Fixed points of polarity type operators

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(joint work with Simeon Reich)

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(50-60 minutes)
The goal

To solve the following geometric fixed point equation:

\[ C = (GC) \circ C. \] (1)

Here:

- \( \emptyset \neq C \) is the unknown subset,
- \( C \) is contained in a real Hilbert space \( X \neq \{0\} \),
- \( G : X \to X \) is a given continuous invertible linear operator,
- \( GC = \{Gc : c \in C\} \).

The polar (or dual) of \( \emptyset \neq S \subseteq X \) is the set \( S^\circ = \{x^* \in X : \langle x^*, s \rangle \leq 1 \forall s \in S\} \).
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To solve the following geometric fixed point equation:

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To solve the following geometric fixed point equation:

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Here:

- $\emptyset \neq C$ is the unknown subset,
- $C$ is contained in a real Hilbert space $X \neq \{0\}$,
- $G : X \to X$ is a given continuous invertible linear operator,
- $GC := \{Gc : c \in C\}$,
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\[ S^\circ := \{ x^* \in X : \langle x^*, s \rangle \leq 1 \ \forall s \in S \}. \]
Considering (1): first motivation

It is a generalization of the equation $C = C \circ$, (2)

The solutions of (2) are the self-polar sets.

Well-known fact: (2) has a unique solution: the unit ball.
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- The solutions of (2) are the **self-polar sets**

- **Well-known fact:** (2) has a unique solution: the unit ball.
Corollary of the well-known fact:
The only norm which can be defined on $\mathbb{R}^n$ so that its unit ball coincides with the unit ball of its dual norm is the Euclidean norm. Here we identify the dual space with $\mathbb{R}^n$ and consider both balls as subsets of $\mathbb{R}^n$.

A clarification related to the corollary: if $C \subseteq X$ is centrally symmetric (i.e., $C = -C$), closed, convex, bounded, and the origin is an interior point of $C$, then $C$ is the unit ball of some norm, $C^\circ$ is the unit ball of the dual norm.
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Illustration of the corollary: $X = \mathbb{R}^2$, $C \neq C^\circ$
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Figure: $C$ is the unit ball of $\ell_1$

Figure: $C^\circ$ is the unit ball of $\ell_\infty \cong \ell_1^*$
Illustration of the corollary: \( X = \mathbb{R}^3, \ C \neq C^\circ \)
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Figure: $C$ is an approximation of the unit ball of $\ell_{1.5}$

Figure: $C^\circ$ is an approximation of the unit ball of $\ell_3 \cong \ell_{1.5}^*(\frac{1}{1.5} + \frac{1}{3} = 1)$
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**Figure:** $C$ is a realistic illustration of the Euclidean unit ball (source: [scienities.com](http://scienities.com))

**Figure:** $C^\circ = C$
Considering (1): second motivation

Originates in a relatively recent branch of research in convex geometry.

In some of the works belonging to this branch, certain order reversing operators acting on various classes of finite-dimensional geometric objects were considered. For instance, Böröczky-Schneider 2008: objects are compact and convex subsets of $\mathbb{R}^n$ containing the origin in their interior, Schneider 2008: objects are closed and convex cones, Artstein-Avidan and Milman 2008, Milman-Segal-Slomka 2011, Slomka 2011: objects are closed and convex subsets of $\mathbb{R}^n$ containing the origin, Artstein-Avidan and Slomka 2012: objects are $n$-dimensional centrally symmetric ellipsoids.

In all of these works $n \in \mathbb{N}$ satisfies either $n \geq 2$ or $n \geq 3$.
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Considering (1): second motivation (Cont.)

A central property that was established there: these operators must have the form $T(C) = AC \circ$ for some invertible linear operator $A: \mathbb{R}^n \to \mathbb{R}^n$.

Equation (1) is directly related to these works because if we denote $T(C) := (GC) \circ$, then a simple verification shows that $T(C) = (G^*)^{-1}C \circ$. 
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Considering (1): third motivation

It has some similarities with the following fixed point equation:

\[ f(x) := \tau f^*(Ex + c) + \langle w, x \rangle + \beta, \quad x \in X \]

Here: \( X \neq \{0\} \) is a real Hilbert space, \( f: X \to [-\infty, \infty] \) is the unknown function, \( \tau > 0 \), \( c \in X \), \( w \in X \), and \( \beta \in \mathbb{R} \) are given, \( E: X \to X \) is a given continuous linear invertible operator, \( f^*(x^*) := \sup \{ \langle x^*, x \rangle - f(x) : x \in X \} \), \( x^* \in X \), is the Legendre-Fenchel transform (the convex conjugate) of \( f \).
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is the Legendre-Fenchel transform (the convex conjugate) of \( f \).
Equation (3) can be thought of as being a convex analytic version of (1). Both have some similarities in their structure, both have several similarities in the properties of the corresponding solution sets. For example, in both cases the solution sets are very sensitive to the various parameters which appear there. Equation (3) was investigated recently (joint work with Iusem and Reich, 2017). Some of the results mentioned in that joint work are useful for deriving some of the results of our work.
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Some of the results mentioned in that joint work are useful for deriving some of the results of our work.
Contributions: The classification theorem

Let $X = \mathbb{R}$ be a real Hilbert space, $G : X \to X$ a continuous and invertible linear operator, and consider equation (1) with an unknown $\emptyset \neq C \subseteq X$:

$$C = (GC) \circ,$$

The following statements hold:

(i) Any solution to (1) must be closed and convex, and must contain 0.

(ii) If $G$ is positive definite, then there exists a unique solution to (1): the ellipsoid of the form $C = \{ x \in X : \langle Gx, x \rangle \leq 1 \}$.

(iii) If $G$ is not positive definite, then there are cases where (1) has several (possibly infinitely many) solutions.

There are cases where (1) does not have any solution which belongs to the class of bounded subsets of $X$ which contain 0 in their interiors.
Theorem

- $X \neq \{0\}$ is a real Hilbert space,
Contributions: The classification theorem

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1. there are cases where (1) has several (possibly infinitely many) solutions
2. there are cases where (1) does not have any solution which belongs to the class of bounded subsets of $X$ which contain 0 in their interiors.
Our analysis is essentially dimension-free. We obtain a few by-products of possible independent interest: results related to coercive bilinear forms and hence to PDE (to the Lax-Milgram theorem), results related to infinite-dimensional convex geometry, and we introduce the class of semi-skew operators.
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We obtain a few by-products of possible independent interest: 

- results related to coercive bilinear forms and hence to PDE (to the LaX-Milgram theorem) 
- results related to infinite-dimensional convex geometry 

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We obtain a few by-products **of possible independent interest:**

- results related to **coercive bilinear forms** and hence to PDE (to the LaX-Milgram theorem)
- results related to **infinite-dimensional convex geometry**
- we introduce the class of **semi-skew operators**.
Example

Now $G : X \to X$ is linear, continuous and invertible but not necessarily positive definite. Assume that there exists an operator $A : X \to X$ positive definite and invertible which satisfies $A = GA^{-1}G^*$. Then the ellipsoid $C := \{x \in X : \langle Ax, x \rangle \leq 1\}$ solves (1).

In particular, if $G$ is an arbitrary unitary operator, then we can take $A := I$ (I is the identity operator) and hence $C := \{x \in X : \|x\| \leq 1\}$ solves (1).
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The existence-non-necessarily-uniqueness part

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- In particular, if $G$ is an **arbitrary unitary operator**, then we can take $A := I$ ($I$ is the identity operator) and hence $C := \{x \in X : \|x\| \leq 1\}$ solves (1).
The non-uniqueness part

Let $X := \mathbb{R}$, $G(x) := -x$, $x \in X$, $\lambda > 0$ be arbitrary. Then $C_\lambda := [-1/\lambda, \lambda]$ solves (1), namely $C_\lambda = (C_\lambda) \circ$. In addition, $C_- := (-\infty, 0]$ and $C_+ := [0, \infty)$ solve (1).
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The non-uniqueness part

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In addition, $C_- := (-\infty, 0]$ and $C_+ := [0, \infty)$ solve (1).
Tools needed in the proof
Tool 1: The Minkowski functional

Let $\emptyset \neq C \subseteq X$, the Minkowski functional: the function $M_C : X \to [0, \infty]$ defined by $M_C(x) := \inf \{ \mu \geq 0 : x \in \mu C \}$, $x \in X$.

Notation: $\mu C := \{ \mu c : c \in C \}$, $\inf \emptyset := \infty$.

$K$ bound $(0)$ $(X)$ is the set of all bounded, convex and closed subsets of $X$ having $0$ in their interior.
Tool 1: The Minkowski functional

- Let $\emptyset \neq C \subseteq X$, 

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- **The Minkowski functional**: the function \( M_C : X \rightarrow [0, \infty] \) defined by

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Properties of $M_C$ for $C \in \mathcal{K}_{\text{bound},(0)}(X)$

Almost a norm (but not necessarily symmetric), Actually, "Equivalent" to the norm:

Let $r_C > 0$ be the radius of any open ball which is contained in $C$ and containing the origin, $\|C\| := \sup\{\|c\| : c \in C\} < \infty$.

Then:

$\|x\|\|C\| \leq M_C(x) \leq \|x\|r_C, \forall x \in X$.

In particular:

$0 = M_C(0) < M_C(x) < \infty$ for $x \neq 0$.

Given $C_1, C_2 \in \mathcal{K}_{\text{bound},(0)}(X)$, if $M_{C_1} = M_{C_2}$, then $C_1 = C_2$.
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Tool 2: the polar of $M_C$ for $C \in \mathcal{K}_{\text{bound},(0)}(X)$
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$$M_C^\circ(x^*) := \sup \left\{ \frac{\langle x^*, x \rangle}{M_C(x)} : 0 \neq x \in X \right\}$$

$$= \sup \{ \langle x^*, x \rangle : x \in X, M_C(x) = 1 \}, \quad x^* \in X,$$
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- $\mathcal{M}_C^\circ$ enjoys similar properties to $\mathcal{M}_C$. 
Tool 3: Duality between $M_C$ and $M_C^\circ$

$M_C$ and $M_C^\circ$ satisfy a dual inequality to $M_C$:

$$r_C \| x^* \| \leq M_C^\circ(x^*) \leq \| C \| \| x^* \| , \quad \forall x^* \in X,$$

If $C = -C$, then $M_C^\circ$ is the dual norm of $M_C$.

Reminder: the polar set:

$$C^\circ := \{ x^* \in X : \langle x^*, c \rangle \leq 1, \quad \forall c \in C \}.$$
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---

**Lemma**

*If $C \in \mathcal{K}_{\text{bound},(0)}(X)$, then $C^\circ \in \mathcal{K}_{\text{bound},(0)}(X)$ and*
Tool 3: Duality between $M_C$ and $M_C^\circ$

- $M_C^\circ$ satisfies a dual inequality to $M_C$:

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**Lemma**

If $C \in \mathcal{K}_{\text{bound},(0)}(X)$, then $C^\circ \in \mathcal{K}_{\text{bound},(0)}(X)$ and

$$M_C^\circ = M_C^C.$$
Tool 4: conjugacy and polarity

Lemma

Let $\phi : \mathbb{R} \rightarrow (-\infty, \infty]$. Assume that $\phi(t) = \infty$ for every $t \in (-\infty, 0)$. Then for each $C \in K$ bound $(0)(X)$ and each $x^* \in X(\phi \circ M C)^* (x^*) = \phi^*(M \circ C(x^*))$. If, in addition, $\phi$ is differentiable over $[0, \infty)$, $\phi'$ is strictly increasing on $[0, \infty)$ and maps it onto itself, and $\phi(0) = \phi'(0) = 0$, then $(\phi \circ M C)^* (x^*) = (\phi')^{-1}(M \circ C(x^*)) M \circ C(x^*) - \phi((\phi')^{-1}(M \circ C(x^*))).$ In particular, for $\phi(t) = \frac{1}{2} t^2$ for $t \in [0, \infty)$, we have $(\frac{1}{2} M^2 C) (x^*) = \frac{1}{2} (M \circ C(x^*))^2.$
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Let $\phi : \mathbb{R} \to (-\infty, \infty]$.
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Lemma

- Let \( \phi : \mathbb{R} \rightarrow (-\infty, \infty] \).
- Assume that \( \phi(t) = \infty \) for every \( t \in (-\infty, 0) \).
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(\phi \circ M_C)^*(x^*) = \phi^*(M_C^*(x^*)).
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Lemma

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  \[(\phi \circ M_C)^*(x^*) = \phi^*(M_C^*(x^*)).\]
- If, in addition, $\phi$ is differentiable over $[0, \infty)$, $\phi'$ is strictly increasing on $[0, \infty)$ and maps it onto itself,
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- Let $\phi : \mathbb{R} \to (-\infty, \infty]$.
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- If, in addition, $\phi$ is differentiable over $[0, \infty)$, $\phi'$ is strictly increasing on $[0, \infty)$ and maps it onto itself, and $\phi(0) = \phi'(0) = 0$, 
Tool 4: conjugacy and polarity

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  $$
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Then for each $C \in \mathcal{H}_{\text{bound},(0)}(X)$ and each $x^* \in X$

$$(\phi \circ M_C)^*(x^*) = \phi^*(M_C^o(x^*)).$$

If, in addition, $\phi$ is differentiable over $[0, \infty)$, $\phi'$ is strictly increasing on $[0, \infty)$ and maps it onto itself, and $\phi(0) = \phi'(0) = 0$, then

$$(\phi \circ M_C)^*(x^*) = (\phi')^{-1}(M_C^o(x^*)) M_C^o(x^*) - \phi((\phi')^{-1}(M_C^o(x^*))).$$
Tool 4: conjugacy and polarity

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- Let $\phi : \mathbb{R} \to (-\infty, \infty]$.
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  \[(\phi \circ M_C)^*(x^*) = \phi^*(M_C^o(x^*))\].
- If, in addition, $\phi$ is differentiable over $[0, \infty)$, $\phi'$ is strictly increasing on $[0, \infty)$ and maps it onto itself, and $\phi(0) = \phi'(0) = 0$, then
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- In particular, for $\phi(t) = \frac{1}{2} t^2$ for $t \in [0, \infty)$, we have
Lemma

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- Then for each $C \in \mathcal{K}_{\text{bound},(0)}(X)$ and each $x^* \in X$
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  (\phi \circ M_C)^*(x^*) = \phi^*((M_C^\circ)(x^*)).
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- In particular, for $\phi(t) = \frac{1}{2} t^2$ for $t \in [0, \infty)$, we have
  \[
  \left(\frac{1}{2} M_C^2\right)^*(x^*) = \frac{1}{2} (M_C^\circ(x^*))^2.
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Tool 5: Ellipsoids

Definition

A : X → X is called positive definite if A is linear, continuous, A∗ = A and ⟨Ax, x⟩ > 0 for each 0 ̸= x ∈ X.

Definition

Given a positive definite operator A : X → X, the centrally symmetric ellipsoid induced by A is D := {x ∈ X : ⟨Ax, x⟩ ≤ 1}.
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Given a positive definite operator A : X → X, the centrally symmetric ellipsoid induced by A is

\[ D := \{ x \in X : \langle Ax, x \rangle \leq 1 \}. \]
Ellipsoids: real world illustrations
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Figure: 2D example (source: Bill Frymire)
Ellipsoids: real world illustrations

Figure: 2D example (source: Bill Frymire)

Figure: 3D example (source: http://rugby1823.blogosfere.it)
Lemma

Let $A : X \to X$ be positive definite and invertible, $D := \{ x \in X : \langle Ax, x \rangle \leq 1 \}$ the ellipsoid induced by $A$.

Then the following statements hold:

(a) $D \in K(0)$. 

(b) $MD(x) = \sqrt{\langle Ax, x \rangle}$ for each $x \in X$. 

(c) $D^{\circ} = \{ x \in X : \langle A^{-1}x, x \rangle \leq 1 \}$. 

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Lemma

Let $A : X \to X$ be positive definite and invertible,

- $D := \{ x \in X : \langle Ax, x \rangle \leq 1 \}$ the ellipsoid induced by $A$.
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  - (a) $D \in K(0, X)$
  - (b) $MD(x) = \sqrt{\langle Ax, x \rangle}$ for each $x \in X$
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Lemma

Let \( A : X \to X \) be positive definite and invertible,

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Then the following statements hold:

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(b) $M_D(x) = \sqrt{\langle Ax, x \rangle}$ for each $x \in X$

(c) $D^\circ = \{ x \in X : \langle A^{-1}x, x \rangle \leq 1 \}$
Lemma

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Tool 6: coercive/elliptic bilinear forms

$B: X^2 \to \mathbb{R}$ is a bilinear form when both $b_1(x) := B(x, y)$ is linear for all $y \in X$, and $b_2(y) := B(x, y)$ is linear for all $x \in X$.

For each continuous bilinear form $B: X^2 \to \mathbb{R}$ there exists a unique continuous linear operator $A: X \to X$ such that $B(x, y) = \langle Ax, y \rangle$ for all $x, y \in X$. 
Tool 6: coercive/elliptic bilinear forms

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A bilinear form $B: X^2 \to \mathbb{R}$ is called coercive (or strongly coercive, or strongly monotone, or elliptic) if there exists $\beta > 0$ such that $B(x, x) \geq \beta \|x\|^2$, $\forall x \in X$.

If $B$ is continuous and symmetric ($B(x, y) = B(y, x)$ for all $x, y \in X$), then $A$ is symmetric; if, in addition, $B$ is coercive, then $A$ is positive definite.

Coercive bilinear forms have various applications in calculus of variations and elliptic partial differential equations, among them Stampacchia's theorem, the Lax-Milgram theorem.
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Coercive bilinear (Cont.)

- A bilinear form \( B : X^2 \to \mathbb{R} \) is called **coercive** (or **strongly coercive**, or **strongly monotone**, or **elliptic**) if there exists \( \beta > 0 \) such that
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Let $D = \{ x \in X : \langle Ax, x \rangle \leq 1 \}$ be the ellipsoid induced by $A$.

The claim $D \in \mathcal{K}(0)(X)$ seems immediate at first glance. However, this is not the case, at least not regarding the boundedness of $D$.

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The proof of the claim \( D \in K_{\text{bound}}(0) \) is based on the following apparently new lemma:

**Lemma**

The bilinear form induced by \( A \) is coercive. In fact,

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\langle Ax, x \rangle \geq \|A - 1\| - 1 \|x\|_2, \quad \forall x \in X.
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The lemma's proof is quite indirect. When \( \dim(X) < \infty \), then a straightforward proof can be given (using eigenvalues and spectral decomposition).
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The lemma is essentially a quantitative converse of the LaX-Milgram theorem: LM state that if \( B(x, y) = \langle Ax, y \rangle \) is continuous and coercive (hence \( \langle Ax, x \rangle > 0 \) when \( x \neq 0 \)), then \( A \) is invertible.

The lemma states that if \( A \) is positive definite and invertible, then \( B \) is coercive with coefficient \( \| A^{-1} \|^{-1} \).

The lemma also gives the optimal (maximal) coercivity coefficient in LM theorem for symmetric \( B \): Any such coefficient must be at most \( \| A^{-1} \|^{-1} \) (simple check), but LM implies that \( A \) is invertible, hence the lemma implies that the coefficient can be \( \| A^{-1} \|^{-1} \).
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Definition

Suppose that \( \dim(X) \geq 2 \). We say that \( E : X \to X \) is a semi-skew operator with respect to the triplet \((u, \alpha_1, \alpha_2)\) (or, briefly, that \( E \) is semi-skew) if the following conditions hold:

1. \( u \in X \) is a unit vector;
2. \( \alpha_1 \) and \( \alpha_2 \) are two real numbers having the same sign (either both are positive or both are negative) and \( \alpha_1 \neq \alpha_2 \);
3. for each \( x \in X \), consider the unique decomposition \( x = x_1 + x_2 \), where \( x_1 \in \mathbb{R}u \) and \( x_2 \in u^\perp \) and identify \( x \) with \( (x_1, x_2) \in \mathbb{R}u \times u^\perp \sim X \) and \( (x_2, x_1) \in u^\perp \times \mathbb{R}u \sim X \); then
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Lemma

Suppose that \( \dim(X) \geq 2 \) and that \( E : X \to X \) is a semi-skew operator. Then:

- \( E \) is linear, continuous and invertible.
- There does not exist any upper semicontinuous \( f : X \to (-\infty, \infty] \) which satisfies \( f(0) \in \mathbb{R} \) and solves the equation \( f(x) = f^*(Ex), x \in X \).

Proposition

Suppose that \( \dim(X) \geq 2 \) and that \( G : X \to X \) is a semi-skew operator. Then (1) does not have any solution \( C \subseteq X \) which is bounded and contains 0 in its interior.
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P.S. The slides are planned to be uploaded online in the not so distant future (within 3 months) to

http://w3.impa.br/~dream/talks