that it is unique.

6. Show that $\ker(f)$ and $\coker(f)$ are unique. Define $\text{Im}(f)$ and $\text{Coim}(f)$ and prove they are unique (if you have made the right definition).

7. Let us be given two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $f \circ g = 0$ in an abelian category. Describe the object $\ker(g) \oplus \text{Im}(f)$.

8. Any short exact sequence $0 \to X \to Y \to Z \to 0$ with $X$ an injective object is split, i.e. $Y \cong X \oplus Z$ and $Y \to Z$ is the projection in the second coordinate and $Y \to X$ is the inclusion map.
Chapter 8

Algebraic de Rham cohomology and Hodge filtration

Inspired by the work of Atiyah and Hodge in [24], Grothendieck in [20] introduced the de Rham cohomology in algebraic geometry.

8.1 De Rham cohomology

Definition 8.1. Let $X$ be any prescheme locally of finite type over a field $k$, and smooth over $k$. We consider the complex $(\Omega^\bullet_{X/k}, d)$ if regular differential forms on $X$. The (algebraic) de Rham cohomology of $X$ is defined to be the hypercohomology

$$H^q_{dR}(X/k) = \mathbb{H}^q(X, \Omega^\bullet_{X/k}), \quad q = 0, 1, 2, \ldots.$$ 

Let $k = \mathbb{C}$ be the field of complex numbers and $X^{an}$ be the underlying complex manifold of $X$. In a similar way we can define the analytic de Rham cohomology of $X^{an}$, i.e.

$$H^q_{dR}(X^{an}) = \mathbb{H}^q(X^{an}, \Omega^\bullet_{X^{an}}), \quad q = 0, 1, 2, \ldots.$$ 

where $\Omega^\bullet_{X^{an}}$ is the complex of holomorphic differential forms on $X^{an}$.

The reader must note that in $X^{an}$ we have considered usual topology of complex manifolds and in $X/\mathbb{C}$ we have considered the Zariski topology. By Poincaré lemma $\Omega^\bullet_{X^{an}}$ is the resoultion of the constant sheaf $\mathbb{C}$ and hence

$$H^q_{dR}(X^{an}) \cong H^q(X^{an}, \mathbb{C}), \quad q = 0, 1, 2, \ldots.$$ 

From another side we have the following:

Theorem 8.1. The canonical map

$$H^q_{dR}(X/\mathbb{C}) \to H^q_{dR}(X^{an}), \quad q = 0, 1, 2, \ldots$$

is an isomorphism of $\mathbb{C}$-vectorspaces.
8.2 Atiyah-Hodge theorem

In order to prove Theorem 8.1 we first consider the case in which $X$ is an affine variety.

**Theorem 8.2.** The complex variety $X^{an}$ is an Stein variety and so by Cartan’s B theorem

$$H^i(X^{an}, \Omega^j_{X^{an}}) = 0, \ i = 1, 2, \ldots, j = 0, 1, 2, \ldots$$

By Proposition 7.13 we conclude that

$$H^i_{dR}(X^{an}) = H^i(\Gamma(\Omega^\bullet_{X^{an}}), d)$$

From the algebraic side we have:

**Theorem 8.3.** (Serre vanishing theorem) Let $X$ be an affine smooth variety over the field $\mathbb{C}$ of complex numbers. Then

$$H^i(X, \Omega^j_X) = 0, \ i = 1, 2, \ldots, j = 0, 1, 2, \ldots$$

By Proposition 7.13 we conclude that

$$H^i_{dR}(X/\mathbb{C}) = H^i(\Gamma(\Omega^\bullet_X), d)$$

We are now going to prove that:

**Theorem 8.4** (Atiyah-Hodge). Let $X$ be an affine smooth variety over the field $\mathbb{C}$ of complex numbers. Then the canonical map

$$H^q(\Gamma(\Omega^\bullet_{X/\mathbb{C}}), d) \to H^q_{dR}(X^{an})$$

is an isomorphism of $\mathbb{C}$-vector spaces.

**Proof.** See [32], p. 86. □

**Corollary 8.1.** Let $X$ be an affine smooth variety over the field $\mathbb{C}$ of complex numbers. Every holomorphic $q$-form $\omega_1$ on $X$ can be written as $\omega_1 = \omega_2 + d\omega_3$, where $\omega_2$ is an algebraic $q$-form on $X$ and $\omega_3$ is a holomorphic $(q - 1)$-form on $X$.

**Proof.** Hint: $d\omega_1$ is zero in the algebraic de Rham cohomology of $X$. □

8.3 Proof of Theorem 8.1

Let $\{U_i\}_{i \in I}$ a covering of $X$ with affine varieties. The intersection of any two affine variety is again an affine variety. This implies that the intersection of any finite number of affine varieties in $\{U_i\}$ is again affine. We use Theorem 8.2 and theorem 8.3 and conclude that $\{U_i\}_{i \in I}$ is a good covering in the sense of the second part of Proposition 7.11. This means that we can take the covering $\{U_i\}_{i \in I}$ (resp. $\{U_i^{an}\}_{i \in I}$) to calculate $H^m_{dR}(X/\mathbb{C})$ (resp. $H^{mar}(X^{an})$). From this point on one has to use Atiyah-Hodge theorem and the diagram (7.3).

If $X$ is a projective variety then Theorem 8.1 follows also from Serre’s GAGA.
8.4 Hodge filtration

For the complex of differential forms $\Omega^\bullet_{X/k}$ we have the complex of truncated differential forms:

$$
\Omega^{\geq i}_{X/k} : \cdots \rightarrow \Omega^i_{X/k} \rightarrow \Omega^{i+1}_{X/k} \rightarrow \cdots
$$

and a natural map

$$
\Omega^{\geq i}_{X/k} \rightarrow \Omega^\bullet_{X/k}
$$

We define the Hodge filtration

$$
0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m_{\text{dr}}(X/k)
$$

as follows

$$
F^q = F^q H^m_{\text{dr}}(X/k) = \text{Im} \left( H^m(X, \Omega^{\geq i}_{X/k}) \rightarrow H^m(X, \Omega^\bullet_{X/k}) \right)
$$

**Proposition 8.1.** We have

$$
F^q / F^{q+1} \cong H^{m-q}(X, \Omega^q)
$$

This proposition follows from Proposition 7.12 and the following:

**Proposition 8.2.** The maps

$$
H^a(X, \Omega^i_X) \rightarrow H^a(X, \Omega^{i+1}_X), \ a \in \mathbb{N}_0, \ i \in \mathbb{Z}
$$

induced by the differential map $d : \Omega^i_X \rightarrow \Omega^{i+1}_X$ are all zero.

*Proof.* See [19].

**Proposition 8.3.** Let $k = \mathbb{C}$. Under the canonical isomorphisms

$$
H^q_{\text{dr}}(X/\mathbb{C}) \rightarrow H^q_{\text{dr}}(X^{\text{an}}), \ q = 0, 1, 2, \ldots
$$

the algebraic Hodge filtration is mapped to the usual Hodge filtration of $H^m(X^{\text{an}}, \mathbb{C})$

*Proof.* This follows from the diagram (7.3) for the complex $\Omega^\bullet_{X^{\text{an}}}$ and the fact that the sheaf of $C^\infty$ differential $(p, q)$-forms is acyclic.
Chapter 9

Lefschetz (1, 1) theorem

Lefschetz theorem is special case of the Hodge conjecture. It is stated and proved in the present chapter.

9.1 Lefschetz (1, 1) theorem

Let \( Y \) be a divisor in \( X \). We associate to \( Y \) a line bundle \( L_Y \) in \( X \) in a natural way. Consider the short exact sequence

\[
0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0
\]

and the corresponding long exact sequence

\[
\cdots \to H^1(X, \mathcal{O}^*) \overset{i^*}{\to} H^2(X, \mathbb{Z}) \overset{\delta}{\to} H^2(X, \mathcal{O}) \to \cdots
\]

**Proposition 9.1.** Let \( Y \) be a divisor in \( X \). We have

\[i^*(L_Y) = \text{Poincaré dual of } [Y].\]

**Proof.** See [18], p. 141 Proposition 1.

In \( H^2(X, \mathbb{C}) \) we have the Hodge filtration

\[
\{0\} = F^3 \subset F^2 \subset F^1 \subset F^0 = H^2(X, \mathbb{C})
\]

By Proposition 8.1 we have a canonical isomorphism \( F^0/F^1 \to H^2(X, \mathcal{O}) \) and so we have a canonical projection

\[\tilde{\delta} : H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O})\]

**Proposition 9.2.** The map \( \delta \) and \( \tilde{\delta} \) are the same.

**Proof.** See [18], p. 163.

**Theorem 9.1.** Let \( X \) be a projective variety. Every class in

\[H^2(X, \mathbb{Z}) \cap F^1(H^2(X, \mathbb{C}))\]

is a Poincaré dual of an analytic variety. Here, we write \( H^2(X, \mathbb{Z}) \) to denote its image in \( H^2(X, \mathbb{C}) \).
Proof. This is a direct consequence of Proposition 9.1 and Proposition 9.2. □

**Proposition 9.3.** The Hodge conjecture is true for cycles

\[ H^{2n-2}(X, \mathbb{Z}) \cap F^{n-1,n-1}(H^{2n-2}(X, \mathbb{C})) \]

Proof. This follows from Lefschetz (1,1)-theorem and hard Lefschetz theorem which says that

\[ H^2(X, \mathbb{C}) \to H^{2n-2}(X, \mathbb{C}), \alpha \mapsto \alpha \cup \bigcup_{i=1}^{n-2} P([Y]) \]

is an isomorphism. □

### 9.2 Some consequences on integrals

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Chapter 10

Deformation of hypersurfaces

For a given smooth hypersurface $M$ of degree $d$ in $\mathbb{P}^{n+1}$ is there any deformation of $M$ which is not embedded in $\mathbb{P}^{n+1}$? We need the answer of this question because it would be essential to us to know that the fibers of a tame polynomial $f$ form the most effective family of affine hypersurfaces. The answer to our question is given by Kodaira-Spencer Theorem which we are going to explain it in this section. For the proof and more information on deformation of complex manifolds the reader is referred to [26], Chapter 5.

Let $M$ be a complex manifold and $M_t, t \in B := (\mathbb{C}^*, 0)$, $M_0 = M$ be a deformation of $M_0$ which is topologically trivial over $B$. We say that the parameter space $B$ is effective if the Kodaira-Spencer map

$$\rho_0 : T_0 B \to H^1(M, \Theta)$$

is injective, where $\Theta$ is the sheaf of vector fields on $M$. It is called complete if any other family which contain $M$ is obtained from $M_t, t \in B$ in a canonical way (see [26], p. 228).

**Theorem 10.1.** If $\rho_0$ is surjective at $0$ then $M_t, t \in B$ is complete.

Let $m = \dim \mathbb{C} \times H^1(M, \Theta)$. If one finds an effective deformation of $M$ with $m = \dim B$ then $\rho_0$ is surjective and so by the above theorem it is complete.

Let us now $M$ be a smooth hypersurface of degree $d$ in the projective space $\mathbb{P}^{n+1}$. Let $T$ be the projectivization of the coefficient space of smooth hypersurfaces in $\mathbb{P}^{n+1}$. In the definition of $M$ one has already $\dim T = (\binom{n+1}{d} - d) - 1$ parameters, from which only

$$m := \left(\frac{n+1+d}{d}\right) - (n+2)^2$$

are not obtained by linear transformations of $\mathbb{P}^{n+1}$.

**Theorem 10.2.** Assume that $n \geq 2$, $d \geq 3$ and $(n,d) \neq (2,4)$. There exists a $m$-dimensional smooth subvariety of $T$ through the parameter of $M$ such that the Kodaira-Spencer map is injective and so the corresponding deformation is complete.

For the proof see [26] p. 234. Let us now discuss the exceptional cases. For $(n,d) = (2,4)$ we have 19 effective parameter but $\dim H^1(M, \Theta) = 20$. The difference comes from a non algebraic deformation of $M$ (see [26] p. 247). In this case $M$ is a $K3$ surface. For $n = 1$, we are talking about the deformation theory of a Riemann surface. According to Riemann’s well-known formula, the complex structure of a Riemann surface of genus $g \geq 2$ depends on $3g - 3$ parameters which is again $\dim H^1(M, \Theta)$ ([26] p. 226).
10.1 Reconstructing the period matrix

We have seen that the period matrix $X = \text{pm}^t$ satisfies the differential equation $dX = A^t \cdot X$. Fix a point $t_0 \in T$ and let $\gamma$ be a path in $T$ which connects $t_0$ to $t \in T$. The analytic continuation of the flat section through $I_\mu \times \mu := X(t_0)^{-1}X(t_0)$ and along $\gamma$ is $X(t_0)^{-1}X(t)$. This gives us the equality

$$(10.1) \quad \text{pm}(t) = (I_\mu \times \mu - \int_\gamma A + \int_\gamma AA - \int_\gamma AAA + \cdots )\text{pm}(t_0),$$

where we have used iterated integrals (for further details see [22]). Note that the above series is convergent and the sum is homotopy invariant but its pieces are not homotopy invariants. The equality (10.1) implies that if we know the value of period matrix for just one point $t_0$ then we can construct the period matrix of other points of $T$ using the Gauss-Manin connection. The calculation of period matrix for examples of tame polynomials in $\mathbb{C}[x]$ is done in S??.
Chapter 11

Hodge filtration of Projective hypersurfaces

11.1 Weighted projective spaces

In this section we recall some terminology on weighted projective spaces. For more information on weighted projective spaces the reader is referred to [13, 34].

Let $n$ be a natural number and $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be a vector of natural numbers whose greatest common divisor is one. The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ in the following way:

$$(X_1, X_2, \ldots, X_{n+1}) \rightarrow (\lambda^{\alpha_1}X_1, \lambda^{\alpha_2}X_2, \ldots, \lambda^{\alpha_{n+1}}X_{n+1}), \lambda \in \mathbb{C}^*.$$ 

We also denote the above map by $\lambda$. The quotient space $\mathbb{P}^\alpha := \mathbb{C}^{n+1}/\mathbb{C}^*$ is called the projective space of weight $\alpha$. If $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ then $\mathbb{P}^\alpha$ is the usual projective space $\mathbb{P}^n$ (Since $n$ is a natural number, $\mathbb{P}^n$ will not mean a zero dimensional weighted projective space). One can give another interpretation of $\mathbb{P}^\alpha$ as follow: Let $G_{\alpha_i} := \{e^{2\pi i m/\alpha_i} \mid m \in \mathbb{Z}\}$. The group $\prod_{i=1}^{n+1} G_{\alpha_i}$ acts discretely on the usual projective space $\mathbb{P}^n$ as follows:

$$(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}), [X_1 : X_2 : \cdots : X_{n+1}] \rightarrow [\epsilon_1 X_1 : \epsilon_2 X_2 : \cdots : \epsilon_{n+1} X_{n+1}].$$ 

The quotient space $\mathbb{P}^n/\prod_{i=1}^{n+1} G_{\alpha_i}$ is canonically isomorphic to $\mathbb{P}^\alpha$. This canonical isomorphism is given by

$$[X_1 : X_2 : \cdots : X_{n+1}] \in \mathbb{P}^n/\prod_{i=1}^{n+1} G_{\alpha_i} \rightarrow [X_1^{\alpha_1} : X_2^{\alpha_2} : \cdots : X_{n+1}^{\alpha_{n+1}}] \in \mathbb{P}^\alpha.$$ 

Let $d$ be a natural number. The polynomial (resp. the polynomial form) $\omega$ in $\mathbb{C}^{n+1}$ is weighted homogeneous of degree $d$ if

$$\lambda^*(\omega) = \lambda^d \omega, \lambda \in \mathbb{C}^*.$$

For a polynomial $g$ this means that

$$g(\lambda^{\alpha_1}X_1, \lambda^{\alpha_2}X_2, \ldots, \lambda^{\alpha_{n+1}}X_{n+1}) = \lambda^d g(X_1, X_2, \ldots, X_{n+1}), \forall \lambda \in \mathbb{C}^*.$$
Let \( g \) be an irreducible polynomial of (weighted) degree \( d \). The set \( g = 0 \) induces a hypersurface \( D \) in \( \mathbb{P}^n \), \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \). If \( g \) has an isolated singularity at \( 0 \in \mathbb{C}^{n+1} \) then Steenbrink has proved that \( D \) is a V-manifold/quasi-smooth variety. A V-manifold may be singular but it has many common features with smooth varieties (see [34, 13]).

For a polynomial form \( \omega \) of degree \( dk \), \( k \in \mathbb{N} \) in \( \mathbb{C}^{n+1} \) we have \( \lambda^* \frac{\omega}{g^\lambda} = \frac{\omega}{g^\lambda} \) for all \( \lambda \in \mathbb{C}^* \).

Therefore, \( \frac{\omega}{g^\lambda} \) induce a meromorphic form on \( \mathbb{P}^n \) with poles of order \( k \) along \( D \). If there is no confusion we denote it again by \( \frac{\omega}{g^\lambda} \). The polynomial form

\[
(11.1) \quad \eta_\alpha = \sum_{i=1}^{n+1} (-1)^{i-1} \alpha_i X_i d\widehat{X}_i,
\]

where \( d\widehat{X}_i = dX_1 \wedge \cdots \wedge dX_{i-1} \wedge dX_{i+1} \wedge \cdots dX_{n+1} \), is of degree \( \sum_{i=1}^{n+1} \alpha_i \).

Let \( \mathbb{P}^{(1,\alpha)} = \{[X_0 : X_1 : \cdots : X_{n+1}] \mid (X_0, X_1, \cdots, X_{n+1}) \in \mathbb{C}^{n+2} \} \) be the projective space of weight \( (1, \alpha) \), \( \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \). One can consider \( \mathbb{P}^{(1,\alpha)} \) as a compactification of \( \mathbb{C}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1})\} \) by putting

\[
(11.2) \quad x_i = \frac{X_i}{X_0^\alpha}, \ i = 1, 2, \cdots, n+1
\]

The projective space at infinity \( \mathbb{P}_\infty^\alpha = \mathbb{P}^{(1,\alpha)} - \mathbb{C}^{n+1} \) is of weight \( \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \).

Let \( f \) be a tame polynomial of degree \( d \) in \( S \) and \( g \) be its last quasi-homogeneous part. We take the homogenization \( F = X_0^d f\left(\frac{X_1}{X_0^\alpha}, \frac{X_2}{X_0^\alpha}, \ldots, \frac{X_{n+1}}{X_0^\alpha}\right) \) of \( f \) and so we can regard \( \{f = 0\} \) as an affine subvariety in \( \{F = 0\} \subset \mathbb{P}^{(1,\alpha)} \).

### 11.2 Complement of hypersurfaces

This section is dedicated to a classic theorem of Griffiths in [19]. Its generalization for quasi-homogeneous spaces is due to Steenbrink in [34].

**Theorem 11.1.** Let \( g(X_1, X_2, \cdots, X_{n+1}) \) be a weighted homogeneous polynomial of degree \( d \), weight \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \) and with an isolated singularity at \( 0 \in \mathbb{C}^{n+1} \) (and so \( D = \{g = 0\} \) is a V-manifold). We have

\[
H^n(\mathbb{P}^\alpha - D, \mathbb{C}) \cong H^0(\mathbb{P}^\alpha, \mathcal{O}^n(D)) / H^0(\mathbb{P}^\alpha, \mathcal{O}^{n-1}(D))
\]

and under the above isomorphism

\[
G^W_{n+1, k} H^{n+1}(\mathbb{P}^\alpha - D, \mathbb{C}) \cong F^{n-k+1} / F^{n-k+2}
\]

\[
H^0(\mathbb{P}^\alpha, \mathcal{O}^n(k D)) / H^0(\mathbb{P}^\alpha, \mathcal{O}^{n-1}((k-1) D)) + H^0(\mathbb{P}^\alpha, \mathcal{O}^{n-1}((k-1) D))
\]

where \( 0 = F^{n+1} \subset F^n \subset \cdots \subset F^1 \subset F^0 = H^n(\mathbb{P}^\alpha - D, \mathbb{C}) \) is the Hodge filtration of \( H^n(\mathbb{P}^\alpha - D, \mathbb{C}) \). Let \( \{X^\beta \mid \beta \in I\} \) be a basis of monomials for the Milnor vector space

\[
\mathbb{C}[X_1, X_2, \cdots, X_{n+1}] / \frac{\partial g}{\partial X_i} \mid i = 1, 2, \ldots, n+1
\]

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A basis of (11.3) is given by

\[(11.4) \quad \frac{X^\beta \eta_\alpha}{g^k}, \; \beta \in I, \; A_\beta = k\]

where

\[\eta_\alpha = \sum_{i=1}^{n+1} (-1)^{i-1} \alpha_i X_i dX_i\]

In the situation of the above theorem \(F^0 = F^1\). The essential ingredient in the proof is Bott’s vanishing theorem for quasi-homogeneous spaces and Proposition 7.13.
Chapter 12

Gauss-Manin connection
Chapter 13

Mixed Hodge structure of affine varieties

13.1 Logarithmic differential forms and mixed Hodge structures of affine varieties

Let $M$ be a projective smooth variety and $D = D_1 + D_2 + \cdots + D_s$ be a normal crossing divisor in $M$, i.e. to each point $p \in M$ there are holomorphic coordinates $z_1, z_2, \ldots, z_n$ around $p$ such that $D = \{z_1 = 0\} + \cdots + \{z_i = 0\}$ for some $i$ depending on $p$. Let also $\Omega^i_M(\log D)$ be the sheaf of meromorphic $i$-forms $\omega$ in $M$ with logarithmic poles along $D$, i.e. $\omega$ and $d\omega$ have poles of order at most one along $D$. This is equivalent to the fact that around each point $p \in M$ the sheaf $\Omega^k_M(\log D)$ is generated by $k$-times wedge products of $\frac{dz_1}{z_1}, \frac{dz_2}{z_2}, \ldots, \frac{dz_i}{z_i}, dz_{i+1}, \ldots, dz_n$, where $i$ is as above. Let $A^\bullet$ be the complex of $C^\infty$ differential forms in $M \setminus D$ and $j : M \setminus D \to M$ be the inclusion. A map between two differential complexes, $p : S_1^\bullet \to S_2^\bullet$ is called to be a quasi-isomorphism if the induced maps

$$H^k(S_{1,x}, d) \to H^k(S_{2,x}, d), \ k = 0, 1, \ldots, \ x \in M$$

are isomorphisms, where $S_k^x$ is the stalk of $S^k$ over $x \in M$.

**Proposition 13.1.** The canonical map

$$\Omega^\bullet(\log D) \to j_* A^\bullet$$

is a quasi isomorphism.

This proposition is proved [19] and [10]. See also [38] Proposition 8.18. The above proposition implies that we have an isomorphism

$$H^k(M \setminus D, \mathbb{C}) \cong \mathbb{H}^k(M, \Omega^\bullet_M), \ k = 0, 1, \ldots.$$ 

The Hodge filtration on $H^m(M \setminus D, \mathbb{C})$ is given by

$$F^k H^m(M \setminus D, \mathbb{C}) := \text{Im}(\mathbb{H}^m(F^k \Omega^\bullet_M(\log D)) \to \mathbb{H}^m(\Omega^\bullet_M(\log D))))$$

where for a differential complex $S^\bullet$

$$F^k S^\bullet := S^\geq k : 0 \to 0 \to \cdots \to 0 \to S^k \to S^{k+1} \to \cdots \to S^n \to \cdots$$

$k$ times 0
is the bete filtration. By definition we have $F^0/F^1 \cong H^n(M, \mathcal{O}_M)$. The weight filtration on $H^m(M \setminus D, \mathbb{C})$ is given by

$$W_k H^m(M \setminus D, \mathbb{C}) := \text{Im}(\mathbb{H}^m(P_k \Omega^m_M(\log(D)) \to \mathbb{H}^m(\Omega^m_M(\log(D))))), \ k \in \mathbb{Z},$$

where $P_k \Omega^m_M(\log(D))$ is the Deligne pole order filtration: A logarithmic differential $m$-form $\omega$ is in $P_k \Omega^m_M(\log(D))$ if in local coordinates, there does not appear a wedge product of more than $k'$-times $\frac{dz_i}{z_i}$, $k' > k$ in $\omega$. The Hodge filtration induces a filtration on $Gr^W_a := W_a/W_{a-1}$ and we set

$$(13.1) \quad Gr^b_a Gr^W_a := F^b Gr^W_a / F^{b+1} Gr^W_a = \frac{(F^b \cap W_a) + W_{a-1}}{(F^{b+1} \cap W_a) + W_{a-1}}, \ a, b \in \mathbb{Z}$$

In the next sections we will introduce two other filtrations which are called again pole order filtrations and have completely distinct nature.

### 13.2 Pole order filtration

For an analytic sheaf $\mathcal{S}$ on $M$ we denote by $\mathcal{S}(\ast D)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of arbitrary order along $D$. For $k = (k_1, k_2, \ldots, k_s) \in \mathbb{N}^s_0$ we denote by $\mathcal{S}(kD)$ the sheaf of meromorphic sections of $\mathcal{S}$ with poles of order at most $k_i$ along $D_i$, $i = 1, 2, \ldots, s$. For $\mathcal{S}(\ast D)$ we have also the pole filtration:

$$P^k \mathcal{S}^\ast(\ast D) : 0 \to 0 \to \cdots \to 0 \to S^k_0 \to S^{k+1}_1 \to \cdots \to S^0_{p-k} \to \cdots,$$

where

$$S^p_{p-k} := \bigcup_{|n| \leq p-k} S^i((n+1)D) \text{ if } p \geq k,$$

$$(n+1) = (n_1 + 1, n_2 + 1, \ldots, n_s + 1), \ |n| = n_1 + n_2 + \cdots + n_s.$$ 

Let $(\mathcal{S}_1, F)$ and $(\mathcal{S}_2, F)$ be two filtered differential complexes and $p : (\mathcal{S}_1, F) \to (\mathcal{S}_2, P)$ a map between them, i.e. we have a collection of maps $p_k : F^k \mathcal{S}_1 \to P^k \mathcal{S}_2$ such that the following diagram commutes

$$\begin{array}{ccc}
F^{k+1} \mathcal{S}_1^\ast & \to & P^{k+1} \mathcal{S}_2^\ast \\
\downarrow & & \downarrow \\
F^k \mathcal{S}_1^\ast & \to & P^k \mathcal{S}_2^\ast
\end{array}, \ k = 0, 1, \ldots$$

The map $p$ is called a quasi-isomorphism of filtered complexes if $p_k, \ k = 0, 1, \ldots$ are quasi-isomorphisms.

**Theorem 13.1.** The inclusion

$$(\Omega^\ast_M (\log D), F) \subset (\Omega^\ast_M (\ast D), P)$$

is a quasi-isomorphisms of filtered differential complexes.

**Proof.** Note that this is a local statement and so we can suppose that $M = (\mathbb{C}^n, 0), \ D = \{z_1 = 0\} + \{z_2 = 0\} + \cdots + \{z_s = 0\}$. The proof can be found in [10] Proposition 3.13, [11] Proposition 3.1.8. See also [39] Proposition 8.18. According to [9] p.647, Deligne has inspired the above theorem from the work of Griffiths .

The above proposition implies that the Hodge filtration on $H^m(M \setminus D, \mathbb{C})$ is also given by

$$F^i H^m(M \setminus D, \mathbb{C}) = \text{Im}(\mathbb{H}^m(P^i \Omega^m_M(\ast D)) \to \mathbb{H}^m(\Omega^m_M(\ast D))).$$
13.3 Another pole order filtration

In this section we assume that $D$ is a positive divisor, i.e., the associated line bundle is positive. From this what we need is the following: For any coherent analytic sheaf $S$ on $M$ we have

$$H^k(M, S(*D)) = 0, \ k = 1, 2, \ldots$$

(13.3)

We do not assume that $D$ is a normal crossing divisor.

**Theorem 13.2.** (Atiyah-Hodge-Grothendieck) If $D$ is positive then

$$H^k(M \setminus D, \mathbb{C}) \cong H^k(M \Omega^*_M(*D), d), \ k = 0, 1, 2, \ldots$$

(13.4)

**Proof.** The proof follows from Proposition 7.13 and (13.3).

From now on assume that $D$ is irreducible. To each cohomology class $\alpha \in H^m(M \setminus D, \mathbb{C})$ we can associate $P(\alpha) \in \mathbb{N}$ which is the minimum number $k$ such that there exists a meromorphic $m$-form in $M$ with poles of order $k$ along $D$ and represents $\alpha$ in the isomorphism (13.4). We have

$$P(\alpha + \beta) \leq \max\{P(\alpha), P(\beta)\}, \ P(k\alpha) = P(\alpha), \ \alpha, \beta \in H^m(M \setminus D, \mathbb{C}), \ k \in \mathbb{C}\{0\}.$$

Using the above facts for a $\mathbb{C}$-basis of $H^m(M \setminus D, \mathbb{C})$, we can find a number $h$ such that for all $\alpha \in H^m(M \setminus D, \mathbb{C})$ we have $P(\alpha) \leq h$. We take the minimum number $h$ with the mentioned property. Now we have the filtration

$$H_0 \subset H_1 \subset \cdots \subset H_{h-1} \subset H_h = H^m(M \setminus D, \mathbb{C})$$

$$H_i = \{\alpha \in H^m(M \setminus D, \mathbb{C}) \mid P(\alpha) \leq i\}, \ i = 0, 1, \ldots, h.$$

We call it the new pole order filtration.

Complementary notes

1. As the reader may have noticed, Theorem 11.1 implies that for the complement of smooth hypersurfaces the pole order filtrations in S13.2 and S13.3 are the same up to reindexing the pieces. In this point the following question arises: Can one find the pieces of the Hodge filtration of $H^m(M \setminus D, \mathbb{C})$ inside the pieces of the new pole order filtration? Using Riemann-Roch Theorem one can find also positive answers to this question for Riemann surfaces. However, the question in general, as far as I know, is open.

13.4 Mixed Hodge structure of affine varieties

Our main examples of modular foliations in Chapters ?? and ?? are associated to a family of affine hypersurfaces and polynomial differential forms in $\mathbb{C}^{n+1}$. Such differential forms have poles at infinity and the corresponding pole order gives us the first numerical invariant to distinguish between differential forms and hence the corresponding modular foliations. Another way to distinguish between differential forms is by looking at their classes in the de Rham cohomology and its Hodge filtration. It is believed that there exists a close relation between the mentioned concepts and the testimonies to this belief are P. Griffiths theorem on the Hodge filtration of the complement of a smooth hypersurface (see S11.2) and some calculations related to Riemann surfaces.
Chapter 14

Lixo de Hodge Theory

The truth in mathematics is a state of satisfaction but not vice versa. The classical way of doing mathematics is to prove, and hence to feel, the truth and then to enjoy the consequent satisfaction. However, with the rapid development of mathematics it seems to be very difficult to transfer to an student all the details leading to a truth. Manytimes we need to use an object and in order to construct it explicitly we spend a lot of time so that the student lose all his/her interest on the subject. In this situation, I think, it is crucial to invest on inducing the state of satisfaction in new learners rather than using the classical methodology of doing mathematics which is defining and proving every thing precisely. In the present text I will try to follow this method in order to introduce one of the main conjectures in Algebraic Geometry, namely the Hodge conjecture.
Bibliography


