Homework 2

Due 27/3/2018*

1 Exercise. Give an example showing that left cosets and right cosets of $GL_2(\mathbb{R})$ in $GL_2(\mathbb{C})$ might not be equal.

2 Exercise. Let $H \subset G$ be a subgroup. Show that there is a bijection $G/H \simeq H\backslash G$.

3 Exercise. Show that the category of vector spaces over the field k has products and coproducts.

4 Exercise. If G, H are groups regarded as a category with only one object and $\phi, \psi : H \to G$ are functors, show that there is a natural transformation $\phi \to \psi$ if and only if ϕ and ψ are conjugate, that is, there exists an element $g \in G$ such that $\phi h = g(\psi h)g^{-1}$ for all $h \in H$.

5 Exercise. Find a category with an arrow which is both monic and epi and it is not an isomorphism.

6 Exercise*.Let k be a field and let $D : \mathbf{Vect}_k \to \mathbf{Vect}_k^{op}$ and $D^{op} : \mathbf{Vect}_k^{op} \to \mathbf{Vect}_k$ be the functors defined as follows. For each vector space $V \in \mathbf{Vect}_k$, both D and D^{op} are given by $V^* := \mathrm{Hom}(V, k)$ with its natural vector space structure. Given a linear map $\phi : V \to W$, we let

$$(D^{op}\phi)(f) = (D\phi)(f) = f \circ \phi.$$

Notice that ϕ is regarded as either an arrow $V \to W$ in $\mathbf{Vect}_k$ or as an arrow $W \to V$ in $\mathbf{Vect}_k^{op}$ above. Show that D is right adjoint to D^{op}. Is it also a left adjoint?

7 Exercise. Show that the category of groups has (small) limits and (small) colimits (it's fine if you only prove for kernels and cokernels).

8 Exercise. Show that in the category of groups there are monomorphisms which are not the kernel of their cokernels.

9 Exercise*.Show that the category of (small) categories has limits. Can you prove it has colimits?

*Starred exercises are optional