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Correction to "Maps of Convex Sets and Invariant Regions for Finite Difference Systems of Conservation Laws"

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The purpose of this note is to correct the second part of Theorem 4 of [1] and its proof. In order to do that, let us simply say that the pair $\mathbf{g}, \mathcal{O}$, formed by a map $\mathbf{g} : \mathcal{O} \to \mathbb{R}^n$ and an open convex set $\mathcal{O} \subseteq \mathbb{R}^n$, has the properties (P1), (P2) or (P3) if it satisfies the following conditions:

(P1) If $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{O}$ with $|\mathbf{u}_n| \to \infty$, then $|\mathbf{g}(\mathbf{u}_n)| \to \infty$.
(P2) For any vector $\mathbf{e} \in \mathbb{R}^n$, $\sup_{\mathbf{u}\in\mathcal{O}} \mathbf{e} \cdot \mathbf{u} < +\infty$ implies $\sup_{\mathbf{u}\in\mathcal{O}} \mathbf{e} \cdot \mathbf{g}(\mathbf{u}) < +\infty$.
(P3) The map $\mathbf{g}(\mathcal{O})$ is simply connected.

Properties (P1) and (P2) are trivially satisfied if $\overline{\mathcal{O}}$ is compact. We also observe that if the pair $\mathbf{g}, \mathcal{O}$ has either of the properties (P1) and (P2), then the pair $\mathbf{g}_\varepsilon, \mathcal{O}$ also has the same property for all $\varepsilon > 0$, where $\mathbf{g}_\varepsilon = \mathbf{g} + \varepsilon\mathbf{I}$.

The statement of the second part of Theorem 4 in [1] should be changed to the following: *Further, if* $\nabla\mathbf{f}(\mathbf{u})$ *satisfies* (9) *and* (10) *with* $\mathcal{S} = \mathcal{U}$, $\overline{\Omega} \subseteq \mathcal{U}$, *then* $\mathbf{f}$ *is injective over* $\overline{\Omega}$, $\mathbf{f}(\overline{\Omega})$ *is convex and the inequality* (11) *holds for all* $\mathbf{u} \in \overline{\Omega}$, $\omega \in \partial\Omega$ *and* $\nu(\omega)$ *in the outer normal cone of* $\partial\Omega$ *at* ω, *provided that, in addition, one of the following is satisfied:*

(i) $\mathcal{U} = \mathbb{R}^n$, *and* $\delta\,\mathbf{I} \le P(\mathbf{u}) \le M\,\mathbf{I}$ *for some* $M, \delta > 0$*;*
(ii) $\mathcal{U} = \mathbb{R}^n$ *and the pair* $\mathbf{f}_\varepsilon, \mathbb{R}^n$ *has the property* (P1)*;*
(iii) *the pair* $\mathbf{f}_\varepsilon, \Omega$ *has the properties* (P1) *and* (P2) *for all* $\varepsilon > 0$*;*
(iv) *the pair* $\mathbf{f}_\varepsilon, \Omega$ *has the properties* (P1) *and* (P3) *for all* $\varepsilon > 0$*, sufficiently small;*
(v) *the pair* $\mathbf{f}, \Omega$ *has the properties* (P1) *and* (P3).

In cases (i) *to* (iv) *above, the inequality* (11) *is still valid when the eigenvalues of* $\nabla\mathbf{f}(\mathbf{u})$ *are just nonnegative for all* $\mathbf{u} \in \mathcal{U}$.

Proof. In case (i), $\mathbf{f}_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and there exists $\alpha > 0$ such that $|\nabla\mathbf{f}(\mathbf{u}) \cdot \boldsymbol{\xi}| \ge \alpha|\boldsymbol{\xi}|$ for all $\mathbf{u}$ and all $\boldsymbol{\xi} \in \mathbb{R}^n$. Hence $\mathbf{f}_\varepsilon$ is a diffeomorphism of $\mathbb{R}^n$ onto itself by a well-known classical result (see, e.g., [2], p 142). In case (ii),

$\mathbf{f}_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ is a local diffeomorphism which is closed and proper, in view of property (P1). Hence, $\mathbf{f}_\varepsilon(\mathbb{R}^n) = \mathbb{R}^n$ and $\mathbf{f}_\varepsilon$ is a covering map from $\mathbb{R}^n$ onto $\mathbb{R}^n$. Since, $\mathbb{R}^n$ is simply connected, it follows that $\mathbf{f}_\varepsilon$ is a diffeomorphism of $\mathbb{R}^n$ onto itself (see, e.g., [2], [3]). In both cases, since $\mathbf{f}$ is a local diffeomorphism and $\mathbf{f}_\varepsilon \to \mathbf{f}$ uniformly over the compacts of $\mathbb{R}^n$, $\mathbf{f}$ itself is a diffeomorphism of $\mathbb{R}^n$ onto itself. It follows from the first part of Theorem 4 in [1] that $\mathbf{f}(\overline{\Omega})$ is convex and so (11) in [1] holds for all $\mathbf{u} \in \overline{\Omega}$, $\omega \in \partial\Omega$ and $\nu(\omega)$ in the outer normal cone of $\partial\Omega$ at ω.

Similarly, in case (iv) (resp., (v)), $\mathbf{f}_\varepsilon$ (resp., $\mathbf{f}$) is a local diffeomorphism which is proper, by property (P1), and so it is a covering map, whose image is simply connected, by property (P3). Hence, again, $\mathbf{f}_\varepsilon$ (resp., $\mathbf{f}$) is a diffeomorphism from Ω onto its image and the assertions follow as above.

As for case (iii), again because of the first part of Theorem 4 in [1], as in the proof of cases (i), (ii), (iv) and (v), it suffices to prove that $\mathbf{f}_\varepsilon$ is a diffeomorphism of Ω over its image. In what follows we drop the subscript ε. First we prove that $\mathbf{f}(\partial\Omega) = \partial\mathbf{f}(\Omega)$. Since $\mathbf{f}$ is a local diffeomorphism, clearly $\mathbf{f}(\partial\Omega) \supset \partial\mathbf{f}(\Omega)$. Therefore, it is enough to prove that there can be no point of $\mathbf{f}(\partial\Omega)$ in the interior of $\mathbf{f}(\overline{\Omega})$. Indeed, suppose $\mathbf{v}_0$ is such a point, and let $\omega_0 \in \partial\Omega$ be such that $\mathbf{f}(\omega_0) = \mathbf{v}_0$, and let $\nu(\omega_0)$ be the outer unit normal to $\partial\Omega$ at ω_0, which we may assume to be well defined by properly choosing $\mathbf{v}_0$. Then $\nu(\omega_0)$ is also local outer normal to $\mathbf{f}(\partial\Omega)$ at $\mathbf{v}_0$ by (C2) in [1]. Since $\mathbf{v}_0$ is in the interior of $\mathbf{f}(\overline{\Omega})$, $\nu(\omega_0)\cdot\mathbf{f}(\mathbf{u})$ cannot assume a maximum at $\mathbf{u} = \omega_0$. Hence, because of the property (P), there exists $\omega_1 \in \partial\Omega$ for which

$$\nu(\omega_0)\cdot\mathbf{f}(\omega_1) = \sup_{\mathbf{u}\in\Omega} \nu(\omega_0)\cdot\mathbf{f}(\mathbf{u}).$$

It then follows that $/\nu(\omega_0)\cdot\mathbf{u} = \nu(\omega_0)\cdot\omega_1$ is a supporting hyperplane to $\overline{\Omega}$ and $\nu(\omega_0)\cdot\mathbf{u} = \nu(\omega_0)\cdot\mathbf{f}(\omega_1)$ is a supporting hyperplane to $\mathbf{f}(\overline{\Omega})$. It follows by convexity that the supporting hyperplanes $\nu(\omega_0)\cdot\mathbf{u} = \nu(\omega_0)\cdot\omega_0$ and $\nu(\omega_0)\cdot\mathbf{u} = \nu(\omega_0)\cdot\omega_1$ must coincide and so both ω_0 and ω_1 must lie in this hyperplane. Again by convexity, the line segment connecting ω_0 to ω_1 is entirely contained in $\partial\Omega$. But then the image by $\mathbf{f}$ of this line segment must be contained in a hyperplane normal to $\nu(\omega_0)$ and containing both $\mathbf{f}(\omega_0)$ and $\mathbf{f}(\omega_1)$, which is an absurd, and so we actually have $\mathbf{f}(\partial\Omega) = \partial\mathbf{f}(\Omega)$. Now, for $\theta \in [0, 1]$, let $\mathbf{f}_\theta = (1-\theta)\,\mathbf{I}+\theta\,\mathbf{f}$; clearly each $\mathbf{f}_\theta$ also satisfies properties (P1) and (P2). We obtain analogously $\mathbf{f}_\theta(\partial\Omega) = \partial\mathbf{f}_\theta(\Omega)$. Let $\mathbf{v}_0 \in \mathbf{f}(\Omega)$ and $\mathbf{u}_0 \in \Omega$ be such that $\mathbf{f}(\mathbf{u}_0) = \mathbf{v}_0$. Define $\mathbf{g}_\theta(\mathbf{u}) = \mathbf{f}_\theta(\mathbf{u})-\mathbf{f}_\theta(\mathbf{u}_0)$. We notice that $0 \notin \mathbf{g}_\theta(\partial\Omega)$, for $\theta \in [0, 1]$. We also observe that the degree $\deg(\mathbf{g}_\theta, \Omega, 0)$ is well defined since, by property (P1), $\mathbf{g}_\theta^{-1}(0)$ is finite, and it coincides with the number of elements of $\mathbf{g}_\theta^{-1}(0)$ because of the positiveness of the spectrum of $\nabla\mathbf{g}_\theta(\mathbf{u})$, everywhere in $\mathcal{U}$. Since $\theta \mapsto \mathbf{g}_\theta$ is a homotopy with $\mathbf{g}_0 = \mathbf{I}-\mathbf{u}_0$ and $\mathbf{g}_1 = \mathbf{f}-\mathbf{v}_0$, we conclude that $\deg(\mathbf{f}-\mathbf{v}_0, \Omega, 0) = 1$, and since this holds for all $\mathbf{v}_0 \in \mathbf{f}(\Omega)$, it follows that $\mathbf{f}$ is a diffeomorphism of Ω over its image, and the proof is finished. The last assertion is trivially proved from the validity of (11) for $\mathbf{f}_\varepsilon$, by making $\varepsilon \to 0$. □

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