Geometria Simplética 2021, Lista 5

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We use Einstein's convention for summations below.

Problem 1: Let us consider the Legendre transform $\mathbb{F}L : TQ \to T^*Q$ associated with a lagrangian $L : TQ \to \mathbb{R}$, let $\alpha \in \Omega^1(T^*M)$ be the tautological 1-form and $\omega_{can} = -d\alpha$. Assume that $\mathbb{F}L$ is a diffeomorphism (i.e., L is hyper-regular).

(a) Show that, in local coordinates x^i, v^i of TQ,

$$\omega_L := \mathbb{F}L^*\omega_{can} = \frac{\partial^2 L}{\partial v^i \partial x^j} dx^i dx^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dx^i dv^j.$$

Verify that the matrix $\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right)$ is invertible.

(b) Let $Z \in \mathfrak{X}(TQ)$, locally written as $Z(x,v) = z^i(x,v)\frac{\partial}{\partial x^i} + w^j(x,v)\frac{\partial}{\partial v^j}$. Show that $i_Z\omega_L = dE$ if and only if $z^i(x,v) = v^i$ and w^j satisfies

$$\frac{\partial^2 L}{\partial v^i v^j}(x,v) w^j(x,v) = \frac{\partial L}{\partial x^i}(x,v) - \frac{\partial^2 L}{\partial v^i \partial x^j}(x,v) v^j.$$

Problem 2: Suppose that L is a hyper-regular lagrangian. Let $H = E \circ \mathbb{F}L^{-1}$, where $E : TQ \to \mathbb{R}$ is the energy ($E(v) = \langle \mathbb{F}L(v), v\rangle - L(v)$). Show that:

(a) If $\gamma(t)$ is a solution of the lagrangian system defined by L, then $\zeta(t) := \mathbb{F}L(\gamma(t), \dot\gamma(t))$ is a solution of the hamiltonian system defined by H.

(b) If $\zeta(t)$ is a solution of the hamiltonian system of H, then $\gamma(t) = \pi \circ \zeta(t)$ (where $\pi : T^*Q \to Q$ is the projection) is a solution of the lagrangian system; moreover, $\zeta(t) = \mathbb{F}L(\gamma(t), \dot\gamma(t))$.

Hints:

- *Note that $\zeta(t)$ is a solution of the hamiltonian system defined by H on T^*Q if and only if $\sigma(t) = \mathbb{F}L^{-1}(\zeta(t))$ is solution of the hamiltonian system defined by E and ω_L in TQ.*

- *Show (use the previous problem) that $\sigma(t) = (\gamma(t), \xi(t))$ is a solution of the hamiltonian system defined by E and ω_L if and only if*

$$\xi(t) = \dot\gamma(t), \quad \textit{and} \quad \frac{\partial^2 L}{\partial v^i v^j}(\gamma, \dot\gamma)\ddot\gamma^j = \frac{\partial L}{\partial x^i}(\gamma, \dot\gamma) - \frac{\partial^2 L}{\partial v^i \partial x^j}(\gamma, \dot\gamma)\dot\gamma^j$$

Problem 3: Let g be a riemannian metric on Q, and consider the lagrangian $L : TQ \to \mathbb{R}$, $L(v) = \frac{1}{2}g(v,v)$. Compute the Legendre transform $\mathbb{F}L$, the energy E, the form ω_L on TQ, and the hamiltonian vector field associated with E; compare the associated hamiltonian system with the Euler-Lagrange equations relative to L.

Problem 4: Let Q be a riemannian manifold, and let $B \in \Omega^1(Q)$. Consider the "electro-magnetic" lagrangian

$$L(x, v) = \frac{1}{2}\|v\|^2 + B_x(v) - V(x).$$

Compute the corresponding hamiltonian. Show that the solution of the lagrangian system is the projection (on Q) of the solution of the system with hamiltonian $H(x, \xi) = \frac{1}{2}\|\xi\|^2 + V$ (independent of B!), but with respect to the symplectic form $\omega - \pi^* dB$ (*Hint: recall lista 2*).