Fixed points of Legendre-Fenchel type transforms and polarity type operators

Daniel Reem
(based on joint works with Alfredo Iusem and Simeon Reich)
Department of Mathematics, The Technion, Haifa, Israel
E-mail: dream@technion.ac.il
http://w3.impa.br/~dream

Perspectives in Modern Analysis, International Conference in honor of Dov Aharonov, Samuel Krushkal, Simeon Reich, and Lawrence Zalcman,
Holon Institute of Technology, Holon, Israel, 29 May 2018
(30 minutes)
History of mathematics

2011-2013: I was a postdoc at IMPA (Brazil)

End-of-Jan-begin-of-Feb 2013: came to a vacation in Israel and gave talks in several seminars. The talk was about a result, based on a joint work with Iusem and Svaiter, regarding a certain characterization of order preserving and order reversing operators acting on the class of lower semicontinuous proper and convex functions in Banach spaces (more details: later).

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Historical background (Cont.)

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July 2017: the paper was posted on the arXiv.


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Some notations and assumptions

From now on, \((X, \langle \cdot, \cdot \rangle)\) is a real Hilbert space, \(X \neq \{0\}\).

Notation: \(C(X)\) is the set of lower semicontinuous proper convex functions \(f: X \rightarrow \mathbb{R} \cup \{+\infty\}\).

Given \(f: X \rightarrow [-\infty, \infty]\), its Legendre-Fenchel transform (namely, the convex conjugate) is the function \(f^*: X \rightarrow [-\infty, \infty]\) defined by \(f^*(x^*) := \sup\{\langle x^*, x \rangle - f(x) : x \in X\}\), \(x^* \in X\).
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First project (Convex Analysis): goal

To solve the following fixed point equation:

\[ f(x) = \tau f^*(E x + c) + \langle w, x \rangle + \beta, \quad x \in X, \]  

Here:

- \( f : X \to [-\infty, \infty] \) is the unknown function,
- \( \tau > 0 \) is given,
- \( c, w \in X \) are given,
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Why considering (1)? First motivation

It is a generalization of the equation $f = f^*$, (2)

The solutions of (2) are the self-conjugate functions. Well-known fact: (2) has a unique solution: the normalized energy function, namely $f(x) = \frac{1}{2} \|x\|_2^2$, $x \in X$. 

Iusem, Reem, Reich

Fixed points, Legendre-Fenchel, polarity

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Why considering (1): second motivation

Let $T_f$ denote the right-hand side of (1), namely $\tau f^*(Ex + c) + \langle w, x \rangle + \beta$, $x \in X$.

Theorem: $T$ is the most general fully order reversing operator acting on $C(X)$, where fully order reversing means that $T$ is invertible and both $T$ and $T^{-1}$ reverse the (pointwise) order.

Original version of this theorem: Artstein-Avidan and Milman, 2009, $X = \mathbb{R}^n$, $n \in \mathbb{N}$.

Generalization to arbitrary infinite-dimensional Banach spaces: joint work with Iusem and Svaiter, 2015 (now $T: C(X) \to C_{w^*}(X^*)$), where $C_{w^*}(X^*)$ is the set of all weak-star lower semicontinuous proper and convex functions $g: X^* \to (-\infty, \infty]$ and $E$ is the adjoint of some continuous, invertible and linear operator). The talks that I gave in 2013 were about this result.
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The classification theorem: an informal version

The nonlinear equation

(1) is very sensitive to the various
parameters which appear in it
and can have no solution, a unique
solution, or several (possibly infinitely many) ones.

If $E$ is positive definite,
then there always exists a solution to (1),
and this solution is quadratic and
strictly convex; sometimes uniqueness can also be established.

If $E$ is not positive definite,
then there can be several (possibly infinitely many) solutions or no solution
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The classification theorem, formal version, I

Suppose that $E$ is positive definite, namely it is continuous, $E^* = E$ and $\langle Ex, x \rangle > 0$ when $x \neq 0$. Then there exists a quadratic solution $f$ to (1), namely

$$f(x) = \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle + \gamma,$$

where $0 \neq A : X \to X$ is linear, self-adjoint and continuous, $b \in X$, $\gamma \in \mathbb{R}$. Actually, the coefficients satisfy the following relations:

$$A = \sqrt{\tau} E,$$

$$b = w + \sqrt{\tau} c_1 + \sqrt{\tau},$$

$$\gamma = \beta (1 + \sqrt{\tau})^2 + 0.5 \sqrt{\tau} \langle c - w, E^{-1}(c - w) \rangle (1 + \sqrt{\tau})^2 (\tau + 1).$$

This solution is strictly convex and it is unique in the class of quadratic solutions having a leading coefficient $A$ which is invertible.
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The classification theorem, II: uniqueness

Suppose that $E$ is positive definite and at least one of the following conditions holds:

1. $\tau = 1$ and $c = w$,
2. $X$ is finite dimensional,
3. $\tau \neq 1$, and $f$ belongs to the class of functions from $X$ to $\mathbb{R}$ which are twice differentiable and their second derivative is continuous at the point $x_0 = (1/(1-\tau))(E^{-1}w - E^{-1}c)$.

Then there exists a unique solution $f$ to (1) in the corresponding classes of functions (first case: all functions from $X$ to $[-\infty, \infty]$; second case: the class of twice differentiable functions having a second derivative which is continuous at $x_0$).

This solution is quadratic and strictly convex and its coefficients satisfy (3).
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3. $\tau \neq 1$,
4. $f$ belongs to the class of functions from $X$ to $R$ which are twice differentiable and their second derivative is continuous at the point $x_0 = (1/(1-\tau))(E^{-1}w - E^{-1}c)$.

Then there exists a unique solution $f$ to (1) in the corresponding classes of functions (first case: all functions from $X$ to $[\sim\infty, \infty]$; second case: the class of twice differentiable functions having a second derivative which is continuous at $x_0$).

This solution is quadratic and strictly convex and its coefficients satisfy (3).
The classification theorem, II: uniqueness

Theorem

Suppose that $E$ is positive definite and at least one of the following conditions holds:

- $\tau = 1$ and $c = w$, 

Then there exists a unique solution $f$ to (1) in the corresponding classes of functions (first case: all functions from $X$ to $[-\infty, \infty]$; second case: the class of twice differentiable functions having a second derivative which is continuous at $x_0 = (1/(1 - \tau))(E - 1w - E - 1c)$).

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Suppose that $E$ is positive definite and at least one of the following conditions holds:

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The classification theorem, III \( (E \text{ is not positive definite: non-uniqueness 1}) \)

Theorem

The equation

\[ f(x_1, x_2) = f^*(x_2, -x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \]

has infinitely many quadratic solutions of the form

\[ f(x) = \frac{1}{2} \langle Ax, x \rangle, \quad x \in \mathbb{R}^2, \]

where \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) is any linear operator the matrix representation of which is symmetric, positive definite, and has determinant 1.
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The classification theorem, IV ($E$ is not positive definite: non-uniqueness 2)

The equation $f(x) = f^*(−x)$, $x ∈ \mathbb{R}$, has infinitely many non-quadratic solutions, among them $f(x) := \begin{cases} \lambda^2 x^2, & x ∈ (-∞, 0] \\ \frac{1}{2} \lambda x^2, & x ∈ [0, ∞) \end{cases}$ for arbitrary $\lambda > 0$, and $f(x) := \begin{cases} \infty, & x ∈ (-∞, 0] \\ -\frac{1}{2} - \log(x), & x ∈ (0, ∞) \end{cases}$.
The classification theorem, IV \((E \text{ is not positive definite: non-uniqueness 2})\)

**Theorem**

The equation

\[ f(x) = f^*(-x), \quad x \in \mathbb{R}, \]

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The classification theorem, IV ($E$ is not positive definite: non-uniqueness 2)

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The classification theorem, V (non-existence)

If $w \neq 0$, then the equation

$$f(x) = f^*(\frac{-x}{\|w\|^2}) + \langle w, x \rangle \quad \forall x \in X,$$

does not have any solution $f : X \to [-\infty, \infty]$.

In addition, if $c \neq 0$, then no $f : X \to [-\infty, \infty]$ satisfies the equation

$$f(x) = f^*(\frac{-x+c}{\|w\|^2}), \quad x \in X.$$
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A by-product of independent interest

Lemma

Assume that $Q : X \to 2^X$ is a monotone operator, i.e.,

$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0, \forall x_1, x_2 \in X, y_1 \in Qx_1, y_2 \in Qx_2.$$  (4)

Suppose that $L : X \to X$ is invertible, strictly monotone (i.e., strict inequality in (4) when $x_1 \neq x_2$) and maximally monotone (i.e., if the graph of $L$ is contained in the graph of some monotone operator $M$, then $L = M$).

If $I \subseteq QL$ or $I \subseteq LQ$, where $I$ is the identity operator, then $Q = L^{-1}$. In particular, $Q$ is single-valued, invertible, strictly monotone and maximally monotone.
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**Lemma**

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$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0, \quad \forall x_1, x_2 \in X, y_1 \in Qx_1, y_2 \in Qx_2.$$  \hspace{1cm} (4)
Lemma

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$$I \subseteq QLQL \quad \text{or} \quad I \subseteq LQLQ,$$

where $I$ is the identity operator, then $Q = L^{-1}$. In particular, $Q$ is single-valued, invertible, strictly monotone and maximally monotone.
The goal

To solve the following geometric fixed point equation:

\[ C = (G \cap C) \circ (X \setminus \{0\}). \quad (5) \]

Here, \( \emptyset \neq C \) is the unknown subset, \( C \) is contained in a real Hilbert space \( X \neq \{0\} \), \( G : X \rightarrow X \) is a given continuous invertible linear operator, \( G \cap C \) is the set of all \( Gc \) for \( c \in C \), and the polar (or dual) of \( \emptyset \neq S \subseteq X \) is the set \( S^\circ := \{ x^* \in X : \langle x^*, s \rangle \leq 1, \forall s \in S \} \). Polar sets are, of course, widely used in geometry and optimization (e.g., the normal cone is the polar of the tangent cone).
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- Polar sets are, of course, **widely used in geometry and optimization** (e.g., the normal cone is the polar of the tangent cone).
Why considering (5)? First motivation

It is a generalization of the equation $C = C \circ$, (6)
The solutions of (6) are the self-polar sets.

Well-known fact: (6) has a unique solution: the unit ball.

Corollary of the well-known fact: If we start with $\mathbb{R}^n$ and want to define on it a norm such that the unit ball induced by this norm will coincide with the unit ball of the dual norm, then we can do this if and only if the norm is Euclidean.

Here we identify the dual space with $\mathbb{R}^n$ and consider both balls as subsets of $\mathbb{R}^n$. 

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Related to a relatively recent branch of research in convex geometry. In some of the works belonging to this branch, certain order reversing operators acting on various classes of finite-dimensional geometric objects were considered. For instance, Böröczky-Schneider 2008: objects are compact and convex subsets of $\mathbb{R}^n$ containing the origin in their interior, Schneider 2008: objects are closed and convex cones, Artstein-Avidan and Milman 2008, Milman-Segal-Slomka 2011, Slomka 2011: objects are closed and convex subsets of $\mathbb{R}^n$ containing the origin, Artstein-Avidan and Slomka 2012: objects are $n$-dimensional centrally symmetric ellipsoids. In all of these works $n \in \mathbb{N}$ satisfies either $n \geq 2$ or $n \geq 3$. 

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- In all of these works $n \in \mathbb{N}$ satisfies either $n \geq 2$ or $n \geq 3$. 
A central property that was established there: these operators must have the form $T(C) = AC \circ$ for some invertible linear operator $A: \mathbb{R}^n \to \mathbb{R}^n$.

Equation (5) is directly related to these works because if we denote $T(C) := (GC) \circ$, then a simple verification shows that $T(C) = (G^* )^{-1}C \circ$. 

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$$T(C) = (G^*)^{-1}C^\circ.$$
Why considering (5)? third motivation

This equation, namely $C = (G\ C) \circ$, has some similarities with the fixed point equation discussed earlier: $f(x) := \tau f^\ast(Ex + c) + \langle w, x \rangle + \beta$, $x \in X$.

Some of the results related to the convex analytic equation are used in the analysis of the convex geometry equation.
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Some of the results related to the convex analytic equation are used in the analysis of the convex geometry equation.
The classification theorem

Theorem

$X \neq \{0\}$ is a real Hilbert space,

$G : X \to X$ is a continuous and invertible linear operator,

Consider equation (5) with an unknown $\emptyset_0 = C \subseteq X$, i.e.,:

$C = (GC) \circ$, 

The following statements hold:

(i) Any solution to (5) must be closed and convex, and must contain 0.

(ii) If $G$ is positive definite, then there exists a unique solution to (5): the ellipsoid of the form $C = \{x \in X : \langle Gx, x \rangle \leq 1\}$.

(iii) If $G$ is not positive definite, then there are cases where (5) has several (possibly infinitely many) solutions.

There are cases where (5) does not have any solution which belongs to the class of bounded subsets of $X$ which contain 0 in their interiors.
The classification theorem

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1. $X \neq \{0\}$ is a real Hilbert space,

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Note: There are cases where (5) does not have any solution which belongs to the class of bounded subsets of $X$ which contain 0 in their interiors.
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(iii) If $G$ is not positive definite, then
  - there are cases where (5) has several (possibly infinitely many) solutions
The classification theorem

**Theorem**

- $X \neq \{0\}$ is a real Hilbert space,
- $G : X \to X$ is a continuous and invertible linear operator,
- Consider equation (5) with an unknown $\emptyset \neq C \subseteq X$, i.e.,:
  \[ C = (GC)^\circ, \]

**The following statements hold:**

(i) Any solution to (5) must be closed and convex, and must contain 0.
(ii) If $G$ is positive definite, then there exists a unique solution to (5): the **ellipsoid** of the form $C = \{ x \in X : \langle Gx, x \rangle \leq 1 \}$.
(iii) If $G$ is not positive definite, then
   1. there are cases where (5) has several (possibly infinitely many) solutions
   2. there are cases where (5) does not have any solution which belongs to the class of bounded subsets of $X$ which contain 0 in their interiors.
Some contributions

Our analysis is essentially dimension-free. We obtain a few by-products of possible independent interest:

1. A convex analytic converse to the celebrated Lax-Milgram theorem from Partial Differential Equations.
2. Results related to infinite-dimensional convex geometry.

Iusem, Reem, Reich

29 May 2018
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  - Results related to **infinite-dimensional convex geometry**
Non-uniqueness: an instructive example

Suppose that $X = \mathbb{R}^n$, $G(x) := -x$, $x \in X$.

Then (5) becomes $C = (-C) \circ$ (7).

This equation has infinitely many solutions, among them:

- The unit ball
- Regular simplices having circumradius $r := \sqrt{n}$ and centroid 0
- Unbounded ice-cream cones with vertex at 0
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Real world illustrations 1: positive definite case
Real world illustrations 1: positive definite case

Figure: 2D example (source: Bill Frymire)
Real world illustrations 1: positive definite case

**Figure:** 2D example  
(source: [Bill Frymire](http://rugby1823.blogosfere.it))

**Figure:** 3D example  
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Real world illustrations 2: $G = -I$
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**Figure:** Ball (source: scienities.com)
Real world illustrations 2: $G = -I$

**Figure:** Ball  *(source: scienities.com)*

**Figure:** Simplex  *(source: AliExpress)*
Real world illustrations 2: \( G = -I \)

**Figure:** Ball (source: scienities.com)

**Figure:** Simplex (source: AliExpress)

**Figure:** Ice cream cones (source: pngpix)
By-product of independent interest

Given a real Hilbert space $X \neq \{0\}$, if $A : X \rightarrow X$ is a positive semidefinite and invertible linear operator, then $A$ is coercive (elliptic, strongly monotone). In particular, $A$ is positive definite. Actually, $\langle Ax, x \rangle \geq \|A^{-1}\|^{-1} \|x\|^2$, $\forall x \in X$ and $\|A^{-1}\|^{-1}$ is the optimal (largest possible) coercivity coefficient.

The lemma is essentially a quantitative converse of the celebrated Lax-Milgram theorem from PDE. LM's theorem states that if $B(x, y) = \langle Ax, y \rangle$ is a continuous and coercive bilinear form (in particular, $\langle Ax, x \rangle > 0$ when $x \neq 0$), then $A$ is invertible. The lemma states that if $A$ is positive definite and invertible, then $B$ is coercive with optimal coercivity coefficient $\|A^{-1}\|^{-1}$.
Lemma

Given a real Hilbert space $X \neq \{0\}$, if $A : X \to X$ is a positive semidefinite and invertible linear operator, then $A$ is coercive (elliptic, strongly monotone). In particular, $A$ is positive definite. Actually,

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The End

The papers and slides can be found online:

- **Legendre-Fenchel:**
  - [http://www.heldermann.de/JCA/JCA26/JCA261/jca26016.htm](http://www.heldermann.de/JCA/JCA26/JCA261/jca26016.htm)

- **Convex geometry:** [https://arxiv.org/abs/1708.09741](https://arxiv.org/abs/1708.09741)

- **Slideshow:** [http://w3.impa.br/~dream/talks](http://w3.impa.br/~dream/talks)