12/13/2021

Dear Movasati,

I am bad at giving references. I can only tell why I believe things are true, and where I would look if I needed a reference. Myself, I wrote things up about the class of a cycle only in the ℓ-adic étale cohomology context [in SGA 4½]

Flatness is natural when you think in term of Chern classes: for $Z \subset X$ (X smooth), <u>up to</u> a sign and a factorial [$(\mathrm{codim}(Z)-1)!$], the class of Z of codimension d is $c_d(\mathcal{O}_Z)$ (Chern class of the structural sheaf of Z, viewed as a coherent $\mathcal{O}_X$-module; this is the lowest c_i here).
If now you have a flat family of subschemes

$$\begin{tikzcd} Z \arrow[r, hook] & X \arrow[d] & X_s \arrow[d] \\ & S & s \arrow[l, hook'] \end{tikzcd}$$

with X smooth over S, flatness ensures that in K-theory $\mathcal{O}_Z$ over X pulls back to $\mathcal{O}_{Z_s}$ over the fiber X_s, so that $c_*(\mathcal{O}_Z)$ in $H^*(X)$ pulls back to $c_*(\mathcal{O}_{Z_s})$ in $H^*(X_s)$. I guess this gives what you want. I hope it also gives cohomology classes with support in the cycle, as $\mathcal{O}_Z$ in K-theory is with support in Z. It also suggest flatness is too much:

flatness controls all $c_*(\mathcal{O}_Z)$ for a subscheme Z, while for a cycle of codimension d, only the class in H^{2d} has a meaning. Subschemes Z_1, Z_2 algebraically equivalent as cycles can give rise to different higher c_i. Example: in $\mathbb{P}^3$, 2 skew lines, versus 2 lines meeting in a point [not in the same component of the Hilbert scheme]

What I was meaning about flatness is the theorem that for any morphism of schemes [assumed not ~~too~~ /terribly bad]

$$\begin{array}{c} X \\ f\downarrow \\ S \end{array}$$

with S reduced, there exists $U \hookrightarrow S$ open and dense above which X is flat.

As often, I could give a reference (in EGA) in algebraic geometry, and I can only hope a parallel statement holds in the analytic case. Care would be needed in the analytic case for a non proper f.

About finding references, I would look in Barlet (and references he might give), Berthelot (he cares about crystalline cohomology, but this includes algebraic de Rham), SGA 6 (about Riemann-Roch, but which however avoids cohomology), perhaps Verdier.

Best

P. D