HOMEWORK 3

1. **Exercise.** Let $C^\bullet, D^\bullet$ be complexes of vector spaces (with differentials d_C and d_D). Let $f : C^\bullet \to D^\bullet$ be a morphism of complexes. Define *cone*(f) to be the graded vector space $C[1] \oplus D$. Define the endomorphism d of *cone*(f) given by the matrix:

$$\begin{pmatrix} d_C & 0 \\ -f & d_D \end{pmatrix}$$

Show that *cone*(f) with d is a complex.

2. **Exercise.** A graded Lie algebra is a graded super vector space $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ with a bracket satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ bilinear and such that

Skew-Symmetry $[a, b] = (-1)^{ij}[b, a]$ for $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$.

Jacobi $[a, [b, c]] = [[a, b], c] + (-1)^{ij}[b, [a, c]]$ for $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$ and $c \in \mathfrak{g}$.

A dgla is a graded Lie algebra with an endomorphism $d : \mathfrak{g} \to \mathfrak{g}[1]$. such that $d^2 = 0$ and $d[a, b] = [da, b] + (-1)^i[a, db]$ for $a \in \mathfrak{g}_i$ and $b \in \mathfrak{g}$.

Let L be a Lie algebra. Notice that it is a dgla with $L = L_0$ and $d = 0$. Show that $L_\dagger := cone(id_L)$ is a dgla.

3. **Exercise.** (a) Let V be a vector space. Show that $\Omega(V) := \operatorname{Sym} V[-1]$ is a graded (super) commutative algebra. Show that if V is finite dimensional so is $\operatorname{Sym} V[-1]$.

(b) Let V be a graded vector space. Show that $\Omega(V) := \operatorname{Sym} V[-1]$ is a graded (super) commutative algebra. Show that if V has one a component with even degree of dimension bigger than 1 then $\Omega(V)$ is infinite dimensional.

Below we will need this definition. A differential graded commutative superalgebra is a graded commutative super algebra as $\Omega(V)$ here together with a differential d raising degree by one, such that $d^2 = 0$ and $d(ab) = (da)b + (-1)^i a(db)$ for a of degree i.

Let A be a dg commutative algebra and let M be any graded vector space. Notice that $A \otimes M$ is naturally a graded vector space and also a graded A-module (here A acts on the first factor). A dg A-module structure on M is a differential d_M on $A \otimes M$ such that this action becomes a dg-action, that is:

$$d_M(a \cdot \zeta) = (d_A a) \cdot \zeta + (-1)^i a \cdot d_M \zeta, \qquad a \in A_i,\ \zeta \in A \otimes M.$$

4. **Exercise.** Let $\mathfrak{g}$ be a graded Lie algebra. A graded module is a graded vector space $V = \oplus V_i$ and an action map $\mathfrak{g} \otimes V \to V$ such that $\mathfrak{g}_i \otimes V_j \mapsto V_{i+j}$ and for $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$ and $v \in V$ we have

$$[a, b] \cdot v = a \cdot b \cdot v - (-1)^{ij} b \cdot a \cdot v.$$

Let L be a Lie algebra and V its module. Show that L naturally acts on $\Omega(V)$ by derivations of the graded commutative algebra structure and that this action makes $\Omega(V)$ a graded module over L.

5. **Exercise.** Let L be a Lie algebra and consider $L^* = \operatorname{Hom}_k(L, k)$ its coadjoint representation. We have the commutative graded algebra $\Omega := \Omega(L^*)$ and L acts on it as in the previous exercise. Show that this action extends to a graded action of $L_\dagger$.

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6. **Exercise.** Let $\mathfrak{g}$ be a dgla as defined in Exercise 2. A dg-module is a complex of vector spaces $V = \oplus V_i$ which is a graded module for $\mathfrak{g}$ as defined in Exercise 4 and satisfying the condition:

$$d_V a \cdot v = (d_{\mathfrak{g}} a) \cdot v + (-1)^i a \cdot d_V v, \qquad a \in \mathfrak{g}_i,\ v \in V.$$

Show that there exists a unique differential $d : \Omega \to \Omega[1]$ such that $d^2 = 0$ and that $d(ab) = (da)b + (-1)^i a(db)$ for $a \in \Omega_i$ and $b \in \Omega$ and such that the action of $L_\dagger$ on Ω makes it into a dg-module.

7. **Exercise.** Return to the setting of Exercise 3. A Lie coalgebra structure on V is a differential $d : \Omega(V) \to \Omega(V)[-1]$ making the commutative algebra $\Omega(V)$ into a dg commutative algebra as in the definition of Exercise 3. In other words. Lie coalgebra structures on V are in correspondence with dg structures on the graded commutative algebra $\Omega(V)$. Notice that this differential is determined from the map $d : V[-1] \to \mathrm{Sym}^2 V[-1]$ by the Leibniz rule.

Show that a Lie algebra structure on L is equivalent to a Lie coalgebra structure on L^* which is equivalent to a dg commutative algebra structure on $\Omega(L^*)$.

8. **Exercise.** Let L^* be a Lie coalgebra and M a vector space. A L^*-module (or a coaction) structure on M is by definition the same as a dg $\Omega(L^*)$-module structure as in Exercise 3.

Let L be a Lie algebra or equivalently L^* a Lie coalgebra. Let M be a vector space. Show that an L-module structure on M is equivalent to a L^*-module structure on M.

9. **Exercise** (The only one actually to compute something)**.** Let L be a Lie algebra and M its module. We have from the previous exercise a dg-module structure on $C(L, M) := \Omega(L^*) \otimes M$. In particular this means that this graded vector space is a complex with its own differential (that I will call d_M as in exercise 3). Define $H^*(L, M)$ as the cohomology of this complex.

Show that if L is a semi-simple Lie algebra and $M = \mathbb{C}$ its trivial module, then $H^1(L, M) = 0$.