Homework 4

Due 26/4/2018*

1 Exercise. Let G be a finite group. Define a ring $\mathbb{Z}[G]$ as the free Abelian group with basis $\{e_g\}_{g\in G}$ and multiplication
$$e_g \cdot e_h = e_{g\cdot h}.$$

Define another ring $\mathbb{Z}'[G]$ as the set of functions $G \to \mathbb{Z}$ with the following operations. Given two functions $\psi, \varphi \in \mathbb{Z}'[G]$ we define
$$(\psi + \varphi)(g) = \psi(g) + \varphi(g), \qquad (\psi \cdot \varphi)(g) = \sum_{h\in G} \psi(h) \cdot \varphi(h^{-1} \cdot g).$$

a) Show that $\mathbb{Z}'[G]$ is a ring.

b) Show that $\mathbb{Z}[G] \simeq \mathbb{Z}[G]$.

2 Exercise. Prove that the maximal ideals of $\mathbb{Z}$ are principal ideals generated by prime integers.

3 Exercise. Let I, J be ideals in R such that $I + J = R$. Show that $IJ = I \cap J$.

4 Exercise. Let R be the ring of continuous functions $[0, 1] \to \mathbb{R}$.

a) Prove that f is a zero divisor if and only if there exists an open interval inside $[0, 1]$ such that f vanishes in this interval.

b) Find the idempotent and nilpotent elements of R.

5 Exercise. Prove that the map $\mathbb{Z} \to \mathbb{Z}_3 \times \mathbb{Z}_5$ sending x to the pair $([x]_3, [x]_5)$ where $[x]_p$ is the reduction mod p, is a surjective homomorphism and find its kernel.

6 Exercise. Does the category of rings have an initial and final object?

*Starred exercises are optional