Geometria Simplética 2021, Lista 4

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Entrega dia 03/10

Problem 1: Let M be a compact, connected, orientable n-dimensional manifold. Let $\Lambda_0, \Lambda_1 \in \Omega^n(M)$ be two volume forms on M such that $\int_M \Lambda_0 = \int_M \Lambda_1$. Show that there is a diffeomorphism $\phi \in \mathrm{Dif}(M)$ such that $\phi^*(\Lambda_1) = \Lambda_0$.

Hint: Show that $\Lambda_t = (1-t)\Lambda_0 + t\Lambda_1$ is a volume form $\forall t \in [0,1]$, and note that the equality of total volumes implies that Λ_0 and Λ_1 define the same cohomology class (justify this fact). Then apply Moser's method.

Problem 2: Give an example of two symplectic forms on $\mathbb{R}^4$ that induce the same orientation, but admit a convex combination that is degenerate. Is it possible to find an example like that, but admitting another path of *symplectic* forms from one to the other? What happens if we consider $\mathbb{R}^2$ instead of $\mathbb{R}^4$?

Problem 3: This problem is useful for Problem 4. (Cf. Problem 7 in lista 1)
Let (V, Ω) be a symplectic vector space (or, more generally, a symplectic vector bundle), and let $W \subseteq V$ be a coisotropic subspace (resp., coisotropic subbundle).

a) Let E be a complement of W^Ω in W (i.e., $W = W^\Omega \oplus E$). Show that the restriction of Ω to E is nondegenerate.

b) Let J be a Ω-compatible complex structure, with g the associated inner product. Show that Ω induces an identification of $J(W^\Omega) = W^\perp$ with $(W^\Omega)^*$. Taking E as the orthogonal complement (with respect to g) to W^Ω in W, show that the identification

$$V \cong E \oplus (W^\Omega \oplus (W^\Omega)^*), \tag{1}$$

is an isomorphism of symplectic vector spaces (resp. bundles) – on the right-hand side, E is equipped with its induced symplectic form (see (a) above) and $W^\Omega \oplus (W^\Omega)^*$ with its canonical symplectic form.

Problem 4: Prove the following generalization of Weinstein's lagrangian neighborhood theorem to coisotropic submanifolds (due to Gotay, 1982): Let (M_0, ω_0) and (M_1, ω_1) be symplectic manifolds, and $\iota_0 : Q \to M_0$, $\iota_1 : Q \to M_1$ be coisotropic embeddings. If $\iota_0^*\omega_0 = \iota_1^*\omega_1$, then there exist open neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of Q, in M_0 and M_1, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi(p) = p$ for all $p \in Q$ and $\varphi^*\omega_1 = \omega_0$,.

Hint: For $i = 0, 1$, use compatible almost complex structures to write $TM_i|_Q$ as in (1); conclude that there is an isomorphism $(TM_0|_Q, \omega_0) \to (TM_1|_Q, \omega_1)$ of symplectic vector bundles, so you can apply Weinstein's theorem.