

# Homework 3

Due 3/4/2018\*

**1 Exercise.** Prove that in any group the orders of  $ab$  and  $ba$  are equal.

**2 Exercise.** Let  $H \subset G$  be the subgroup generated by two elements  $a, b \in G$  (that is the smallest subgroup of  $G$  containing both  $a, b$ ). Show that if  $ab = ba$  then  $H$  is Abelian.

**3 Exercise.** Prove that every subgroup of index 2 is normal and find a subgroup of index 3 that is not normal.

In the following exercises, let  $k$  be a field,  $GL_n(k)$  is the group of invertible  $n \times n$  matrices with entries in  $k$ .  $SL_n(k)$  is the subgroup of matrices with determinant 1.  $PGL_n(k)$  and  $PSL_n(k)$  are the respective quotient groups by the central subgroups of matrices which are multiple of the identity.

**4 Exercise.** Prove that the group  $GL_2(\mathbb{F}_2)$  of two by two invertible matrices with entries in the field with two elements  $\mathbb{F}_2$  is isomorphic to the symmetric group  $S_3$ .

**5 Exercise\*.** Prove that the group  $PSL_2(\mathbb{F}_7)$  is isomorphic to the group  $GL_3(\mathbb{F}_2)$ . If you can't solve this in the first few hours, just take a look at <https://math.stackexchange.com/questions/1401>

**6 Exercise.** Let  $S$  be a set with a right action of a group  $G$ . Define the subset

$$H = \cap_{s \in S} G_s.$$

Show that  $H$  is a normal subgroup of  $G$ .

**7 Exercise.** Let  $G$  be the group of rotational symmetries of a cube (defined the same way as for the tetrahedron in the first homework). Find the stabilizer of a big diagonal line.

**8 Exercise.** The quaternion group  $H$  is a group of order 8 with elements

$$H = \{\pm 1, \pm i, \pm j, \pm k\},$$

And multiplication as follows:

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j.$$

where 1 is the identity of the group and the usual rule of signs for multiplication is used (eg.  $(-i)j = -(ij) = -k$ ).

Compute  $\text{Aut } H / \text{Inn } H$ .

**9 Exercise.** Consider  $PGL_3(\mathbb{F}_2)$  acting on  $\mathbb{P}_{\mathbb{F}_2}^2$ . Find the stabilizer of a point and a line.

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\*Starred exercises are optional