Geometria Simplética 2019, Lista 6

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Problem 1: Let N be a manifold of dimension $2n-1$. Show that $\zeta \in \Omega^1(N)$ is a contact form if and only if $\zeta \wedge (d\zeta)^{n-1}$ is a volume form.

Problem 2:

(a) Let $\zeta \in \Omega^1(N)$ be a contact form. Consider $N \times \mathbb{R}$ equipped with the 2-form $d(e^t\zeta)$ (where t is the coordinate on $\mathbb{R}$). Verify that this 2-form is symplectic, and conclude that any (N, ζ) can be viewed as a hypersurface of contact type of a symplectic manifold.

The symplectic manifold $(N \times \mathbb{R}, d(e^t\zeta))$ *is called the* **simplectization** *of* N.

(b) On the other hand: Let (M, ω) be symplectic and $\iota : S \hookrightarrow M$ a hypersurface of contact type, with contact form ζ. Suppose that S is compacta. Show that there is a neighborhood U of S in M that is symplectomorphic to a neighborhood of S in its symplectication, $(S \times (-\epsilon, \epsilon), d(e^t\zeta))$, for an $\epsilon > 0$. [Note that this follows directly from Lista 4, problem 4, but we can be more explicit here]

Hints:

- *Consider the conformally symplectic vector field X in a neighborhood of S (such that $\zeta = \iota^*(i_X\omega)$); use the tubular neighborhood theorem to obtain an identification $\psi : U \xrightarrow{\sim} S \times (-\epsilon, \epsilon)$, in such a way that X corresponds to $\frac{\partial}{\partial t}$.*
- *Let Y be the Reeb vector field on S. For each $x \in S$, write $T_xM = W_1 \oplus W_2$, where $W_1 = \ker(\zeta) \subset T_xM$ and $W_2 \subset T_xM$ is the subspace generated by $X(x)$ and $Y(x)$. Verify that W_1 and W_2 are symplectically orthogonal with respect to ω, and also with respect to $\psi^*d(e^t\zeta)$.*
- *Use the decomposition $T_xM = W_1 \oplus W_2$ to verify that ω and $\psi^*d(e^t\zeta)$ coincide on $TM|_S$. Use the Darboux-Weinstein theorem.*

(c) Let $\xi \in \Omega^1(N)$ be a contact form on N and $D = \ker(\xi) \subset TN$. Let $L \subset N$ be a submanifold such that $TL \subset D|_L$. (1) Check that T_xL is an isotropic subspace of the symplectic vector space $(D_x, d\xi|_x)$ for all $x \in L$, so $\dim(L) \leq \frac{1}{2}\text{rank}(D)$. In case of equality, we call L **legendrian**. (2) Verify that L is legendrian iff $L \times \mathbb{R}$ is a lagrangian submanifold of the symplectization $N \times \mathbb{R}$.

(d) We saw that S^{2n-1} has contact structure $\zeta = \iota^*(\frac{1}{2}\sum_{i=1}^n q^i dp_i - p_i dq^i)$, where $\iota : S^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ is the inclusion. Show that its symplectization is symplectomorphic to $\mathbb{R}^{2n} - \{0\}$ (with the canonical symplectic form).

Problem 3: Show the Darboux theorem for contact manifolds: Given contact manifold (N^{2n-1}, ζ), around any point there exist local coordinates $q^1, \dots, q^{n-1}, p_1, \dots, p_{n-1}$, z, such that $\zeta = \sum_i q^i dp_i + dz$.

Hint: One can, for example, use symplectization and adapt thesymplectic Darboux theorem

Problem 4:

(a) Let (M, ω) be a symplectic manifold and suppose that $\iota_C : C \hookrightarrow M$ is a submanifold such that $\iota_C^* \omega = d\eta$ for a 1-form η in C. Show that there is no compact, symplectic submanifold N of M with $N \subseteq C$.

(b) Let N be c ompact sympletic submanifold (e.g. S^2), and let $M = N \times \mathbb{R}^2$, equipped with the product symplectic structure. Show that the hypersurface $C = N \times \{z \in \mathbb{R}^2 \mid \|z\| = 1\} \hookrightarrow M$ is not of contact type.

Problem 5: Consider, in $\mathbb{R}^{2n}$, the ellipsoid

$$S_{(r_1,\ldots,r_n)} = \{(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} \frac{q_j^2 + p_j^2}{r_j^2} = 1\},$$

where $r_1, \ldots, r_n$ are positive. We saw that S is a hypersurface of contact type. Compute its Reeb vector field and describe its flow (hint: complex coordinates $z_j = q_j + \sqrt{-1} p_j$ help solving the ODE). Show that the Reeb flow admits *at least* n periodic orbits.
Assume, for simplicity, that $n = 2$. Describe general conditions on r_1, r_2 so that: (1) there are only 2 periodic orbits; (2) there are infinitely many periodic orbits.