Geometria Simplética 2021, Lista 10

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Problem 1: Let (M, J) be an almost complex manifold. We say that $Q \hookrightarrow M$ is an *almost complex submanifold* if $J(TQ) = TQ$ (if (M, J) is complex, then Q is a *complex submanifold*). Show that an almost complex submanifold Q of an almost Kähler manifold (M, J, ω) inherits an almost Kähler structure (in particular, it is symplectic). Also, if (M, J, ω) is Kähler, then Q is Kähler.

Problem 2: Let N_J be the Nijenhuis tensor associated to an almost complex structure J on M:

$$N_J(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

a) Check that $N_J(fX, gY) = fgN_J(X, Y)$, where $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$. Hence N_J is a tensor (i.e., the value $N_J(X, Y)$ at a point $x \in M$ only depends on $X_x, Y_x \in T_xM$).

b) Show that $N_J(X, JX) = 0$, and deduce that $N_J \equiv 0$ if M is a surface. Conclude (using the Newlander-Nirenberg theorem) that every orientable surface admits a complex/Kähler structure.

Problem 3: Check the "easy" direction of the Newlander-Nirenberg theorem:

a) Let (M_1, J_1) and (M_2, J_2) be almost complex manifolds. Let $\phi : M_1 \to M_2$ satisfy $d\phi \circ J_1 = J_2 \circ d\phi$. Show that if X, Y are vector fields on M_1, X', Y' are vector fields on M_2, and $X \sim_\phi X'$, $Y \sim_\phi Y'$, then $N_{J_1}(X, Y) \sim_\phi N_{J_2}(X', Y')$.

b) Verify that, if J_0 is the canonical complex structure on $\mathbb{R}^{2n}$, then $N_{J_0} \equiv 0$. If (M, J) is a complex manifold, conclude that $N_J \equiv 0$.

Problem 4: Let J be an almost complex structure on M and let $T_{10} \subset TM \otimes \mathbb{C}$ (complexification of TM) be the subbundle defined, pointwise, as the $+i$-eigenspace of J. Check that $T_{10} = \{X - iJX \mid X \in TM\}$, and show that $N_J = 0$ if and only if T_{10} is involutive with respect to the Lie bracket (extended to complex vector fields).

Problem 5: Show that $\mathbb{C}P^1$ is diffeomorphic (as a 2-dim real manifold) to S^2. *(Hint: stereographic projection on S^2.)*
Verify that the Fubini-Study form on the chart $\mathcal{U}_0 = \{[z_0, z_1] \in \mathbb{C}P^1 \mid z_0 \neq 0\}$ is given by:

$$\omega_{\text{FS}} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2},$$

where $\frac{z_1}{z_0} = z = x + iy$ (usual coordinate on $\mathbb{C}$). Use this expression to calculate the total area of $\mathbb{C}P^1$ with respect to ω_{FS}:

$$\int_{\mathbb{C}P^1} \omega_{\text{FS}} = \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}.$$

Check that $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{area}}$, where ω_{area} is the area form on S^2.

Problem 6: Consider two Kähler forms ω_0 and ω_1 on a compact complex manifold (M, J). Show that (M, ω_0) and (M, ω_1) are symplectomorphic.
(*Hint: Check that* $\omega_t = t\omega_1 + (1 - t)\omega_0$ *is symplectic for all* $0 \leq t \leq 1$).

Problem 7: Consider the hamiltonian action of S^1 on $\mathbb{C}^n$ from lista 8 (problem 5). For each $t > 0$, show that the reduced space $\mu^{-1}(-t/2)/S^1$ is $\mathbb{C}P^{n-1}$ with the symplectic form $\omega_{red} = t\omega_{FS}$ (ω_{FS} is the Fubini-Study form). (*Hint: show that* $\pi^*\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log(|z|^2)$, *where* $\pi : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}P^{n-1}$ *is the natural projection, and that the pullback of this form to the level sets of* μ *agree with the pullback of the canonical symplectic form.*)

Problem 8: The usual action of $U(n+1)$ on $\mathbb{C}^{n+1}$ induces an action of $U(n+1)$ on $\mathbb{C}P^n$. Show that this action is hamiltonian and find a formula for the moment map. *(Hint:* $\mathbb{C}^{n+1}$ *carries (hamiltonian) actions of* $U(n+1)$ *and* S^1*; regarding* $\mathbb{C}P^n$ *as a symplectic reduction by* S^1 *(as above), show that the moment map for the action of* $U(n+1)$ *on* $\mathbb{C}^{n+1}$ *(see lista 8, problem 6) induces a moment map for the action of* $U(n+1)$ *on* $\mathbb{C}P^n$*).*