2 Limits

In this lecture we continue with our study of the basic objects in category theory. In this lecture we will deal with the existence of objects satisfying some universal properties. But let us first define some properties of morphisms and some canonical functors that were left out in the previous lecture.

2.1 Definition. Let $\phi \in \mathrm{Hom}_{\mathscr{C}}(a,b)$ be a morphism in a category. For any object c it produces two maps of sets:

$$\phi \circ \cdot : \mathrm{Hom}_{\mathscr{C}}(c,a) \to \mathrm{Hom}_{\mathscr{C}}(c,b), \qquad \cdot \circ \phi : \mathrm{Hom}_{\mathscr{C}}(b,c) \to \mathrm{Hom}_{\mathscr{C}}(a,c).$$

We will say that ϕ is a *monomorphism* if the first map is an injection of sets and ϕ is an *epimorphism* if the second map is a surjection of sets.

2.2 Remark. In the category of sets these notions agree with the usual notions for maps of sets. However, in the category of topological spaces for example, an epimorphism may not be surjective.

A morphism may be both a monomorphism and an epimorphism but not an isomorphism. Take for example the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$ in the category of Hausdorff topological spaces.

A more radical example is to consider the category depicted by the graph

$$\bullet \to \bullet$$

The only non-identity arrow is both an epimorphism and a monomorphism and it is not an isomorphism.

2.3 Definition. Let $F : I \to \mathscr{C}$ be a functor between categories (the notation is suggestive as we will often take the source of F to be given by a partially ordered set). A *limit* of F is an object a of $\mathscr{C}$ together with morphisms $\phi_i : a \to F(i)$ for each object i of I satisfying the following properties

a) For any morphism $\psi_{ij} : i \to j$ in I, the following diagram commutes

$$\begin{tikzcd} & a \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\ F(i) \arrow[rr, "F(\psi_{ij})"'] & & F(j) \end{tikzcd} \tag{2.3.1}$$

b) For any other object a' together with morphisms $\phi'_i : a \to F(i)$ making the analogous diagram to (2.3.1) commute there exists a unique morphism $a' \to a$ making the following diagram commute

$$\begin{tikzcd} a' \arrow[drr, dotted, "\exists!"] \arrow[ddr, "\phi'_i"'] \arrow[dddrrr, bend left, "\phi'_j"] & & & \\ & & a \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\ & F(i) \arrow[rr, "F(\psi_{ij})"'] & & F(j) \end{tikzcd}$$

The limit if it exists is denoted by $\lim F$, $\varprojlim F$ or even $\varprojlim_i F(i)$. Notice that it may not be unique, if a and a' are limits for F then there exists a *unique isomorphism* $a \simeq a'$.

2.4. Turning around arrows one arrives at the notion of a *colimit*, in other words, let $F : I \to \mathscr{C}$ be a functor. A *colimit* for F if it exists is an object a of $\mathscr{C}$ together with morphisms $\phi^i : F(i) \to a$ for each object i of I satisfying the following properties:

a) For any morphism $\psi_{ij} : i \to j$ in I the following diagram commutes:

$$\begin{tikzcd} & a & \\ F(i) \arrow[ur, "\phi_i"] \arrow[rr, "F(\psi_{ij})"'] & & F(j) \arrow[ul, "\phi_j"'] \end{tikzcd} \tag{2.4.1}$$

b) For any other object a' together with morphisms $\phi'_i : a \to F(i)$ making the analogous diagram to (2.3.1) commute there exists a unique morphism $a' \to a$ making the following diagram commute

$$\begin{tikzcd} a' & & & \\ & & a \arrow[ull, dotted, "\exists!"'] & \\ & F(i) \arrow[uul, "\phi'_i"] \arrow[ur, "\phi_i"] \arrow[rr, "F(\psi_{ij})"'] & & F(j) \arrow[ul, "\phi_j"'] \arrow[uulll, bend right, "\phi'_j"'] \end{tikzcd}$$

The colimit if it exists is denoted by colim F, $\varinjlim F$ or even $\varinjlim_i F(i)$. Notice that it may not be unique, if a and a' are limits for F then there exists a *unique isomorphism* $a \simeq a'$.

2.5 Examples.

a) If the category I is the empty category with no objects and no morphisms. Then a colimit (for the empty functor) is called a *final object* of $\mathscr{C}$. It is an object $*$ of $\mathscr{C}$ such that for every other object a of $\mathscr{C}$ there exists a unique morphism $a \to *$ in $\mathscr{C}$. The category of sets or topological spaces for example has the set $*$ with only one element as a final object. The category of fields however does not have a final object.

b) Similarly, a colimit for the empty category I is called an *initial object* in $\mathscr{C}$. It is an object $*$ of $\mathscr{C}$ such that for any other object a of $\mathscr{C}$ there is a unique morphism $* \to a$. The category of sets has the empty set as an initial object. A partially ordered set, when viewed as a category, has an initial object only if it has a *least element.*

c) If $I = [0]$ is the category with only one object and one morphism, then a functor $I \to \mathscr{C}$ is simply the datum of an object a of $\mathscr{C}$. A limit for this functor is the same thing as a final object in the category $\mathscr{C}_{/a}$. Similarly a colimit of this functor is the same thing as an initial object in the category $\mathscr{C}_{a/}$.

d) If $I = [0] \amalg [0]$ is the disjoint union of two categories with one object and one morphism, then a functor $I \to \mathscr{C}$ is the same thing as two objects, a, b of $\mathscr{C}$. A limit for this functor is called a *product* and is denoted by $a \times b$ a colimit is called a *coproduct* and its denoted by $a \amalg b$. In the category of topological spaces for example, the product of spaces with the product topology is a product in the categorical sense, while the disjoint union of topological spaces is a coproduct. In the category $\mathbf{Vect}_k$ of vector spaces, the direct sum is both a product and a coproduct.

e) More generally, consider I the category given by the graph

$$\bullet \to \bullet \leftarrow \bullet.$$

A functor from I to $\mathscr{C}$ is specified by three objects a, b, c together with morphisms $a \to b$ and $c \to b$. A limit for this functor is called a *fibered product* and it is denoted by $a \times_b c$. The corresponding diagram

$$\begin{tikzcd} a \times_b c \arrow[r] \arrow[d] & c \arrow[d] \\ a \arrow[r] & b \end{tikzcd}$$

is called a *Cartesian diagram.* The universal property satisfied by this product is that any object d of $\mathscr{C}$ together with two morphisms $d \to a$ and $d \to c$ making the solid part of the following diagram commute, then the dotted arrow making the full diagram commute exists and is unique:

$$\begin{tikzcd} d \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & a \times_b c \arrow[r] \arrow[d] & c \arrow[d] \\ & a \arrow[r] & b \end{tikzcd}$$

In the category of topological spaces, the fibered product of two morphisms $g : a \to b$, $f : c \to b$ is given by the subspace of the product $a \times b$ consisting on pairs of points $x \in a$, $y \in c$ such that $g(x) = f(y)$ equipped with the subspace topology. When $a = *$ is the topological space consisting of one point, the map $g :\to b$ is determined by it's image $z \in b$. The fiber product in this case is naturally homeomorphic to $f^{-1}(z)$ hence the name *fibered product.*

If the category $\mathscr{C}$ has a final object $*$, then the fiber product when $b = *$ is the same as a product.

f) Dually, if I is given as the category given by the graph

$$\bullet \leftarrow \bullet \to \bullet,$$

then a functor $I \to \mathscr{C}$ is specified by two morphisms $b \to a$, $b \to c$. A colimit for this functor is called a *fibered coproduct* and is denoted by $a \amalg_b c$. It suffices to say that the *universal property* satisfied by it are summarized in the following commuting diagram

$$\begin{tikzcd} d & & \\ & a \amalg_b c \arrow[ul, dotted, "\exists!"'] & c \arrow[l] \arrow[ull, bend right] \\ & a \arrow[u] \arrow[uul, bend left] & b \arrow[l] \arrow[u] \end{tikzcd}$$

The notation is suggestive: for the category of sets, with two maps of sets $f : T \to S$, $g : T \to U$, a fibered coproduct is a *quotient* set of the disjoint union $S \amalg U$, by the equivalence relation where we declare $S \ni f(t) \sim g(t) \in U$.

g) Consider the case when I is given by the following graph

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}.$$

A functor $I \to \mathscr{C}$ consists of two objects a, b of $\mathscr{C}$ and two morphisms $\phi, \psi \in \mathrm{Hom}_{\mathscr{C}}(a, b)$. A limit for this functor is called an *equalizer* of ϕ and ψ. Indeed if k is a limit, then k comes equipped with a morphism $\sigma : k \to a$ such that $\phi \circ \sigma = \psi \circ \sigma$. Any other object k' with a morphism $\sigma' : k' \to a$ such that both compositions to b are equal, factors through k.

Dually, a limit for this diagram is called a *coequalizer* of ϕ and ψ. This is an object c together with a morphism $\tau : b \to c$ such that both compositions $\tau \circ \phi = \tau \circ \psi$.

2.6. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, then we may produce another category $\mathscr{C} \times \mathscr{D}$ whose objects are pairs (a, b) of objects an object a of $\mathscr{C}$ and an object b of $\mathscr{D}$ and morphisms $(a, b) \to (a', b')$ are pairs of morphisms in $\mathrm{Hom}_{\mathscr{C}}(a, a') \times \mathrm{Hom}_{\mathscr{D}}(b, b')$ with the obvious composition. We have an obvious functor

$$\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}, \qquad a \mapsto (a, a), \qquad \phi \mapsto (\phi, \phi).$$

This is called the *diagonal functor.*

We also have the *coproduct category* $\mathscr{C} \amalg \mathscr{D}$ whose objects are the disjoint union of the objects of $\mathscr{C}$ and of $\mathscr{D}$ and morphism are given by

$$\mathrm{Hom}_{\mathscr{C}\amalg\mathscr{D}}(a, b) = \begin{cases} \mathrm{Hom}_{\mathscr{C}}(a, b) & \text{if both } a \text{ and } b \text{ are objects of } \mathscr{C} \\ \mathrm{Hom}_{\mathscr{D}}(a, b) & \text{if both } a \text{ and } b \text{ are objects of } \mathscr{D} \\ \varnothing & \text{otherwise} \end{cases}$$

We have an obvious functor $\nabla : \mathscr{C} \amalg \mathscr{C} \to \mathscr{C}$ which is the identity on objects and morphisms.

2.7. Notice that if the category $\mathscr{C}$ has products, then we have a unique isomorphism $a \times (b \times c) \simeq (a \times b) \times c$. Indeed, by the universal property of the products we have maps $a \times (b \times c) \to b \times c \to c$ and $a \times (b \times c) \to b \times c \to b$ and $a \times (b \times c) \to a$. Using the last two maps we obtain a map $a \times (b \times c) \to a \times b$, which in combination with the first map gives the required morphism $a \times (b \times c) \to (a \times b) \times c$. The same argument is used to produce the inverse.

It makes sense therefore to write the products $a \times b \times c$ instead of using parenthesis. This does not mean that any parenthesis location would produce the same objct, but it will be unique modulo a unique isomorphism.

In fact, this latter product is the same as the limit of the functor from the category I given by the graph

$$\bullet \qquad \bullet \qquad \bullet$$

which corresponds to the three objects a,b and c.

2.8. Let $\mathscr{C}$ be a category with an initial and final object $*$. Then we have a marked morphism[1] $* \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ in for every pair of objects, given as the composition of the unique morphisms $a \to * \to b$. Notice also that in this case we have $\mathrm{Hom}_{\mathscr{C}}(*, *) = *$, the set with only one element.

Let $\mathscr{C}$ be a category such that it has products, coproducts and they are isomorphic. For example the category of vector spaces or Abelian groups. Then for every pair of objects a, b the set $\mathrm{Hom}_{\mathscr{C}}(a, b)$ has an associative operation. Indeed for $\phi, \psi \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ we define $\phi + \psi$ as given by the compositio

$$a \xrightarrow{\Delta} a \times a \xrightarrow{\phi \times \psi} a \times a \simeq a \amalg a \xrightarrow{\nabla} a. \tag{2.8.1}$$

[1]here we are abusing notation and using the same letter for the object and the morphism

Notice also that in this case we have canonical isomorphisms $a \times * \simeq a \amalg * \simeq a$. Indeed we have the projection $\pi : a \times * \to a$. On the other hand the maps Id_a and the unique morphism $a \to *$ produce a morphism $a \to a \times *$ which is easily seen to be an inverse to π. The situation for the coproduct is similar.

2.9 Proposition. *Let $\mathscr{C}$ be a category with an initial and final object* 0 *and with products and coproducts that are moreover isomorphic. Then for any two objects a, b the set of morphisms is a commutative associative monoid with the sum defined by* (2.8.1) *and the identity being the* 0 *morphism $a \to 0 \to b$.*

Proof. Let us check first commutativity. Let us call π_1 an π_2 the two canonical projections $a \times a \to a$. Then using the maps $a \times a \to a$ in different order, that is π_2, π_1, we obtain a morphism $\sigma : a \times a \to a \times a$ which is the *exchange of the two factors*. It is clear that $\sigma^2 = \mathrm{Id}_{a \times a}$ hence σ is invertible. The same argument shows that we have an isomorphism (also denoted by σ) of $a \amalg a$ that corresponds to the exchange of the factors.

Now consider the commuting diagram:

$$\begin{tikzcd}
 & a \times a \arrow[r, "\psi \times \phi"] \arrow[d, "\sigma"] & a \amalg a \arrow[d, "\sigma"] \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & & a \\
 & a \times a \arrow[r, "\phi \times \psi"'] & a \amalg a \arrow[ur] &
\end{tikzcd}$$

The composition on the top is $\psi + \phi$ and in the bottom is $\phi + \psi$.

We now proceed to check associativity. The two maps Id_a and Δ produce by the universal property a morphism $a \to a \times (a \times a)$. On the other hand the two maps Δ, Id_a produce a map $a \to (a \times a) \times a$. Since the product is associative these two maps are identified and we have a unique morphism $\Delta : a \to a \times a \times a$. This morphism is therefore obtained as either composition

$$\begin{tikzcd}
 & a \times a \arrow[dr, "(\mathrm{Id}_a \times \Delta)"] & \\
a \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & a \times a \times a \\
 & a \times a \arrow[ur, "(\Delta \times \mathrm{Id}_a)"'] &
\end{tikzcd}$$

Dually, the two compositions in the following diagram commute:

$$\begin{tikzcd}
 & a \amalg a \arrow[dr, "\nabla"] & \\
a \amalg a \amalg a \arrow[ur, "\nabla \amalg \mathrm{Id}_a"] \arrow[dr, "\mathrm{Id}_a \amalg \nabla"'] & & a \\
 & a \amalg a \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

We now consider the diagram

$$\begin{tikzcd}
 & a \times a \times a \arrow[r, "\psi\times\phi\times\zeta"] & a \times (a \amalg a) \arrow[r, "\mathrm{Id}_a \times \nabla"] \arrow[dd, "\sim"] & a \amalg a \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & & & a \\
 & a \times a \times a \arrow[r, "\psi\times\phi\times\zeta"'] & (a \amalg a) \times a \arrow[r, "\nabla\times id_a"'] & a \amalg a \arrow[ur] &
\end{tikzcd}$$

The composition on the top is $\psi + (\phi + \zeta)$ and at the bottom is $(\psi + \phi) + \zeta$. Commutativity of the diagram implies associativity.

Finally if we let $\psi = 0$ in (2.8.1) we have the following commuting diagram

$$\begin{tikzcd}
 & a \times a \arrow[r, "\phi\times 0"] & a \amalg a \arrow[dr, "\nabla"] & \\
a \arrow[ur, "\Delta"] \arrow[r, "\mathrm{Id}_a"'] & a \times 0 \arrow[r, "\phi"'] & a \amalg 0 \arrow[r, "\mathrm{Id}_a"'] & a
\end{tikzcd}$$

proving that the operation is unital. □

2.10 Definition. A category $\mathscr{C}$ with the conditions of the previous proposition and such that for every pair of objects a, b the monoid $\mathrm{Hom}_{\mathscr{C}}(a, b)$ has inverses is called a *additive* category. It is customary to write $\oplus$ instead of $\times$ or $\amalg$ in this case.

2.11 Definition. Let $\mathscr{C}$ be a category with an initial and final object $*$ (for example if $\mathscr{C}$ is an additive category). Then a limit (resp. colimit) in Example 2.5 g) applied to the case when $\psi = * \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ is called a *kernel* (resp. *cokernel*) of ϕ.

2.12 Exercises

2.12.1. Prove the assertion in 2.5 c).

2.12.2. Check that the direct sum is both a product and a coproduct in the category of vector spaces.

2.12.3. Find a product in the category $\mathbf{Set}_*$.

2.12.4. Prove that the product category defined in 2.6 is a product in the category of categories (at least in the case when they are both small).

2.12.5. Let $\mathscr{C}$ be a category with an initial and final object $*$. Show that $\mathrm{Hom}_{\mathscr{C}}(*, *)$ is the set with only one element.

2.12.6. Find a category which is not an additive category but that satifies the conditions of Proposition 2.9.

2.12.7. Let $\mathscr{C}$ be a category with an initial and final object $*$. Show that every kernel is a monomorphism and that every cokernel is an epimorphism[2].

[2] The converse is not true, namely there are categories admitting a (mono/epi)-morphism which is not a kernel/cokernel of a morphism.