A few exercises more

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Please try solving these exercises without looking their solutions up. The ones marked with one skull (☠) are harder. The ones marked with a skull and a question mark (☠?) may or may not be hard (Let me know.)

1. ☠ Is there a "color translation function" m that allows us to convert between the results of alpha blending in gamma and linear spaces? I.e., is there m such that, for all linear f, α_f and b, α_b, we have

$$\gamma\big(m(f, \alpha_f) \oplus m(b, \alpha_b)\big) = \gamma(f, \alpha_f) \oplus \gamma(b, \alpha_b) \tag{1}$$

   Why would such a function be useful?

2. Find the formula for the curvature $\kappa(0)$ at the first endpoint of a *rational* Bézier curve segment with first control points p_0, p_1, p_2.

3. Show that inflections are invariant under projective transformations.

4. ☠? Given an integral quadratic Bézier curve with control points p_0, p_1, p_2, is there an implicit formula $f(p) = 0$ for the curve that is guaranteed to vanish at p_0 and p_2?

5. Show that a monotonic curve segment with no inflections cannot cross the line that connects its endpoints.

6. Show that the intersection of the tangents at the endpoints of a monotonic curve segment with no inflections happens inside the bounding box.

7. You are now familiar with the representation of Bézier curve segments as sums in the form

$$\gamma(t) = \sum_{i=0}^{n} p_i \, b_i^n(t),$$

   where p_i are the control points and $b_i^n(t)$ are the Bernstein basis polynomials. We have seen that this representation is affine invariant. I.e., to apply an affine transformation to the curve, we can simply apply the same transformation to the control points. Now let's consider other polynomials bases.

   (a) Prove that a curve given by control points in a polynomial basis $c_0^n(t), \ldots, c_n^n(t)$ is affine invariant if and only if the basis forms a partition of unity. I.e., if and only if

$$\sum_{i=0}^{n} c_i^n(t) = 1, \quad \text{for all } t.$$

(b) The centered B-spline basis function $\beta^n(t)$ can be defined recursively as

$$\beta^0(t) = \begin{cases} 1, & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta^n(t) = \int_{-\infty}^{\infty} \beta^{n-1}(u)\, \beta^0(t-u)\, du, \quad n > 0.$$

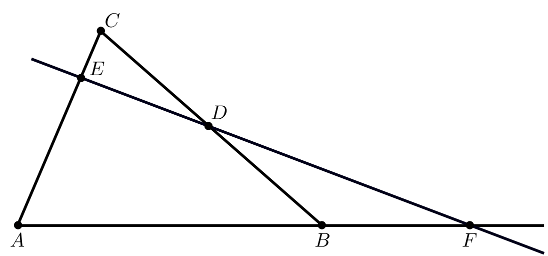
Given control points $\ldots, p_{-1}, p_0, p_1, \ldots$, a B-spline curve is then given by the sum

$$\gamma(t) = \sum_{i=-\infty}^{\infty} p_i\, \beta^n(t-i).$$

Prove that the representation is also affine invariant.

8. Prove Menelaus' theorem (which was, in fact, known before Menelaus). I.e., prove that, given a triangle ABC and a transversal line that crosses BC, AC, and AB at points D, E, and F, respectively, with D, E, and F distinct from A, B, and C, then the ratios of signed distances between these points satisfy

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = -1.$$



Tip: Recall that affine transformations can take any 3 non-collinear points into any other 3 non-collinear points. They also preserve collinearity, intersections, and ratios of distances along lines. Be smart when picking your triangle and the expression for the transversal line.