The Geometric Stability of Voronoi Diagrams with Respect to Small Changes of the Sites

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Voronoi diagrams: a short reminder

A decomposition of a given space $X$ into cells induced by given sites and a given distance function $d$. To each site $P_k$ one associates the subset (cell) $R_k$ defined as follows:

$$R_k = \{ x \in X : d(x, P_k) \leq d(x, P_j) \forall j \neq k \}.$$
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- VD appear in a huge number of places in science and technology and have diverse applications.

- They have been the subject of extensive investigation during the last 40 years.
A question

Consider the following question:

Does a small change of the sites, e.g., of their position or shape, yield a small change in the corresponding Voronoi cells?
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A question (cont.)

A natural and fundamental question because in practice, no matter which algorithm is being used for computing the cells, one approximates the sites for various reasons: lack of exact information about them, inevitable numerical errors in their representation, for simplification purposes, and so on: imprecision is inherent.

But are we sure that the resulting Voronoi cells approximate well the real (ideal) ones?
A question (cont.)

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Example
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Figure: 10 ideal shopping centers or post offices a flat city.
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Figure: A more realistic situation.
Does this example describe a general phenomenon?

More precisely, does a small change in the position or the shape of the sites yield a small change in the shapes of the Voronoi cells?
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Does a small change in the position or the shape of the sites yield a small change in the shapes of the Voronoi cells?
What is known about this phenomenon?

Surprisingly, almost nothing!

The discussion in the literature is:
- very brief
- intuitive and without proofs
- limited to very few places
- restricted (low) dimensional Euclidean spaces, mainly point sites

However, there is some discussion in a few places on the combinatorial stability of VD for moving point sites in finite dimensional Euclidean spaces:
- not stable in general, but is stable most of the time.
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The setting of the main result

- A closed and convex subset $X$ of a uniformly convex normed space (examples of such spaces: Euclidean spaces, $\ell^p$ spaces, $L^p(\Omega)$ spaces, $1 < p < \infty$);
- There is a positive lower bound on the distance between the sites;
- A certain boundedness condition holds, e.g., when $X$ is bounded or when the sites form a (distorted) lattice;
- The distance to each site is attained;
- The changes are measured w.r.t. the Hausdorff distance $D$.

Notation:
- $P'_k$ is the perturbed site corresponding to $P_k$;
- $R'_k$ is the perturbed cell corresponding to $R_k$.
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The main result

Under this setting, for any $\epsilon > 0$ there exists $\Delta > 0$ such that for each $k \in K$, if $D(P_k, P'_k) < \Delta$, then $D(R_k, R'_k) < \epsilon$.

Moreover, explicit bounds on the changes are given. There are counterexamples which show that the assumptions imposed above are crucial.
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Clarifications

The Hausdorff distance indeed measures changes between shapes (see the paper for mathematical+physical motivations).

Infinitely many sites of a general form are allowed. The space can be infinite dimensional: no curse of dimensionality (pure geometric result).

In general, $\Delta = O(\epsilon^2)$; if the sites are strictly inside $X$, then $\Delta = O(\epsilon)$.

The constants inside $O$ are explicit and independent of the dimension.
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Illustration
Figure: 5 sites in a square in $(\mathbb{R}^2, \ell_p)$, $p = 3.14159$. 
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Figure: The sites have been slightly perturbed; the cells have been slightly perturbed.
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Real-world and theoretical Examples
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- **Robot motion, collision detection**: sites = static/dynamic obstacles

Solid state physics:
sites = atoms in an infinite and unbounded lattice (crystal)

Signal processing:
sites = (distorted) signals;

  - **Discrete signal**: in a (very) high dimensional space.
  - **Continuous signal**: in the infinite dimensional space $L^2$. 
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Numerical simulations: sites = parts in a (continuously changing) simulated phenomenon
Molecular biology: sites (points/spheres) = atoms/amino acids
Computational geometry: existence of zone diagrams
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And so on.
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Counterexamples

Figure: Four sites in a square in $(\mathbb{R}^2, \ell_\infty)$. The cell of $P_1 = \{(0,0)\}$ is displayed.

Figure: Now either $P_1 = \{(\beta, \beta)\}$, $\beta > 0$ arbitrary small, or $P_4$ (the lower site) is a small square.
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The full diagram of Figure 5.

Each cell is represented as a union of rays. Note the large intersection between cells 1, 2, and 3.

The full diagram of Figure 6 after $P_1$ has moved. Cells 2, 3 have been (significantly) changed too.
Counterexamples (Cont.)

Figure: The full diagram of Figure 5. Each cell is represented as a union of rays. Note the large intersection between cells 1, 2, and 3.
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Long and technical (unfortunately)

Based on a non-standard approach

Some ingredients in the proof:

- A new representation theorem for the cells
- New geometric estimates
- The forgotten strong triangle inequality of James Clarkson
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Other approaches for proving the result?

The main difficulty in using familiar approaches is the generality of the result. Problems may occur with:

- Euclidean arguments (norm=more general);
- compactness arguments (when dim=infinite);
- sites of a general+complicated form;
- lower envelopes (complicated cells with non-algebraic boundaries);
- possible infinite accumulated error due to $\infty$ sites or $\infty$ dimension;

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The geometric stability of Voronoi diagrams

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Other approaches? (Cont.)

And if we restrict ourselves to a more concrete setting?

Open Problem

To use a different approach for proving the result and obtaining the explicit bounds ($\Delta = O(\epsilon)$, etc.) in specific but important settings: $\dim = 2$ and $3$ distance = Euclidean Sites = points, discs, balls

Such a proof may illuminate and simplify the current one in these settings and may also improve the bounds.
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