Geometria Simplética 2021, Lista 1

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Problem 1: Let V be a symplectic vector space ($\dim(V) = 2n$), and $\Omega \in \wedge^2 V^*$ be a skew-symmetric bilinear form. Show that Ω is nondegenerate iff $\Omega^n \neq 0$.

Problem 2: Let (V, Ω) be a symplectic vector space, and let $W \subseteq V$ be any linear subspace.

a) Show that $V_W := \frac{W}{W \cap W^\Omega}$ inherits a natural symplectic structure Ω_W uniquely determined by the condition $\pi^* \Omega_W = \Omega|_W$ (here $\pi : W \to W/(W \cap W^\Omega)$ is the quotient projection).

   (*The space* (V_W, Ω_W) *is called the "reduced space".*)

b) Suppose that W is *coisotropic*, and let $L \subset V$ be lagrangian. Show that the image of $L \cap W$ via $\pi : W \to V_W$ is lagrangian in the reduced space.

Problem 3: We saw in class that any symplectomorphism $T : V_1 \to V_2$ defines a lagrangian subspace by its graph: $\Gamma_T := \{(Tu, u), u \in V_1\} \subset V_2 \oplus \overline{V}_1$. So we think of lagrangian subspaces of $V_2 \oplus \overline{V}_1$ as generalizations of symplectomorphisms. We now see how to generalize their composition.

Consider symplectic vector spaces V_1, V_2, V_3, and $E = V_3 \oplus \overline{V}_2 \oplus V_2 \oplus \overline{V}_1$.

a) Show that $\Delta := \{(v_3, v_2, v_2, v_1) \in E\}$ is coisotropic in E and its reduction E_Δ can be identified with $V_3 \oplus \overline{V}_1$.

b) Given lagrangian subspaces $L_1 \subset V_2 \oplus \overline{V}_1$ and $L_2 \subset V_3 \oplus \overline{V}_2$, define the *composition* of L_2 and L_1 by

$$L_2 \circ L_1 := \{(v_3, v_1) \,|\, \exists v_2 \in V_2 \text{ s.t. } (v_3, v_2) \in L_2, (v_2, v_1) \in L_1\}.$$

   Show that $L_2 \circ L_1$ is a lagrangian subspace of $V_3 \oplus \overline{V}_1$. (*Hint: show that the composition can be identified with the reduction of* $L_2 \times L_1 \subset E$ *with respect to* Δ).

c) Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be simplectomorphisms. Show that $\Gamma_{T_2 \circ T_1} = \Gamma_{T_2} \circ \Gamma_{T_1}$.

Problem 4: Let (V, J) be a complex vector space, let Ω be a symplectic structure on V. Show that J and Ω are compatible iff there exists a hermitian inner product $h : V \times V \to \mathbb{C}$ such that Ω is its imaginary part. Show that any (complex) orthonormal basis of (V, h) can be extended to a symplectic bases of (V, Ω).

Problem 5: Consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega_0)$, where $\Omega_0(u, v) = -u^t J_0 v$ (same notation as in class). Check that its group of linear symplectomorphisms is given by $Sp(2n) = \{A \in GL(2n) \,|\, A^t J_0 A = J_0\}$. Show that $Sp(2n)$ is a smooth submanifold of $GL(2n)$ and that its tangent space at the identity $I \in GL(2n)$ is given by $T_I Sp(2n) = \{A : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \mid A^t J_0 + J_0 A = 0\}$. Conclude that $Sp(2n)$ has dimension $2n^2 + n$. Verify also that $Sp(2n)$ is not compact.

Problem 6: Consider the standard compatible triple (Ω_0, J_0, g_0) on $\mathbb{R}^{2n}$ (as in class). Let $\mathrm{O}(2n)$ be the linear orthogonal group of $\mathbb{R}^{2n}$ (i.e, linear transformations preserving the canonical inner product g_0), and let $\mathrm{Sp}(2n)$ be the symplectic linear group. Through the identifiction $\mathbb{R}^{2n} \cong \mathbb{C}^n$ (as complex vector spaces), we may see $\mathrm{GL}(n, \mathbb{C})$ (the group of linear automorphisms of $\mathbb{C}^n$) as a subgroup of $\mathrm{GL}(2n, \mathbb{R})$: a complex matrix $A + iB$ is identified with the real $2n \times 2n$ matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Let now $\mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$ be the group of linear transformation preserving the natural hermitian inner product of $\mathbb{C}^n$. Show that the intersection of any two of the groups

$$\mathrm{Sp}(2n), \mathrm{O}(2n), \mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$$

is $U(n)$.

Problem 7: Let (V, Ω) be a symplectic vector space, let $W \subseteq V$. Let J be a Ω-compatible complex structure, and g the corresponding inner product. We have the relation $J(W^{\Omega}) = W^{\perp_g}$ (verify it if you have not done so). (a) Use this fact to show that any coisotropic subspace of V has an isotropic complement. In particular, any lagrangian subspace $L \subset V$ has a lagrangian complement L', $V = L \oplus L'$. (b) Show that there is a natural identification $L' \cong L^*$, that induces a symplectomorphism $V \cong L \oplus L^*$ (where $L \oplus L^*$ has the natural symplectic structure $((l, \alpha), (l', \alpha')) \mapsto \alpha(l') - \alpha'(l)$).

Problem 8: Let V be a real vector space and $\pi \in \wedge^2 V$ a Poisson structure. Consider $\pi^\sharp : V^* \to V$ defined by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$, and let $R = \pi^\sharp(V^*) \subseteq V$. Show that there is a unique symplectic form Ω on R given by $\Omega(u, v) = \pi(\alpha, \beta)$, where $u = \pi^\sharp(\alpha)$ and $v = \pi^\sharp(\beta)$. Conversely, show that given a pair (R, Ω), where $R \subseteq V$ is a subspace and $\Omega \in \wedge^2 R^*$ is a symplectic form on R, there is a unique Poisson structure π on V such that $R = \pi^\sharp(V^*)$ and Ω is defined as above.

Bonus problem: Prove the following generalizations of the problem 7 about lagrangian complements:

(1) Let $W_1, \ldots, W_k$ be lagrangian subspaces of V. Show that there is a lagangian subspace $L \subset V$ satisfying $L \cap W_j = \{0\}$ for all j. [*Hint: problem 2 may help...*]

(2) Let $E \subseteq V^{2n}$ be an arbitrary subspace of dimension n. Show that there is a *lagrangian* subspace L such that $E \oplus L = V$.