1 Categories

1.1. Throughout these lectures we will use the language of *category theory*. The treatment will be very informal and We will not dwell in the subtleties nor technicalities of set theory and I recommend the reader to look into the bibliography, specially [1].

1.2. A *Category* $\mathscr{C}$ consists of the following data

a) A collection $Ob(\mathscr{C})$ of *objects*.

b) For each pair of objects a, b a *set* of *morphisms* $\mathrm{Hom}_{\mathscr{C}}(a, b)$.

c) For each object a an element $\mathrm{Id}_a \in \mathrm{Hom}_{\mathscr{C}}(a, a)$

d) For each three objects a, b, c a map of sets (called the *composition*)

$$\circ : \mathrm{Hom}_{\mathscr{C}}(b, c) \times \mathrm{Hom}_{\mathscr{C}}(a, b) \to \mathrm{Hom}_{\mathscr{C}}(a, c), \qquad (\phi, \psi) \mapsto \phi \circ \psi.$$

subject to the following axioms:

a) $\mathrm{Id}_a \circ \phi = \phi$ and $\phi \circ \mathrm{Id}_b = \phi$ for all objects a, b and every $\phi \in \mathrm{Hom}_{\mathscr{C}}(b, a)$.

b) Composition is associative, that is for every four objects a, b, c, d and morphisms $\phi \in \mathrm{Hom}_{\mathscr{C}}(a, b)$, $\psi \in \mathrm{Hom}_{\mathscr{C}}(b, c)$ and $\theta \in \mathrm{Hom}_{\mathscr{C}}(c, d)$ we have

$$\theta \circ (\psi \circ \phi) = (\theta \circ \psi) \circ \phi \in \mathrm{Hom}_{\mathscr{C}}(a, d).$$

1.3. For a given category $\mathscr{C}$ and two objects a, b. We often times will denote a morphism $\phi \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ by $\phi : a \to b$ even though there is no *map of sets* involved.

The collection of all morphisms will be denoted by $\mathrm{Mor}(\mathscr{C})$. We have two maps $s, t : \mathrm{Mor}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ *source* and *target*, that is for ϕ as above we have $s(\phi) = a$ and $t(\phi) = b$.

1.4. A morphism $\phi \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ is said to be an *isomorphism* if there exists $\psi \in \mathrm{Hom}_{\mathscr{C}}(b, a)$ such that $\phi \circ \psi = \mathrm{Id}_b$ and $\psi \circ \phi = \mathrm{Id}_a$. In this case we say that a and b are *isomorphic*.

1.5 Definition. A group is a category with only one object $*$ and such that every morphism is an isomorphism.

1.6 Examples.

a) $\mathbf{Set}$ is the category whose objects are sets and morphisms are maps of sets.

b) $\mathbf{Top}$ is the category whose objects are topological spaces and morphisms are continuous maps of topological spaces

c) $[\mathbf{n}]$ is the category with objects the set $\{0, \cdots, n\}$ and morphisms given by

$$\mathrm{Hom}_{[\mathbf{n}]}(i, j) = \begin{cases} \varnothing & i > j \\ * & i \leq j \end{cases}$$

where $*$ is the set with only one element.

d) $\mathbf{Set}_*$ is the category of *pointed* sets, that is pairs (S, s) of a set S and an element $s \in S$. Morphisms $(S, s) \to (T, t)$ are given by Maps of sets $S \to T$ such that $s \mapsto t$.

e) Let k be a field. The category $\mathbf{Vect}_k$ as k-vector spaces as objects and k-linear maps as morphisms.

f) The category Δ has as objects the non-negative integer numbers $0, 1, 2, \cdots$ and for each pair of objects n, m a morphism $\phi : n \to m$ is a non-decreasing map of sets $\{0, \cdots, n\} \to \{0, \cdots, m\}$.

g) The category $\mathbf{Set}^f$ has as object *finite sets* and as morphisms maps of sets.

h) The category $\mathcal{S}$ has as objects finite non-empty, linearly ordered sets and as morphisms order-preserving maps.

i) Any partially ordered set $\{i, j, \cdots\}$ can be viewed as a category, with only one morphism for each $i \leq j$.

j) An oriented graph gives rise to a category with one object for each vertex and one morphism for each arrow (one needs to add compositions and identity morphisms).

k) $\mathbf{Grp}$ is the category whose objects are groups (definition below) and morphisms are homomorphisms of groups.

1.7 Remark. Notice that for each pair of objects, there is a *set* of morphisms. The collection of all objects (and consequently the collection of all morphisms) might be large enough not to be a set, for example the category $\mathbf{Grp}$ or even $\mathbf{Set}$. The categories as defined in these lectures are called *locally small* in the literature. We will not deal with general categories where the morphism spaces need not be *small sets.*

1.8 Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* $F : \mathscr{C} \to \mathscr{D}$ are assignments $\mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ and $\mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{D})$, $a \mapsto F(a)$, $\mathrm{Hom}_{\mathscr{C}}(a, b) \ni \phi \mapsto F(\phi) \in \mathrm{Hom}_{\mathscr{D}}(F(a), F(b))$ such that

a) $F(\mathrm{Id}_a) = \mathrm{Id}_{F(a)}$ for all objects a of $\mathscr{C}$.

b) For every three objects a, b, c of $\mathscr{C}$ and morphisms $\phi \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ and $\psi \in \mathrm{Hom}_{\mathscr{C}}(b, c)$ we have $F(\psi \circ_{\mathscr{C}} \phi) = F(\psi) \circ_{\mathscr{D}} F(\phi) \in \mathrm{Hom}_{\mathscr{D}}(F(a), F(c))$.

1.9 Examples.

a) For any category $\mathscr{C}$ we have a functor $\mathrm{Id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ which is the identity on objects and morphisms.

b) For any category $\mathscr{C}$ there is a unique functor $\mathscr{C} \to [0]$ which sends any object a of $\mathscr{C}$ to the unique object of $[0]$ and any morphism in $\mathscr{C}$ to the unique morphism in $[0]$.

c) A functor from $[0]$ to a category $\mathscr{C}$ is equivalent to the datum of an object a of the category $\mathscr{C}$.

d) Let G and H be two groups. A functor $H \to G$ will be called a *homomorphism* of groups. We let Grp be the category whose objects are groups and morphisms are homomorphisms of groups.

e) The functor $F : \mathbf{Vect}_k \to \mathbf{Set}$ that assigns to each vector space k its underlying set and each linear map its underlying map of sets. Notice that this functor in fact can be thought of as a functor $\mathbf{Vect}_k \to \mathbf{Set}_*$ since each vector space has a marked point (the zero vector).

f) The functor $\Delta \to \mathcal{S}$ assigns to n the linearly ordered set $\{0, \cdots, n\}$ and to each map $n \to m$ the corresponding map of sets.

g) A functor from the category given by the graph

to $\mathbf{Vect}_k$ consists of a vector space V and a linear endomorphism of V.

h) A functor from the category given by the linearly ordered set $\mathbb{Z}$ (see example 1.6 i) above) to $\mathbf{Top}$ consists of a sequence $\{X_i\}_{i\in\mathbb{Z}}$ of topological spaces and a continuous map $X_i \to X_{i+1}$ for each $i \in \mathbb{Z}$.

i) More generally a functor from the category given by a partially ordered set I to a category $\mathscr{C}$ consists of a family $\{a_i\}_{i\in I}$ of objects in $\mathscr{C}$ parametrized by I and for each ordered pair $i \le j$ a morphism $\phi_{ij} : a_i \to a_j$ such that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for each $i \le j \le k$. A particular case for the set I with only one element is the example c) above.

1.10. Given two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ we obtain a functor $G \circ F : \mathscr{C} \to \mathscr{E}$ by letting $G \circ F(a) = G(F(a))$ and $G \circ F(\phi) = G(F(\phi))$. This composition is associative in the obvious way.

1.11 Definition. Given two categories $\mathscr{C}$ and $\mathscr{D}$ and two functors $F, G : \mathscr{C} \to \mathscr{D}$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of a collection of morphisms $\alpha_a : F(a) \to G(a)$ for each object a of $\mathscr{C}$ such that for each morphism $\phi \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ the following diagram commutes:

$$\begin{tikzcd} F(a) \arrow[r, "\alpha_a"] \arrow[d, "F(\phi)"'] & G(a) \arrow[d, "G(\phi)"] \\ F(b) \arrow[r, "\alpha_b"'] & G(b) \end{tikzcd}$$

For any functor F we have a natural transformation $\mathrm{Id}_F : F \Rightarrow F$ given simply by $\alpha_a = \mathrm{Id}_{F(a)}$.

1.12. Natural transformations can be composed in different ways. Let $F, G, H : \mathscr{C} \to \mathscr{D}$ be three functors and let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be two natural transformations. We have a *vertical* composition $\beta \circ \alpha : F \Rightarrow H$ given by $(\beta \circ \alpha)_a := \beta_{G(a)} \circ \alpha_a$.

We also have the following horizontal composition. Suppose we have $F, F' : \mathscr{C} \to \mathscr{D}$ and $G, G' : \mathscr{D} \to \mathscr{E}$ functors. Suppose we also have natural transformations $\alpha : F \Rightarrow F'$ and $\beta : G \Rightarrow G'$. Then we have the natural transformation $\beta \circ \alpha : G \circ F \Rightarrow G' \circ F'$ given by $(\beta \circ \alpha)_a = G'(\alpha_a) \circ \beta_{F(a)}$.

1.13 Definition. Given two functors $F, G : \mathscr{C} \to \mathscr{D}$ we say that a natural transformation $\alpha : F \Rightarrow G$ is an *isomorphism* if there exists a $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = \mathrm{Id}_F$ and $\alpha \circ \beta = \mathrm{Id}_G$. We say that two functors are *isomorphic* if there exists an isomorphism between them. We denote $F \simeq G$ in this case.

1.14 Definition. Two categories $\mathscr{C}$ and $\mathscr{D}$ are said to be *equivalent* if there exists two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and isomorphisms of functors $G \circ F \simeq \mathrm{Id}_{\mathscr{C}}$ and $F \circ G \simeq \mathrm{Id}_{\mathscr{D}}$. In this case the functor G is said to be a *inverse* to F and vice-versa. Two categories are said to be isomorphic if these isomorphisms are the identity natural transformation, namely $G \circ F = \mathrm{Id}_{\mathscr{C}}$ and $F \circ G = \mathrm{Id}_{\mathscr{D}}$.

1.15 Remark. Here is the big departure from categories to sets. One is tempted to think of categories as sets (of their objects for example) and functors as maps between these sets. When comparing two maps between sets, one says that $f : S \to T$ is invertible if there exists $g : T \to S$ such that $f \circ g = \mathrm{Id}_T$ and $g \circ f = \mathrm{Id}_S$, that is, on each side of the equal side we have endomorphisms of a given set and we have to elements of this set, which we may ask if they are equal or not. On the other hand, the compositions like $F \circ G$ or the identity functor $\mathrm{Id}_{\mathscr{D}}$ are *endofunctors* of a given category, not really elements of a given set, but rather *objects of another category!*. As such we may ask if these objects are isomorphic or not.

Notice that inverses are not uniquely defined. But given two inverses, say G and G' for F there exists a *canonical* isomorphism $G \simeq G'$ (Exercise 1.21.3)

1.16 Example. *The category Δ of 1.6 f) is equivalent to the category $\mathcal{S}$ of 1.6 h). In fact an* inverse G *for the functor in 1.9 f) is given as follows. To the finite non-empty linearly ordered set I (that is an object of $\mathcal{S}$) we assign $G(I) = [|I| - 1]$. Notice that there exists a unique order preserving bijection of sets $I \simeq [|I| - 1]$. Identifying the set I with the set $[|I| - 1]$ with these isomorphisms we see that for each order preserving map $I \to J$ we have a unique order preserving map $[|I| - 1] \to [|J| - 1]$, that is a morphism in Δ. If we call F the functor $\Delta \to \mathcal{S}$ it is clear that $G \circ F = \mathrm{Id}_\Delta$. However the composition in the other direction is not the identity: $F \circ G(I)$ is not* equal *to the set I, but it is isomorphic within the category $\mathcal{S}$, that is, the set $[|I| - 1]$ is an object of $\mathcal{S}$ and there exists a (unique) isomorphism $\alpha_I : I \xrightarrow{\sim} [|I| - 1]$. The collection of all these α_I defines the natural transformation $\mathrm{Id}_{\mathcal{S}} \Rightarrow F \circ G$ which is not the identity, but it is an isomorphism.*

1.17 Example. *Perhaps a more brutal example is to consider the category with two objects a and b and exactly one morphism between any two objects. This category is equivalent to the category* $[0]$ *with only one object.*

1.18. Natural transformations allow us to construct new categories from given ones. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. We would like to consider a *category* $\mathrm{Funct}(\mathscr{C}, \mathscr{D})$ with objects functors $F : \mathscr{C} \to \mathscr{D}$ and with morphisms $\mathrm{Hom}_{\mathrm{Funct}(\mathscr{C},\mathscr{D})}(F, G)$ natural transformations $F \Rightarrow G$. We have seen that we have identity natural transformation and that the composition (vertical) is associative. The only problem is that with this definition the Hom spaces may fail to be (small) sets!. As mentioned at the beginning I will not deal with these issues in these lectures, but you can read about this in the bibliography. It suffices to say that if we define a category to be *small* if the collection of morphisms is a set (in particular the collection of objects is also a set) then if $\mathscr{C}$ is equivalent to a small category, then the category of functors above defined is a category.

1.19. Particular examples of the category of functors defined above are important. For example, if we let $\mathscr{C}$ be given by a partially ordered set I. Then the objects of $\mathrm{Func}(\mathscr{C}, \mathscr{D})$ consist of families of objects $\{a_i\}_{i \in I}$ parametrized by I as in Example 1.9 i). A morphism $\{a_i\} \to \{b_i\}$ consists of a family of morphisms $\psi_i \in \mathrm{Hom}_{\mathscr{C}}(a_i, b_i)$ such that for every $i \leq j$ the following diagram commutes

$$\begin{tikzcd} a_i \arrow[r, "\psi_i"] \arrow[d, "\phi^a_{ij}"'] & b_i \arrow[d, "\phi^b_{ij}"] \\ a_j \arrow[r, "\psi_j"'] & b_j . \end{tikzcd}$$

where ϕ^a_{ij} and ϕ^b_{ij} are the defining morphisms as in Example 1.9 i).

1.20. There are other ways of constructing new categories from given ones. For example if $\mathscr{C}$ is a category, then we define $\mathscr{C}^{op}$ as the category with the same objects of $\mathscr{C}$ but with morphisms

$$\mathrm{Hom}_{\mathscr{C}^{op}}(a, b) := \mathrm{Hom}_{\mathscr{C}}(b, a),$$

and obvious compositions.

We may define the category of arrows in $\mathscr{C}$ with objects the morphisms of $\mathscr{C}$ for a given pair of morphisms $\phi : a \to b$ and $\psi : c \to d$ we define a morphism $\phi \to \psi$ to be a pair of morphisms ξ, υ such that the following diagram commutes

$$\begin{tikzcd} a \arrow[r, "\phi"] \arrow[d, "\xi"'] & b \arrow[d, "\upsilon"] \\ c \arrow[r, "\psi"'] & d \end{tikzcd}$$

For a given object a of $\mathscr{C}$ we may define in a similar way the categories $\mathscr{C}_{/a}$ and $\mathscr{C}_{a/}$ to be the categories with objects given by morphisms in $\mathscr{C}$ $b \to a$ and $a \to b$ respectively. I'll let you think about morphisms.

1.21 Exercises

1.21.1. Prove that the *vertical composition* of 1.12 is associative and that the identity natural transformation is a two sided identity for this composition.

1.21.2. Prove that the *horizontal composition* of 1.12 is indeed a natural transformation. Find a compatibility satisfied by the horizontal and vertical compositions of natural transformations.

1.21.3. Let F and G be inverse functors between two categories $\mathscr{C}$ and $\mathscr{D}$. Let G' be another inverse for F. Show that there exists a *canonical* isomorphism $G \simeq G'$ (don't worry now about the word *canonical*, just find one isomorphism).

1.21.4. Prove the statement in Example 1.17.

1.21.5. Check that the category of arrows defined in 1.20 is indeed a category.

References

[1] S. Mac Lane. Categories for the working mathematician. In *Graduate text in mathematics*, volume 5. Springer, 1971.