Maximal entropy measures for Viana maps

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Abstract

In this note we construct measures of maximal entropy for a certain class of maps with critical points called Viana maps. The main ingredients of the proof are the non-uniform expansion features and the slow recurrence (to the critical set) of generic points with respect to the natural candidates for attaining the topological entropy.

1 Introduction

Nowadays, it is well-known that a dynamical system can be understood from the study of its invariant measures (this study is called ergodic theory). However, there are in general many invariant measures for a given system, and it is necessary to make a selection of some “interesting” measures. Motivated by statistical physics, some candidates called equilibrium states, are the measures which satisfy a variational principle.

The theory of equilibrium states in the case of uniformly hyperbolic systems (developed by Bowen, Ruelle, Sinai among others) is now a classical theory with completely satisfactory results. But, in the non-uniform (higher dimensional) context only few results are known (see, for instance, Oliveira [O] for a recent progress in this direction; for a random version of it, see Arbieto, Matheus, Oliveira [AMO]).

The central theme of this note is to prove the existence of maximal entropy measures for a class of non-uniformly expanding maps with critical set introduced by Viana [V]. Some features of these maps (which we would like to call Viana maps) include the existence of a unique absolutely continuous invariant measure $\mu_0$ with non-uniform expansion, slow recurrence to
the critical set and positive Lyapounov exponents (see [V]). Moreover, $\mu_0$ is stochastically stable$^1$.

On the other hand, an obstruction to construct general equilibrium states for Viana maps (among other non-uniformly expanding maps) is the \textit{a priori} dependence on the Lebesgue measure as reference in the concepts of non-uniform expansion and slow recurrence to the critical set. To overcome this difficulty we apply the basic strategy of Oliveira [O]:

- We select a certain set $K$ of invariant measures which are natural candidates to realize the variational principle (for nearly constant potentials);
- We prove that any measure outside $K$ can not be an equilibrium measure and any measure inside $K$ is \textit{expanding} (see definition 3.11 below);
- We show that any expanding measure admits generating finite partitions. This leads us to the semicontinuity of the entropy$^2$, and hence, by a standard argument, to the existence of equilibrium states for nearly constant potentials.

However, it is worth pointing out an extra difficulty of the case of Viana maps: the presence of critical points is an obstacle to obtaining infinitely many times of uniform expansion directly from the non-uniform expansion condition (indeed, some slow recurrence to the critical set should be proven)$^3$. Since the abundance of hyperbolic times is fundamental to prove that expanding measures admit generating partitions (see lemma 3.14), we need to solve this problem.

$^1$Here, the stochastic stability is restricted to the random perturbations with the same critical set $\mathcal{C}$ and whose derivative at any point $x \notin \mathcal{C}$ (i.e., in the complement of the critical set) is equal to the derivative of the unperturbed system at the same point $x$ (see Alves, Araújo [AA]).

$^2$The idea of semicontinuity of the entropy on a “good” subset of measures goes back to the fundamental work of Newhouse [N], where a quantity measuring the lack of semicontinuity is introduced and it is showed that this quantity is related to the growth of volumes of smooth submanifolds. As a consequence, Newhouse obtain that all smooth ($C^\infty$) self-maps of a compact manifold have a maximal entropy measure. On the other hand, this general result is false for some $C^r$ maps, $1 \leq r < \infty$ and so it is necessary to rework the details of this idea in order to get equilibrium states for Viana maps.

$^3$In the context of Oliveira [O], this problem does not occur since the transformations are local diffeomorphisms, and so infinitely many hyperbolic times can be obtained directly from non-uniform expansion.
Fortunately, the slow recurrence to the critical set can be obtained from the integrability of the distance function to the critical set $C$. Because the Viana maps behave like a power of the distance to $C$, it is reasonable that this integrability problem is related to the regularity of the Lyapounov exponents (i.e., the Lyapounov exponents are bounded away from $-\infty$). Now, the regularity of the Lyapounov exponents of measures with a non-trivial chance to attain the supremum of the variational principle (i.e., whose entropy is not $-\infty$) is an easy consequence of Ruelle’s inequality. Thus, the presence of critical points is not a great trouble and we still can use the basic strategy of Oliveira [O].

Now we are going to state our main result. In order to do this, let us recall some definitions and notations.

In general, given a continuous map $f : M \to M$ of a compact metric space $M$ and a continuous function $\phi : M \to \mathbb{R}$, we call an $f$-invariant Borel probability measure $\mu$ an equilibrium state of $(f, \phi)$ if $\mu$ realizes the variational principle

$$h_\mu(f) + \int \phi d\mu = \sup_{\eta \in \mathcal{I}} \left( h_\eta(f) + \int \phi d\eta \right),$$

where $\mathcal{I}$ is the set of $f$-invariant Borel probabilities.

Now we are in position to state our main result:

**Theorem A.** Viana maps admit equilibrium states for any nearly constant potential $\phi$ (in the sense of the condition (3) below). In particular, Viana maps have maximal entropy measure (since they are equilibrium states for the constant potential $\phi = 0$). Moreover, any equilibrium state of the Viana maps (associated to nearly constant potentials) are hyperbolic measures with all Lyapounov exponents greater than some $\lambda_0 > 0$.

To close the introduction, let us comment about the organization of the paper: section 2 contains the definition of Viana maps, some of their properties and a precise statement of what does “$\phi$ is nearly constant” means. In section 3 we present the proof of the theorem A. Finally, in section 4, we point out some generalizations and questions concerning the theorem A.

## 2 Viana maps

Let $a_0 \in (1, 2)$ such that $x = 0$ is pre-periodic for the quadratic map $h(x) = a_0 - x^2$. Denote by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and let $b : \mathbb{S}^1 \to \mathbb{R}$ be a Morse function, e.g.,
\[ b(\theta) = \sin(2\pi \theta). \] Fix some \( \alpha > 0 \) sufficiently small and put

\[
\tilde{f}(\theta, x) := (g(\theta), a(\theta) - x^2) := (g(\theta), Q(\theta, x)),
\]

where \( g \) is the expanding map \( g(\theta) = d\theta \) of \( S^1 \), for some integer \( d \geq 16 \) and \( a(\theta) = a_0 + \alpha b(\theta) \). \(^4\) Since \( a_0 < 2 \), it is easy to check that for \( \alpha \) sufficiently small there is \( I \subset (-2, 2) \) such that the closure of \( \tilde{f}(S^1 \times I) \) is contained in the interior of \( S^1 \times I \). Indeed, \( h(x) = a_0 - x^2 \) has a unique fixed point \( x_0 = \frac{1}{2}(-1 - \sqrt{1 + 4a_0}) < 0 \) which is a repeller. In particular, if we take \( \beta > 0 \) slightly smaller than \( -x_0 \), the interval \( I = [-\beta, \beta] \) satisfies \( h(I) \subset \text{int}(I) \) and \( |h'| > 1 \) on \( \mathbb{R} \setminus \text{int}(I) \). Hence, if \( \alpha \) is sufficiently small we still have \( \tilde{f}(S^1 \times I) \subset \text{int}(S^1 \times I) \). Notice also that, if \( f \) is \( C^0 \)-close to \( \tilde{f} \) then \( f \) also has \( S^1 \times I \) as a forward invariant region, i.e., any \( f \) close to \( \tilde{f} \) may be considered as a map \( f : S^1 \times I \to S^1 \times I \).

In what follows, we consider the parameters \( d \geq 16 \) and \( a_0 \in (1, 2) \) fixed and \( \alpha \) is sufficiently small depending on \( d \) and \( a_0 \). Under these conditions, Viana proved that

**Theorem 2.1** (Viana [V]). For \( \alpha \) is sufficiently small, there exists a positive constant \( c_0 > 0 \) such that any map \( f \) sufficiently close to \( \tilde{f} \) in the \( C^3 \) topology has both Lyapounov exponents greater than \( c_0 \) at Lebesgue almost every point, that is,

\[
\liminf_{n \to \infty} \frac{1}{n} \log \| Df^n(\theta, x) \| > c_0, \quad \text{for Lebesgue a.e. } (\theta, x) \in S^1 \times I.
\]

The central problem overcome by [V] in order to get two positive Lyapounov exponents is to control the recurrence of generic orbits to the critical set \( C \) of the maps \( f \) \( C^3 \)-close to \( \tilde{f} \). In fact, if one defines the critical set \( C \) to be the set of points \( (\theta, x) \) such that \( \det Df(\theta, x) = 0 \), then it is not hard to show that any orbit spending a lot of time near \( C \) can not fit the conclusion of theorem 2.1. Here, the slow recurrence of generic orbits to the critical set depends on a large deviations argument, and a non-trivial role is played by the geometry of the critical set (which is the graph of a \( C^2 \) function \( \eta : S^1 \to I \) arbitrarily \( C^2 \)-close to zero). For further details, see [V].

\(^4\)The assumption \( d \geq 16 \) doesn’t play crucial role in our results and it is present here only for sake of simplicity. Indeed, the results of Buzzi, Sester and Tsujii [BST] can be applied to replace \( d \geq 16 \) by \( d \geq 2 \), at least when we consider the \( C^\infty \)-topology. See the section 4 for details.
A very common fact in smooth ergodic theory is: “the positivity of Lyapounov exponents acts as a strong evidence of the existence of absolutely continuous invariant measures”. Indeed, this general principle works in several examples of maps with positive Lyapounov exponents, including Viana maps:

**Theorem 2.2** (Alves [A]). Any map $f$ sufficiently $C^3$ close to $\tilde{f}$ admits an unique absolutely continuous invariant measure $\mu_0$.

Next, let us study the tangent bundle dynamics of the Viana maps. Denote by $f_0$ the product map $f_0 = g \times h$, i.e., $f_0 : S^1 \times I \to S^1 \times I$, $f_0(\theta, x) = (d\theta, a_0 - x^2)$. Since $Df_0 \cdot \frac{\partial}{\partial \theta} = d \cdot \frac{\partial}{\partial \theta}$, $Df_0 \cdot \frac{\partial}{\partial x} = -2x \frac{\partial}{\partial x}$, $d \geq 16 > 4$ and $I \subset (-2, 2)$, it is not hard to see that the splitting $T(\theta, x) = E_u \oplus E_c$ is a dominated splitting of the tangent bundle of $S^1 \times I$ into two invariant (one-dimensional) subbundles where $E_u$ is uniformly expanding and $E_c$ is dominated by $E_u$. In other words, $f_0$ is a partially hyperbolic map of type $E_u \oplus E_c$. From the theory of partial hyperbolicity we know that, if $\alpha$ is sufficiently small, $\tilde{f}$ (as any $C^1$ nearby map) also admits a partially hyperbolic splitting of type $E_u \oplus E_c$ which varies continuously. As a consequence, we get

**Proposition 2.3.** Given $\epsilon > 0$, the Lyapounov exponent

$$\lambda^u(p, f) := \lim_{n \to \pm \infty} \frac{1}{n} \log \| Df^n(p)_{|E^u} \|$$

associated to $E^u$ at every point $p$ verifies

$$\log(d - \epsilon) \leq \lambda^u(p, f) \leq \log(d + \epsilon),$$

for any $f$ $C^1$-close to $\tilde{f}$ (if $\alpha$ is sufficiently small).

*Proof.* Since $E^u$ is a continuous function of $f$ and the derivative of $f_0$ expands $\frac{\partial}{\partial \theta}$ by the constant factor $d$, it is clear that the proposition holds for any $f$ sufficiently $C^1$-close to $f_0$, which is certainly true in the context of Viana maps, provided $\alpha$ is small. \qed

Before ending this section, we apply the previous proposition to obtain a lower bound on the entropy of the SRB measure $\mu_0$. Using Pesin’s formula we know that the entropy $h_{\mu_0}(f)$ of $\mu_0$ is the integrated sum of the positive
Lyapounov exponents of $\mu_0$. Hence, if we use theorem 2.1 and proposition 2.3, it follows that

$$h_{\mu_0}(f) \geq \log(d - \epsilon) + c_0.$$  \hfill (2)

Once this notation is established, we are able to state precisely our condition on the potential $\phi$.

Definition 2.4. We say that the potential $\phi$ is nearly constant if

$$\max \phi - \min \phi < \frac{1}{2}(c_0 - \log \frac{d + \epsilon}{d - \epsilon}).$$  \hfill (3)

Note that the right-hand side of (3) is positive for $\epsilon > 0$ sufficiently small depending on $d$ and $c_0$, the lower bound on the Lyapounov exponents of the Viana maps.

After these preparation, we are ready to prove our main result.

3 Proof of theorem A

From now on, we consider Viana maps $f$, which are, by definition, all $C^3$ close maps to $\tilde{f}$ for any small $\alpha$.

Definition 3.1. Let $K$ be the set of ergodic measures $\mu$ whose central Lyapounov exponent $\lambda_c(\mu)$ is greater than $\frac{1}{4}(c_0 - \log \frac{d + \epsilon}{d - \epsilon})$. We define $K$ to be the set of invariant measures $\mu$ whose ergodic decomposition $(\mu_p)$ belongs to $K$ for $\mu$-almost every $p$. The sets $K$ and $\mathcal{K}$ are not empty since $\mu_0$ belongs to both of them. See theorem 2.1.

It is interesting to consider $K$ since it contains any measure liable of satisfying the variational principle:

Lemma 3.2. There exists a constant $\kappa_0 > 0$ such that every measure $\eta \notin K$ satisfies

$$h_\eta(f) + \int \phi d\eta + \kappa_0 < \sup_\mu \left(h_\mu(f) + \int \phi d\mu\right).$$  \hfill (4)

Proof. The idea is to compare $h_\eta(f)$ with $h_{\mu_0}(f)$, where $\mu_0$ is the unique SRB measure of $f$ (see theorem 2.2). Without loss of generality, we can suppose that $\eta$ is ergodic. In this case, $\eta \notin K$ means $\lambda_c(\eta) \leq \frac{1}{4}(c_0 - \log \frac{d + \epsilon}{d - \epsilon})$. Ruelle’s inequality combined with the estimate (1) of proposition 2.3 gives

$$h_\eta(f) \leq \log(d + \epsilon) + \frac{1}{4}\left(c_0 - \log \frac{d + \epsilon}{d - \epsilon}\right).$$

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Hence, the condition on the potential (3) and estimate (2) imply
\[ h_\eta(f) + \int \phi \, d\eta < h_{\mu_0}(f) + \int \phi \, d\mu_0 - \frac{1}{4} \log(d + \epsilon). \]
Taking \( \kappa_0 = \frac{1}{4} \log d < \frac{1}{4} \log(d + \epsilon) \), for instance, concludes the proof.

Next, we fix a sequence \( \mu_k \) of invariant ergodic measures such that \( h_{\mu_k}(f) + \int \phi \, d\mu_k \to \sup\eta (h_\eta(f) + \int \phi \, d\eta) \) when \( k \to \infty \). Taking a subsequence if necessary, we also suppose that \( \mu_k \to \mu_\infty \) (in the weak * topology) as \( k \to \infty \). By the lemma 3.2 we may assume that \( \mu_k \in K \). We are going to prove that \( \mu_\infty \) is an equilibrium measure and every equilibrium measure belongs to \( K \). Clearly, these claims finish the proof of our main result.

The heart of the strategy to show that \( \mu_\infty \) attains the variational principle is to construct \( \mathcal{P} \) a fixed generating partition with respect to all measures \( (\mu_k) \) in a subsequence \( \mu_k \) such that \( \mu_\infty(\partial \mathcal{P}) = 0 \). Indeed, if we are able to exhibit such a partition \( \mathcal{P} \), it is a standard matter to use Kolmogorov-Sinai’s theorem to conclude that \( \mu_\infty \) is an equilibrium state. Nevertheless, the main technical issue related to the existence of generating partitions is the presence of infinitely many hyperbolic times, i.e., the abundance of certain times where the dynamics of the Viana maps exhibits robust expanding features.

Before proceeding further, we recall the concept of hyperbolic time for maps with critical points, which is the fundamental concept behind the expanding properties of the measures \( (\mu_k) \). The setting of this definition is as follows.

Consider \( f : M \to M \) a \( C^2 \) map which is local diffeomorphism except at a zero Lebesgue measure set \( C \subset M \). Assume \( f \) behaves like a power of the distance to the critical set \( C \), i.e., there are constants \( B > 1 \) and \( \ell > 0 \) such that, for every \( p, q \notin C \) with \( 2 \operatorname{dist}(p, q) < \operatorname{dist}(p, C) \) and \( v \in T_p M \) we have:

- \( \frac{1}{B} \operatorname{dist}(p, C)\ell \leq \frac{\|Df(p)v\|}{\|v\|} \leq B \operatorname{dist}(p, C)\ell; \)
- \( |\log \|Df(p)^{-1}\| - \log \|Df(q)^{-1}\|| \leq B \frac{\operatorname{dist}(p, q)}{\operatorname{dist}(p, C)\ell}; \)
- \( |\log |\det Df(p)^{-1}| - \log |\det Df(q)^{-1}|| \leq B \frac{\operatorname{dist}(p, q)}{\operatorname{dist}(p, C)\ell}; \)

Remark 3.3. As the reader can easily verify, Viana maps satisfy all the previous assumptions, since \( \tilde{f} \) (and any \( C^2 \)-nearby map \( f \)) satisfies them.

For the definition of hyperbolic times, we fix \( 0 < b < \min\{1/2, 1/(2\ell)\} \).

\[ \frac{1}{4} \log d < \frac{1}{4} \log(d + \epsilon) \]
Definition 3.4. Given $0 < \sigma < 1$ and $\delta > 0$, we say that $n$ is a $(\sigma, \delta)$-hyperbolic time for $p$ if, for all $1 \leq k \leq n$,

$$\prod_{j=n-k}^{n-1} \|Df^j(p)^{-1}\| \leq \sigma^k \quad \text{and} \quad \text{dist}_\delta(f^{n-k}(p), C) \geq \sigma^{bk}.$$ 

The usefulness of hyperbolic times is the key property:

**Proposition 3.5.** Given $\sigma < 1$ and $\delta > 0$, there exists $\delta_1 > 0$ such that if $n$ is a $(\sigma, \delta)$-hyperbolic time of $p$, then there exists a neighbourhood $V_p$ of $p$ such that:

- $f^n$ maps $V_p$ diffeomorphically onto the ball $B_{\delta_1}(f^n(p))$;
- For every $1 \leq k \leq n$ and $y, z \in V_p$,
  $$\text{dist}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \text{dist}(f^n(y), f^n(z)).$$

**Proof.** See Alves, Bonatti and Viana [ABV, p.377].

**Notation.** In the sequel, we write $\sigma_0 = \exp(-\frac{1}{12} \zeta)$, where $\zeta := c_0 - \log \frac{\text{dist}}{\delta} > 0$.

The relevant expanding feature of the measures $\mu_k \in \mathcal{K}$ is contained in the following proposition, which gives a quantitative criterion for the existence of infinitely many hyperbolic times for $\mu_k$-generic points:

**Proposition 3.6.** There are $\nu > 0$, $\delta > 0$ (depending only on $\sigma_0 > 0$, $f$ and $\mu_\infty$) and a subsequence $(\mu_n) := (\mu_{k_j})$ of $(\mu_k)$ such that, for each $n \in \mathbb{N}$, given $p \in H(\sigma_0)$ a $\mu_n$-generic point and any sufficiently large $N \geq 1$, there exist $(\sigma_0, \delta)$-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq N$ for $p$ with $l \geq \nu N$.

In order to prove this proposition, we should keep some control of the recurrence (to the critical set $\mathcal{C}$) of generic points of the invariant ergodic measures $\mu_k$ (whose Lyapounov exponents are regular) in an uniform fashion.

The first step in this direction is to prove the following abstract lemma:

**Lemma 3.7.** Let $\eta$ be an invariant ergodic measure such that $\lambda^c(\eta) \geq -L > -\infty$, where $\lambda^c$ is the Lyapounov exponent of $\eta$ associated to $E^c$. Then,

$$\int |\log \text{dist}(p, C)|d\eta \leq L + \log 12 < \infty.$$
Proof. Since $\eta$ is ergodic, $\lambda^c(\eta) = \int |\log DF|_E d\eta > -\infty$. On the other hand, the definition of $f_0$ implies that, for $\alpha$ sufficiently small, $\frac{\alpha}{3} \|DF(p)\|_E \leq \text{dist}(p, C) \leq 3 \|DF(p)\|_E$. Also, the phase space of the Viana maps is $S^1 \times I$, where $I \subset (-2, 2)$, so that $\text{dist}(p, C) \leq 2$ for any $p \in S^1 \times I$. These three facts together finish the proof.

An easy consequence of this lemma is that the critical set has zero $\mu_\infty$-measure:

**Corollary 3.8.** It holds $\mu_\infty(C) = 0$.

**Proof.** Recall that $(\mu_k) \in K$ are ergodic invariant measures. Hence, $\lambda^c(\mu_k) \geq \zeta/4$. Fix $\delta > 0$ and denote by $V_\delta(C)$ the $\delta$-neighborhood of the critical set $C$. The lemma 3.7 applied to the sequence $(\mu_k)$ gives that

$$\int_{V_\delta(C)} |\log \text{dist} (p, C)| d\mu_k \leq \int |\log \text{dist} (p, C)| d\mu_k \leq -\zeta/4 + \log 12.$$  

In particular, if $\delta < 1$, we obtain

$$\log \left( \frac{1}{\delta} \right) \cdot \mu_k(V_\delta(C)) \leq \int_{V_\delta(C)} |\log \text{dist} (p, C)| d\mu_k \leq \log 12.$$  

This completes the proof, since by weak * convergence we have $\mu_\infty(V_\delta(C)) \leq \liminf_{k \to \infty} \mu_k(V_\delta(C))$, for all $\delta > 0$.

Once we know that the critical set $C$ has zero $\mu_\infty$-mass, it is possible to take more advantage of the weak * convergence $\mu_k \to \mu_\infty$ to get the desired control of the recurrence to the critical set via the following lemma:

**Lemma 3.9.** Given $\delta \in (0, 1)$ such that $\mu_\infty(\partial V_\delta(C)) = 0$, we have

$$\liminf_{k \to \infty} \int_{V_\delta(C)} |\log \text{dist} (p, C)| d\mu_k \leq \int_{V_\delta(C)} |\log \text{dist} (p, C)| d\mu_\infty.$$  

**Proof.** Since the function $p \mapsto \log \text{dist}(p, C)$ is continuous outside $C$, $\mu_k \to \mu_\infty$
(when $k \to \infty$) in the weak * topology and $\mu_\infty(\partial V_\delta(C)) = 0$, it follows
\[
\int \log \text{dist}(p, C)d\mu_\infty - \int_{V_\delta(C)} \log \text{dist}(p, C)d\mu_\infty \\
= \int_{(S^1 \times I) - V_\delta(C)} \log \text{dist}(p, C)d\mu_\infty \\
= \lim_{k \to \infty} \int_{(S^1 \times I) - V_\delta(C)} \log \text{dist}(p, C)d\mu_k \\
= \lim_{k \to \infty} \left( \int \log \text{dist}(p, C)d\mu_k - \int_{V_\delta(C)} \log \text{dist}(p, C)d\mu_k \right) \\
\geq \liminf_{k \to \infty} \int \log \text{dist}(p, C)d\mu_k + \liminf_{k \to \infty} \int_{V_\delta(C)} - \log \text{dist}(p, C)d\mu_k.
\]
Therefore,
\[
\liminf_{k \to \infty} \int_{V_\delta(C)} - \log \text{dist}(p, C)d\mu_k \leq \int_{V_\delta(C)} - \log \text{dist}(p, C)d\mu_\infty \\
+ \int \log \text{dist}(p, C)d\mu_\infty - \liminf_{k \to \infty} \int \log \text{dist}(p, C)d\mu_k.
\] (5)

However, the lemma 3.7 applied to the sequence $\mu_k \in K$ and the corollary 3.8 implies $\mu_k(C) = 0 = \mu_\infty(C)$. Since the set $(S^1 \times I) - C$ is open and the function $p \mapsto \log \text{dist}(p, C)$ is continuous on $(S^1 \times I) - C$, we get
\[
\int \log \text{dist}(p, C)d\mu_\infty = \int_{(S^1 \times I) - C} \log \text{dist}(p, C)d\mu_\infty \\
\leq \liminf_{k \to \infty} \int_{(S^1 \times I) - C} \log \text{dist}(p, C)d\mu_k = \liminf_{k \to \infty} \int \log \text{dist}(p, C)d\mu_k.
\]

Putting this information into the inequality (5) finish the desired result. \hfill \Box

As we promised above, an immediate corollary of the lemma 3.9 is the uniform control of the recurrence to the critical set:

**Corollary 3.10.** Given $\gamma > 0$ and $\delta > 0$ with $\mu_\infty(V_\delta(C)) = 0$ and
\[
\int_{V_\delta(C)} |\log \text{dist}(p, C)|d\mu_\infty < \gamma,
\]
there exists a subsequence \((\mu_j)\) of \((\mu_k)\) such that, for each \(j\), it holds

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log \text{dist}_\delta(f^j(p), C) \leq \gamma, \quad \text{for } \mu_j \text{-a.e. point } p. \quad (6)
\]

Here \(\text{dist}_\delta\) is the \(\delta\)-truncated distance defined by

\[
\text{dist}_\delta(p, C) = \begin{cases} 
1 & \text{if } \text{dist} (p, C) \geq \delta, \\
\text{dist} (p, C) & \text{otherwise}
\end{cases} \quad (7)
\]

Proof. This follows from the lemma 3.9 and a simple application of Birkhoff’s theorem (since \(\mu_k\) is ergodic for all \(k \in \mathbb{N}\)). \(\square\)

Next, we study the expanding behavior of the derivative of the Viana maps with respect to generic points of the measures belonging to the set \(\mathcal{K}\).

Notation. For a fixed \(\sigma < 1\), let \(H(\sigma)\) be the set of points \(p \in M\) such that:

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(p))^{-1}\| \leq 3 \log \sigma < 0
\]

We introduce the following definition:

**Definition 3.11.** \(\mu\) is called a \(\sigma^{-1}\)-expanding measure if \(p \in H(\sigma)\) for \(\mu\)-almost every point \(p\).

We now prove that any measure \(\mu \in \mathcal{K}\) is expanding:

**Lemma 3.12.** Any \(\mu \in \mathcal{K}\) is \(\sigma_0^{-1}\)-expanding (with \(\sigma_0 = \exp(-\frac{1}{12} \zeta)\), where \(\zeta := c_0 - \log \frac{d+\epsilon}{d-\epsilon} > 0\)).

Proof. Without loss of generality we can assume \(\mu \in \mathcal{K}\) ergodic. This implies that

\[
\lambda^c(p) = \lambda^c(\mu) = \int \log \|Df|_{E^c}\|d\mu \geq \frac{1}{4} \left( c_0 - \log \frac{d+\epsilon}{d-\epsilon} \right)
\]
for \(\mu\)-a.e. \(p\). Because \(\mu\) is ergodic and the central direction \(E^c\) is one-dimensional, this is equivalent to

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(p))^{-1}\| \leq -3 \frac{1}{12} \left( c_0 - \log \frac{d+\epsilon}{d-\epsilon} \right) = 3 \log \sigma_0 < 0.
\]

This fact gives us the assumptions in the definition of \(H(\sigma)\) at \(\mu\)-almost every point. \(\square\)
For later reference, we recall the Pliss lemma:\footnote{This combinatorial lemma turns out to be one of the most important tools in the current proofs of the existence of hyperbolic times}

**Lemma 3.13** (Pliss lemma). Given $A \geq c_2 > c_1 > 0$, define $\nu_0 := (c_2 - c_1)/(A - c_1)$. Then, for any real numbers $a_1, \ldots, a_N$ such that

$$\sum_{j=1}^{N} a_j \geq c_2 \cdot N \quad \text{and} \quad a_j \leq A \text{ for every } 1 \leq j \leq N,$$

there are $l > \nu_0 \cdot N$ and $1 \leq n_1 < \cdots < n_l \leq N$ so that

$$\sum_{j=n_i+1}^{n_i} a_j \geq c_1(n_i - n) \text{ for every } 0 \leq n \leq n_i \text{ and } i = 1, \ldots, l.$$

**Proof.** See [ABV, p.365]. \hfill \Box

Finally, we are ready to complete the proof of our quantitative criterion for the abundance of hyperbolic times:

**Proof of proposition 3.6.** The basic idea is to follow the proof of lemma 5.4 of [ABV], i.e., to use the Pliss lemma twice, firstly for the sequence $a_j = -\log \|Df(f^j(p))^{-1}\|$ and secondly for the sequence $a_j = \log \text{dist}_{\delta}(f^{j-1}p, C)$ (with some appropriate $\delta > 0$). Let us work out the details.

Fix $\rho > \beta$. Since $f$ behaves like a power of the distance, we obtain

$$|\log \|Df(p)^{-1}\|| \leq \rho |\log \text{dist}(p, C)|$$

for every $p$ in a fixed neighborhood\footnote{Here $\xi = e^{-\log B/(\rho - \beta)}$ suffices.} $V = V_\xi(C)$ of $C$. Fix $\gamma_1 \leq |\log \sigma_0|/2\rho$. Taking $\delta_1 > 0$ small such that $\mu_\infty(\partial V_{\delta_1}(C)) = 0$ and

$$\int_{V_{\delta_1}(C)} |\log \text{dist}(p, C)|d\mu_\infty < \gamma_1,$$

it follows from the corollary 3.10 that there exists a subsequence $(\mu_t)$ of $(\mu_k)$ such that, for each $t$, it holds

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_{\delta_1}(f^j(p), C) \leq \gamma_1, \quad \text{for } \mu_t - \text{a.e. point } p.$$
Note that $\delta_1$ depends only on $\sigma_0$, $f$ and $\mu_\infty$, i.e., $\delta_1 = \delta_1(\sigma_0, f, \mu_\infty)$. Fix $H_1 \geq \rho|\log \delta_1|$ large enough so that $-\log \|Df(p)^{-1}\| \leq H_1$ for any $p \in (S^1 \times I) - V$.

Define $\nu_1 = (H_1 - |\log \sigma_0|)/(|\log \sigma_0|/2)$ and fix $\gamma_2 < \nu_1(b|\log \sigma_0|)$. Taking $\delta_2 > 0$ small so that $\mu_\infty(\partial V_{\delta_2}(C)) = 0$ and
\[
\int_{V_{\delta_2}(C)} |\log \text{dist}(p, C)| \, d\mu_\infty < \gamma_2,
\]
we obtain from the corollary 3.10 that there exists a subsequence $(\mu_m)$ of $(\mu_t)$ such that, for each $m$, it holds
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \|Df(f^j(p))^{-1}\| \geq 2|\log \sigma_0|N,
\]
and
\[
\sum_{j=0}^{N-1} -\log \text{dist}_{\delta_1}(f^j(p), C) \leq \gamma_1 \cdot N,
\]
for every large $N$. Let $E$ be the subset of times $1 \leq j \leq N$ such that $-\log \|Df(f^{j-1}(p))^{-1}\| > H_1$ and define
\[
a_j := \begin{cases} 
-\log \|Df(f^{j-1}(p))^{-1}\| & \text{if } j \notin E, \\
0 & \text{if } j \in E.
\end{cases}
\]
By construction, $a_j \leq H_1$ for $1 \leq j \leq N$. Note that $j \in E$ implies $f^{j-1}(p) \in V$ and
\[
\rho|\log \delta_1| \leq H_1 < -\log \|Df(f^{j-1}(p))^{-1}\| < \rho|\log \text{dist}(p, C)|,
\]
i.e., $\text{dist}(f^{j-1}(p), C) < \delta_1$. In particular, $\text{dist}_{\delta_1}(f^{j-1}(p), C) = \text{dist}(f^{j-1}(p), C) < \delta_1$. Hence,
\[
\sum_{j \in E} -\log \|Df(f^{j-1}(p))^{-1}\| \leq \rho \sum_{j \in E} |\log \text{dist}(f^{j-1}(p))| \leq \rho \gamma_1 N.
\]
Because of our choice of $\gamma_1$, we get
\[
\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} -\log \| Df(f^{j-1}(p))^{-1} \|
- \sum_{j \in E} -\log \| Df(f^{j-1}(p))^{-1} \| \geq \frac{3|\log \sigma_0|}{2} N.
\]
Applying the Pliss lemma to $a_j$ with $c_1 = |\log \sigma_0|$, $c_2 = 3|\log \sigma_0|/2$ and $A = H_1$, we obtain that, for some $l_1 \geq \nu_1 \cdot N$, there are $1 \leq p_1 < \cdots < p_{l_1} \leq N$ such that
\[
\sum_{j=n+1}^{p_i} -\log \| Df(f^{j-1}(p))^{-1} \| \geq \sum_{j=n+1}^{p_i} a_j \geq (p_i - n) |\log \sigma_0| \quad (8)
\]
for every $0 \leq n < p_i$ and $1 \leq i \leq l_1$.

Now, we recall that
\[
\sum_{j=0}^{N-1} \log \text{dist}_{\delta_2}(f^{j-1}(p), \mathcal{C}) \geq -\gamma_2 N.
\]
Put $c_1 = b \log \sigma_0$, $c_2 = -\gamma_2$, $A = 0$. Define
\[
\nu_2 := \frac{c_2 - c_1}{A - c_1} = 1 - \frac{\gamma_2}{b|\log \sigma_0|}.
\]
The Pliss lemma with $a_j = \log \text{dist}_{\delta_2}(f^{j-1}(p), \mathcal{C})$ ensures the existence of $l_2 \geq \nu_2 N$ and $1 \leq q_1 < \cdots < q_{l_2} \leq N$ such that
\[
\sum_{j=n}^{q_i-1} \log \text{dist}_{\delta_2}(f^{j}(p), \mathcal{C}) \geq b \log \sigma_0(q_i - n) \quad (9)
\]
for every $0 \leq n < q_i$ and $1 \leq i \leq l_2$.

Observe that our choice of $\gamma_2$ forces $\nu_1 + \nu_2 > 1$. Defining $\nu := \nu_1 + \nu_2 - 1 > 0$, we conclude that there exist $l = (l_1 + l_2 - N) \geq \nu N$ and $1 \leq n_1 < \cdots < n_l \leq N$ such that the conditions (8) and (9) occurs simultaneously:
\[
\sum_{j=n+1}^{n_i} -\log \| Df(f^{j-1}(p))^{-1} \| \geq \sum_{j=n+1}^{n_i} a_j \geq (n_i - n) |\log \sigma_0|
\]
and
\[ \sum_{j=n}^{n_i-1} \log \text{dist}_{\delta_2}(f^j(p), C) \geq b \log \sigma_0(n - n_i) \]
for every \(0 \leq n < n_i\) and \(1 \leq i \leq l\). So, we obtain
\[ \prod_{j=n_i-n+1}^{n_i} \| Df(f^j(p))^{-1} \| \leq \sigma_0^n \text{ and dist}_{\delta_2}(f^{n_i-n}(p), C) \geq \sigma_0^m. \]

In other words, for each \(n\) and any \(\mu_n\)-generic point \(p\), all these \(n_i\) are \((\sigma_0, \delta)\)-hyperbolic times for \(p\) with \(\delta = \delta_2\) depending only on \(\sigma_0, f\) and \(\mu_\infty\). \(\square\)

At this point, we can derive the existence of generating partitions:

**Lemma 3.14.** There exists \(\delta_1 > 0\) (depending only on \(\sigma_0\) and \(\delta\)) such that any partition \(\mathcal{P}\) with diameter less than \(\delta_1\) is a generating partition for any element \(\mu_n\) of the subsequence of \((\mu_k)\) provided by proposition 3.6.

**Proof.** Define
\[ A_\epsilon(p) := \{ y \in M : \text{dist}(f^j(y), f^j(p)) \leq \epsilon \text{ for every } j \geq 0 \}. \]
First we prove that, for each \(n\), \(A_\epsilon(p) = \{p\}\) for \(\mu_n\)-a.e. \(p\), and then we show how this can be used to finish the proof.

**Claim 3.15.** There exists \(\delta_1 > 0\) such that for any \(\epsilon < \delta_1\), any \(\mu_n\) and \(\mu_n\)-a.e. \(p\),
\[ A_\epsilon(p) = \{p\}. \]

**Proof of the claim.** Proposition 3.6 guarantees that, for every \(\mu_n\), a given \(\mu_n\)-generic point \(p\) has infinitely many \((\sigma_0, \delta)\)-hyperbolic times \(n_i(p)\). Hence, applying the proposition 3.5 we conclude that there exists some \(\delta_1 > 0\) (depending only on \(\sigma_0\) and \(\delta\)) such that if \(z \in A_\epsilon(p)\) with \(\epsilon < \delta_1\) then for any \(n_i\) we have
\[ \text{dist}(z, p) \leq \sigma_0^{n_i/2} \text{dist}(f^{n_i}(z), f^{n_i}(p)) \leq \sigma_0^{n_i/2}. \]
Since \(n_i \to \infty\) as \(i \to \infty\), we deduce that \(z = p\). \(\square\)

It is now a more or less standard matter to show that the previous claim implies that any finite partition \(\mathcal{P} = \{P_1, \ldots, P_l\}\) with \(\text{diam}(\mathcal{P}) < \delta_1\) is a generating partition for every \(\mu_n\). Indeed, for each \(\mu_n\), given any measurable
set $A$ and given $\kappa > 0$, consider $K_1 \subset A$ and $K_2 \subset M - A$ compact sets with $\mu(A \setminus K_1) \leq \kappa$ and $\mu(A^c \setminus K_2) \leq \kappa$. Let $r := \text{dist}(K_1, K_2) > 0$. Claim 3.15 shows that, for $m$ large enough, $\text{diam}\mathcal{P}(m)(p) \leq r/2$ for every $p$ in a set of $\mu$-measure greater than $1 - \kappa$, where

$$\mathcal{P}(m)(p) := \{C^{(m)} := P_{i_1} \cap \cdots \cap f^{-m+1}(P_{i_n}) \text{ with } p \in C^{(m)}\}.$$ 

Consider the sets $C^{(m)}_1, \ldots, C^{(m)}_q \in \mathcal{P}(m)$ that intersect $K_1$. Then, it is not difficult to see that $\mu(\cup C^{(n)}_1 \Delta A) \leq 3\kappa$.

Since $A$ is an arbitrary measurable set and $\kappa > 0$ is an arbitrary positive real number, this proves that $\mathcal{P}$ is a generating partition (for every $\mu_n$).

Closing the discussion of this section, we conclude the proof of theorem A: 

Proof of theorem A. Fix a finite partition $\mathcal{P}$ with diameter less than $\delta_1 > 0$, the constant of lemma 3.14, such that $\nu_\infty(\partial P) = 0$ for any $P \in \mathcal{P}$. Observe that Kolmogorov-Sinai’s theorem implies $h_{\mu_k}(f) = h_{\mu_k}(f, \mathcal{P})$ by the lemmas 3.14 and 3.12. Because the function $\nu \rightarrow h_\nu(f, \mathcal{P})$ is upper-semicontinuous at every measure $\nu$ with $\nu(\partial \mathcal{P}) = 0$, we get

$$\sup_{\eta}(h_\eta(f) + \int \phi d\eta) = \lim_{k} \sup(h_{\mu_k}(f) + \int \phi d\mu_k)$$

$$= \lim_{k} \sup(h_{\mu_k}(f, \mathcal{P}) + \int \phi d\mu_k)$$

$$\leq h_\mu(f, \mathcal{P}) + \int \phi d\mu$$

$$\leq h_\mu(f) + \int \phi d\mu.$$ 

In other words, $\mu$ is an equilibrium state. Finally, the fact that every equilibrium state $\eta$ belongs to $\mathcal{K}$ follows directly from the lemma 3.2. This concludes the proof.

\section{Remarks}

Let us make few comments about the hypothesis $d \geq 16$ in the definition of the Viana maps, about the generalization of the theorem A in the context
of random perturbations and about some natural questions associated to its statement.

The hypothesis $d \geq 16$ in the definition of the maps $\tilde{f}$ was used by Viana to ensure that $C^3$ small perturbations have two positive Lyapounov exponents. However, Buzzi, Sester and Tsujii [BST] noticed that if one is willing to consider $C^\infty$ small perturbations (instead of $C^3$), then the theorems of Viana and Alves (see theorems 2.1, 2.2) still hold. Also, it is proved in [BST] that the vertical expansion on the dynamical strip $I$ of the derivative of $f$ is uniformly less than 2. In particular, our comments on the tangent bundle dynamics of $f$ are true even if $2 \leq d \leq 16$. Summarizing,

*Theorem A also holds for $C^\infty$ small perturbations of Viana maps with $d \geq 2$(instead of stronger requirement $d \geq 16$).

Also, it is not difficult to check that the arguments of this note apply to the context of small random perturbations of Viana maps. Indeed, the techniques of Alves and Araújo [AA] allow one to consider hyperbolic times, expanding measures, slow recurrence and non-uniform expansion in the non-deterministic setting
\footnote{Although the statements in Alves and Araújo [AA] are for Viana maps (and its $C^3$ perturbations) with $d \geq 16$, the results of Buzzi, Sester and Tsujii [BST] ensure their proof can be carried out for $d \geq 2$ (at the cost of considering only $C^\infty$ small perturbations).}, and prove the analogues (in the random context) of Viana’s result (see theorem 2.1) and Alves’ result (see theorem 2.2) about the existence of a unique physical measure with all Lyapounov exponents larger that some positive number. We obtain a version of theorem A for small random perturbations of Viana maps by considering the modified definition of the set of measures $\mathcal{K}$ (as defined in Arbieto, Matheus, Oliveira [AMO]).

Finally, a natural question motivated by the existence of equilibrium measures is the uniqueness and ergodic properties, including decay of correlations, for the equilibrium state obtained in theorem A. Since the approach of Young towers is not easy in the context of general equilibrium measures
\footnote{This occurs essentially because the known constructions of Young towers depends on the Lebesgue measure as a reference measure, but the general equilibrium measures are singular with respect to Lebesgue, even in the uniformly hyperbolic case.} and Ruelle’s thermodynamical formalism via the spectral properties of the transfer operator is delicate in the presence of critical points, this question presents an interesting problem for the extension of the theory of equilibrium states beyond uniform hyperbolicity. For results in this direction for one dimensional maps, see [BK], [DU], [PS1], [PS2], [Y], for Hénon-like maps, see
[WY], for countable Markov shifts and piecewise expanding maps, see [BS], among others.

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