Geometria Simplética 2021, Lista 9

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Problem 1: Let $L(t)$ and $A(t)$ be smooth families of matrices in $M_n(\mathbb{R})$ such that the equation

$$L'(t) = [A(t), L(t)] = (AL - LA)(t), \tag{1}$$

called *Lax's equation*, holds.

(a) Let $U(t)$ be the solution to the equation

$$U'(t) = A(t)U(t), \quad U(0) = \mathrm{Id}.$$

Using ODE theory, verify that the solution to this equation exists and is unique, and $U(t)$ is invertible. If $V = U^{-1}$, show that $V(t)$ satisfies $V'(t) = -V(t)A(t)$.

(b) Show that $(V(t)L(t)U(t))' = 0$, and conclude that $L(t)$ (solution to (1)) is given by $L(t) = U(t)L(0)U(t)^{-1}$.

It follows that L follows an "isospectral evolution", so functions of L that are invariant by conjugation are conserved quantities, for example $\mathrm{tr}(L^k)$, $k = 1, \ldots, n$

We will see below how to use Lax's equation to find conserved quantities for hamiltonian systems.

Problem 2: We consider the simplest example: the harmonic oscillator. The phase space is $\mathbb{R}^2 = \{(q, p)\}$, with canonical form $dq \wedge dp$, and hamiltonian $H(q, p) = \frac{1}{2}(\lambda^2 q^2 + p^2)$. Consider the matrices

$$L = \begin{pmatrix} p & \lambda q \\ \lambda q & -p \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\lambda/2 \\ \lambda/2 & 0 \end{pmatrix}$$

(a) Show that in this example Hamilton's equations are equivalent to Lax's equation and $H = \frac{1}{4}\mathrm{Tr}(L^2)$.

(b) Consider $\mathcal{U} = \mathbb{R}^2 - \{0\}$, let θ be the usual angle coordinate on $\mathcal{U}$. Show that $(H, \theta) : \mathcal{U} \to \mathbb{R}_+ \times S^1$ is a diffeomorphism that is not symplectic (equipping $\mathbb{R}_+ \times S^1$ with the canonical symplectic form,as we did in class for the model $B \times \mathbb{T}^n$).

(c) Find action-angle coordinates on $\mathcal{U}$.

The next two problems illustrate important integrable systems in $\mathbb{R}^{2n}$; both can be described by Lax's equation. For simplicity we assume that $n = 2$.

Problem 3: Consider on $T^*\mathbb{R}^2 = \mathbb{R}^4$, with coordinates (q_1, q_2, p_1, p_2), the hamiltonian

$$H(q,p) := \frac{1}{2}(p_1^2 + p_2^2) + e^{2(q_1 - q_2)}.$$

(a) Find the corresponding hamiltonian system (this is called the *Toda lattice* and models a linear molecule with 2 atoms subject to an exponential interaction).

(b) Show that $f = p_1 + p_2$ is a conserved quantity ($\{H, f\}$=0).

(c) Are H and f independent?

Now consider the 2×2 matrices L, A given by

$$L_{11} = p_1,\ L_{22} = p_2,\ L_{12} = L_{21} = e^{q_1 - q_2}, \quad A_{11} = A_{22} = 0,\ A_{12} = -A_{21} = -e^{q_1 - q_2}.$$

Show that solving the hamiltonian system is equivalent to solving Lax's equation (1). Note that $f = \mathrm{tr}(L)$ and $H = \frac{1}{2}\mathrm{tr}(L^2)$.

Problem 4: Consider the hamiltonian (defined for $q_1 \neq q_2$)

$$H(q,p) := \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{(q_1 - q_2)^2}.$$

Verify that the corresponding system, called *Calogero-Moser*, is

$$\dot{q}_j = p_j, \qquad \dot{p}_j = 2(q_j - q_k)^{-3}, \ \ k \neq j, \ \ j = 1, 2. \tag{2}$$

Consider 2×2 matrices L and A given by

$$L_{jj} = p_j,\ \ L_{kj} = i(q_k - q_j)^{-1}, \quad A_{jj} = -i(q_j - q_k)^{-2},\ \ A_{jk} = i(q_j - q_k)^{-2}.$$

for $j = 1, 2$ and $k \neq j$ (here $i = \sqrt{-1}$). As in the previous problem, show the equivalence of solving (2) and (1). Find another conserved quantity for this system (other than H).