

Large-Time Behavior of Entropy Solutions of Conservation Laws

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We are concerned with the large-time behavior of discontinuous entropy solutions for hyperbolic systems of conservation laws. We present two analytical approaches and explore their applications to the asymptotic problems for discontinuous entropy solutions. These approaches allow the solutions of arbitrarily large oscillation without a priori assumption on the ways from which the solutions come. The relation between the large-time behavior of entropy solutions and the uniqueness of Riemann solutions leads to an extensive study of the uniqueness problem. We use a direct method to show the large-time behavior of large L^∞ solutions for a class of $m \times m$ systems including a model in multicomponent chromatography; we employ the uniqueness of Riemann solutions and the convergence of self-similar scaling sequence of solutions to show the asymptotic behavior of large BV solutions for the 3×3 system of Euler equations in thermoelasticity. These results indicate that the Riemann solution is the unique attractor of large discontinuous entropy solutions, whose initial data are $L^\infty \cap L^1$ or $BV \cap L^1$ perturbation of the Riemann data, for these systems. These approaches also work for proving the large-time behavior of approximate solutions to hyperbolic conservation laws. © 1999 Academic Press

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1. INTRODUCTION

Consider a hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^m, \quad (1.1)$$

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where the flux function $f(u)$ is a nonlinear mapping from \mathbb{R}^m to \mathbb{R}^m . The condition of strict hyperbolicity requires that the Jacobian $\nabla f(u)$ have m real distinct eigenvalues $\lambda_j(u)$, and m linearly independent left and right eigenvectors $l_j(u)$, $r_j(u)$, $1 \leq j \leq m$:

$$l_j(u) \nabla f(u) = \lambda_j(u) l_j(u), \quad \nabla f(u) r_j(u) = \lambda_j(u) r_j(u),$$

respectively. That is, the Jacobian $\nabla f(u)$ is diagonalizable for any value of u .

We are concerned with the large-time behavior of any discontinuous entropy solution $u(t, x)$ of (1.1) taking its initial data,

$$u|_{t=0} = u_0(x) \equiv R_0(x) + P_0(x), \quad (1.2)$$

where

$$R_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \quad (1.3)$$

and $P_0(x)$ is its perturbation satisfying

$$P_0(x) \in L^\infty \cap L^1(\mathbb{R}) \quad \text{or} \quad BV \cap L^1(\mathbb{R}). \quad (1.4)$$

Let $R(x/t)$ be a Riemann solution governed by (1.1) and

$$u|_{t=0} = R_0(x). \quad (1.5)$$

The problem we consider here is whether, for a certain $R(\xi)$,

$$u(t, \xi t) \rightarrow R(\xi)$$

in some topological sense as $t \rightarrow \infty$ (see Definition 2.1) for any initial perturbation $P_0(x)$ satisfying (1.4). Then the function $R(x/t)$ must be a self-similar Riemann solution. That is, our problem is whether the Riemann solution $R(x/t)$ is the unique attractor of any entropy solution as long as its initial data satisfy (1.2)–(1.4). This implies the asymptotic stability of the Riemann solution with respect to the initial perturbation $P_0(x)$ in the topological sense. The significance of (2.4) in Section 2 is its equivalence to the L^1_{loc} -convergence of the whole self-similar scaling sequence of the entropy solution, whose formal argument has motivated many results on the large-time behavior of solutions for viscous conservation laws in recent decades. Furthermore, for any system endowed with a strictly convex entropy, the stability in the sense of (2.4) implies actually the stability in the strong sense of (2.5).

The main objective of this paper is to present two analytical approaches and explore their applications to studying the large-time behavior of

discontinuous entropy solutions to hyperbolic systems of conservation laws.

In Section 2, we introduce a rigorous mathematical sense for the large-time behavior of discontinuous entropy solutions via the convergence along the rays emanating from the origin, both in time-average (*weak asymptotics*) and in a usual sense in time (*strong asymptotics*). The equivalence between time-average and scale-invariance is shown. Possible analytical approaches for studying the asymptotic problems are discussed. Theorem 2.3 indicates that, for the systems endowed with a strictly convex entropy, any weak asymptotics in the sense of (2.4) implies the corresponding strong asymptotics in the sense of (2.5). This result is achieved by using a method motivated from the arguments in [40]. The situation here is similar to the one in [4]: The convergence in time-average along rays implies the convergence in the usual sense in time with the aid of the entropy inequality. Because of Theorem 2.3 in Section 2, the discussions in the other sections will be centered in the sense of weak asymptotics, since all the results contained therein can be immediately strengthened to the sense of strong asymptotics.

In Section 3, we provide a direct application of the approaches, introduced in Section 2, to the scalar conservation laws. We show that any Riemann solution of multidimensional scalar conservation laws is asymptotically stable with large $L^\infty \cap L^1$ initial perturbation in the sense of (2.4). In particular, any planar Riemann solution is asymptotically stable with respect to any large multidimensional $L^\infty \cap L^1$ initial perturbation in the strong sense of (2.5), provided that the corresponding flux function, which determines the planar Riemann solution, contains only isolated reflexion points.

In Section 4, we present a direct analytical approach, the ray method, through a class of hyperbolic systems for studying the large-time behavior of entropy solutions. Such a class includes a model in multicomponent chromatography (see [36]). We prove in Theorem 4.1 that any entropy solution of (1.1)–(1.4) with arbitrarily large data asymptotically tends to the Riemann solution. This means that the Riemann solution is the unique attractor of any L^∞ discontinuous entropy solution whose initial data are *arbitrarily large* perturbation of the Riemann data (1.3) in the sense of Definition 2.1. We remark that, for general $m \times m$ systems, $m \geq 3$, neither a convergence result for the whole scaling sequence in L^1_{loc} nor a uniqueness result for Riemann solutions, in the class of general L^∞ entropy solutions, is now available, although some partial results are known. The direct method is the only one to make this possible so far. For a particular system, such a compactness is known [23]. Some partial uniqueness results for small solutions in BV , which do not cover all types of Riemann solutions, have been obtained (cf. [9, 19]). We also refer to [20] in which a theorem established implies the uniqueness of the Riemann solutions in the

class of L^∞ self-similar entropy solutions assuming values in a small neighborhood of a constant state. The latter result would be useful for the present investigation if one could show the self-similar structure of the limits of subsequences of the scaling sequence associated with a given entropy solution, which requires further analysis. To handle the case of L^∞ solutions, we need to use some basic facts about divergence-measure fields (see [5–7]).

In Section 5 we present another approach, through several classes of systems, with the aid of Theorem 2.1 that the uniqueness of a Riemann solution plus the compactness of the self-similar scaling sequence of entropy solutions implies the asymptotic stability of the Riemann solution with initial $L^\infty \cap L^1$ or $BV \cap L^1$ perturbation in the sense of Definition 2.1. These results indicate that, for these systems, the Riemann solution is the *unique* attractor for such an initial perturbation. The classes of systems we consider include 2×2 strictly hyperbolic and genuinely nonlinear systems and the 3×3 Euler system (5.34) and (5.38) in thermoelasticity. We focus mainly on the uniqueness of Riemann solutions for these systems to achieve their asymptotic stability by following the approach. We develop the ideas in [12] to obtain the uniqueness of Riemann solutions in several different situations.

Our first result is the uniqueness of a Riemann solution in the class of large L^∞ entropy solutions for the p -system, provided that its Riemann data u_L and u_R are connected only by rarefaction wave curves. Then it is extended to general 2×2 strictly hyperbolic and genuinely nonlinear systems for L^∞ solutions of small oscillation, with the aid of the basic facts of divergence-measure fields (see [5–7]). Combining these results with a compactness theorem in [10] yield the asymptotic stability of Riemann solutions for the Cauchy problem of such systems in the sense of Definition 2.1. For the p -system, arbitrarily large initial perturbation is allowed.

We also recall a uniqueness result of Riemann solutions in the class of BV solutions by DiPerna [12] for 2×2 systems whose characteristic fields are either genuinely nonlinear or linearly degenerate. Combining this result with the compactness of bounded sets in BV implies the asymptotic stability of Riemann solutions in the class of BV entropy solutions of (1.1)–(1.2) with $O(T_0)$ growth of total variation over $[0, T_0] \times \mathbb{R}$ and small oscillation (not necessary for the p -system). This growth of total variation is natural for the solutions obtained from the Glimm method [16].

We then come to the main part of Section 5 for the 3×3 Euler equations in thermoelasticity. The first is the uniqueness of Riemann solutions in the class of L^∞ entropy solutions, provided the initial left and right states of the Riemann data are connected only by rarefaction curves of the first and third families and, possibly, a contact discontinuity curve of the second family. No assumption of small oscillation is required here. Once more some basic facts of divergence-measure fields in [5–7] are used. Combining

this uniqueness result with a compactness result in [3] yields the asymptotic stability of shock-free Riemann solutions with respect to the initial perturbation $P_0(x)$ satisfying (1.4) (with the entropy function $s(t, x)$ in a weaker sense). The second is the uniqueness of general Riemann solutions in the class of BV solutions. Again, no assumption of small oscillation is required for this case. This result together with the compactness of bounded sets in BV implies the asymptotic stability of Riemann solutions in the class of BV entropy solutions whose initial data $u_0(x)$ satisfy (1.2)–(1.4) with $O(T_0)$ growth of its total variation over $[0, T_0] \times \mathbb{R}$.

In Section 6, we discuss how to apply the approaches we developed in Sections 2–5 to studying the asymptotic problems for approximate solutions. We show this for the viscous case.

Finally, we comment on some essential differences between our asymptotic results obtained from the approaches developed here and earlier results on related problems. First, there has been a large literature on the asymptotic stability of viscous shock profiles and rarefaction waves (see, e.g., [22, 18, 34, 24, 30, 43], [35, 44], and references cited therein). In general, their analysis is based on energy estimates and gives more precise information about the large-time behavior of the solutions, besides implying the asymptotic stability in the sense of Definition 2.1. However, they are suitable only for viscous equations and, as far as we know, it has not been possible to treat general large perturbation of Riemann data with both shock and rarefaction waves for such systems by a similar analysis. There is also an important analysis of large-time behavior of Glimm solutions of hyperbolic conservation laws introduced by Liu (see [31, 32]), which is designed specifically for solutions obtained from the Glimm method. In his analysis the asymptotic approach to the Riemann solution is obtained in terms of a norm, which is equivalent to the total variation for small initial data. It is not difficult to see that the results obtained for 2×2 systems in [31] imply the asymptotic stability of the Riemann solution in the class of solutions, obtained from the Glimm method, in the sense of Definition 2.1. The main motivation of this paper is to develop new approaches that are applicable to general large entropy solutions, constructed by any method, for hyperbolic systems of conservation laws.

Some results in this paper have been announced in [5].

2. SCALE-INVARIANCE, TIME-AVERAGE, AND LARGE-TIME BEHAVIOR

In this section we discuss the relation between time-average used widely in many fields and self-similar scale-invariance of underlying conservation laws to understand the large-time behavior of entropy solutions.

A bounded measurable function $u(t, x)$ is said an entropy (or admissible) solution of (1.1)–(1.2) in $\Pi_T = [0, T) \times \mathbb{R}$ if, for any C^2 convex entropy pair $(\eta(u), q(u))$, $\nabla\eta(u) \geq 0$, determined by

$$\nabla q(u) = \nabla\eta(u) \nabla f(u), \tag{2.1}$$

$u(t, x)$ satisfies

$$\partial_t \eta(u) + \partial_x q(u) \leq 0 \tag{2.2}$$

in the sense of distributions. That is,

$$\iint_{\Pi_T} \{ \eta(u(t, x)) \phi_t + q(u(t, x)) \phi_x \} dx dt + \int_{-\infty}^{\infty} \eta(u_0(x)) \phi(0, x) dx \geq 0, \tag{2.3}$$

for any nonnegative smooth function ϕ with compact support contained in $[0, T) \times \mathbb{R}$.

3.1. Scale-Invariance, Time-Average, and Weak Asymptotics

The problem we want to understand is whether any entropy solution $u(t, \xi t)$ of (1.1)–(1.4) converges to a certain function $R(\xi)$ in some topological sense as $t \rightarrow \infty$.

DEFINITION 2.1. An entropy solution of (1.1)–(1.2) has a function $R(x/t)$ as its *weak asymptotics* provided

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)| dt = 0, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{2.4}$$

The function $R(x/t)$ is said to be the *strong asymptotics* of $u(t, x)$ provided

$$\text{ess } \lim_{t \rightarrow \infty} |u(t, t\xi) - R(\xi)| = 0, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{2.5}$$

In either case we say that $R(x/t)$ is an attractor or an asymptotic equilibrium of $u(t, x)$. If any entropy solution $u(t, x)$ of (1.1)–(1.4) with $P(x) \in L^\infty \cap L^1(\mathbb{R}^n)$ or $BV \cap L^1(\mathbb{R}^n)$ asymptotically tends to the same function $R(x/t)$, we say $R(x/t)$ is *asymptotically stable* with respect to initial perturbation $P(x)$ or the unique attractor for such solutions in the sense of (2.4) or (2.5).

Remark 2.1. We observe that (2.4) is equivalent to

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{\theta T}^T |u(t, t\xi) - R(\xi)| dt = 0, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n), \tag{2.6}$$

for some $\theta \in (0, 1)$. Indeed, the fact that (2.4) implies (2.6) is obvious. On the other hand, assuming that (2.6) holds, using

$$\begin{aligned} \frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)| dt &\leq \theta \frac{1}{\theta T} \int_0^{\theta T} |u(t, t\xi) - R(\xi)| dt \\ &\quad + \frac{1}{T} \int_{\theta T}^T |u(t, t\xi) - R(\xi)| dt, \end{aligned}$$

and taking the lim sup of both sides, one arrives at

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)| dt \leq \theta \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)| dt,$$

which yields the reverse implication.

Remark 2.2. The strong asymptotics in the sense of (2.5) in connection with conservation laws was considered earlier by Weinberger [48], where the strong asymptotics of the viscous solution to the Riemann solution was obtained for a one-dimensional scalar conservation law with flux function containing only isolated inflexion points.

For the systems considered here, $R(x/t)$ in Definition 2.1 will be the classical self-similar Riemann solution of (1.1) and (1.5) (see [27]). This can be better explained through the self-similar scaling sequence of $u(t, x)$:

$$u^T(t, x) = u(Tt, Tx), \quad T > 0.$$

The following theorem holds for any dimension of space variables.

THEOREM 2.1. *Let $v(t, x)$ be a measurable function defined on $(0, \infty) \times \mathbb{R}^n$ satisfying*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T |v(t, \xi t)| dt \leq C, \quad (2.7)$$

on any compact set in \mathbb{R}^n , for some $C > 0$ independent of ξ . Then its scaling sequence v^T converges to 0 in $L^1_{\text{loc}}(\mathbb{R}_+^{n+1})$ as $T \rightarrow +\infty$ if and only if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |v(t, \xi t)| dt = 0, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n). \quad (2.8)$$

Proof. Suppose v^T converges to 0 in $L^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$. Fix $N > 0$ and $T > 0$. Clearly, we have

$$\begin{aligned} \frac{1}{T} \int_0^T |v(t, \xi t)| dt &= \frac{1}{T} \int_0^{T/N} |v(t, \xi t)| dt + \frac{1}{T} \int_{T/N}^T |v(t, \xi t)| dt \\ &\leq \frac{1}{N} \left(\frac{N}{T} \int_0^{T/N} |v(t, \xi t)| dt \right) + \frac{N^n}{T^{n+1}} \int_0^T |v(t, \xi t)| t^n dt. \end{aligned}$$

Integrate each term of the inequality in ξ over a given compact $K \subset \mathbb{R}^n$ and then take the lim sup when T goes to infinity. The second term, after the change of variables $t' = t/T$, transforms into $N^n \int_K \int_0^1 |v^T(t, \xi t)| t^n dt d\xi$, which goes to 0, when $T \rightarrow \infty$, by assumption. For the first term we have an estimate of the form \tilde{C}/N , for some positive constant \tilde{C} , because of (2.7). Since $N > 0$ is arbitrary, we make $N \rightarrow \infty$ to get (2.8). This proves the direct implication. The converse is straightforward. ■

Set $v(t, x) = u(t, x) - R(x/t)$. We clearly see the equivalence between the asymptotic behavior of $u(t, x)$, given by (2.4), and the convergence of the scaling sequence u^T to $R(x/t)$ in $L^1_{\text{loc}}(\mathbb{R}^2_+)$. This equivalence motivates several different approaches to solve the asymptotic problem of entropy solutions. One is a direct approach, as we will see in Section 4, to understand directly the asymptotic behavior of the solution through the rays $\xi = x/t$, $\xi \in \mathbb{R}$, without resorting to the equivalence. Another approach, which makes use of the equivalence, is to invoke the compactness of the scaling sequence of the perturbed solution, when it is a priori known or else to prove, and the uniqueness of the Riemann solution in a class of solutions which includes all possible limits of the scaling sequence. We will use this approach in Section 5 for several classes of systems. Both cases will yield the L^1_{loc} -convergence of the *whole* self-similar scaling sequence of the entropy solution.

Besides the approaches just mentioned, there is also a situation in which the asymptotic stability of Riemann solutions in the sense of (2.4) is immediately verified. This is given by the following theorem, which is stated for the general case of several space variables.

THEOREM 2.2. *Assume that $u(t, x)$, $R(x/t) \in L^\infty(\mathbb{R}^{n+1}_+)$ satisfy that there exist $C > 0$, independent of t , and $\gamma \in [0, 1)$ such that*

$$\|u - R\|_{L^p(\mathbb{R}^n)}^p(t) \leq Ct^{n-1+\gamma}, \quad \text{for some } 1 \leq p < \infty. \quad (2.9)$$

Then $u(t, x)$ tends to the values of $R(\xi)$ along almost all rays $x/t = \xi$:

$$\frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)| dt \rightarrow 0, \quad T \rightarrow \infty, \quad \text{for almost all } \xi \in \mathbb{R}^n. \quad (2.10)$$

Proof. For any $r > 0$, condition (2.9) implies

$$\int_{|\xi| \leq r} |u(t, t\xi) - R(\xi)|^p d\xi \leq C(1+t)^{\gamma-1}, \quad \text{for any } t > 1,$$

that is,

$$\int_{|\xi| \leq r} \frac{|u(t, t\xi) - R(\xi)|^p}{(1+t)^\theta} d\xi \leq C(1+t)^{\gamma-1-\theta}, \quad \text{for } t > 1 \text{ and } \gamma < \theta < 1.$$

One concludes that

$$\int_0^\infty \int_{|\xi| \leq r} \frac{|u(t, t\xi) - R(\xi)|^p}{(1+t)^\theta} d\xi dt < \infty.$$

Then, for almost all ξ with $|\xi| \leq r$, one has

$$\int_0^\infty \frac{|u(t, t\xi) - R(\xi)|^p}{(1+t)^\theta} dt \leq M(\xi) < \infty.$$

Since $r > 0$ is arbitrary, for almost all $\xi \in \mathbb{R}^n$ and $T > 1$, we have

$$\begin{aligned} & \frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)|^p dt \\ & \leq \frac{2^\theta}{T^{1-\theta}} \int_0^T \frac{|u(t, t\xi) - R(\xi)|^p}{(1+t)^\theta} dt \rightarrow 0, \quad \text{when } T \rightarrow \infty. \end{aligned}$$

This implies, using Jensen's inequality, that

$$\left(\frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)| dt \right)^p \leq \frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)|^p dt \rightarrow 0,$$

for a.e. $\xi \in \mathbb{R}^n$, when $T \rightarrow \infty$, which is (2.10). ■

2.2. Weak Asymptotics Implies Strong Asymptotics

For the systems endowed with a strictly convex entropy, we now show how the weak asymptotics of any entropy solution can be automatically strengthened to allow the passage from the notion of weak to that of strong asymptotics for the same solution. This goal is achieved using a method

motivated from the arguments in Serre–Xiao [40]. The strategy is similar to the one for obtaining the decay of periodic solutions in L^p , $1 \leq p < \infty$, from the decay along rays in time-average (see [4]): The convergence in time-average implies the convergence in the usual sense in time with the aid of the entropy inequality.

THEOREM 2.3. *Consider (1.1) endowed with a strictly convex entropy. Let $u(t, x)$ be an L^∞ entropy solution of (1.1)–(1.2). Let $R(\xi)$, $\xi = x/t$, be the self-similar entropy solution of the Riemann problem (1.1) and (1.3), which is piecewise Lipschitz in the variable ξ with only a finite number of points of jump discontinuities. Suppose u is weakly asymptotic to R in the sense of (2.4). Then u is strongly asymptotic to R in the sense of (2.5).*

Proof. Let $(\eta(u), q(u))$ be a strictly convex entropy pair of (1.1). Denote $(\alpha(u, v), \beta(u, v))$ a family of entropy pairs, parametrized by v and formed by the quadratic parts of η and q at v :

$$\begin{aligned} \alpha(u, v) &= \eta(u) - \eta(v) - \nabla\eta(v)(u - v), \\ \beta(u, v) &= q(u) - q(v) - \nabla q(v)(f(u) - f(v)). \end{aligned}$$

Since u is an L^∞ entropy solution of (1.1), one has

$$\partial_t \eta(u) + \partial_x q(u) \leq 0 \tag{2.11}$$

in the sense of distributions.

Let $I = (\xi_1, \xi_2)$ be any open interval where $R(\xi)$ is Lipschitz continuous. For (t, x) in the wedge $\xi_1 < x/t < \xi_2$, one has

$$\partial_t R + \partial_x f(R) = 0, \tag{2.12}$$

$$\partial_t \eta(R) + \partial_x q(R) = 0. \tag{2.13}$$

Then we obtain

$$\partial_t \alpha(u, R) + \partial_x \beta(u, R) \leq -\nabla^2 \eta(R)(\partial_x R, Qf(u, R)) \tag{2.14}$$

in the sense of distributions, where $Qf(u, v) = f(u) - f(v) - \nabla f(v)(u - v)$ is the quadratic part of f at u .

Now, since u is just an L^∞ function, we consider a mollifying kernel $\omega \in C_0^\infty(-1, 1)$, $\omega \geq 0$, $\int_{\mathbb{R}} \omega(t) dt = 1$, and set $\omega_\delta(t) = \delta^{-1} \omega(\delta^{-1}t)$, $\delta > 0$. We will use the notation $h^\delta = h * w_\delta$, for any function h depending on t . Then, from (2.14), we get

$$\partial_t \alpha^\delta(u, R) + \partial_x \beta^\delta(u, R) \leq -(\nabla^2 \eta(R)(\partial_x R, Qf(u, R)))^\delta. \tag{2.15}$$

We now use the change of coordinates $(t, x) \mapsto (t, \xi)$, $\xi = x/t$. Inequality (2.15) then becomes

$$\partial_t \alpha^\delta(u, R) - \frac{\xi}{t} \partial_\xi \alpha^\delta(u, R) + \frac{1}{t} \partial_\xi \beta^\delta(u, R) \leq - \left(\frac{1}{t} \nabla^2 \eta(R) (\partial_\xi R, Qf(u, R)) \right)^\delta.$$

The derivatives with respect to ξ in (2.16) should be taken in the sense of distributions. We consider a nonnegative smooth function of ξ , $\phi \in C_0^\infty(\xi_1, \xi_2)$, such that $\phi(\xi) = 1$, for $\xi_1 + \varepsilon < \xi < \xi_2 - \varepsilon$, $\varepsilon > 0$ sufficiently small. Applying (2.16) to the test function $\phi(\xi)$ yields

$$\frac{d}{dt} Y_\phi^\delta \leq \frac{C(\|\phi\|_\infty + \text{Var}\{\phi\})}{t}, \quad (2.17)$$

for some constant $C > 0$, where we denote

$$Y_\phi^\delta = \int_I \alpha^\delta(u, R) \phi(\xi) d\xi.$$

The fact that u is weakly asymptotic to R translates into

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T Y(t) dt = 0. \quad (2.18)$$

We will prove that

$$\text{ess lim}_{t \rightarrow \infty} Y(t) = 0. \quad (2.19)$$

Indeed, we have

$$\left(t - \frac{T}{2}\right) Y_\phi^\delta(t)^2 = 2 \int_{T/2}^t \left(s - \frac{T}{2}\right) (Y_\phi^\delta)'(s) Y_\phi^\delta(s) ds + \int_{T/2}^t Y_\phi^\delta(s)^2 ds,$$

and thus use (2.17) to get

$$\frac{T}{2} Y_\phi^\delta(T)^2 \leq C \int_{T/2}^T \frac{t - T/2}{t} Y_\phi^\delta(t) dt + \int_{T/2}^T Y_\phi^\delta(t)^2 dt.$$

Now, in the above inequality, we can make $\phi \rightarrow 1$ in (ξ_1, ξ_2) , keeping $\|\phi\|_\infty$ and $\text{Var}\{\phi\}$ bounded, and then make $\delta \rightarrow 0$ to get

$$\frac{T}{2} Y(T)^2 \leq C \int_{T/2}^T \frac{t - T/2}{t} Y(t) dt + \int_{T/2}^T Y(t)^2 dt, \quad (2.20)$$

assuming that T is a Lebesgue point of $Y(t)$. Inequality (2.20), valid for all Lebesgue point T of $Y(t)$, immediately leads to (2.19) by using (2.18) and the boundedness of $Y(t)$.

To extend (2.19) to the case where I is any bounded interval, possibly containing points of jump discontinuity of R , we observe that I is the union of a finite number of open intervals, in where R is Lipschitz continuous, plus a finite number of points. Then, the integral of $|u(t, t\xi) - R(\xi)|$ over I is equal to the sum of the integrals of this function over these intervals, each of which, as has been proved, goes to zero when $t \rightarrow +\infty$. Hence, we arrive at the strong asymptotics for u . ■

3. LARGE-TIME BEHAVIOR OF ENTROPY SOLUTIONS TO SCALAR CONSERVATION LAWS

The next theorem provides a simple application of Theorem 2.2 with the aid of the equivalence in Theorem 2.1. Consider the Cauchy problem of scalar conservation laws in several space variables:

$$u_t + \operatorname{div}_x f(u) = 0, \tag{3.1}$$

$$u|_{t=0} = u_0(x) \equiv R_0\left(\frac{x}{|x|}\right) + P_0(x), \quad P_0(x) \in L^\infty \cap L^1(\mathbb{R}^n). \tag{3.2}$$

In this context, a Riemann solution means an entropy solution of (3.1)–(3.2) with self-similar initial data $R_0(x/|x|)$. It is clear that Definition 2.1 can be generalized to that for any number of space variables.

THEOREM 3.1. *Any Riemann solution $R(x/t)$ of the scalar conservation law (3.1) with Riemann data $R_0(x/|x|)$ is asymptotically stable with respect to $L^\infty \cap L^1$ perturbation $P_0(x)$ in the sense of (2.10). In particular, the Riemann solution $R(x/t)$ is the unique attractor for any entropy solution $u(t, x)$ of (3.1)–(3.2) with any $P_0(x) \in L^\infty \cap L^1(\mathbb{R}^n)$.*

Proof. Indeed, Kruzkov’s uniqueness theorem [25] indicates that, given any two L^∞ entropy solutions $u(t, x)$ and $\bar{u}(t, x)$ with initial data $u_0(x)$ and $\bar{u}_0(x)$, respectively, one has

$$\int_{|x| \leq r} |u(t, x) - \bar{u}(t, x)| \, dx \leq \int_{|x| \leq r + \kappa t} |u_0(x) - \bar{u}_0(x)| \, dx,$$

for any $r > 0$ and some constant K independent of both t and r . Hence, if $u_0 - \bar{u}_0$ belongs to $L^1(\mathbb{R}^n)$, one gets

$$\|u - \bar{u}\|_{L^1(\mathbb{R}^n)}(t) \leq \|u_0 - \bar{u}_0\|_{L^1(\mathbb{R}^n)}.$$

Then Theorem 2.2 yields the stability in the sense of (2.10). ■

Combining Theorem 3.1 and Theorem 2.3 with the fact that the Riemann solutions are always piecewise Lipschitz, we conclude

THEOREM 3.2. *Let the flux function $f_1(u)$ contains only isolated reflexion points. Then the planar Riemann solution with the Riemann data:*

$$R_0(x) = \begin{cases} u_L, & x_1 < 0, \\ u_R, & x_1 > 0, \end{cases}$$

is asymptotically stable with respect to any multidimensional $L^\infty \cap L^1$ perturbation $P_0(x)$ in the strong sense of (2.5).

Remark 3.1. The strong asymptotics of viscous solutions to the corresponding Riemann solutions was showed by Weinberger [48] for the one-dimensional viscous conservation laws with flux function containing only isolated inflexion points. Theorem 3.2 holds even for more general multi-dimensional cases. See [49] for the details.

Remark 3.2. The same result is true for the viscous scalar conservation laws by using either Kruzkov's arguments in [25] or DiPerna's theorem on the uniqueness of the measure-valued solutions in [11] with the aid of Theorem 2.1.

4. LARGE-TIME BEHAVIOR VIA DIRECT METHOD

In this section we present a direct method, the ray method, through a class of hyperbolic systems for studying the large-time behavior of solutions of (1.1)–(1.2) with the aid of entropy analysis. The systems are $m \times m$ hyperbolic systems endowed with affine characteristic hypersurfaces, identified by Temple [46], which arise from many important areas such as multicomponent chromatography (cf. [36, 23]).

DEFINITION 4.1. (cf. [38]) A hyperbolic system (1.1) is said a Temple system in a domain $V \subset \mathbb{R}^m$ if, for any $i \in \{1, \dots, m\}$, it satisfies the following:

1. there exists an i -Riemann invariant $w_i(u)$, i.e., $\nabla w_i(u) \nabla f(u) = \lambda_i(u) \nabla w_i(u)$;

2. the level sets $\{u \in V \mid w_i(u) = \text{const.}\}$ are intersections of affine hyperplanes with V .

A well-known example of such systems is the $m \times m$ chromatography system for Langmuir isotherms (cf. [36])

$$\partial_t u_i + \partial_x \left(\frac{k_i u_i}{1 + \sum_{j=1}^m u_j} \right) = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \quad 1 \leq i \leq m, \quad (4.1)$$

where $0 < k_1 < k_2 < \dots < k_m$ are given numbers.

Let $H(s)$ denote the Heaviside function and $s_{\pm} \equiv H(\pm s)s$. Such systems have several distinguished features (see [38, 19]).

LEMMA 4.1. *For the Temple systems,*

$$l_i(v) \cdot (u - v) = 0 \Rightarrow l_i(v) \cdot (f(u) - f(v)) = 0, \quad \text{for all } (u, v) \in V \times V. \quad (4.2)$$

For any $v \in V$, the following pairs of functions are entropy pairs,

$$\begin{aligned} (\eta_i^{\pm}, q_i^{\pm})(u, v) &\equiv ((l_i(v) \cdot (u - v))_{\pm}, \pm H(\pm l_i(v) \cdot (u - v)) l_i(v) \cdot (f(u) - f(v))), \\ (\eta_i, q_i)(u, v) &\equiv (|l_i(v) \cdot (u - v)|, \text{sgn}(l_i(v) \cdot (u - v)) l_i(v) \cdot (f(u) - f(v))), \end{aligned}$$

that is, each of them satisfies the entropy equations (2.1) for any $v \in V$.

Proof. As observed in [19], for (4.2), we just use the mean-value formula

$$f(u) - f(v) = \int_0^1 A(v + s(u - v)) \cdot (u - v) ds$$

and notice that every hyperplane $\Pi_i(a) = \{u \in K \mid w_i(u) = a\}$ is invariant under $A(u) = \nabla f(u)$ in the sense that $A(z)(u - v) \parallel \Pi_i(a)$, for any $u, v, z \in \Pi_i(a)$. Now, the fact that (η_i^{0+}, q_i^{0+}) , $(\eta_i^{\pm}, q_i^{\pm})$, and (η_i, q_i) are entropy pairs is an immediate consequence of (4.2). ■

The following lemma, due to Heibig [19], indicates another feature of the Temple systems, which will be the key for the quasidecoupling property obtained in the proof of Theorem 4.1 below.

LEMMA 4.2. *For any Temple system (1.1), there exists a unique function $\bar{A}: \mathbb{R}^m \times \mathbb{R}^m \rightarrow M_{m,m}(\mathbb{R})$ such that*

1. *for all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m$, $f(u) - f(v) = \bar{A}(u, v) \cdot (u - v)$, $\bar{A}(u, u) = A(u)$;*
2. *for all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m$, $A(v)$ and $\bar{A}(u, v)$ have the same left and right eigenvectors;*
3. *the matrix function $\bar{A}(u, v)$ is smooth.*

The next lemma serves as a complement to Lemma 4.2. Its role in the proof of Theorem 4.1 lies in the fact that, through it, solutions with large oscillation are allowed.

LEMMA 4.3. *Let $\bar{\lambda}_i(u, v)$ be the i th eigenvalue of $\bar{A}(u, v)$ determined by Lemma 3.2. Suppose (1.1) is a Temple system on a compact set*

$$\mathbf{O} = \bigcap_{i=1}^m \{u \in \mathbb{R}^m : |w_i(u) - w_i(\bar{u})| \leq M_i\}, \quad (4.3)$$

for certain $M_i > 0$, $i = 1, \dots, m$. Then,

$$\min_{u \in \mathbf{O}} \lambda_i(u) \leq \bar{\lambda}_i(u, v) \leq \max_{u \in \mathbf{O}} \lambda_i(u), \quad (u, v) \in \mathbf{O} \times \mathbf{O}. \quad (4.4)$$

Proof. Let $(u, v) \in \mathbf{O} \times \mathbf{O}$. Assume for the moment that $w_i(u) \neq w_i(v)$. Let v^* be the point in \mathbf{O} given by $w_i(v^*) = w_i(v)$ and $w_j(v^*) = w_j(u)$, for $j \neq i$. By Lemma 4.1, we have

$$\begin{aligned} l_i(v) \cdot (f(u) - f(v)) &= l_i(v) \cdot (f(u) - f(v^*)) \\ &= l_i(v) \cdot (u - v^*) \int_0^1 \lambda_i(su + (1-s)v^*) ds. \end{aligned} \quad (4.5)$$

On the other hand, by Lemma 4.2, we also have

$$l_i(v) \cdot (f(u) - f(v)) = \bar{\lambda}_i(u, v) l_i(v) \cdot (u - v) = \bar{\lambda}_i(u, v) l_i(v) \cdot (u - v^*). \quad (4.6)$$

Then equations (4.5)–(4.6) give

$$\bar{\lambda}_i(u, v) = \int_0^1 \lambda_i(su + (1-s)v^*) ds, \quad (4.7)$$

from which (4.4) follows. If $w_i(u) = w_i(v)$, then (4.4) holds by continuity. ■

We assume that the eigenvalues of (1.1) satisfy

$$\kappa_i \leq \lambda_i(u) \leq \kappa_{i+1}, \quad i = 1, \dots, m, \quad \text{for some } \kappa_1 < \kappa_2 < \dots < \kappa_{m+1}; \quad (4.8)$$

$$\frac{\partial \lambda_i}{\partial w_i}(u) > 0, \quad i = 1, \dots, m, \quad \text{for all } u \in \mathbf{O}. \quad (4.9)$$

Here and in the discussion which follows, \mathbf{O} is a region of the type given in (4.3). Observe that (4.8) allows the loss of strict hyperbolicity in some

points. Condition (4.9) says that all fields are genuinely nonlinear in the sense of Lax [27].

We recall that the existence of entropy solutions of the Cauchy problem for Temple systems, with large initial data in $BV(\mathbb{R})$, was proved by Serre [39] and LeVeque–Temple [29]. In this case, the solution is also in $BV([0, T] \times \mathbb{R})$, for any $T > 0$. The existence of entropy solutions in the case of initial data in $L^\infty(\mathbb{R})$ is known at least in the case of the chromatography system (4.1) (see [23]). We also recall that any region \mathbf{O} is invariant under the viscous flow, or some numerical schemes (e.g. Godunov, Lax–Friedrichs, and Glimm) for the systems. Therefore, if the initial data are in \mathbf{O} , any entropy solution obtained by one of these approximations also takes its values in \mathbf{O} .

THEOREM 4.1. *Suppose (1.1) is a Temple system satisfying (4.8)–(4.9). Assume that $u_0(x) \in L^\infty(\mathbb{R})$ satisfies (1.2)–(1.4) and takes its values in a region \mathbf{O} . Then, any entropy solution $u(t, x) \in L^\infty(\mathbb{R}_+^2)$ of (1.1)–(1.4) with its values in \mathbf{O} asymptotically tends to the Riemann solution $R(x/t)$ of (1.1) and (1.5) in the sense of (2.5). This implies that $R(x/t)$ is the unique attractor of any L^∞ entropy solution of (1.1)–(1.4).*

Proof. It suffices from Theorem 2.3 to show that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u(t, t\xi) - R(\xi)| dt = 0, \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{4.10}$$

We divide the proof into two steps.

Step 1. We first assume $u(t, x) \in BV([0, T] \times \mathbb{R})$, for any $T > 0$. Then, for any convex entropy pair, $u(t, x)$ satisfies

$$\partial_t \eta(u) + \partial_x q(u) \leq 0 \tag{4.11}$$

in the sense of Radon measures. We set

$$E_j^{\xi, T} = \{(t, x) \in \mathbb{R}_+^2 \mid 0 \leq t \leq T, (-1)^j (x/t - \xi) > 0\}, \quad j = 1, 2.$$

From (4.8) and Lemma 4.3, we have $\kappa_i \leq \bar{\lambda}_i(u, v) \leq \kappa_{i+1}$, $i = 1, \dots, m$, where $\bar{\lambda}_i(u, v)$ are the eigenvalues of $\bar{A}(u, v)$ in Lemma 4.2. We divide into two cases.

Case 1. We first consider $-\infty < \xi < \kappa_1$ as well as $\kappa_{m+1} < \xi < \infty$. For the former, we take $\eta(u) = \eta_j(u, u_L)$ and $q(u) = q_j(u, u_L)$ in (4.11), with $j = 1, \dots, m$, and integrate (4.11) over

$$E_1^{\xi, T} \cap \{(t, x) \mid x > -X - C(T - t), 0 < t < T\}, \tag{4.12}$$

for some $X > |\xi|T$ and $C > 0$ chosen so that $C > \max_{u, v \in \mathbf{O}} \{q_j(u, v)/\eta_j(u, v), j = 1, \dots, m\}$. We then apply the Green Theorem and observe that the resultant terms, corresponding to the integrations over the lines $x = -X - C(T - t)$ and $t = T$, are both nonnegative. We first throw out these terms and then make $X \rightarrow \infty$ to obtain

$$\int_0^T (-\xi\eta_j + q_j)(u(t, \xi t), u_L) dt \leq \int_{-\infty}^0 \eta_j(u_0(x), u_L) dx. \quad (4.13)$$

We will repeat the same procedure several times in what follows, where the same details henceforth will be omitted. We will refer to it only as an integration of (4.11) over $E_j^{\xi, T}, j = 1, 2$, for the particular entropy pair that we use.

Define the probability Radon measures μ_T^ξ on \mathbf{O} by

$$\langle \mu_T^\xi, h(u) \rangle = \frac{1}{T} \int_0^T h(u(t, \xi t)) dt, \quad \text{for any } h \in C(\mathbf{O}).$$

By the weak compactness in $\mathcal{M}(\mathbf{O})$ (the space of Radon measures on \mathbf{O}), there exist a subsequence (still denoted) μ_T^ξ and a probability measure μ^ξ such that $\mu_T^\xi \rightharpoonup \mu^\xi$ when $T \rightarrow \infty$. Now, dividing (4.13) by T and letting $T \rightarrow \infty$, we obtain

$$\begin{aligned} \langle \mu^\xi, -\xi |l_j(u_L) \cdot (u - u_L)| \\ + \operatorname{sgn}(l_j(u_L) \cdot (u - u_L))(l_j(u_L) \cdot (f(u) - f(u_L))) \rangle \leq 0. \end{aligned} \quad (4.14)$$

Then, applying Lemma 4.2 to (4.14), we find

$$\langle \mu^\xi, |l_j(u_L) \cdot (u - u_L)| (-\xi + \bar{\lambda}_j(u, u_L)) \rangle \leq 0.$$

Thus, since $\xi < \kappa_1 \leq \bar{\lambda}_j(u, v), j = 1, \dots, m$, for $u, v \in \mathbf{O}$, we get

$$\langle \mu^\xi, |l_j(u_L) \cdot (u - u_L)| (-\xi + \bar{\lambda}_j(u, u_L)) \rangle = 0.$$

Therefore, one must have

$$\operatorname{supp} \mu^\xi \subseteq \{u \in \mathbf{O} \mid l_j(u_L) \cdot (u - u_L) = 0, j = 1, \dots, m\} = \{u_L\}.$$

We then conclude that $\mu^\xi = \delta_{u_L}$, where δ_{u_L} denotes the Dirac measure concentrated in u_L . Since this holds no matter which weakly convergent subsequence of μ_T^ξ we take, we have $\mu_T^\xi \rightharpoonup \delta_{u_L}$, for $\xi \leq \kappa_1$. Consequently,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u(t, \xi t) - u_L| dt = 0, \quad \text{for } \xi < \kappa_1.$$

This is (4.10) for $\xi < \kappa_1$. Analogously, using $\eta_j(u, u_R)$, $j = 1, \dots, m$, and integrating (4.11) over $E_2^{\xi, T}$, we get $\mu_T^\xi \rightarrow \delta_{u_R}$, and so (4.10), if $\kappa_{m+1} < \xi < \infty$.

Case 2. We now consider $\xi \in (\kappa_q, \kappa_{q+1})$, for some $q \in \{1, \dots, m\}$. Define μ_T^ξ and μ^ξ as above. Let $u_1 = u_L$, $u_{m+1} = u_R$, $u_j = R(\kappa_j)$, $j = 2, \dots, m$, be the constant states in the Riemann solution, so that u_j is connected to u_{j+1} on the right by either a j -rarefaction wave or a j -shock wave, $j = 1, \dots, m$. Let $r_j(u)$ denote the j th right eigenvector of $A(u) = \nabla f(u)$, where l_j and r_j are normalized so that $l_j(u) \cdot r_j(u) = 1$, $\nabla \lambda_j(u) \cdot r_j(u) > 0$, $j = 1, \dots, m$. We claim that

$$\text{supp } \mu^\xi \subseteq L_q \equiv \{u \in \mathbf{O} \mid u = u_q + sr_q(u_q), s \in \mathbb{R}\}. \tag{4.15}$$

This can be seen as follows. Since, by hypothesis, (1.1) is a Temple system and $R(x/t)$ is the Riemann solution with left state u_L and right state u_R , we have

$$\begin{aligned} L_q &= \{u \in \mathbf{O} \mid u = u_q + sr_q(u_q), s \in \mathbb{R}\} \\ &= \bigcap_{i=1}^{q-1} \{u \in \mathbf{O} \mid l_i(u_R) \cdot (u - u_R) = 0\} \cap \bigcap_{j=q+1}^m \{u \in \mathbf{O} \mid l_j(u_L) \cdot (u - u_L) = 0\}. \end{aligned}$$

Thus we take in (4.11) $(\eta, q)(u) = (\eta_i, q_i)(u, u_R)$, for $i \in \{1, \dots, q-1\}$, and, after integrating over $E_2^{\xi, T}$, we divide the resultant inequality by T and let $T \rightarrow \infty$ to get

$$\langle \mu^\xi, |l_i(u_R) \cdot (u_R - u)| (\xi - \bar{\lambda}_i(u, u_R)) \rangle \leq 0, \tag{4.17}$$

$i = 1, \dots, q-1$, applying Lemma 4.2. Similarly, we take in (4.11) $(\eta, q)(u) = (\eta_j, q_j)(u, u_L)$, for $j \in \{q+1, \dots, m\}$, integrate over $E_1^{\xi, T}$, and follow the same procedure as above to get

$$\langle \mu^\xi, |l_j(u_L) \cdot (u - u_L)| (-\xi + \bar{\lambda}_j(u, u_L)) \rangle \leq 0, \quad j = q+1, \dots, m, \tag{4.18}$$

applying Lemma 4.2 again. Now, since $\bar{\lambda}_1(u, v) \leq \dots \leq \bar{\lambda}_{q-1}(u, v) \leq \kappa_q < \xi < \kappa_{q+1} \leq \bar{\lambda}_{q+1}(u, v) \leq \dots \leq \bar{\lambda}_m(u, v)$, for $u, v \in \mathbf{O}$, the inequalities in (4.17)–(4.18) can be replaced by the equalities. Then (4.17)–(4.18) together with (4.16) imply (4.15).

Subcase 1. $\kappa_q < \xi < \min\{\lambda_q(u_q), \lambda_q(u_{q+1})\}$ as well as $\max\{\lambda_q(u_q), \lambda_q(u_{q+1})\} < \xi < \kappa_{q+1}$.

If $\xi < \inf_{u \in \mathbf{O}} \lambda_q(u)$, we take the entropy pair $(\eta_q, q_q)(u, u_q)$, which satisfies $\eta_q(u_L, u_q) = 0$ since $w_q(u_L) = w_q(u_q)$, and integrate (4.11) over $E_1^{\xi, T}$.

We divide the resultant inequality by T and consider μ_T^ξ and μ^ξ as above. Using (4.15), we find

$$\langle \mu^\xi, |l_q(u_q) \cdot (u - u_q)| (-\xi + \lambda_q(u, u_q)) \rangle \leq 0, \quad (4.19)$$

where $\lambda_q(u, v) = \int_0^1 \lambda_q(\theta u + (1 - \theta)v) d\theta$ is the q th eigenvalue of $\int_0^1 A(\theta u + (1 - \theta)v) d\theta$, when $u, v \in L_q$. Now, for $u \in L_q \cap \mathbf{O}$, inequality (4.19) is possible only if the equality holds, which implies $\mu^\xi = \delta_{u_q}$. Since this holds for any weakly convergent subsequence of μ_T^ξ , we get $\mu_T^\xi \rightarrow \delta_{u_q}$ as desired.

If $\xi \in (\inf_{u \in \mathbf{O}} \lambda_q(u), \min\{\lambda_q(u_q), \lambda_q(u_{q+1})\})$, let $u_\xi \in L_q$ be such that $\lambda_q(u_\xi) = \xi$. We take the entropy pair $(\eta_q^-, q_q^-)(u, u_\xi)$, satisfying $\eta_q^-(u_R, u_\xi) = (l_q(u_\xi) \cdot (u_\xi - u_R))_+ = 0$. The latter holds because the vectors $l_q(u_\xi)$ and $u_R - u_\xi$ point to the same half-space determined by the hyperplane $w_q = w_q(u_\xi)$. To see this, we recall that (w_1, \dots, w_m) is a coordinate system in \mathbf{O} , $w_q = \text{const.}$ are hyperplanes, $l_q(u_\xi)$ is the normal to the hyperplane $w_q = w_q(u_\xi)$, and $w_q(u_R) = w_q(u_{q+1}) > w_q(u_\xi)$, since λ_q is an increasing function of w_q . With this entropy pair in (4.11), we integrate (4.11) over $E_2^{\xi, T}$ and consider the measures μ_T^ξ and μ^ξ . We then find

$$\langle \mu^\xi, (l_q(u_\xi) \cdot (u_\xi - u))_+ (\xi - \lambda_q(u, u_\xi)) \rangle \leq 0.$$

Notice that $(l_q(u_\xi) \cdot (u_\xi - u))_+ = (s_\xi - s)_+$, where s_ξ is given by $u_\xi = u_q + s_\xi r_q(u_q)$. On the other hand, $\lambda_q(u, u_\xi) \leq \xi$ if $u = u_q + sr_q(u_q)$ and $s \leq s_\xi$. These facts imply

$$\text{supp } \mu^\xi \subseteq \{u \in L_q \mid u = u_q + sr_q(u_q), s \geq s_\xi\}. \quad (4.20)$$

Now, we again take the pair $(\eta_q, q_q)(u, u_q)$ and integrate (4.11) over $E_1^{\xi, T}$. For the probability measures μ_T^ξ and μ^ξ as above, we obtain (4.19). Since $\lambda_q(u, u_q) \geq \xi$ if $u = u_q + sr_q(u_q)$ and $s \geq s_\xi$, we conclude $\mu^\xi = \delta_{u_q}$ and then also $\mu_T^\xi \rightarrow \delta_{u_q}$ in this case. Analogously we obtain $\mu_T^\xi \rightarrow \delta_{u_{q+1}}$ if $\xi \in (\max\{\lambda_q(u_q), \lambda_q(u_{q+1})\}, \kappa_{q+1})$.

Subcase 2. If $\xi \in (\min\{\lambda_q(u_q), \lambda_q(u_{q+1})\}, \max\{\lambda_q(u_q), \lambda_q(u_{q+1})\})$, we consider two different cases:

- (i) $\lambda_q(u_q) \leq \lambda_q(u_{q+1})$, the q -wave is a rarefaction wave;
- (ii) $\lambda_q(u_q) > \lambda_q(u_{q+1})$, the q -wave is a shock wave.

For (i), $\lambda_q(R(\xi)) = \xi$. Consider the entropy pair $(\eta_q^+, q_q^+)(u, R(\xi))$. Then $\eta_q^+(u_L, R(\xi)) = 0$. Integrating (4.11) over $E_1^{\xi, T}$ and considering the probability measures μ_T^ξ and μ^ξ , we obtain

$$\langle \mu^\xi, (l_q(R(\xi)) \cdot (u - R(\xi)))_+ (-\xi + \lambda_q(u, R(\xi))) \rangle \leq 0.$$

Now $(l_q(R(\xi)) \cdot (u - R(\xi)))_+ = (s - s_\xi)_+$, where $R(\xi) = u_q + s_\xi r_q(u_q)$ and $\lambda_q(u, R(\xi)) \geq \xi$ if $u = u_q + sr_q(u_q)$ with $s \geq s_\xi$. Hence, we conclude

$$\text{supp } \mu^\xi \subseteq \{u \in L_q \mid u = u_q + sr_q(u_q), s \leq s_\xi\}. \tag{4.21}$$

On the other hand, integrating (4.11) over $E_2^{\xi, T}$ with $\eta(u) = \eta_q^-(u, R(\xi))$, which satisfies $\eta(u_R) = \eta_q^-(u_R, R(\xi)) = 0$, and proceeding as usual, we obtain again (4.20) where now s_ξ is as in (4.21). Then (4.20)–(4.21) imply $\mu^\xi = \delta_{R(\xi)}$. Hence $\mu_T^\xi \rightarrow \delta_{R(\xi)}$.

For (ii), the q -wave is a shock wave. We first show that

$$\text{supp } \mu^\xi \subseteq \{u \in \mathbf{O} \mid u = u_q + sr_q(u_q), s(u_{q+1}) \leq s \leq 0\}, \tag{4.22}$$

where μ^ξ is the limit of a weakly convergent subsequence of μ_T^ξ , and $s(u_{q+1})$ is such that $u_{q+1} = u_q + s(u_{q+1})r_q(u_q)$. To arrive at this, we first consider the entropy pair $(\eta_q^+, q_q^+)(u, u_q)$ and notice that $\eta_q^+(u_L, u_q) = 0$, since $w_q(u_L) = w_q(u_q)$. We then integrate (4.11) over $E_1^{\xi, T}$. In the same way, we eventually find

$$\langle \mu^\xi, (l_q(u_q) \cdot (u - u_q))_+ (-\xi + \lambda_q(u, u_q)) \rangle \leq 0.$$

Now $(l_q(u_q) \cdot (u - u_q))_+ = s_+$ for $u = u_q + sr_q(u_q)$, and $\lambda_q(u, u_q) \geq \lambda_q(u_q) > \xi$ if $s \geq 0$. Then we must have

$$\text{supp } \mu^\xi \subseteq \{u \in L_q \mid u = u_q + sr_q(u_q), s \leq 0\}.$$

Analogously, using the entropy pair $(\eta_q^-, q_q^-)(u, u_{q+1})$ satisfying $\eta_q^-(u_R, u_{q+1}) = 0$ since $w_q(u_{q+1}) = w_q(u_R)$, we arrive at

$$\text{supp } \mu^\xi \subseteq \{u \in \mathbf{O} \mid u = u_q + sr_q(u_q), s \geq s(u_{q+1})\},$$

which implies (4.22). Now, observe that, if σ_q is the speed of the shock wave connecting u_q with u_{q+1} , we must have

$$\lambda_q(u_q, u_{q+1}) = \lambda_q(u_{q+1}, u_q) = \sigma_q,$$

which easily follows from the properties of the systems and the Rankine–Hugoniot relations. Thus, for $\lambda_q(u_{q+1}) < \xi < \sigma_q$, we take the entropy pair $(\eta_q, q_q)(u, u_q)$ and integrate (4.11) over $E_1^{\xi, T}$. We then arrive at (4.19) by the same procedure. Since $\xi < \sigma_q = \lambda_q(u_{q+1}, u_q) \leq \lambda_q(u, u_q)$, if $u = u_q + sr_q(u_q)$ with $s \geq s(u_{q+1})$, we must have $\mu^\xi = \delta_{u_q}$ and, consequently, $\mu_T^\xi \rightarrow \delta_{u_q}$.

Analogously, for $\sigma_q < \xi < \lambda_q(u_q)$, taking the entropy pair $(\eta_q, q_q)(u, u_{q+1})$, integrating (4.11) over $E_2^{\xi, T}$, and following the usual procedure, we have

$$\langle \mu^\xi, |l_q(u_{q+1}) \cdot (u - u_{q+1})| (\xi - \lambda_q(u, u_{q+1})) \rangle \leq 0.$$

Now we have $\xi > \sigma_q = \lambda_q(u_q, u_{q+1}) \geq \lambda_q(u, u_{q+1})$ if $u = u_q + sr_q(u_q)$ with $s \leq 0$. Therefore, we conclude $\mu^\xi = \delta_{u_{q+1}}$ and $\mu_T^\xi \rightharpoonup \delta_{u_{q+1}}$. Hence we have proved (4.10) for almost all $\xi \in \mathbb{R}$.

Step 2. We now consider a general L^∞ solution $u(t, x) \in \mathbf{O}$, a.e. Actually, we will show that the procedure carried out for BV solutions can be also applied to the case of L^∞ solutions as long as one can define solution values on $x/t = \xi$, except for a certain set of measure zero of ξ . More specifically, our problem reduces to justifying the use of the Gauss–Green Formula in Step 1 for the L^∞ solution, for which the theory of divergence-measure fields meets the need (see [5–7]; also [1]). It has been also shown in [6–7] that, for any L^∞ entropy solution $u(t, x)$, the normal traces of any entropy pair $(\eta(u), q(u))$ on the lines $x/t = \xi$, $\xi \in \mathbb{R}$, are given by the usual scalar product of the restrictions of those fields with the normal to those lines as long as ξ does not belong to a set of measure zero in \mathbb{R} . By restriction of the fields we mean the restriction to those lines of a precise representative for the fields $(\eta(u(t, x)), q(u(t, x)))$, defined on \mathbb{R}_+^2 .

We fix a particular representative of the class of functions that coincide with $u(t, x)$ almost everywhere, which is still denoted by $u(t, x)$. Let $\mathcal{N} \subset \mathbb{R}_+^2$ be a Borel set of measure zero containing the set of points that are not Lebesgue points of u . We redefine u on \mathcal{N} setting $u(t, x) = \bar{u}$, for $(t, x) \in \mathcal{N}$, where \bar{u} is the constant state in the definition of \mathbf{O} . Actually, the particular value of u over \mathcal{N} is irrelevant. In this way, $u(t, x)$ is a Borel map $\mathbb{R}_+^2 \rightarrow \mathbb{R}^m$.

LEMMA 4.4. *Let $v: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a bounded Borel map and $h: \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuous function. If $y \in \Omega$ is a Lebesgue point of v , then y is a Lebesgue point of $h(v)$.*

By Lemma 4.4, for any continuous function $h: \mathbb{R}^m \rightarrow \mathbb{R}$, (t, x) is a Lebesgue point of $h(u)$, provided that (t, x) is a Lebesgue point of u . We also recall that, as a corollary of Schwartz’s lemma on nonnegative distributions [37], we have

$$\operatorname{div}_{t,x}(\eta(u(t, x)), q(u(t, x))) \in \mathcal{M}(\mathbb{R}_+^2),$$

for any entropy pair (η, q) . That is, $(\eta(u(t, x)), q(u(t, x)))$ is locally a divergence-measure field over \mathbb{R}_+^2 . To apply the results in [6–7], we need the following lemma.

LEMMA 4.5. *There exists a set of measure zero $\tilde{\mathcal{N}} \subset \mathbb{R}$ such that, for all continuous function $h: \mathbb{R}^m \rightarrow \mathbb{R}$,*

$$\operatorname{meas}\{t \in \mathbb{R}_+ : (t, \xi t) \text{ is not a Lebesgue point of } h(u(t, x))\} = 0,$$

provided that $\xi \in \mathbb{R} - \tilde{\mathcal{N}}$. Similarly, there is a set of measure zero $\tilde{\mathcal{F}} \subset \mathbb{R}_+$ such that, for all continuous function $h: \mathbb{R}^m \rightarrow \mathbb{R}$,

$$\text{meas}\{x \in \mathbb{R}: (t, x) \text{ is not a Lebesgue point of } h(u(t, x))\} = 0,$$

provided that $t \in \mathbb{R}_+ - \tilde{\mathcal{F}}$.

We continue the proof of Theorem 4.1. It suffices to justify the use of the Gauss–Green Formula over the domains $E_j^{\xi, T}, j = 1, 2$, since all the remainder follow exactly those in the proof for BV solutions given above. Now we know from Lemma 4.5 that, for a.e. $\xi \in \mathbb{R}, t_0, T > 0$, the set of non-Lebesgue points of $u(t, x)$, contained in the boundaries of the domains

$$E_{j, t_0}^{\xi, T} = \{(t, x) \in \mathbb{R}_+^2 \mid t_0 < t < T, (-1)^j (x/t - \xi) > 0\}, \quad j = 1, 2,$$

has \mathcal{H}^1 measure zero. That is, for a.e. $\xi \in \mathbb{R}, t_0, T \in \mathbb{R}_+, \mathcal{H}^1(\mathcal{N} \cap \partial E_{j, t_0}^{\xi, T}) = 0, j = 1, 2$. Given any entropy pair $(\eta(u), q(u))$, we can apply the usual Gauss–Green Formula to the field $(\eta(u), q(u))$ over the domains $E_{j, t_0}^{\xi, T}, j = 1, 2$, for ξ out of a set of measure zero in \mathbb{R} and $t_0 < T$, both out of a set of measure zero in \mathbb{R}_+ , with the aid of Lemma 3.4, Theorems 1–2 in [6]. Furthermore, by the fact that the initial data are assumed in the sense of limit as $t \rightarrow 0$ in L^1_{loc} (Theorem 4 of [6]), we can make $t_0 \rightarrow 0$ and then get the usual Gauss–Green Formula for the domains $E_j^{\xi, T}, j = 1, 2$, for a.e. $\xi \in \mathbb{R}, T > 0$. In particular, if η is nonnegative and convex, from (1.2), (1.4), and the finiteness of propagation speed of the solution (Theorem 5 of [6]), we obtain the estimates on the weak limits μ^ξ of the measures μ^ξ_T , defined as above, applied to $\xi\eta(u) - q(u)$, as long as ξ does not belong to a certain set of measure zero in \mathbb{R} . This completes the proof. ■

Remark 4.1. For system (4.1), the compactness in $L^1_{\text{loc}}(\mathbb{R}_+^2)$ of uniformly bounded solution sequence is established in [23]. The compactness of the self-similar scaling sequence from Theorem 4.1 is stronger than the one in [23] for (4.1), since it gives the convergence of the *whole* sequence, which cannot be obtained by [23] without a general uniqueness theorem.

Remark 4.2. Uniqueness results for entropy solutions of (1.1)–(1.2) in $BV([0, \infty) \times \mathbb{R})$ have been obtained in [9] for a specific 2×2 Temple system and in [19] for a class of $m \times m$ Temple systems including (4.1). Both of these results are valid only for solutions of small variation and small oscillation, and impose further restrictions which exclude some standard Riemann solutions. In [9] the existence of solutions in that class was proved, provided the corresponding restrictions on the initial data are imposed. Remark that Theorem 4.1 does not impose any restriction on the L^∞ solution of $m \times m$ Temple systems, obtained by any of the Glimm’s,

Godunov's, or Lax-Friedrichs' scheme, or else the vanishing viscosity method.

5. LARGE-TIME BEHAVIOR VIA UNIQUENESS OF RIEMANN SOLUTIONS

In this section we consider some classes of systems to present another method for studying the asymptotic behavior of entropy solutions. This method is based on the following observation.

PROPOSITION 5.1. *Let $\mathcal{S}(\mathbb{R}_+^2)$ denote a class of functions defined on \mathbb{R}_+^2 . Assume that the Cauchy problem (1.1)–(1.2) satisfies the following.*

- (i) *The Riemann solution is unique in the class $\mathcal{S}(\mathbb{R}_+^2)$;*
- (ii) *Given an entropy solution of (1.1)–(1.2), $u \in \mathcal{S}(\mathbb{R}_+^2)$, the sequence $u^T(t, x)$ is compact in $L_{loc}^1(\mathbb{R}_+^2)$, and any limit function of its subsequences is still in $L_{loc}^1 \cap \mathcal{S}(\mathbb{R}_+^2)$.*

Then the Riemann solution $R(x/t)$ is asymptotically stable in $\mathcal{S}(\mathbb{R}_+^2)$ with respect to the corresponding initial perturbation $P_0(x)$.

Proof. Take any subsequence $\{u^{T_k}(t, x)\}_{k=1}^\infty \subset \{u^T(t, x)\}_{T>0}$. Condition (ii) implies that there exists a further subsequence converging in L_{loc}^1 to $\tilde{u}(t, x) \in L_{loc}^1 \cap \mathcal{S}(\mathbb{R}_+^2)$ satisfying the same data of $R(x/t)$. Condition (i) then ensures that $\tilde{u}(t, x) = R(x/t)$ a.e., which is unique. This indicates that the whole sequence $\{u^T(t, x)\}_{T>0}$ converges to the Riemann solution $R(x/t)$ in $L_{loc}^1(\mathbb{R}_+^2)$. For any $0 < r < \infty$, we have

$$\begin{aligned} & \frac{2}{T^2} \int_0^T \int_{|\xi| \leq r} |u(t, \xi) - R(\xi)| t \, d\xi \, dt \\ &= \frac{2}{T^2} \int_0^T \int_{|x| \leq rt} |u(t, x) - R(x/t)| \, dx \, dt \\ &= 2 \int_0^1 \int_{|x| \leq rt} |u^T(t, x) - R(x/t)| \, dx \, dt \rightarrow 0, \quad \text{when } T \rightarrow \infty. \quad \blacksquare \end{aligned}$$

In this section, the class $\mathcal{S}(\mathbb{R}_+^2)$ will be always either an open subset of $BV_{loc}(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$, or an open subset of $L^\infty(\mathbb{R}_+^2)$.

Proposition 5.1 indicates that the compactness of scaling sequences and the uniqueness of Riemann solutions imply the asymptotic stability of Riemann solutions in the sense of (2.4). The systems we consider here include the 2×2 strictly hyperbolic equations and the 3×3 Euler equations in thermoelasticity.

For BV solutions, the compactness of the scaling sequence is obtained through the following observation.

LEMMA 5.1. *Assume that $u(t, x) \in BV_{loc}(\mathbb{R}_+^{n+1})$ satisfies*

$$\iint_{(0, T_0) \times \{|x| \leq cT_0\}} |\nabla_{(t,x)} u| \leq CT_0^n, \quad (5.1)$$

for any $c > 0$, $T_0 > 0$, and some $C > 0$ independent of T_0 . Then $u^T(t, x)$ also satisfies (5.1) with the same constant C . In particular, for $u \in L^\infty(\mathbb{R}_+^{n+1})$, then the sequence u^T is compact in $L^1_{loc}(\mathbb{R}_+^{n+1})$.

This condition is satisfied by the entropy solutions possessing total variation in x uniformly bounded for all $t > 0$, which is the case for the solutions constructed by Glimm's method (see [16, 17]). Hence, the compactness follows from Helly's theorem for bounded sets in BV .

For L^∞ solutions of the systems considered here, the method of compensated compactness has been applied successfully and yields the compactness of uniformly bounded sequences of entropy solutions: in [10], for 2×2 strictly hyperbolic and genuinely nonlinear systems, and, in [3], for the 3×3 Euler equations in thermoelasticity.

The uniqueness of Riemann solutions in the class of BV solutions for the 2×2 systems is due to DiPerna [12]. The main results of this section are uniqueness theorems for Riemann solutions in the following contexts: (i) L^∞ solutions of the p -system with large oscillation and initial Riemann states connected only by rarefaction wave curves; (ii) L^∞ solutions of the 2×2 systems with small oscillation and initial Riemann states connected only by rarefaction wave curves; (iii) L^∞ solutions of the 3×3 Euler equations with large oscillation and initial Riemann states connected only by rarefaction wave curves of the first and third families, and, possibly, a contact discontinuity curve of the second family; (iv) BV solutions of the 3×3 equations with large oscillation and general Riemann initial states.

As we indicated above, once we have the compactness of the scaling sequence, the asymptotic problem reduces to the uniqueness problem of Riemann solutions of (1.1) and (1.5). Therefore, in what follows, we mainly study the uniqueness problem with the aid of entropy analysis. We start with the 2×2 case.

5.1. Uniqueness and Stability of Rarefaction Waves in L^∞ for 2×2 Systems

We first treat the case that the Riemann solution consists of two rarefaction waves. That is, there exists $u_M \in \mathbb{R}^2$ such that the Riemann solution satisfies

$$v(t, x) = R(x/t) \equiv \begin{cases} u_L, & x/t \leq \lambda_1(u_L), \\ R_1(x/t), & \lambda_1(u_L) < x/t < \lambda_1(u_M), \\ u_M, & \lambda_1(u_M) \leq x/t \leq \lambda_2(u_M), \\ R_2(x/t), & \lambda_2(u_M) < x/t < \lambda_2(u_R), \\ u_R, & x/t \geq \lambda_2(u_R), \end{cases} \quad (5.2)$$

where $R_1(\zeta)$ and $R_2(\zeta)$ are the solutions of the boundary value problems

$$\begin{cases} \frac{d}{d\zeta} R_1(\zeta) = r_1(R_1(\zeta)), & \lambda_1(u_L) < \zeta, \\ R_1(\lambda_1(u_L)) = u_L, \\ \lambda_1(R_1(\zeta)) = \zeta, \end{cases} \quad (5.3)$$

and

$$\begin{cases} \frac{d}{d\zeta} R_2(\zeta) = r_2(R_2(\zeta)), & \zeta < \lambda_2(u_R), \\ R_2(\lambda_2(u_R)) = u_R, \\ \lambda_2(R_2(\zeta)) = \zeta. \end{cases} \quad (5.4)$$

Here $r_1(u)$ and $r_2(u)$ are right eigenvectors of $\nabla f(u)$ corresponding to the eigenvalues $\lambda_1(u)$ and $\lambda_2(u)$, respectively. We observe that the third equation of both (5.3) and (5.4) normalize r_1 and r_2 , respectively, so that $R_1(\zeta)$ and $R_2(\zeta)$ (and consequently u_M) are completely determined by the first two equations in (5.3) and (5.4), respectively.

Let $u(t, x)$ be any solution of (1.1) and (1.5) such that $u \in L^\infty(\mathbb{R}_+^2)$. By the Schwartz lemma on nonnegative distributions [37] and the theory of divergence-measure fields [6, 7], it follows that, given any convex entropy pair (η, q) of (1.1), $(\eta(u(t, x)), q(u(t, x))) \in \mathcal{D}\mathcal{M}((0, \infty) \times \mathbb{R})$. For strictly hyperbolic systems, Lax's theory [27] indicates that, given any constant state \tilde{v} , there always exists a neighborhood of \tilde{v} such that one can find a strictly convex entropy pair (η_*, q_*) of (1.1) defined in that neighborhood. For the p -system

$$\partial_t u_1 + \partial_x p(u_2) = 0, \quad \partial_t u_2 - \partial_x u_1 = 0, \quad (5.5)$$

where $p \in C^2 \cap L^1((a, b))$ satisfies $p' < 0$ and $p'' > 0$, it is well known that

$$\eta_*(u_1, u_2) = \frac{1}{2}u_1^2 + \int_{u_2}^b p(s) ds, \quad q_*(u_1, u_2) = u_1 p(u_2) \quad (5.6)$$

is a strictly convex entropy pair in any compact subset of $\mathbb{R} \times (a, b)$.

We assume that (η_*, q_*) is a strictly convex entropy for (1.1). Following [2], one has that, for any C^2 entropy η , there exists a constant C_η such that $\eta + C_\eta \eta_*$ is a convex entropy. Consequently, we have $(\eta(u(t, x)), q(u(t, x))) \in \mathcal{DM}((0, \infty) \times \mathbb{R})$ for any entropy pair of (1.1).

Consider the family of entropy pairs $(\alpha(u, v), \beta(u, v))$, parameterized by v , formed by the quadratic parts of η_* and q_* at v :

$$\alpha(u, v) = \eta_*(u) - \eta_*(v) - \nabla \eta_*(v)(u - v), \tag{5.7}$$

$$\beta(u, v) = q_*(u) - q_*(v) - \nabla q_*(v)(f(u) - f(v)). \tag{5.8}$$

It follows from Theorem 3 in [6] that, if $u(t, x)$ is an L^∞ entropy solution of (1.1) and $v(t, x)$ is a $BV_{\text{loc}} \cap L^\infty$ entropy solution of (1.1), then $(\alpha(u, v), \beta(u, v))(t, x) \in \mathcal{DM}((0, \infty) \times \mathbb{R})$. As in [12], we consider the measures

$$\begin{aligned} \theta &= \partial_t \eta_*(u(t, x)) + \partial_x q_*(u(t, x)), \\ \gamma &= \partial_t \alpha(u(t, x), v(t, x)) + \partial_x \beta(u(t, x), v(t, x)). \end{aligned}$$

Set

$$\begin{aligned} \Omega_1 &= \{(t, x) \mid \lambda_1(u_L) < x/t < \lambda_1(u_M), t > 0\}, \\ \Omega_2 &= \{(t, x) \mid \lambda_2(u_M) < x/t < \lambda_2(u_R), t > 0\}. \end{aligned}$$

Recall that

$$\nabla^2 \eta(u)(r_1(u), r_2(u)) = 0, \tag{5.9}$$

for any entropy η (see [12]). We notice that, because of (5.9),

$$l_j(v) = r_j(v) \nabla^2 \eta_*(v) \tag{5.10}$$

is a left eigenvector of $\nabla f(v)$ corresponding to the eigenvalue $\lambda_j(v)$, $j = 1, 2$. We also easily see that, for $(t, x) \in \Omega_j$, one has

$$\frac{\partial v(t, x)}{\partial x} = \frac{1}{t} r_j(v(t, x)), \quad j = 1, 2.$$

Then, by (5.2) and Theorem 3 in [6], for any Borel set $E \subset \Omega_j$, $j = 1, 2$, we have

$$\gamma(E) = \theta(E) - \iint_E \frac{1}{t} l_j(v) Qf(u, v) \, dx \, dt, \tag{5.11}$$

where $Qf(u, v) = f(u) - f(v) - \nabla f(v)(u - v)$ is the quadratic part of f at v . As a direct consequence of (5.11), we have the following result for the p -system with $p' < 0$ and $p'' > 0$.

THEOREM 5.1. *Let $v(t, x)$ be the classical Riemann solution of (5.5) and (1.5), consisting of two rarefaction waves. Let $u(t, x)$ be any L^∞ entropy solution of (5.5) and (1.5) in $[0, T) \times \mathbb{R}$. Then $u(t, x) = v(t, x)$, a.e. in $[0, T) \times \mathbb{R}$.*

Proof. Let Π_t denote the strip $\{(s, x) \mid x \in \mathbb{R}, 0 < s < t\}$ and $\Omega_j(t) = \Omega_j \cap \Pi_t$, $j = 1, 2$. By the Gauss-Green Formula (Theorem 2 of [6]) and the finiteness of propagation speed of the solution (Theorem 5 of [6]), we have

$$\gamma\{\Pi_t\} = \int_{\mathbb{R}} \alpha(u(t, x), v(t, x)) dx, \quad \text{for a.e. } t > 0. \quad (5.12)$$

On the other hand, we have from (5.11)

$$\gamma\{\Omega_j(t)\} = \theta\{\Omega_j(t)\} - \iint_{\Omega_j(t)} \frac{1}{s} l_j(v) Qf(u, v) dx ds, \quad j = 1, 2.$$

Since $v(t, x)$ is constant in each component of $\Pi_t - \{\Omega_1(t) \cup \Omega_2(t)\}$, one has

$$\gamma\{\Pi_t\} = \theta\{\Pi_t\} - \sum_{j=1,2} \iint_{\Omega_j(t)} \frac{1}{s} l_j(v) Qf(u, v) dx ds. \quad (5.13)$$

Then, since $\theta \leq 0$ as a Radon measure, it suffices to prove that $l_j(v) Qf(u, v) \geq 0$. For the p -system, $l_j(v)$ is a positive multiple of $(1, \pm \sqrt{-p'(v_2)})$ and

$$Qf(u, v) = (p(u_2) - p(v_2) - p'(v_2)(u_2 - v_2), 0)^\top,$$

and hence we actually have $l_j(v) Qf(u, v) \geq 0$, $j = 1, 2$. This completes the proof. ■

Remark 5.1. It is clear from the proof above that, for the p -system, the rarefaction waves are not only unique but also L^2_{loc} stable in the class of L^∞ entropy solutions of (1.1)–(1.2), satisfying the entropy inequality (2.3) only for a strictly convex entropy pair (η_*, q_*) of (5.5). In the general case, as we will see below, we must assume that the entropy inequality is satisfied for all convex entropies of (1.1). This is always true for solutions obtained by the vanishing viscosity method or by numerical schemes such as the Lax-Friedrichs', Godunov's, and Glimm's scheme.

We now return to general 2×2 systems. We will prove the following result in the class of L^∞ solutions, which is an extension of DiPerna's theorem [12] in the class of BV_{loc} solutions.

THEOREM 5.2. *For every $\tilde{v} \in \mathbb{R}^2$, there exists a constant $\delta > 0$ depending only on f and \tilde{u} with the following property: If $v(t, x)$ is the classical Riemann solution of (1.1) and (1.5), consisting of two rarefaction waves, and $u(t, x)$ is any L^∞ entropy solution of (1.1) and (1.5) in $[0, T) \times \mathbb{R}$ such that $\|u - \tilde{v}\|_\infty \leq \delta$ and $\|v - \tilde{v}\|_\infty \leq \delta$, then $u(t, x) = v(t, x)$, a.e. in $[0, T) \times \mathbb{R}$.*

Proof. *Step 1.* We consider a pair of Riemann invariants $w = (w_1(v), w_2(v))$ for (1.1) satisfying

$$\nabla w_i(v) \cdot \tilde{r}_j(v) = \delta_{i+j,3}, \quad i, j = 1, 2, \tag{5.14}$$

where $\tilde{r}_i(v) = a_i(v) r_i(v)$ for some smooth functions $a_i(v) > 0$, and $\delta_{k,l}$ is the Kronecker symbol ($\delta_{k,l} = 1$, if $k = l$; 0 , if $k \neq l$). The existence of such a pair for the 2×2 systems in any compact subset of \mathbb{R}^2 is well-known (see [27, 41]).

We easily see from (5.9)–(5.10) and (5.14) that $l_j(v) = m_j(v) \nabla w_j(v)$ for some smooth functions $m_j(v) > 0$, $j = 1, 2$. The following lemma is proved in [12].

LEMMA 5.2. *Given $M > 0$, there exist positive constants c_1, c_2 , and δ such that, if $|u|$ and $|v|$ are less than M and $|u - v| \leq \delta$, then*

$$l_j(v) Qf(u, v) \geq c_1(w_j(u) - w_j(v))^2 - c_2(w_i(u) - w_i(v))^2, \quad i \neq j. \tag{5.15}$$

To continue the proof of Theorem 5.2, we observe that Lemma 5.2 has the following corollary.

COROLLARY 5.1. *Let $v(t, x)$ be given by (5.2). Let $u(t, x)$ be any L^∞ entropy solution of (1.1) and (1.5). There exists $\delta > 0$ such that, if $\|u - v\|_\infty \leq \delta$ and E is any Borel set with $E \subset \Omega_j$,*

$$\gamma(E) \leq \theta(E) + \iint_E \frac{\text{const.}}{t} (w_i(u) - w_i(v))^2 dx dt, \quad i \neq j, \quad i, j = 1, 2. \tag{5.16}$$

Step 2. To overcome the difficulty represented by the singularity $1/t$ in the integrals in (5.16), one idea is to use a couple of auxiliary entropies so that some part of the nonpositive measure θ can be used to control the effects of that singularity (see [12]). This is done in the following lemma for L^∞ entropy solutions (without BV structure).

LEMMA 5.3. Given $M > 0$ and $\varepsilon > 0$, one can find $\delta(\varepsilon, M) > 0$ such that, if $v(t, x)$ is given by (5.2), $u(t, x)$ is an L^∞ entropy solution of (1.1) and (1.5) in Π_T , and

$$\|u\|_\infty + \|v\|_\infty \leq M, \quad \text{osc}(u) + \text{osc}(v) \leq \delta,$$

then, for $0 \leq t < T$,

$$\begin{aligned} & \int_{L_t^j} (w_i(u(t, x)) - w_i(v(t, x)))^2 dx \\ & \leq \text{const. } \varepsilon |\theta\{\Pi_t^j\}|, \quad i \neq j, \quad i, j = 1, 2, \end{aligned} \quad (5.17)$$

where

$$\begin{aligned} L_t^j &= \{x \mid (-1)^j (x - \lambda_j(u_M)t) > 0\}, \\ \Pi_t^j &= \{(s, x) \mid x \in L_s^j, 0 < s < t\}, \quad j = 1, 2. \end{aligned}$$

Proof. We find a couple of entropies $\eta_j(u)$, $j = 1, 2$, satisfying the properties

$$\begin{aligned} c_1(w_i(u) - w_i(v))^2 \leq \eta_j(u) \leq c_2(w_i(u) - w_i(v))^2, \\ \text{if } w_i(v) = w_i(u_M), \quad i \neq j, \end{aligned} \quad (5.18)$$

$$\nabla^2 \eta_j(u_M) \geq 0, \quad j = 1, 2, \quad (5.19)$$

$$\begin{aligned} (-1)^j (\lambda_j(v) \eta_j(u) - q_j(u)) \geq 0, \\ \text{if } w_i(v) = w_i(u_M), \quad i \neq j, \end{aligned} \quad (5.20)$$

where the constants c_j , $j = 1, 2$, in (5.18) are positive. Such entropies exist (see Appendix).

We consider the distributions

$$\mu_j \equiv \partial_t \eta_j(u) + \partial_x q_j(u), \quad j = 1, 2.$$

As mentioned above, for suitable constants C_{η_j} , we have that $\tilde{\eta}_j = \eta_j + C_{\eta_j} \eta_*$, $j = 1, 2$, are convex entropy functions. Therefore, by the assumption, $\partial_t \tilde{\eta}_j(u) + \partial_x \tilde{q}_j(u)$, $j = 1, 2$, are nonpositive distributions (both satisfy (2.3)). Thus, by the Schwartz lemma [37], they are actually Radon measures over $(0, T) \times \mathbb{R}$. Since this is also true for $\partial_t \eta_*(u) + \partial_x q_*(u)$, one has that μ_j , $j = 1, 2$, are (signed) Radon measures over $(0, T) \times \mathbb{R}$. Moreover, given $\varepsilon > 0$, one can take $C_{\eta_j} < \varepsilon$, $j = 1, 2$, so that

$$\mu_j \leq -\varepsilon \theta, \quad j = 1, 2, \quad (5.21)$$

by property (5.19), provided that $\delta > 0$ is sufficiently small. Now, using the Gauss-Green Formula and the finiteness of propagation speeds for L^∞ solutions, and setting $L(t) \equiv \{x = \lambda_1(u_M)s, 0 < s < t\}$, one has

$$\mu_1\{\Pi_t^1\} = \int_{-\infty}^{\lambda_1(u_M)t} \eta_1(u(t, x)) dx + \int_{L(t)} \Phi((\eta_1(u), q_1(u)) | L(t)) dH^1(s), \tag{5.22}$$

for a.e. $t \in (0, T)$, namely, those t such that H^1 -almost all points of the line $s = t$ are Lebesgue points of $u(s, x)$. Also, the second term in the right-hand side of (5.22) is nonnegative. This can be seen by the following procedure. Extend the field $(\eta_1(u), q_1(u))$ to all \mathbb{R}^2 by setting it as 0 outside Π_T . Consider the open set $\Omega = \{x < \lambda_1(u_M)s, s \in \mathbb{R}\}$ and the deformation of $\partial\Omega$ given by $\Psi((s, \lambda_1(u_M)s), \tau) = (s, \lambda_1(u_M)s - \tau)$. Since $\Phi((\eta_1(u), q_1(u)) | \partial\Omega_\tau) \geq 0$, for a.e. τ by property (5.20), we conclude that it also holds for $\tau = 0$ in $x = \lambda_1(u_M)s$. On the other hand, we have from (5.21)

$$\mu_1\{\Pi_t^1\} \leq \varepsilon |\theta\{\Pi_t^1\}|.$$

This fact, together with (5.22), property (5.18), and the above observation about the flux term, gives (5.17). The proof for $j = 2$ is similar. ■

Step 3. Again, by the Gauss-Green Formula for \mathcal{DM} fields and the finiteness of propagation speeds of the solutions, we have

$$\gamma\{\Pi_t\} = \int_{-\infty}^{\infty} \alpha(u(t, x), v(t, x)) dx, \tag{5.23}$$

for a.e. $t \in (0, T)$. Also, from (5.16), we have

$$\gamma\{\Pi_t\} \leq \theta\{\Pi_t\} + \sum_{j=1, 2; i \neq j} \iint_{\Omega_j(t)} \frac{C}{s} (w_i(u) - w_i(v))^2 dx ds. \tag{5.24}$$

Now, it follows from Lemma 5.3 that

$$\theta \left\{ \bigcup_{j=1}^2 \Pi_t^j \right\} \leq \frac{c}{\varepsilon} \sum_{j=1, 2; i \neq j} \int_{L_t^j} (w_i(u(t, x)) - w_i(v(t, x)))^2 dx, \tag{5.25}$$

where c is the positive constant given by Lemma 5.3. Combining (5.23)–(5.24) with (5.25), we finally arrive at

$$\begin{aligned} & \frac{1}{\varepsilon} \sum_{j=1, 2; i \neq j} \int_{L_t^j} (w_i(u) - w_i(v))^2 dx + \int_{-\infty}^{\infty} \sum_{j=1, 2} (w_j(u) - w_j(v))^2(t, x) dx \\ & \leq \sum_{j=1, 2; i \neq j} \iint_{\Omega_j(t)} \frac{C}{s} (w_i(u) - w_i(v))^2 dx ds. \end{aligned} \tag{5.26}$$

Denote by $g(t)$ the sum of the first two integrals on the left-hand side of (5.26). With this inequality, one easily obtains

$$0 \leq g(t) \leq \int_0^t \frac{C\varepsilon}{s} g(s) ds. \quad (5.27)$$

It follows directly from the finiteness of propagation speeds of the solutions that $g(t) \leq Ct$. Plugging the latter in (5.27), for $\varepsilon > 0$ sufficiently small, we obtain that $g(t) \leq \lambda Ct$, for the same “const.” taken before plugging and for some λ , $0 < \lambda < 1$. Since we can keep plugging as many times as we wish, we must have $g(t) \equiv 0$, for a.e. $t \in (0, T)$. Now, returning to inequality (5.26), we obtain that the second integral on the left-hand side of (5.26) must vanish for a.e. $t \in (0, T)$. This gives the desired result. ■

Hence, combining a compactness theorem in [10] with Theorems 5.1–5.2 yields the following.

THEOREM 5.3. *Let (1.1) be a strictly hyperbolic and genuinely nonlinear system. Let $R(x/t)$ be a Riemann solution of (1.1) with Riemann data $R_0(x)$ whose left and right states are connected by one or two rarefaction waves. Let $u(t, x)$ be an L^∞ entropy solution of (1.1)–(1.2) with $u_0(x)$ satisfying (1.2)–(1.4). For $\delta > 0$ sufficiently small, if $\|R - \bar{u}\|_\infty \leq \delta$ and $\|u - \bar{u}\|_\infty \leq \delta$, where $\bar{u} \in \mathbb{R}^2$ is a constant, then the Riemann solution is asymptotically stable with respect to the initial perturbation $P_0(x)$. For the p -system, the restriction of small oscillation on R and u can be removed.*

5.2. Uniqueness and Stability of Riemann Solutions in BV

In this section we recall a theorem in [12] for the uniqueness in BV of general Riemann solutions of the 2×2 systems, whose characteristic fields are either genuinely nonlinear or linearly degenerate, with small oscillation. The restriction of small oscillation can be removed for the p -system. We also recall some other points in [12] which will be used in our study on the 3×3 Euler equations in Subsections 5.3–5.4.

We assume now that $v(t, x)$ is the classical Riemann solution of (1.1) and (1.5). For concreteness, we may suppose that $v(t, x)$ consists of a 1-shock wave connecting u_L to some state u_M and a 2-rarefaction wave connecting u_M to u_R . That is,

$$v(t, x) = \begin{cases} u_L, & x/t < \sigma = \sigma(u_L, u_M), \\ u_M, & \sigma < x/t \leq \lambda_2(u_M), \\ R_2(x/t), & \lambda_2(u_M) < x/t < \lambda_2(u_R), \\ u_R, & x/t \geq \lambda_2(u_R), \end{cases} \quad (5.28)$$

where σ is the shock speed, determined by the Rankine-Hugoniot relations

$$\sigma(u_L - u_M) - f(u_L) + f(u_M) = 0, \tag{5.29}$$

and $R_2(\xi)$ is the solution of the boundary value problem (5.4). We assume that the 1-shock wave connecting u_L and u_M satisfies the Lax entropy conditions

$$\lambda_1(u_M) < \sigma < \lambda_1(u_L), \quad \sigma < \lambda_2(u_M).$$

The second inequality in (5.30) is automatically satisfied when

$$\lambda_1(u) < k_0 < \lambda_2(u), \tag{5.31}$$

for some constant k_0 , for all u in the region under consideration. For the p -system, (5.31) holds for $k_0 = 0$ and all $u \in \mathbb{R}^2$ for which the system is defined. The solutions considered here will always take values in a neighborhood of a constant state where (5.31) is satisfied for some k_0 . This fact was necessary in the above proof of the uniqueness of Riemann solutions consisting of two rarefaction waves.

To deal with shock waves, DiPerna [12] used the concept of generalized characteristics (see Dafermos [8]). A generalized j -characteristic associated with a solution $u(t, x) \in BV$ is defined as a trajectory of the equation

$$\dot{x}(t) = \lambda_j(u(t, x)), \tag{5.32}$$

where (5.32) is interpreted in the sense of Filippov [14]. Thus, a j -characteristic is a Lipschitz continuous curve $(t, x(t))$ whose speed of propagation $\dot{x}(t)$ satisfies

$$\dot{x}(t) \in [m_x\{\lambda_j(u(t, x(t)))\}, M_x\{\lambda_j(u(t, x(t)))\}], \tag{5.33}$$

where $m_x\{\lambda_j(u(t, x(t)))\}$ and $M_x\{\lambda_j(u(t, x(t)))\}$ denote the essential minimum and the essential maximum of $\lambda_j(u(t, \cdot))$ at the point $x(t)$. As it was proved by Filippov [14], among all solutions of (5.33) passing through a point (t_0, x_0) , there is an upper solution $\bar{x}(t)$ and a lower solution $\underline{x}(t)$, that is, solutions of (5.33) such that any other solution $x(t)$ of (5.33) satisfies the inequality $\underline{x}(t) \leq x(t) \leq \bar{x}(t)$. The lower and upper solutions, for $t > t_0$, are called the minimal and maximal forward j -characteristics, respectively. An essential feature about solutions in $BV(\Pi_T)$ is that, given any generalized i -characteristic $y(t)$, it must propagate either with shock speed or with characteristic speed (cf. [8]). This allows one to treat $(t, y(t))$ simply as a shock curve of $u(t, x)$ in the (t, x) -plane.

One of the main lemmas in the proof of [12], which will be used in Subsection 5.4, is following.

LEMMA 5.4. *Let (1.1) be an $m \times m$ strictly hyperbolic system endowed with a strictly convex entropy pair, whose characteristic fields are either genuinely nonlinear or linearly degenerate. Suppose $u(t, x)$ is a BV_{loc} entropy solution of (1.1) and (1.5) in $[0, T) \times \mathbb{R}$. Let $x_{max}^m(t)$ denote the maximal forward m -characteristic through $(0, 0)$. Let $x_{min}^1(t)$ denote the minimal forward 1-characteristic passing through $(0, 0)$. Then $u(t, x) = u_L$, for a.e. (t, x) with $x < x_{min}^1(t)$, $0 \leq t < T$, and $u(t, x) = u_R$, for a.e. (t, x) with $x > x_{max}^m(t)$, $0 \leq t < T$.*

Combining Lemma 5.1 and the L^1_{loc} -compactness of bounded subsets of BV , with DiPerna's uniqueness theorem in BV for general Riemann solutions, we immediately obtain the following theorem.

THEOREM 5.4. *Given $\tilde{u} \in \mathbb{R}^2_+$, there exists $\delta > 0$ for which the following hold. Let $u(t, x) \in BV_{loc}(\mathbb{R}^2_+)$ be an entropy solution of (1.1)–(1.2), with $u_0(x)$ satisfying (1.2)–(1.4). Assume u satisfies (5.1), for any $c > 0$, $T_0 > 0$, and some $C > 0$ independent of T_0 , and $\|u - \tilde{u}\|_\infty < \delta$. Then u asymptotically tends to the Riemann solution $R(x/t)$. For the p -system, the restriction of small oscillation on u can be removed.*

Remark 5.2. One can easily obtain the results analogous to Theorem 5.4 for general $m \times m$ strictly hyperbolic systems whose characteristic fields are either genuinely nonlinear or linearly degenerate, in the case where the Riemann solution consists of only extreme shocks, that is, the Riemann states u_L and u_R are connected by shock curves of the first and m th characteristic families. This is an immediate consequence of the proof of the uniqueness theorem given in [12]. As a corollary, one concludes that the Riemann solution is the unique attractor of BV entropy solutions of the full 3×3 Euler system of conservation of mass, momentum, and energy, provided that these solutions satisfy (5.1), with sufficiently small oscillation, and their initial data satisfy (1.2)–(1.4), where u_L and u_R can be connected only by shock curves (of the first and third families). In the following subsection we consider this system with a special class of constitutive relations (see (5.38)). In this case, we obtain more general results.

5.3. 3×3 Euler Equations: Shock-Free Riemann Solutions

Our objective here is to establish the uniqueness of Riemann solutions of the Euler equations in thermodynamics to obtain consequently their asymptotic stability. As for the p -system above, we first prove the uniqueness of large Riemann solutions in the class of L^∞ solutions, when the Riemann solutions do not contain shock waves. Now, besides rarefaction waves of the first and third families, it may contain a contact discontinuity of the second family.

The balance laws of mass, momentum, and energy for inviscid elastic media are expressed, in Lagrangian coordinates, by the equations

$$\partial_t \tau - \partial_x v = 0, \quad \partial_t v + \partial_x p = 0, \quad \partial_t (e + \frac{1}{2}v^2) + \partial_x (vp) = 0, \quad (5.34)$$

where τ , v , p , and e denote respectively the deformation gradient (the specific volume for fluids, the strain for solids), the velocity, the pressure, and the internal energy. Other relevant physical variables are the entropy s and the temperature θ . The system (5.34) is complemented by the Clausius inequality

$$\partial_t s \geq 0, \quad (5.35)$$

which expresses the second law of thermodynamics.

Selecting (τ, v, s) as the state vector, we write constitutive equations

$$e = \hat{e}(\tau, s), \quad p = \hat{p}(\tau, s), \quad \theta = \hat{\theta}(\tau, s), \quad (5.36)$$

satisfying the conditions

$$\hat{p} = -\hat{e}_\tau, \quad \hat{\theta} = \hat{e}_s, \quad (5.37)$$

which are the consequence of the first law of thermodynamics and ensure that (5.35) holds as an identity for any smooth solution of (5.36). Under the standard assumptions $\hat{p}_\tau < 0$ and $\hat{\theta} > 0$, the system is strictly hyperbolic.

We consider the following class of constitutive relations for the new state vector (w, v, s) with the form

$$\tau = w + \kappa_1 s, \quad p = h(w), \quad e = H(w) + \kappa_2 s, \quad \theta = \kappa_1 h(w) + \kappa_2, \quad (5.38)$$

where κ_1 and κ_2 are positive constants, and $H(w) = -\int^w h(\omega) d\omega$. Throughout the following, we assume that $h(w)$ in (5.38) satisfies

$$h \in C^3, \quad h(w) > 0, \quad h'(w) < 0, \quad \text{and} \quad h''(w) > 0, \quad (5.39)$$

for w in the region of interest. Also, (5.38) can be written into the form (5.36) as

$$e = H(\tau - \kappa_1 s) + \kappa_2 s, \quad p = h(\tau - \kappa_1 s), \quad \theta = \kappa_1 h(\tau - \kappa_1 s) + \kappa_2 \quad (5.40)$$

and (5.37) holds. The model (5.38) can be regarded as a "first-order correction" to general constitutive relations (5.36) (see [3] for details).

Write $u = (\tau, v, E)$, where $E = e + \frac{1}{2}v^2$ is the energy, and consider the Cauchy problem for (5.34) with initial data

$$u|_{t=0} = u_0(x) \equiv (\tau_0(x), v_0(x), E_0(x)), \quad (5.41)$$

satisfying (1.2)–(1.4), with the Riemann problem for (5.34):

$$u|_{t=0} = R_0(x) \equiv \begin{cases} u_L \equiv (\tau_L, v_L, E_L), & x < 0, \\ u_R \equiv (\tau_R, v_R, E_R), & x > 0. \end{cases} \quad (5.42)$$

As usual, we say that $u(t, x) = (\tau(t, x), v(t, x), E(t, x))$ is a weak solution of (5.34)–(5.41) in Π_T if, for all $\phi \in C^1(\Pi_T)$ with compact support in Π_T , one has

$$\iint_{\Pi_T} \{u\phi_t + f(u)\phi_x\} dx dt + \int_{\mathbf{R}} u_0(x)\phi(0, x) dx = 0, \quad (5.43)$$

where $f(u) = (-v, p(\tau, s), vp(\tau, s))$. Since the mapping from (τ, v, E) to (w, v, s) is one-to-one, we will not distinguish these two coordinates in terms of solutions.

Let $\bar{u}(t, x)$ denote the classical Riemann solution. We start with the case where the classical Riemann solution of (5.34) and (5.42) is shock-free, that is, u_L and u_R can be connected only by rarefaction wave curves and possibly a contact discontinuity curve of the second family with linear degeneracy.

THEOREM 5.5. *Let $\bar{u}(t, x)$ be the classical shock-free Riemann solution of (5.34) and (5.42). Let $u(t, x) = (\tau(t, x), v(t, x), E(t, x)) \in L^\infty(\Pi_T; \mathbb{R}^3)$ be any weak solution of (5.34) and (5.42) in Π_T , satisfying (5.35) in the sense of distributions. Assume (5.38)–(5.39) hold. Then $u(t, x) = \bar{u}(t, x)$, a.e. in Π_T .*

Proof. Let $W(t, x)$ and $\bar{W}(t, x)$ be the projections of $u(t, x)$ and $\bar{u}(t, x)$ on the $w-v$ plane. We notice that \bar{W} is a Lipschitz solution of

$$\partial_t w - \partial_x v = 0, \quad \partial_t v + \partial_x p(w) = 0, \quad (5.44)$$

for $t > 0$. Indeed, by assumption, $\bar{u}(t, x)$ does not contain any shock discontinuities and s is constant along rarefaction wave curves (see, e.g., [41]), while v and p (hence, also w) are constant along the contact discontinuity wave curves. We consider the strictly convex entropy pair for (5.44):

$$(\eta_*, q_*)(w, v) = \left(\frac{1}{2}v^2 + H(w), vh(w)\right). \quad (5.45)$$

Then $W(t, x)$ is a weak solution of

$$\partial_t w - \partial_x v = -\kappa_1 \partial_t s = \kappa(\partial_t \eta_*(w, v) + \partial_x q_*(w, v)), \quad \partial_t v + \partial_x p(w) = 0, \quad (5.46)$$

from (5.34), where $\kappa = \kappa_1/\kappa_2$.

Next, we consider the family of quadratic entropy pairs, parameterized by $\bar{W} = (\bar{W}, \bar{v})$, given by

$$\begin{aligned} \alpha(W, \bar{W}) &= \eta_*(W) - \eta_*(\bar{W}) - \nabla \eta_*(\bar{W})(W - \bar{W}), \\ \beta(W, \bar{W}) &= q_*(W) - q_*(\bar{W}) - \nabla \eta_*(\bar{W})(f(W) - f(\bar{W})), \end{aligned}$$

where $f(W) = (-v, h(w))$. We again use Theorem 3 in [6] to conclude

$$(\alpha(W(t, x), \bar{W}(t, x)), \beta(W(t, x), \bar{W}(t, x))) \in \mathcal{DM}(\Pi_T),$$

and the validity of the product rule, since $\bar{W}(t, x)$ is locally Lipschitz in Π_T . Consider the measures

$$\begin{aligned} \theta &= \partial_t \eta_*(W(t, x)) + \partial_x q_*(W(t, x)), \\ \gamma &= \partial_t \alpha(W(t, x), \bar{W}(t, x)) + \partial_x \beta(W(t, x), \bar{W}(t, x)), \end{aligned}$$

where the fact that θ is a nonpositive measure is granted by the entropy condition (5.35). We have

$$\begin{aligned} \gamma &= \partial_t \alpha(W, \bar{W}) + \partial_x \beta(W, \bar{W}) \\ &= \partial_t \eta_*(W) + \partial_x q_*(W) - \kappa \partial_w \eta_*(\bar{W})(\partial_t \eta_*(W) + \partial_x q_*(W)) \\ &\quad - \nabla^2 \eta_*(\bar{W})(\partial_t \bar{W}(W - \bar{W}) + \partial_x \bar{W}(f(W) - f(\bar{W}))) \\ &\leq \theta - \nabla^2 \eta_*(\bar{W})(f(W) - f(\bar{W}) - \nabla f(\bar{W})(W - \bar{W})) \partial_x \bar{W}, \end{aligned}$$

where we used again the fact that $\nabla^2 \eta_* \nabla f$ is symmetric and that $\partial_w \eta_*$ is negative by (5.45) and (5.39). Therefore, the same arguments as in the proof of Theorem 5.2 yield $W(t, x) = \bar{W}(t, x)$. To conclude the proof, we notice that, by the first equation in (5.34), we must have $\partial_t(s(t, x) - \bar{s}(t, x)) = 0$, a.e. in Π_T . It then follows that $s(t, x) = \bar{s}(t, x)$, a.e. in Π_T , from $s(0, x) = \bar{s}(0, x)$, $x \in \mathbb{R}$. Hence we obtain $u(t, x) = \bar{u}(t, x)$, a.e. in Π_T , as desired. ■

Although we have assumed $u \in L^\infty(\Pi_T; \mathbb{R}^3)$ through the proof of Theorem 5.5, we only used the property $(w, v) \in L^\infty(\Pi_T; \mathbb{R}^2)$. Hence the same proof gives the uniqueness of Riemann solutions in the class of weak solutions satisfying (5.35), with $(w, v) \in L^\infty(\Pi_T; \mathbb{R}^2)$ and $s \in \mathcal{M}(\Pi_T)$, where the definition of weak solution should be adapted in an obvious way. The results in [3] can be used to prove the compactness of weak solutions (w^T, v^T, s^T) of (5.34) and (5.41), satisfying (5.35), with (w^T, v^T) uniformly bounded in $L^\infty(\mathbb{R}_+^2)$ and s^T uniformly bounded in $\mathcal{M}_{loc}(\mathbb{R}_+^2)$. They also imply the convergence of the vanishing viscosity method to prove the existence of a weak solution (w, v, s) of (5.34), (5.41), satisfying (5.35), and

$$|s| \{ [0, T_0] \times [-cT_0, cT_0] \} \leq CT_0^2, \tag{5.47}$$

for any $c > 0$, $T_0 > 0$, and some $C > 0$ independent of T_0 , where $|s|$ denotes the variation of the measure s . Therefore, combining Theorem 5.5 with the compactness result in [3] yields the following asymptotic result.

THEOREM 5.6. *Suppose $u(t, x)$ is a weak solution of (5.34) and (5.41) such that $(w, v) \in L^\infty(\mathbb{R}_+^2)$, $(s, s_t) \in \mathcal{M}_{loc}(\mathbb{R}_+^2)$, satisfying (5.35) and (5.47), and $(w_0, v_0, s_0) \in L^\infty(\mathbb{R})$ satisfies (1.2)–(1.4). Assume that u_L is connected to u_R by a Riemann solution $(W, V, S)(t, x)$ consisting of only rarefaction waves of the first and third families and possibly a contact discontinuity of the second linearly degenerate family. Then $(w, v)(t, x)$ asymptotically tends to $(W, V)(x/t)$ in the sense of (2.3). Moreover, for any $\phi \in C_0^\infty(\Omega)$,*

$$\langle s^T, \phi \rangle \rightarrow \langle S, \phi \rangle.$$

The Riemann solution (W, V, S) is the unique attractor.

Proof. The only thing to be observed is that, if (w^T, v^T, s^T) is the scaling sequence associated with the weak solution (w, v, s) , where the scaling of s must be taken in the sense of distributions, then s^T also satisfies (5.47) with the same constant $C > 0$. Hence, the theorem follows from the compactness result in [3] and is the straightforward extension of Theorem 5.5 to the case where $(w, v) \in L^\infty(\Pi_T; \mathbb{R}^2)$ and $s \in \mathcal{M}(\Pi_T)$. ■

5.4. 3×3 Euler Equations: General Riemann Solutions

We now investigate the uniqueness of general Riemann solutions in the class of BV solutions. The existence of BV solutions can be obtained by the Glimm scheme for BV initial data with moderate oscillation. The idea is to prove first the uniqueness of solutions of the corresponding Cauchy problem for the subsystem (5.46). The difficulty is now that the projection of any Riemann solution in the $w-v$ plane no longer satisfies the entropy identity: $\partial_t \eta_*(\bar{W}) + \partial_x q_*(\bar{W}) = 0$ in the sense of distributions. Therefore, more careful analysis is needed.

THEOREM 5.7. *Let $u(t, x) = (\tau(t, x), v(t, x), E(t, x)) \in BV(\Pi_T; \mathbb{R}^3)$ be a weak entropy solution of (5.34) and (5.42) in Π_T , satisfying the entropy condition (5.35) in the sense of distributions. Assume (5.38) and (5.39). Then $u(t, x) = \bar{u}(t, x)$, a.e. in Π_T .*

Proof. As in the proof of Theorem 5.5, the strategy will be to consider first the subsystem (5.46) to get the coincidence of the projections on the $w-v$ plane, and then to conclude immediately the coincidence of $u(t, x)$ and the Riemann solution $\bar{u}(t, x)$ a.e.. For concreteness, we consider only a generic Riemann solution $\bar{u}(t, x)$ consisting of the constant state \bar{u}_L connected on the right by a 1-shock to the constant state \bar{u}_M , a stationary

contact discontinuity connecting \bar{u}_M to \bar{u}_N on the right, and a rarefaction wave connecting \bar{u}_N on the right to \bar{u}_R . Using the method in [12], we consider the auxiliary function

$$\bar{u}(t, x) = \begin{cases} \bar{u}_L, & x < x(t), & 0 \leq t < T, \\ \bar{u}_M, & x(t) < x < \max\{x(t), \sigma t\}, & 0 < t < T, \\ \bar{u}(t, x), & x > \max\{x(t), \sigma t\}, & 0 < t \leq T, \end{cases}$$

where $x(t)$ is the minimal 1-characteristic of $u(t, x)$, and $x = \sigma t$ is the line of 1-shock discontinuity in $\bar{u}(t, x)$. We then consider the measure

$$\tilde{\gamma} = \partial_t \alpha(W(t, x), \tilde{W}(t, x)) + \partial_x \beta(W(t, x), \tilde{W}(t, x)),$$

where \tilde{W} is the projection of \tilde{u} over the $w-v$ plane, and $\alpha(W, \bar{W})$ and $\beta(W, \bar{W})$ are defined as above. Our problem essentially reduces to analyzing the measure $\tilde{\gamma}$ over the region where the Riemann solution experiments a rarefaction wave and over the curve $x(t)$, which for simplicity may be taken as the jump set of $\tilde{W}(t, x)$.

Again, using the Gauss-Green formula for BV functions and the finiteness of propagation speeds of the solutions, we have

$$\tilde{\gamma}\{II_t\} = \int_{-\infty}^{+\infty} \alpha(W(t, x), \tilde{W}(t, x)) dx. \tag{5.48}$$

On the other hand,

$$\tilde{\gamma}\{II_t\} = \tilde{\gamma}\{L(t)\} + \gamma\{\bar{\Omega}_2(t)\} + \theta\{II_t - (L(t) \cup \bar{\Omega}_2(t))\}, \tag{5.49}$$

where $L(t) = \{(s, x(s)) \mid 0 < s < t\}$, since $\tilde{\gamma}$ reduces to the measure θ on the open sets where \tilde{W} is a constant, and $\tilde{W}(t, x) = \bar{W}(t, x)$ over $\bar{\Omega}_2$. Hence, if one shows that

$$\tilde{\gamma}\{L(t)\} \leq 0, \tag{5.50}$$

the problem will reduce to the same verification as in the shock-free case. Thus, we consider the functional

$$D(\sigma, W_-, W_+, \bar{W}_-, \bar{W}_+) = \sigma[\alpha(W, \bar{W})] - [\beta(W, \bar{W})],$$

where the square bracket denotes the left limit minus the right limit of shock wave curve in the (t, x) -plane for the function inside the bracket. We will prove that

$$D(\sigma, W_-, W_+, \bar{W}_-, \bar{W}_+) \leq 0, \tag{5.51}$$

if W_- , W_+ are projections over the w - v plane of states u_- , u_+ , respectively, which are connected by a 1-shock of speed σ , and \bar{W}_- , \bar{W}_+ are projections over the same plane of states \bar{u}_- , \bar{u}_+ , respectively, which are connected by a 1-shock of speed $\bar{\sigma}$, and also either $u_- = \bar{u}_-$ or $u_+ = \bar{u}_+$. Using Theorem 5.4, it is then clear that (5.51) immediately implies (5.50). We will verify (5.51) assuming $u_- = \bar{u}_-$; the case where $u_+ = \bar{u}_+$ follows by the same procedure. Thus, when $u_- = \bar{u}_-$, an easy calculation shows that

$$\begin{aligned} & D(\sigma, W_-, W_+, \bar{W}_-, \bar{W}_+) \\ &= d(\sigma, W_-, W_+) - d(\bar{\sigma}, \bar{W}_-, \bar{W}_+) \\ & \quad - (\sigma - \bar{\sigma}) \alpha(W_-, \bar{W}_+) + \kappa_1 \eta_w(\bar{W}_+) (\sigma(s_- - s_+) - \bar{\sigma}(s_- - \bar{s}_+)), \end{aligned} \quad (5.52)$$

where $d(\sigma, W, \bar{W}) = \sigma[\eta(W)] - [q(W)]$, and $(\eta, q) = (\eta_*, q_*)$ is the entropy pair in (5.45). From the Rankine-Hugoniot relation for (5.34), we may view the states $u_+ = (w_+, v_+, s_+)$ connected on the right by a 1-shock to a state $u_- = (w_-, v_-, s_-)$ as parameterized by the shock speed σ , with $\sigma \leq \lambda_1(u_-) < 0$. We recall that, through this parameterization, $s(\sigma)$ satisfies (see [27, 41])

$$s(\sigma) = s(\lambda_1(u_-)) - \frac{\ddot{s}(\lambda_1(u_-))}{6} (\lambda_1(u_-) - \sigma)^3 + O((\lambda_1(u_-) - \sigma)^4), \quad (5.53)$$

and

$$\ddot{s}(\lambda_1(u_-)) < 0. \quad (5.54)$$

According to the parameterization, we set $(W_+, s_+) = (W_+, s_+)(\sigma)$ and $(\bar{W}_+, \bar{s}_+) = (W_+, s_+)(\bar{\sigma})$ in (5.52). For concreteness, we assume $\bar{\sigma} > \sigma$. Now, we have

$$\begin{aligned} & \kappa_1 \eta_w(\bar{W}_+) (\sigma(s_- - s_+(\sigma)) - \bar{\sigma}(s_- - s_+(\bar{\sigma}))) \\ &= \kappa_1 \eta_w(\bar{W}_+) (\sigma - \bar{\sigma}) \left(s_- - s_+(\bar{\sigma}) - \sigma \frac{s_+(\bar{\sigma}) - s_+(\sigma)}{\bar{\sigma} - \sigma} \right) \\ &= \kappa_1 \eta_w(\bar{W}_+) (\sigma - \bar{\sigma}) (s_- - s_+(\bar{\sigma}) - \sigma \hat{s}(\hat{\sigma})), \end{aligned}$$

where $\hat{\sigma}$ satisfies $\sigma \leq \hat{\sigma} \leq \bar{\sigma} \leq \lambda_1(\bar{u}_-)$. Define $b(\sigma) \equiv d(\sigma, W, \bar{W}) = \sigma[\eta(W)] - [q(W)]$. One easily verifies that

$$\hat{b}(\bar{\sigma}) = \alpha(W_-, W_+(\bar{\sigma})) - \kappa_1 \eta_w(\bar{W}_+) (s_- - s_+(\bar{\sigma}) - \bar{\sigma} \hat{s}(\bar{\sigma})).$$

Now, from $0 > \lambda_1(u_-) \geq \bar{\sigma} \geq \sigma$ and $0 \geq \dot{s}(\bar{\sigma}) \geq \dot{s}(\hat{\sigma})$, it follows that $\eta_w(\bar{W}_+) \bar{\sigma} \dot{s}_+(\bar{\sigma}) \geq \eta_w(\bar{W}_+) \sigma \dot{s}(\hat{\sigma})$ and, hence, we obtain $\dot{b}(\bar{\sigma}) \geq \alpha(W_-, \bar{W}_+) - \kappa_1 \eta_w(\bar{W}_+) (s_- - s_+ - \sigma \dot{s}(\hat{\sigma}))$. Therefore, we have

$$D(\sigma, W_-, W_+, \bar{W}_-, \bar{W}_+) \leq b(\sigma) - b(\bar{\sigma}) - \dot{b}(\bar{\sigma})(\sigma - \bar{\sigma}).$$

Observe that the above inequality is also true in the case where $\sigma > \bar{\sigma}$. Now, it is not difficult to verify that

$$\ddot{b}(\lambda) \leq 0, \quad \text{for all } \lambda \leq \lambda_1(u_-).$$

Indeed, one has

$$\begin{aligned} \dot{b}(\lambda) &= -(\dot{W}_+(\lambda))^\top \nabla^2 \eta(W_+(\lambda))(W_- - W_+(\lambda)) \\ &\quad - \kappa_1 p(w_+(\lambda))(-2\dot{s}_+(\lambda) - \lambda \ddot{s}_+(\lambda)) \\ &\quad - \kappa_1 p'(w_+(\lambda)) \dot{w}_+(\lambda)(s_- - s_+(\lambda) - \lambda \dot{s}_+(\lambda)). \end{aligned}$$

Hence, since $p'(w) < 0$, $\dot{w}_+(\lambda) = \dot{v}_+(\lambda) - \kappa_1 \dot{s}_+(\lambda) > 0$, for $\lambda < \lambda_1(u_-)$ (see [27, 41]), (5.55) follows. We conclude (5.51).

As we already said, from (5.51) and the arguments in the shock-free case, we get that $W(t, x) = \tilde{W}(t, x)$, a.e. in Π_T . From the last equality and the Rankine–Hugoniot relations for (5.46), we conclude that $W(t, x) = \bar{W}(t, x)$, a.e. in Π_T . Now, by the same arguments in the proof of Theorem 5.5, we conclude $u(t, x) = \bar{u}(t, x)$, a.e. in Π_T . This completes the proof of Theorem 5.7. ■

Again, as an immediate consequence of Theorem 5.7 and the L^1_{loc} -compactness of bounded sets in BV , we have the following theorem.

THEOREM 5.8. *Suppose that $u(t, x) \in BV_{\text{loc}}((0, \infty) \times \mathbb{R}; \mathbb{R}^3)$ is an entropy solution of (5.34) and (1.2)–(1.4), satisfying (5.1) and the entropy condition (5.35) in the sense of distributions. Then $u(t, x)$ asymptotically tends to the Riemann solution of (5.34) and (5.42), the unique attractor.*

Remark 5.3. The same approach as above can be applied to proving the asymptotic stability of Riemann solutions for the degenerate 4×4 system of Maxwell equations for plane waves in electromagnetism and the $m \times m$ systems with symmetry as models for magnetohydrodynamics and elastic strings. It can also be applied to studying the large-time behavior of solutions of hyperbolic systems with relaxation for the same type of initial data. For these and other correlated results, see [5, 7].

6. LARGE-TIME BEHAVIOR OF APPROXIMATE SOLUTIONS

We are concerned with the asymptotic behavior of approximate solutions, generated from a dissipative mechanism, such as viscosity and relaxation, or from a numerical scheme. For concreteness, in this section we consider the Cauchy problem for viscous conservation laws:

$$\partial_t u + \partial_x f(u) = \partial_x^2 u, \quad (6.1)$$

$$u|_{t=0} = u_0(x). \quad (6.2)$$

The second approach in Section 5 can be directly adapted into the one for the approximate solutions to (1.1). The compactness of the self-similar scaling sequence $u^T(t, x) = u(Tt, Tx)$ can be achieved in the same way. For the viscous case (6.1), $u^T(t, x)$ satisfy

$$\partial_t u^T + \partial_x f(u^T) = \frac{1}{T} \partial_x^2 u^T, \quad u^T|_{t=0} = u_0(Tx). \quad (6.3)$$

In the same fashion for the systems with certain nonlinearity, one can show that $u^T(t, x)$ is compact in L^1_{loc} . The other ingredient is the uniqueness of Riemann solutions for the inviscid systems (1.1), which has been discussed in Section 4.

In this section, we show that the direct method in Section 4 can be employed to understand the large-time behavior of viscous solutions, approximation to entropy solutions, of the Cauchy problem (6.1)–(6.2). The following general discussions hold for any parabolic system under the only assumption that f be smooth, say, C^2 .

First, using standard parabolic arguments (e.g., [15, 21, 26]), we have

LEMMA 6.1. *Let $u(t, x)$ be the classical solution of (6.1) and (1.2) with uniform bound in R^2_+ . Then $u(t, x)$ has the following properties:*

1. *There exist $t_0 > 0$ and $C = C(t_0) > 0$ such that*

$$\|\partial_x u(t)\|_\infty \leq C \sqrt{\frac{t+t_0}{t}} \|u_0\|_\infty, \quad \text{for } 0 < t \leq \infty; \quad (6.4)$$

2. *For any bounded interval $I \subset \mathbb{R}$,*

$$\lim_{t \rightarrow 0} \|u(t, \cdot) - u_0(\cdot)\|_{L^1(I)} = 0; \quad (6.5)$$

3. *If $u_0(x)$ satisfies (1.2)–(1.4), then $u(t, \cdot) - u_0(\cdot) \in L^p(\mathbb{R})$, for all $1 \leq p \leq \infty$, $t > 0$;*

4. For any $0 < t_1 < T < \infty$,

$$\partial_x^k u \in L^p([t_1, T] \times \mathbb{R}), \quad k = 1, 2, \dots, \quad 1 \leq p \leq \infty. \quad (6.6)$$

LEMMA 6.2. Let system (6.1) be endowed with a strictly convex entropy pair $(\eta_*(u), q_*(u))$. Let $u(t, x) \in L^\infty(\mathbb{R}_+^2)$ be a solution of (6.1). Then, given $\xi_1, \xi_2 \in \mathbb{R}$ and $0 < \theta < 1$,

$$\int_0^\infty \int_{\xi_1 t}^{\xi_2 t} \frac{|\partial_x u(t, x)|^2}{(1+t)^{1+\theta}} dx dt < +\infty. \quad (6.7)$$

Proof. As is well known, we may assume $\eta_*(u) \geq c_0 |u|^2 \geq 0$ without loss of generality. Then

$$\partial_t \eta_*(u) + \partial_x q_*(u) = \partial_x (\nabla \eta_*(u) \partial_x u) - (\partial_x u)^\top \nabla^2 \eta_*(u) \partial_x u.$$

Dividing the above identity by $(1+t)^{1+\theta}$, one has

$$\begin{aligned} \partial_t \left(\frac{\eta_*(u)}{(1+t)^{1+\theta}} \right) + \frac{(1+\theta) \eta_*(u)}{(1+t)^{2+\theta}} + \partial_x \left(\frac{q_*(u)}{(1+t)^{1+\theta}} + \frac{\nabla \eta_*(u) \partial_x u}{(1+t)^{1+\theta}} \right) \\ = - \frac{(\partial_x u)^\top \nabla^2 \eta_*(u) \partial_x u}{(1+t)^{1+\theta}}. \end{aligned}$$

Integrating over $0 < t < T$, $\xi_1 t < x < \xi_2 t$, yields

$$\begin{aligned} \int_{\xi_1 T}^{\xi_2 T} \frac{\eta_*(u(T, x))}{T^{1+\theta}} dx + \int_0^T \int_{\xi_1 t}^{\xi_2 t} \frac{\eta_*(u(t, x))}{(1+t)^{2+\theta}} dx dt \\ + \int_0^T \frac{[-\xi \eta_*(u(t, \xi t)) + q_*(u(t, \xi t))]}{(1+t)^{1+\theta}} \Big|_{\xi_1}^{\xi_2} dt \\ = \int_0^T \frac{[\nabla \eta_*(u(t, \xi t)) \partial_x u(t, \xi t)]_{\xi_1}^{\xi_2}}{(1+t)^{1+\theta}} dt - \int_0^T \int_{\xi_1 t}^{\xi_2 t} \frac{(\partial_x u)^\top \nabla^2 \eta_*(u) \partial_x u}{(1+t)^{1+\theta}} dx dt. \end{aligned}$$

Therefore, using Lemma 6.1, the uniform boundedness of u , and the strict convexity of η , we have

$$\int_0^T \int_{\xi_1 t}^{\xi_2 t} \frac{|\partial_x u(t, x)|^2}{(1+t)^{1+\theta}} dx dt \leq A,$$

for some $A > 0$, independent of T . Lemma 6.2 follows. ■

LEMMA 6.3. Let $u(t, x) \in L^\infty(\mathbb{R}_+^2)$ be the solution of (6.1)–(6.2). Then,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\partial_x u(t, \xi t)| dt = 0, \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Proof. Indeed, from Lemma 6.2, we have

$$\int_{\xi_1}^{\xi_2} \int_0^\infty \frac{|\partial_x u(t, \xi t)|^2}{(1+t)^{1+\theta}} t dt d\xi < \infty.$$

Then, for a.e. $\xi \in \mathbb{R}$, one has

$$\int_0^\infty \frac{|\partial_x u(t, \xi t)|^2}{(1+t)^\theta} dt < \infty.$$

Therefore, for T sufficiently large,

$$\frac{1}{T} \int_0^T |\partial_x u(t, \xi t)|^2 dt \leq \frac{2^\theta}{T^{1-\theta}} \int_0^T \frac{|\partial_x u(t, \xi t)|^2}{(1+t)^\theta} dt \leq \frac{2^\theta}{T^{1-\theta}} \int_0^\infty \frac{|\partial_x u(t, \xi t)|^2}{(1+t)^\theta} dt,$$

and then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\partial_x u(t, \xi t)|^2 dt = 0, \quad \text{a.e. } \xi \in \mathbb{R}.$$

Now, by Jensen's inequality, one has

$$\left(\frac{1}{T} \int_0^T |\partial_x u(t, \xi t)| dt \right)^2 \leq \frac{1}{T} \int_0^T |\partial_x u(t, \xi t)|^2 dt.$$

This implies (6.3). ■

Now we show through a class of systems, the Temple systems, how to combine Lemmas 6.1–6.3 with the first approach in Section 4 to study the large-time behavior of solutions of (6.1)–(6.2) with general initial data.

THEOREM 6.1. *Assume that (6.1) is a viscous Temple system satisfying (4.8)–(4.9). Suppose that $u_0(x) \in L^\infty(\mathbb{R})$ satisfies (1.2)–(1.4) and takes its values in a region \mathbf{O} given by (4.3). Then the Cauchy problem (6.1) and (1.2) has a unique global bounded smooth solution $u(t, x) \in \mathbf{O}$. Furthermore,*

$$\lim_{t \rightarrow \infty} \int |u(t, \xi t) - R(\xi)| dt = 0, \quad L^1_{loc}(\mathbb{R}), \quad (6.8)$$

where $R(x/t)$ is the Riemann solution of (1.1) and (1.5). This means that the Riemann solution is the unique attractor of any solution $u(t, x)$ whose initial data are a perturbation to (1.5), as in (1.4).

Proof. The existence of a global bounded smooth solution of (6.1) and (6.2) can be obtained by following the standard arguments (see, e.g., [15, 21, 26]).

Given any convex nonnegative entropy pair (η, q) , $\eta \in C^2$, we multiply (6.1) by $\nabla\eta(u)$ to obtain

$$\partial_t \eta(u) + \partial_x q(u) = \partial_x (\nabla\eta(u) \partial_x u) - (\partial_x u)^\top \nabla^2 \eta(u) \partial_x u. \tag{6.9}$$

Because of Lemma 6.1, we can proceed as in the proof of Theorem 4.1. Namely, integrating (6.9) over the regions obtained as intersections of $E_j^{\xi, T}$, $j = 1, 2$, with $|x| < X + C(T - t)$, $0 < t < T$, for sufficiently large $C > 0$, and using the Green Theorem, one has by Lemma 6.1 that, for each fixed $t > 0$,

$$\lim_{|x| \rightarrow \infty} \partial_x u(t, x) = 0. \tag{6.10}$$

Also using (6.4), (6.9)–(6.10), and applying the Dominate Convergence Theorem yields

$$\begin{aligned} & - \int_0^\infty \eta(u_0(x)) \, dx + \int_{\xi T}^\infty \eta(u(T, x)) \, dx - \int_0^T (-\xi\eta + q)(u(t, \xi t)) \, dt \\ & \leq \int_0^T \nabla\eta(u(t, \xi t)) \partial_x u(t, \xi t) \, dt, \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} & - \int_{-\infty}^0 \eta(u_0(x)) \, dx + \int_{-\infty}^{\xi T} \eta(u(T, x)) \, dx + \int_0^T (-\xi\eta + q)(u(t, \xi t)) \, dt \\ & \leq \int_0^T \nabla\eta(u(t, \xi t)) \partial_x u(t, \xi t) \, dt, \end{aligned} \tag{6.12}$$

provided that $\eta(u_R) = 0$ for the first case and $\eta(u_L) = 0$ for the second one.

If we show that the entropy functions η_j^\pm and η_j can be obtained as pointwise limits of C^2 nonnegative convex entropy functions, then, following the proof of Theorem 4.1 with the aid of Lemma 6.3, (6.11), and (6.12), we conclude that, for a.e. $\xi \in \mathbb{R}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u(t, \xi t) - R(\xi)| \, dt = 0. \tag{6.13}$$

Now we easily see that

$$\begin{aligned} \eta_j(u, v) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} |L_j(v(w)) \cdot (u - v(w))| \psi^\varepsilon(w(v) - w) \, dw, \\ \eta_j^\pm(u, v) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} (L_j(v(w)) \cdot (u - v(w)))_\pm \psi^\varepsilon(w(v) - w) \, dw, \end{aligned}$$

where $\psi^\varepsilon(w) = \varepsilon^{-m} \psi(w/\varepsilon)$ with $\psi \in C_0^\infty((-1, 1)^m)$, $\psi \geq 0$, and $\int_{\mathbb{R}^m} \psi(w) dw = 1$. Furthermore, the integral expressions in the above limits are entropy functions of (1.1), and a short calculation as in [19] shows that they are convex.

Finally, we show that the convergence in the sense of (6.13) implies the convergence in the sense of (6.8). Let $(\eta(u), q(u))$ denote a strictly convex entropy pair for (1.1), and $(\alpha(u, v), \beta(u, v))$ be defined by (5.7), (5.8) obtained from (η, q) . Multiplying (6.1) by $\nabla \eta(u)$, one easily obtains

$$\partial_t \eta(u) + \partial_x q(u) \leq \partial_x^2 \eta(u). \quad (6.14)$$

Let $I = (\xi_1, \xi_2)$ be any open interval where $R(\xi)$ is Lipschitz continuous. For (t, x) in the wedge $\xi_1 < x/t < \xi_2$, one has

$$\partial_t R + \partial_x f(R) = 0, \quad (6.15)$$

$$\partial_t \eta(R) + \partial_x q(R) = 0. \quad (6.16)$$

Then we have

$$\begin{aligned} \partial_t \alpha(u, R) + \partial_x \beta(u, R) &\leq \partial_x^2 \eta(u) + \partial_x \nabla \eta(R) \partial_x u - \nabla^2 \eta(R) (\partial_x R, \partial_x u) \\ &\quad - \nabla^2 \eta(R) (\partial_x R, Qf(u, R)), \end{aligned} \quad (6.17)$$

where Qf is as in (5.11). We now use the change of coordinates $(t, x) \mapsto (t, \xi)$, $\xi = x/t$. Inequality (6.17) then becomes

$$\begin{aligned} \partial_t \alpha(u, R) - \frac{\xi}{t} \partial_\xi \alpha(u, R) + \frac{1}{t} \partial_\xi \beta(u, R) \\ \leq \frac{1}{t} \partial_\xi (\partial_x \eta(u)) + \frac{1}{t} \partial_\xi (\nabla \eta(R) \partial_x u) - \frac{1}{t} \partial_\xi (\nabla \eta(R) \partial_x u) \\ - \frac{1}{t} \nabla^2 \eta(R) (\partial_\xi R, \partial_x u) - \frac{1}{t} \nabla^2 \eta(R) (\partial_\xi R, Qf(u, R)). \end{aligned}$$

Integrating the above inequality in the variable ξ over I and using Lemma 6.1 which guarantees the uniform boundedness of u_x for $t \geq t_0 > 0$, one obtains

$$\frac{d}{dt} Y(t) \leq \frac{C}{t}, \quad (6.18)$$

for some constant $C > 0$, where $Y(t) = \int_I \alpha(u(t, t\xi), R(\xi)) d\xi$. Then the same arguments as in the proof of Theorem 2.3 to conclude that (6.18) and the fact that u is weakly asymptotic to R , which translates into

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T Y(t) dt = 0, \tag{6.19}$$

together imply

$$\lim_{t \rightarrow \infty} Y(t) = 0. \tag{6.20}$$

The remaining follows exactly as in the inviscid case. The proof is complete. ■

Remark 6.1. The compactness in $L^1_{loc}(\mathbb{R}^2_+)$ of uniformly bounded vanishing viscosity sequences for the particular Temple system (4.1) is established in [23]. For the special case of scaling sequences, the compactness from Theorem 6.1 is stronger than the one from the result in [23] for the viscous system, since it gives the convergence of the whole sequence, which cannot be obtained by [23] without a uniqueness theorem.

7. APPENDIX

7.1. Proof of Lemma 3.5

Let \mathcal{N} be the Borel set of measure zero in the definition of the precise representative for $u(t, x)$. Integrating $\chi_{\mathcal{N}}$, the indicator function of \mathcal{N} , over $[N^{-1}, N] \times \mathbb{R}$ for some positive integer N , changing the variables, and using Fubini's theorem, we obtain

$$\begin{aligned} 0 &= \int_{[N^{-1} \times N] \times \mathbb{R}} \chi_{\mathcal{N}}(t, x) dx dt = \int_{\mathbb{R}} \left\{ \int_{[N^{-1}, N]} \chi_{\mathcal{N}}(t, \xi t) t dt \right\} d\xi \\ &\geq N^{-1} \int_{\mathbb{R}} \left\{ \int_{[N^{-1}, N]} \chi_{\mathcal{N}}(t, \xi t) dt \right\} d\xi \geq 0. \end{aligned}$$

We conclude that, for almost all $\xi \in \mathbb{R}$, $\text{meas}\{t \in [N^{-1}, N]: (t, \xi t) \in \mathcal{N}\} = 0$. Making N run over all positive integers and using Lemma 4.4, we obtain the first assertion. The second is proved in the same way.

7.2. *Proof of Existence of Entropies $\eta_j(u)$, $j = 1, 2$, Satisfying (5.18)–(5.20)*

To find these entropies, we first recall the entropy equations in the coordinate system of Riemann invariants

$$\frac{\partial q}{\partial w_j} = \lambda_j \frac{\partial \eta}{\partial w_j}, \quad j = 1, 2. \quad (7.1)$$

In particular, the entropies are solutions of the second-order linear hyperbolic equation

$$\frac{\partial^2 \eta}{\partial w_1 \partial w_2} + \frac{\partial_{w_2} \lambda_1}{\lambda_1 - \lambda_2} \frac{\partial \eta}{\partial w_1} - \frac{\partial_{w_1} \lambda_2}{\lambda_1 - \lambda_2} \frac{\partial \eta}{\partial w_2} = 0. \quad (7.2)$$

We define η_j as the solution of the Goursat problem for (7.2), on characteristic lines $w_j = w_j^M \equiv w_j(u_M)$, with Goursat data

$$\eta_j|_{w_j=w_j^M} = (w_i - w_i^M)^2, \quad \eta_j|_{w_i=w_i^M} \equiv 0, \quad i \neq j, \quad i, j = 1, 2. \quad (7.3)$$

We recall that, by using the Riemann method [42], one can write the solution of (7.2)–(7.3) as an integral expression of the form

$$\eta_j(w) = \int_{w_i^M}^{w_i} H(w; s) (\rho_i'(s) + \alpha(s) \rho_i(s)) ds, \quad (7.4)$$

where $H(w; s)$ and $\alpha(s)$ are smooth functions depending only on the coefficients in (7.2) and $\rho_i(s) = (s - w_i^M)^2$.

We can check property (5.18) directly from (7.4) expanding $\eta_j(w)$ up to the second order in w_i at $w_i = w_i^M$, $i \neq j$. Indeed, we see from (7.4) that $\partial \eta_j / \partial w_i |_{w_i=w_i^M} = 0$, for all $w_j \in \mathbb{R}$, $i \neq j$. Since $(\partial^2 \eta_j / \partial w_i^2)(w_M) = 2$, (5.18) follows provided that $|w_j - w_j^M|$, $j \neq i$ is small.

Property (5.19) follows from (5.9) and

$$\nabla^2 \eta_i(u_M)(\tilde{r}_j(u_M), \tilde{r}_j(u_M)) = \frac{\partial^2 \eta_i}{\partial w_j^2}(w_M) = 2, \quad i \neq j,$$

$$\nabla^2 \eta_j(u_M)(\tilde{r}_j(u_M), \tilde{r}_j(u_M)) = \frac{\partial^2 \eta_j}{\partial w_j^2}(w_M) = 0, \quad j = 1, 2.$$

We now verify property (5.20). We normalize the fluxes q_j associated to the entropies η_j , $j = 1, 2$, respectively, by setting $q_j(w_M) = 0$, $j = 1, 2$. Let

$$\pi_j(u, v) = \lambda_j(v) \eta_j(u) - q_j(u), \quad j = 1, 2.$$

We will show that $(-1)^j \pi_j(u, v) \geq 0$ if $w_i(v) = w_i^M$, $i \neq j$. We first show the case $j = 1$. Fix v , satisfying $w_2(v) = w_2^M$, and regard π_1 as a function of u

only. We observe that π_1 vanishes on the line $w_2 = w_2^M$, since $\eta_1(u)$ vanishes by definition and $q_1(u)$ vanishes because of $q_1(u_M) = 0$ and $\partial q_1 / \partial w_1 = \lambda_1 \partial \eta_1 / \partial w_1 = 0$ on this line. Moreover, the derivative of π_1 with respect to w_2 also vanishes on the line $w_2 = w_2^M$. Indeed, we have $\partial \eta_1 / \partial w_2 \equiv 0$ on $w_2 = w_2^M$, as one can see from (7.4). Then, by (7.1), we also have $\partial q_1 / \partial w_2 \equiv 0$ and thus $\partial \pi_1 / \partial w_2 \equiv 0$ over the line $w_2 = w_2^M$.

Now, for given u , let u' be such that $w_1(u') = w_1(u)$ and $w_2(u') = w_2^M$. By the Taylor expansion, one has

$$\eta_1(u) = \frac{1}{2} \frac{\partial^2 \eta_1}{\partial w_2^2}(u')(w_2 - w_2^M)^2 + O(|w_2 - w_2^M|^3),$$

$$q_1(u) = \frac{1}{2} \frac{\partial^2 q_1}{\partial w_2^2}(u')(w_2 - w_2^M)^2 + O(|w_2 - w_2^M|^3).$$

Differentiating the entropy identity (7.1) ($j=2$) with respect to w_2 gives that the first terms in the right-hand side of the two expansions above differ by a factor $\lambda_2(u')$, since $\partial \eta_1 / \partial w_2 \equiv 0$ over $w_2 = w_2^M$. Then we find

$$\lambda_2(u') \eta_1(u) - q_1(u) = O(|w_2 - w_2^M|^3).$$

We conclude that

$$\begin{aligned} \pi_1(u, v) &= (\lambda_1(v) - \lambda_2(u')) \eta_1(u) + \lambda_2(u') \eta_1(u) - q_1(u) \\ &= (\lambda_1(v) - \lambda_2(u')) \eta_1(u) + O(|w_2 - w_2^M|^3) \leq 0, \end{aligned}$$

if u and v are sufficiently close. This completes the verification of (5.18)–(5.20). The verification of the properties for η_2 and q_2 is similar.

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