

GLOBAL WELL-POSEDNESS FOR A NLS-KDV SYSTEM ON \mathbb{T}

C. MATHEUS

ABSTRACT. We prove that the Cauchy problem of the Schrödinger - Korteweg - deVries (NLS-KdV) system on \mathbb{T} is globally well-posed for initial data (u_0, v_0) below the energy space $H^1 \times H^1$. More precisely, we show that the non-resonant NLS-KdV is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 11/13$ and the resonant NLS-KdV is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 8/9$. The idea of the proof of this theorem is to apply the I-method of Colliander, Keel, Staffilani, Takaoka and Tao in order to improve the results of Arbieto, Corcho and Matheus concerning the global well-posedness of the NLS-KdV on \mathbb{T} in the energy space $H^1 \times H^1$.

1. INTRODUCTION

We consider the Cauchy problem of the Schrödinger-Korteweg-deVries system

$$(1.1) \quad \begin{cases} i\partial_t u + \partial_x^2 u = \alpha uv + \beta |u|^2 u, \\ \partial_t v + \partial_x^3 v + \frac{1}{2} \partial_x (v^2) = \gamma \partial_x (|u|^2), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad t \in \mathbb{R}. \end{cases}$$

The Schrödinger-Korteweg-deVries (NLS-KdV) system naturally appears in fluid mechanics and plasma physics as a model of interaction between a short-wave $u = u(x, t)$ and a long-wave $v = v(x, t)$.

In this paper we are interested in global solutions of the NLS-KdV system for rough initial data. Before stating our main results, let us recall some of the recent theorems of local and global well-posedness theory of the Cauchy problem (1.1).

For continuous spatial variable (i.e., $x \in \mathbb{R}$), Corcho and Linares [5] recently proved that the NLS-KdV system is locally well-posed for initial data $(u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})$ with $k \geq 0$, $s > -3/4$ and

- $k - 1 \leq s \leq 2k - 1/2$ if $k \leq 1/2$,
- $k - 1 \leq s < k + 1/2$ if $k > 1/2$.

Furthermore, they were able to prove the global well-posedness of the NLS-KdV system in the energy $H^1 \times H^1$ using three conserved quantities discovered by M. Tsutsumi [7], whenever $\alpha\gamma > 0$.

Also, Pecher [6] improved this global well-posedness result by an application of the I-method of Colliander, Keel, Staffilani, Takaoka and Tao (for instance, see [3]) combined with some refined bilinear estimates. In particular, Pecher proved that, if $\alpha\gamma > 0$, the NLS-KdV

Date: November 16, 2005.

Key words and phrases. Global well-posedness, Schrödinger-Korteweg-de Vries system, I-method.

system is globally well-posed for initial data $(u_0, v_0) \in H^s \times H^s$ with $s > 3/5$ in the resonant case $\beta = 0$ and $s > 2/3$ in the non-resonant case $\beta \neq 0$.

On the other hand, in the periodic setting (i.e., $x \in \mathbb{T}$), Arbieto, Corcho and Matheus [1] proved the local well-posedness of the NLS-KdV system for initial data $(u_0, v_0) \in H^k \times H^s$ with $0 \leq s \leq 4k - 1$ and $-1/2 \leq k - s \leq 3/2$. Also, using the same three conserved quantities discovered by M. Tsutsumi, one obtains the global well-posedness of NLS-KdV on \mathbb{T} in the energy space $H^1 \times H^1$ whenever $\alpha\gamma > 0$.

Motivated by this scenario, we combine the new bilinear estimates of Arbieto, Corcho and Matheus [1] with the I-method of Tao and his collaborators to prove the following result

Theorem 1.1. *The NLS-KdV system (1.1) on \mathbb{T} is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 11/13$ in the non-resonant case $\beta \neq 0$ and $s > 8/9$ in the resonant case $\beta = 0$, whenever $\alpha\gamma > 0$.*

The paper is organized as follows. In the section 2, we discuss the preliminaries for the proof of the theorem 1.1: Bourgain spaces and its properties, linear estimates, standard estimates for the non-linear terms $|u|^2u$ and $\partial_x(v^2)$, the bilinear estimates of Arbieto, Corcho and Matheus [1] for the coupling terms uv and $\partial_x(|u|^2)$, the I-operator and its properties. In the section 3, we apply the results of the section 2 to get a variant of the local well-posedness result of [1]. In the section 4, we recall some conserved quantities of (1.1) and its modification by the introduction of the I-operator; moreover, we prove that two of these modified energies are almost conserved. Finally, in the section 5, we combine the almost conservation results in section 4 with the local well-posedness result in section 3 to conclude the proof of the theorem 1.1.

2. PRELIMINARIES

A successful procedure to solve some dispersive equations (such as the nonlinear Schrödinger and KdV equations) is to use the Picard's fixed point method in the following spaces:

$$\begin{aligned} \|f\|_{X^{k,b}} &:= \left(\int \sum_{n \in \mathbb{Z}} \langle n \rangle^{2k} \langle \tau + n^2 \rangle^{2b} |\widehat{f}(n, \tau)| d\tau \right)^{1/2} \\ &= \|U(-t)f\|_{H_t^b(\mathbb{R}, H_x^k)} \\ \|g\|_{Y^{s,b}} &:= \left(\int \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \langle \tau - n^3 \rangle^{2b} |\widehat{g}(n, \tau)| d\tau \right)^{1/2} \\ &= \|V(-t)f\|_{H_t^b(\mathbb{R}, H_x^s)} \end{aligned}$$

where $\langle \cdot \rangle := 1 + |\cdot|$, $U(t) = e^{it\partial_x^2}$ and $V(t) = e^{-it\partial_x^3}$. These spaces are called Bourgain spaces. Also, we introduce the restriction in time norms

$$\|f\|_{X^{k,b}(I)} := \inf_{\widetilde{f}|_I = f} \|\widetilde{f}\|_{X^{k,b}} \quad \text{and} \quad \|g\|_{Y^{s,b}(I)} := \inf_{\widetilde{g}|_I = g} \|\widetilde{g}\|_{Y^{s,b}}$$

where I is a time interval.

The interaction of the Picard method has been based around the spaces $Y^{s,1/2}$. Because we are interested in the continuity of the flow associated to (1.1) and the $Y^{s,1/2}$ norm do not control the $L_t^\infty H_x^s$ norm, we modify the Bourgain spaces as follows:

$$\|u\|_{X^k} := \|u\|_{X^{k,1/2}} + \|\langle n \rangle^k \widehat{u}(n, \tau)\|_{L_n^2 L_\tau^1} \quad \text{and} \quad \|v\|_{Y^s} := \|v\|_{Y^{s,1/2}} + \|\langle n \rangle^s \widehat{v}(n, \tau)\|_{L_n^2 L_\tau^1}$$

and, given a time interval I , we consider the restriction in time of the X^k and Y^s norms

$$\|u\|_{X^k(I)} := \inf_{\tilde{u}|_I = u} \|\tilde{u}\|_{X^k} \quad \text{and} \quad \|v\|_{Y^s(I)} := \inf_{\tilde{v}|_I = v} \|\tilde{v}\|_{Y^s}$$

Furthermore, the mapping properties of $U(t)$ and $V(t)$ naturally leads one to consider the companion spaces

$$\|u\|_{Z^k} := \|u\|_{X^{k,-1/2}} + \left\| \frac{\langle n \rangle^k \widehat{u}(n, \tau)}{\langle \tau + n^2 \rangle} \right\|_{L_n^2 L_\tau^1} \quad \text{and} \quad \|v\|_{W^s} := \|v\|_{Y^{s,-1/2}} + \left\| \frac{\langle n \rangle^s \widehat{v}(n, \tau)}{\langle \tau - n^3 \rangle} \right\|_{L_n^2 L_\tau^1}$$

In the sequel, ψ denotes a non-negative smooth bump function supported on $[-2, 2]$ with $\psi = 1$ on $[-1, 1]$ and $\psi_\delta(t) := \psi(t/\delta)$ for any $\delta > 0$.

Next, we recall some properties of the Bourgain spaces:

Lemma 2.1. $X^{0,3/8}([0, 1]), Y^{0,1/3}([0, 1]) \subset L^4(\mathbb{T} \times [0, 1])$. *More precisely,*

$$\|\psi(t)f\|_{L_{xt}^4} \lesssim \|f\|_{X^{0,3/8}} \quad \text{and} \quad \|\psi(t)g\|_{L_{xt}^4} \lesssim \|g\|_{Y^{0,1/3}}.$$

Proof. See [2]. □

Another basic property of these spaces are their stability under time localization:

Lemma 2.2. *Let* $X_{\tau=h(\xi)}^{s,b} := \{f : \langle \tau - h(\xi) \rangle^b \langle \xi \rangle^s |\widehat{f}(\tau, \xi)| \in L^2\}$. *Then,*

$$\|\psi(t)f\|_{X_{\tau=h(\xi)}^{s,b}} \lesssim_{\psi,b} \|f\|_{X_{\tau=h(\xi)}^{s,b}}$$

for any $s, b \in \mathbb{R}$. *Moreover, if* $-1/2 < b' \leq b < 1/2$, *then for any* $0 < T < 1$, *we have*

$$\|\psi_T(t)f\|_{X_{\tau=h(\xi)}^{s,b'}} \lesssim_{\psi,b',b} T^{b-b'} \|f\|_{X_{\tau=h(\xi)}^{s,b}}.$$

Proof. First of all, note that $\langle \tau - \tau_0 - h(\xi) \rangle^b \lesssim_b \langle \tau_0 \rangle^{|b|} \langle \tau - h(\xi) \rangle^b$, from which we obtain

$$\|e^{it\tau_0} f\|_{X_{\tau=h(\xi)}^{s,b}} \lesssim_b \langle \tau_0 \rangle^{|b|} \|f\|_{X_{\tau=h(\xi)}^{s,b}}.$$

Using that $\psi(t) = \int \widehat{\psi}(\tau_0) e^{it\tau_0} d\tau_0$, we conclude

$$\|\psi(t)f\|_{X_{\tau=h(\xi)}^{s,b}} \lesssim_b \left(\int |\widehat{\psi}(\tau_0)| \langle \tau_0 \rangle^{|b|} \right) \|f\|_{X_{\tau=h(\xi)}^{s,b}}.$$

Since ψ is smooth with compact support, the first estimate follows.

Next we prove the second estimate. By conjugation we may assume $s = 0$ and, by composition it suffices to treat the cases $0 \leq b' \leq b$ or $\leq b' \leq b \leq 0$. By duality, we may take

$0 \leq b' \leq b$. Finally, by interpolation with the trivial case $b' = b$, we may consider $b' = 0$. This reduces matters to show that

$$\|\psi_T(t)f\|_{L^2} \lesssim_{\psi,b} T^b \|f\|_{X_{\tau=h(\xi)}^{0,b}}$$

for $0 < b < 1/2$. Partitioning the frequency spaces into the cases $\langle \tau - h(\xi) \rangle \geq 1/T$ and $\langle \tau - h(\xi) \rangle \leq 1/T$, we see that in the former case we'll have

$$\|f\|_{X_{\tau=h(\xi)}^{0,0}} \leq T^b \|f\|_{X_{\tau=h(\xi)}^{0,b}}$$

and the desired estimate follows because the multiplication by ψ is a bounded operation in Bourgain's spaces. In the latter case, by Plancherel and Cauchy-Schwarz

$$\begin{aligned} \|f(t)\|_{L_x^2} &\lesssim \|\widehat{f(t)}(\xi)\|_{L_\xi^2} \lesssim \left\| \int_{\langle \tau - h(\xi) \rangle \leq 1/T} |\widehat{f}(\tau, \xi)| d\tau \right\|_{L_\xi^2} \\ &\lesssim_b T^{b-1/2} \left\| \int \langle \tau - h(\xi) \rangle^{2b} |\widehat{f}(\tau, \xi)|^2 d\tau \right\|_{L_\xi^2}^{1/2} = T^{b-1/2} \|f\|_{X_{\tau=h(\xi)}^{s,b}}. \end{aligned}$$

Integrating this against ψ_T concludes the proof of the lemma. \square

Also, we have the following duality relationship between X^k (resp., Y^s) and Z^k (resp., W^s):

Lemma 2.3. *We have*

$$\left| \int \chi_{[0,1]}(t) f(x, t) g(x, t) dx dt \right| \lesssim \|f\|_{X^s} \|g\|_{Z^{-s}}$$

and

$$\left| \int \chi_{[0,1]}(t) f(x, t) g(x, t) dx dt \right| \lesssim \|f\|_{Y^s} \|g\|_{W^{-s}}$$

for any s and any f, g on $\mathbb{T} \times \mathbb{R}$.

Proof. See [4, p. 182–183] (note that, although this result is stated only for the spaces Y^s and W^s , the same proof adapts for the spaces X^k and Z^k). \square

Now, we recall some linear estimates related to the semigroups $U(t)$ and $V(t)$:

Lemma 2.4 (Linear estimates). *It holds*

- $\|\psi(t)U(t)u_0\|_{Z^k} \lesssim \|u_0\|_{H^k}$ and $\|\psi(t)V(t)v_0\|_{W^s} \lesssim \|v_0\|_{H^s}$;
- $\|\psi_T(t) \int_0^t U(t-t')F(t')dt'\|_{X^k} \lesssim \|F\|_{Z^k}$ and $\|\psi_T(t) \int_0^t V(t-t')G(t')dt'\|_{Y^s} \lesssim \|G\|_{W^s}$.

Proof. See [3], [4] or [1]. \square

Furthermore, we have the following well-known multilinear estimates for the cubic term $|u|^2u$ of the nonlinear Schrödinger equation and the nonlinear term $\partial_x(v^2)$ of the KdV equation:

Lemma 2.5. $\|uv\bar{w}\|_{Z^k} \lesssim \|u\|_{X^{k, \frac{3}{8}}} \|v\|_{X^{k, \frac{3}{8}}} \|w\|_{X^{k, \frac{3}{8}}}$ for any $k \geq 0$.

Proof. See [2] and [1]. \square

Lemma 2.6. $\|\partial_x(v_1v_2)\|_{W^s} \lesssim \|v_1\|_{Y^{s,\frac{1}{3}}}\|v_2\|_{Y^{s,\frac{1}{2}}} + \|v_1\|_{Y^{s,\frac{1}{2}}}\|v_2\|_{Y^{s,\frac{1}{3}}}$ for any $s \geq -1/2$, if $v_1 = v_1(x, t)$ and $v_2 = v_2(x, t)$ are x -periodic functions having zero x -mean for all t .

Proof. See [2], [3] and [1]. □

Next, we revisit the bilinear estimates of mixed Schrödinger-Airy type of Arbieto, Corcho and Matheus [1] for the coupling terms uv and $\partial_x(|u|^2)$ of the NLS-KdV system.

Lemma 2.7. $\|uv\|_{Z^k} \lesssim \|u\|_{X^{k,\frac{3}{8}}}\|v\|_{Y^{s,\frac{1}{2}}} + \|u\|_{X^{k,\frac{1}{2}}}\|v\|_{Y^{s,\frac{1}{3}}}$ whenever $s \geq 0$ and $k - s \leq 3/2$.

Lemma 2.8. $\|\partial_x(u_1\overline{u_2})\|_{W^s} \lesssim \|u_1\|_{X^{k,3/8}}\|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}}\|u_2\|_{X^{k,3/8}}$ whenever $1 + s \leq 4k$ and $k - s \geq -1/2$.

Remark 2.1. Although the lemmas 2.7 and 2.8 are not stated as above in [1], it is not hard to obtain them from the calculations of Arbieto, Corcho and Matheus.

Finally, we introduce the I-operator: let $m(\xi)$ be a smooth non-negative symbol on \mathbb{R} which equals 1 for $|\xi| \leq 1$ and equals $|\xi|^{-1}$ for $|\xi| \geq 2$. For any $N \geq 1$ and $\alpha \in \mathbb{R}$, denote by I_N^α the spatial Fourier multiplier

$$\widehat{I_N^\alpha f}(\xi) = m\left(\frac{\xi}{N}\right)^\alpha \widehat{f}(\xi).$$

For latter use, we recall the following general interpolation lemma:

Lemma 2.9 (Lemma 12.1 of [4]). Let $\alpha_0 > 0$ and $n \geq 1$. Suppose Z, X_1, X_2, \dots, X_n are translation-invariant Banach spaces and T is a translation invariant n -linear operator such that

$$\|I_1^\alpha T(u_1, \dots, u_n)\|_Z \lesssim \prod_{j=1}^n \|I_1^\alpha u_j\|_{X_j},$$

for all u_1, \dots, u_n and $0 \leq \alpha \leq \alpha_0$. Then,

$$\|I_N^\alpha T(u_1, \dots, u_n)\|_Z \lesssim \prod_{j=1}^n \|I_N^\alpha u_j\|_{X_j}$$

for all u_1, \dots, u_n , $0 \leq \alpha \leq \alpha_0$ and $N \geq 1$. Here the implicit constant is independent of N .

After these preliminaries, we can proceed to the next section where a variant of the local well-posedness of Arbieto, Corcho and Matheus is obtained.

In the sequel we take $N \gg 1$ a large integer and denote by I the operator $I = I_N^{1-s}$ for a given $s \in \mathbb{R}$.

3. A VARIANT LOCAL WELL-POSEDNESS RESULT

This section is devoted to the proof of the following proposition:

Proposition 3.1. *For any $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $\int_{\mathbb{T}} v_0 = 0$ and $s \geq 1/3$, the periodic NLS-KdV system (1.1) has a unique local-in-time solution on the time interval $[0, \delta]$ for some $\delta \leq 1$ and*

$$(3.1) \quad \delta \sim \begin{cases} (\|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1})^{-\frac{16}{3}-}, & \text{if } \beta \neq 0, \\ (\|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1})^{-8-}, & \text{if } \beta = 0. \end{cases}$$

Moreover, we have $\|Iu\|_{X^1} + \|Iv\|_{Y^1} \lesssim \|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1}$.

Proof. We apply the I-operator to the NLS-KdV system (1.1) so that

$$\begin{cases} iIu_t + Iu_{xx} = \alpha I(uv) + \beta I(|u|^2 u), \\ Iv_t + Iv_{xxx} + I(vv_x) = \gamma I(|u|^2)_x, \\ Iu(0) = Iu_0, \quad Iv(0) = Iv_0. \end{cases}$$

To solve this equation, we seek for some fixed point of the integral maps

$$\begin{aligned} \Phi_1(Iu, Iv) &:= U(t)Iu_0 - i \int_0^t U(t-t') \{ \alpha I(u(t')v(t')) + \beta I(|u(t')|^2 u(t')) \} dt', \\ \Phi_2(Iu, Iv) &:= V(t)Iv_0 - \int_0^t V(t-t') \{ I(v(t')v_x(t')) - \gamma I(|u(t')|^2)_x \} dt'. \end{aligned}$$

The interpolation lemma 2.9 applied to the linear and multilinear estimates in the lemmas 2.4, 2.5, 2.6, 2.7 and 2.8 yields, in view of the lemma 2.2,

$$\begin{aligned} \|\Phi_1(Iu, Iv)\|_{X^1} &\lesssim \|Iu_0\|_{H^1} + \alpha \delta^{\frac{1}{8}-} \|Iu\|_{X^1} \|Iv\|_{Y^1} + \beta \delta^{\frac{3}{8}-} \|Iu\|_{X^1}^3, \\ \|\Phi_2(Iu, Iv)\|_{Y^1} &\lesssim \|Iv_0\|_{H^1} + \delta^{\frac{1}{6}-} \|Iv\|_{Y^1}^2 + \gamma \delta^{\frac{1}{8}-} \|Iu\|_{X^1}^2. \end{aligned}$$

In particular, these integrals maps are contractions provided that $\beta \delta^{\frac{3}{8}-} (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^2 \ll 1$ and $\delta^{\frac{1}{8}-} (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}) \ll 1$. This completes the proof of the proposition 3.1. \square

4. MODIFIED ENERGIES

Consider the following three quantities:

$$(4.1) \quad M(u) := \|u\|_{L^2},$$

$$(4.2) \quad L(u, v) := \alpha \|v\|_{L^2}^2 + 2\gamma \int \Im(u\overline{u_x}) dx,$$

$$(4.3) \quad E(u, v) := \alpha \gamma \int v|u|^2 dx + \gamma \|u_x\|_{L^2}^2 + \frac{\alpha}{2} \|v_x\|_{L^2}^2 - \frac{\alpha}{6} \int v^3 dx + \frac{\beta \gamma}{2} \int |u|^4 dx.$$

In the sequel, we suppose $\alpha \gamma > 0$. Note that

$$(4.4) \quad |L(u, v)| \lesssim \|v\|_{L^2}^2 + M \|u_x\|_{L^2}$$

and

$$(4.5) \quad \|v\|_{L^2}^2 \lesssim |L| + M\|u_x\|_{L^2}.$$

Also, the Gagliardo-Nirenberg and Young inequalities implies

$$(4.6) \quad \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 \lesssim |E| + |L|^{\frac{5}{3}} + M^8 + 1$$

and

$$(4.7) \quad |E| \lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + |L|^{\frac{5}{3}} + M^8 + 1$$

In particular, combining the bounds (4.4) and (4.7),

$$(4.8) \quad |E| \lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + \|v\|_{L^2}^{\frac{10}{3}} + M^{10} + 1.$$

Moreover, from the bounds (4.5) and (4.6),

$$(4.9) \quad \|v\|_{L^2}^2 \lesssim |L| + M|E|^{1/2} + M^6 + 1$$

and hence

$$(4.10) \quad \|u\|_{H^1}^2 + \|v\|_{H^1}^2 \lesssim |E| + |L|^{5/3} + M^8 + 1$$

$$(4.11) \quad \begin{aligned} \frac{d}{dt}L(Iu, Iv) &= 2\alpha \int Iv(IvIv_x - I(vv_x))dx + 2\alpha\gamma \int Iv(I(|u|^2) - |Iu|^2)_x dx \\ &\quad + 4\alpha\gamma\Re \int I\bar{u}_x(IuIv - I(uv))dx + 4\beta\gamma\Re \int ((Iu)^2 I\bar{u} - I(u^2\bar{u}))I\bar{u}_x dx \\ &=: \sum_{j=1}^4 L_j. \end{aligned}$$

$$\begin{aligned}
(4.12) \quad \frac{d}{dt}E(Iu, Iv) &= \alpha \int (I(vv_x) - IvIv_x)Iv_{xx}dx + \frac{\alpha}{2} \int (Iv)^2(I(vv_x) - IvIv_x)dx + \\
&+ 2\beta\gamma\Im \int (I(|u|^2u)_x - (Iu)^2I\bar{u})_x I\bar{u}_x dx \\
&+ \alpha\gamma \int |Iu|^2(IvIv_x - I(vv_x))dx + \alpha\gamma \int (|Iu|^2 - I(|u|^2))IvIv_x dx \\
&+ \alpha\gamma \int Iv_{xx}(|Iu|^2 - I(|u|^2))_x dx - 2\alpha\gamma\Im \int Iu_x(I(\bar{u}v) - I\bar{u}Iv)_x dx \\
&+ \alpha\gamma^2 \int (I(|u|^2) - |Iu|^2)_x |Iu|^2 dx + 2\alpha^2\gamma\Im \int IvIu(I(\bar{u}v) - I\bar{u}Iv) dx \\
&+ 2\beta^2\gamma\Im \int Iu(I\bar{u})^2(I(|u|^2u) - (Iu)^2I\bar{u}) dx \\
&- 2\alpha\beta\gamma\Im \int IvIu(I(|u|^2\bar{u}) - Iu(I\bar{u}))^2 dx - 2\alpha\beta\gamma\Im \int (Iu)^2 I\bar{u}(I(\bar{u}v) - I\bar{u}Iv) dx \\
&=: \sum_{j=1}^{12} E_j
\end{aligned}$$

4.1. Estimates for the modified L-functional.

Proposition 4.1. *Let (u, v) be a solution of (1.1) on the time interval $[0, \delta]$. Then, for any $N \geq 1$ and $s > 1/2$,*

$$(4.13) \quad |L(Iu(\delta), Iv(\delta)) - L(Iu(0), Iv(0))| \lesssim N^{-1+\delta^{\frac{19}{24}-}} (\|Iu\|_{X^{1,1/2}} + \|Iv\|_{Y^{1,1/2}})^3 + N^{-2+\delta^{\frac{1}{2}-}} \|Iu\|_{X^{1,1/2}}^4.$$

Proof. Integrating (4.11) with respect to $t \in [0, \delta]$, it follows that we have to bound the (integral over $[0, \delta]$ of the) four terms on the right hand side. To simplify the computations, we assume that the Fourier transform of the functions are non-negative and we ignore the appearance of complex conjugates (since they are irrelevant in our subsequent arguments). Also, we make a dyadic decomposition of the frequencies $|n_i| \sim N_j$ in many places. In particular, it will be important to get extra factors N_j^{0-} everywhere in order to sum the dyadic blocks.

We begin with the estimate of $\int_0^\delta L_1$. It is sufficient to show that

$$(4.14) \quad \int_0^\delta \sum_{n_1+n_2+n_3=0} \left| \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |\widehat{v}_1(n_1, t)| |n_2| |\widehat{v}_2(n_2, t)| |\widehat{v}_3(n_3, t)| \lesssim N^{-1} \delta^{\frac{5}{6}-} \prod_{j=1}^3 \|v_j\|_{Y^{1,1/2}}$$

- $|n_1| \ll |n_2| \sim |n_3|$, $|n_2| \gtrsim N$. In this case, note that

$$\begin{cases} \left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left| \frac{\nabla m(n_2) \cdot n_1}{m(n_2)} \right| \lesssim \frac{N_1}{N_2}, \text{ if } |n_1| \leq N, \text{ and} \\ \left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_1}{N} \right)^{1/2}, \text{ if } |n_1| \geq N. \end{cases}$$

Hence, using the lemmas 2.1 and 2.2 we obtain

$$\left| \int_0^\delta L_1 \right| \lesssim \frac{N_1}{N_2} \|v_1\|_{L^4} \|(v_2)_x\|_{L^4} \|v_3\|_{L^2} \lesssim N^{-2+\delta \frac{5}{6}} N_{\max}^{0-} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}$$

if $|n_1| \leq N$, and

$$\left| \int_0^\delta L_1 \right| \lesssim \left(\frac{N_1}{N} \right)^{1/2} \frac{1}{N_1 N_3} \delta^{\frac{5}{6}} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}} \lesssim N^{-2+\delta \frac{5}{6}} N_{\max}^{0-} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}.$$

- $|n_2| \ll |n_1| \sim |n_3|$, $|n_1| \gtrsim N$. This case is similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. The multiplier is bounded by

$$\left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_1}{N} \right)^{1-}.$$

In particular, using the lemmas 2.1 and 2.2,

$$\left| \int_0^\delta L_1 \right| \lesssim \left(\frac{N_1}{N} \right)^{1-} \|v_1\|_{L^2} \|(v_2)_x\|_{L^4} \|v_3\|_{L^4} \lesssim N^{-1+\delta \frac{5}{6}} N_{\max}^{0-} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}.$$

Now, we estimate $\int_0^\delta L_2$. Our task is to prove that

$$(4.15) \quad \int_0^\delta \sum_{n_1+n_2+n_3=0} \left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |n_1+n_2| \widehat{u}_1(n_1, t) \widehat{u}_2(n_2, t) \widehat{v}_3(n_3, t) \lesssim N^{-1+\delta \frac{19}{24}} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}$$

- $|n_2| \ll |n_1| \sim |n_3| \gtrsim N$. We estimate the multiplier by

$$\left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left\langle \left(\frac{N_2}{N} \right)^{\frac{1}{2}} \right\rangle.$$

Thus, using $L_{xt}^2 L_{xt}^4 L_{xt}^4$ Hölder inequality and the lemmas 2.1 and 2.2

$$\begin{aligned} \int_0^\delta L_2 &\lesssim \left\langle \left(\frac{N_2}{N} \right)^{\frac{1}{2}} \right\rangle \frac{1}{\langle N_2 \rangle N_3} \delta^{\frac{19}{24}} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}} \\ &\lesssim N^{-1+\delta \frac{19}{24}} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}. \end{aligned}$$

- $|n_1| \ll |n_2| \sim |n_3|$. This case is similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. Estimating the multiplier by

$$\left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_2}{N} \right)^{1-}$$

we conclude

$$\begin{aligned} \int_0^\delta L_2 &\lesssim \left(\frac{N_2}{N}\right)^{1-} \frac{1}{N_1 N_2} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}} \\ &\lesssim N^{-2+} \delta^{\frac{19}{24}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}. \end{aligned}$$

Next, let us compute $\int_0^\delta L_3$. We claim that

$$(4.16) \quad \begin{aligned} \int_0^\delta \sum_{n_1+n_2+n_3=0} &\left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |\widehat{u}_1(n_1, t) \widehat{v}_2(n_2, t) |n_3| \widehat{u}_3(n_3, t) \\ &\lesssim N^{-2+} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \end{aligned}$$

- $|n_2| \ll |n_1| \sim |n_3|$, $|n_1| \gtrsim N$. The multiplier is bounded by

$$\begin{cases} \left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left| \frac{\nabla m(n_1) \cdot n_2}{m(n_1)} \right| \lesssim \frac{N_2}{N_1}, & \text{if } |n_2| \leq N, \text{ and} \\ \left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_2}{N}\right)^{1/2}, & \text{if } |n_2| \geq N. \end{cases}$$

So, it is not hard to see that

$$\int_0^\delta L_3 \lesssim N^{-2+} \delta^{\frac{19}{24}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}$$

- $|n_1| \ll |n_2| \sim |n_3|$, $|n_2| \gtrsim N$. This case is completely similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. Since the multiplier is bounded by N_2/N , we get

$$\int_0^\delta L_3 \lesssim N^{-2+} \delta^{\frac{19}{24}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.$$

Finally, it remains to estimate the contribution of $\int_0^\delta L_4$. It suffices to see that

$$(4.17) \quad \begin{aligned} \int_0^\delta \sum_{n_1+n_2+n_3+n_4=0} &\left| \frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \right| |n_4| \prod_{j=1}^4 \widehat{u}_j(n_j, t) \lesssim \\ &N^{-2+} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \end{aligned}$$

- $N_1, N_2, N_3 \gtrsim N$. Since the multiplier verifies

$$\left| \frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \right| \lesssim \left(\frac{N_1}{N} \frac{N_2}{N} \frac{N_3}{N}\right)^{\frac{1}{2}},$$

the application of $L_{xt}^4 L_{xt}^4 L_{xt}^4$ Hölder inequality and the lemmas 2.1, 2.2 yields

$$\int_0^\delta L_4 \lesssim \left(\frac{N_1}{N} \frac{N_2}{N} \frac{N_3}{N}\right)^{\frac{1}{2}} \frac{\delta^{\frac{1}{2}-}}{N_1 N_2 N_3} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-3+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_2 \gtrsim N$ and $N_3, N_4 \ll N_1, N_2$. Here the multiplier is bounded by $\left(\frac{N_1}{N} \frac{N_2}{N}\right)^{\frac{1}{2}} \left\langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \right\rangle$. Hence,

$$\int_0^\delta L_4 \lesssim \left(\frac{N_1}{N} \frac{N_2}{N}\right)^{\frac{1}{2}} \left\langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \right\rangle \frac{\delta^{\frac{1}{2}-}}{N_1 N_2 \langle N_3 \rangle} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+\delta^{\frac{1}{2}-}} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_4 \gtrsim N$ and $N_2, N_3 \ll N_1, N_4$. In this case we have the following estimates for the multiplier

$$\left| \frac{m(n_1 + n_2 + n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \right| \lesssim \begin{cases} \left| \frac{\nabla m(n_1)(n_2+n_3)}{m(n_1)} \right| \lesssim \frac{N_2+N_3}{N_1}, & \text{if } N_2, N_3 \leq N \\ \left(\frac{N_1}{N} \frac{N_2}{N}\right)^{\frac{1}{2}} \left\langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \right\rangle, & \text{if } N_2 \geq N, \\ \left(\frac{N_1}{N} \frac{N_3}{N}\right)^{\frac{1}{2}} \left\langle \left(\frac{N_2}{N}\right)^{\frac{1}{2}} \right\rangle, & \text{if } N_3 \geq N. \end{cases}$$

Therefore, it is not hard to see that, in any of the situations $N_2, N_3 \leq N$, $N_2 \geq N$ or $N_3 \geq N$, we have

$$\int_0^\delta L_4 \lesssim N^{-2+\delta^{\frac{1}{2}-}} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_2 \sim N_4 \gtrsim N$ and $N_3 \ll N_1, N_2, N_4$. Here we have the following bound

$$\int_0^\delta L_4 \lesssim \left(\frac{N_1}{N} \frac{N_2}{N}\right)^{\frac{1}{2}} \left\langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \right\rangle \frac{\delta^{\frac{1}{2}-}}{N_1 N_2 N_3} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

At this point, clearly the bounds (4.14), (4.15), (4.16) and (4.17) concludes the proof of the proposition 4.1. \square

4.2. Estimates for the modified E-functional.

Proposition 4.2. *Let (u, v) be a solution of (1.1) on the time interval $[0, \delta]$ such that $\int_{\mathbb{T}} v = 0$. Then, for any $N \geq 1$, $s > 1/2$,*

$$(4.18) \quad \begin{aligned} & |E(Iu(\delta), Iv(\delta)) - E(Iu(0), Iv(0))| \lesssim \\ & \left(N^{-1+\delta^{\frac{1}{6}-}} + N^{-\frac{2}{3}+\delta^{\frac{3}{8}-}} + N^{-\frac{3}{2}+\delta^{\frac{1}{8}-}} \right) (\|Iu\|_{X^1} + \|Iv\|_{Y^1})^3 + \\ & N^{-1+\delta^{\frac{1}{2}-}} (\|Iu\|_{X^1} + \|Iv\|_{Y^1})^4 + N^{-2+\delta^{\frac{1}{2}-}} \|Iu\|_{X^1}^4 (\|Iu\|_{X^1}^2 + \|Iv\|_{Y^1}). \end{aligned}$$

Proof. Again we integrate (4.12) with respect to $t \in [0, \delta]$, decompose the frequencies into dyadic blocks, etc., so that our objective is to bound the (integral over $[0, \delta]$ of the) E_j for each $j = 1, \dots, 12$.

For the expression $\int_0^\delta E_1$, apply the lemma 2.3. We obtain

$$\left| \int_0^\delta E_1 \right| \lesssim \|Iv_{xx}\|_{Y^{-1}} \|IvIv_x - I(vv_x)\|_{W^1} \lesssim \|Iv\|_{Y^1} \|IvIv_x - I(vv_x)\|_{W^1}$$

Writing the definition of the norm W^1 , it suffices to prove the bound

$$(4.19) \quad \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} +$$

$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \lesssim$$

$$N^{-1+\delta^{\frac{1}{6}}} \|v_1\|_{Y^{1,1/2}} \|v_2\|_{Y^{1,1/2}}.$$

Recall that the dispersion relation $\sum_{j=1}^3 \tau_j - n_j^3 = -3n_1n_2n_3$ implies that, since $n_1n_2n_3 \neq 0$, if we put $L_j := |\tau_j - n_j^3|$ and $L_{\max} = \max\{L_j; j = 1, 2, 3\}$, then $L_{\max} \gtrsim \langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle$.

- $|n_2| \sim |n_3| \gtrsim N$, $|n_1| \ll |n_2|$. The multiplier is bounded by

$$\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \begin{cases} \frac{N_1}{N_2}, & \text{if } |n_1| \leq N, \\ \left(\frac{N_1}{N}\right)^{\frac{1}{2}}, & \text{if } |n_1| \geq N. \end{cases}$$

Thus, if $|\tau_3 - n_3^3| = L_{\max}$, we have

$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2}$$

$$\lesssim \begin{cases} \frac{N_1}{N_2} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}}} \|v_1\|_{L_{xt}^4} \|(v_2)_x\|_{L_{xt}^4} \lesssim N^{-1+\delta^{\frac{1}{3}}-N_{\max}^{0-}} \|v_1\|_{Y^{1,1/2}} \|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1| \leq N, \\ \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \frac{N_3}{N_1} \frac{1}{(N_1 N_2 N_3)^{\frac{1}{2}}} \|v_1\|_{L_{xt}^4} \|(v_2)_x\|_{L_{xt}^4} \lesssim N^{-\frac{3}{2}+\delta^{\frac{1}{3}}-N_{\max}^{0-}} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}, & \text{if } |n_1| \geq N. \end{cases}$$

and

$$\left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1}$$

$$\lesssim \begin{cases} \frac{N_1}{N_2} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}}} \|v_1\|_{L_{xt}^4} \|(v_2)_x\|_{L_{xt}^4} \lesssim N^{-1+\delta^{\frac{1}{3}}-N_{\max}^{0-}} \|v_1\|_{Y^{1,1/2}} \|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1| \leq N, \\ \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \frac{N_3}{N_1} \frac{\delta^{\frac{1}{3}-}}{(N_1 N_2 N_3)^{\frac{1}{2}}} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-\frac{3}{2}+\delta^{\frac{1}{3}}-N_{\max}^{0-}} \|v_1\|_{Y^{1,\frac{1}{2}}} \|v_2\|_{Y^{1,\frac{1}{2}}}, & \text{if } |n_1| \geq N. \end{cases}$$

If either $|\tau_1 - n_1^3| = L_{\max}$ or $|\tau_2 - n_2^3| = L_{\max}$, we have

$$\begin{aligned} & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} \\ & \lesssim \begin{cases} \frac{N_1}{N_2} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}}} \frac{\delta^{\frac{1}{6}-}}{N_1} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}} \lesssim N^{-1} + \delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1, 1/2}} \|v_2\|_{Y^{1, 1/2}}, & \text{if } |n_1| \leq N, \\ \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \frac{N_3}{N_1} \frac{1}{(N_1 N_2 N_3)^{\frac{1}{2}}} \delta^{\frac{1}{6}-} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}} \lesssim N^{-\frac{3}{2}+} \delta^{\frac{1}{3}-} N_{\max}^{0-} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}}, & \text{if } |n_1| \geq N. \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \\ & \lesssim \begin{cases} \frac{N_1}{N_2} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}-}} \frac{\delta^{\frac{1}{6}-}}{N_1} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}} \lesssim N^{-1} + \delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1, 1/2}} \|v_2\|_{Y^{1, 1/2}}, & \text{if } |n_1| \leq N, \\ \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \frac{N_3}{N_1} \frac{\delta^{\frac{1}{6}-}}{(N_1 N_2 N_3)^{\frac{1}{2}-}} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}} \lesssim N^{-\frac{3}{2}+} \delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}}, & \text{if } |n_1| \geq N. \end{cases} \end{aligned}$$

- $|n_1| \sim |n_2| \gtrsim N$. Estimating the multiplier by

$$\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_1}{N}\right)^{1-},$$

we have that, if $|\tau_3 - n_3^3| = L_{\max}$,

$$\begin{aligned} & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} + \\ & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \\ & \lesssim \left\{ \left(\frac{N_1}{N}\right)^{1-} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}}} \frac{\delta^{\frac{1}{3}-}}{N_1} + \left(\frac{N_1}{N}\right)^{1-} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}-}} \frac{\delta^{\frac{1}{3}-}}{N_1} \right\} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}} \\ & \lesssim N^{-\frac{3}{2}+} \delta^{\frac{1}{3}-} N_{\max}^{0-} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}} \end{aligned}$$

and, if either $|\tau_1 - n_1^3| = L_{\max}$ or $|\tau_2 - n_2^3| = L_{\max}$,

$$\begin{aligned} & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} + \\ & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v}_1(n_1, \tau_1) n_2 \widehat{v}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \\ & \lesssim \left\{ \left(\frac{N_1}{N} \right)^{1-} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}}} \frac{\delta^{\frac{1}{6}-}}{N_1} + \left(\frac{N_1}{N} \right)^{1-} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}-}} \frac{\delta^{\frac{1}{6}-}}{N_1} \right\} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}} \\ & \lesssim N^{-\frac{3}{2}+} \delta^{\frac{1}{6}-} N_{\max}^{0-} \|v_1\|_{Y^{1, \frac{1}{2}}} \|v_2\|_{Y^{1, \frac{1}{2}}}. \end{aligned}$$

For the expression $\int_0^\delta E_2$, it suffices to prove that

$$(4.20) \quad \begin{aligned} & \left| \int_0^\delta \sum \frac{m(n_3 + n_4) - m(n_3)m(n_4)}{m(n_3)m(n_4)} \widehat{v}_1(n_1, t) \widehat{v}_2(n_2, t) \widehat{v}_3(n_3, t) n_4 \widehat{v}_4(n_4, t) \right| \lesssim \\ & N^{-2+} \delta^{\frac{2}{3}-} \prod_{j=1}^4 \|v_j\|_{Y^{1, 1/2}}. \end{aligned}$$

Since at least two of the N_i are $\gtrsim N$, we can assume that $N_1 \geq N_2 \geq N_3$ and $N_1 \gtrsim N$. Hence,

$$\begin{aligned} \int_0^\delta E_2 & \lesssim \\ & \begin{cases} \left(\frac{N_1}{N} \right)^{1-} \frac{\delta^{\frac{2}{3}-}}{N_1 N_2 N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1, 1/2}} \lesssim N^{-2+} \delta^{\frac{2}{3}-} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1, 1/2}}, & \text{if } |n_3| \sim |n_4| \gtrsim N, \\ \frac{N_3}{N_4} \frac{\delta^{\frac{2}{3}-}}{N_1 N_2 N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1, 1/2}} \lesssim N^{-2+} \delta^{\frac{2}{3}-} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1, 1/2}}, & \text{if } |n_3| \ll |n_4|, |n_3| \leq N |n_4| \gtrsim N, \\ \left(\frac{N_3}{N} \right)^{\frac{1}{2}} \frac{\delta^{\frac{2}{3}-}}{N_1 N_2 N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1, \frac{1}{2}}} \lesssim N^{-2+} \delta^{\frac{2}{3}-} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1, \frac{1}{2}}}, & \text{if } |n_3| \ll |n_4|, |n_3| \geq N, |n_4| \gtrsim N. \end{cases} \end{aligned}$$

Next, we estimate the contribution of $\int_0^\delta E_3$. We claim that

$$(4.21) \quad \begin{aligned} & \int_0^\delta \sum \frac{m(n_1 n_2 n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \widehat{u}_1(n_1, t) \widehat{u}_2(n_2, t) \widehat{u}_3(n_3, t) |n_4|^2 \widehat{u}_4(n_4, t) \lesssim \\ & N^{-1+} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1, 1/2}}. \end{aligned}$$

- $|n_1| \sim |n_2| \sim |n_3| \sim |n_4| \gtrsim N$. Since the multiplier satisfies

$$\frac{m(n_1 n_2 n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \lesssim \left(\frac{N_1}{N} \right)^{\frac{3}{2}}$$

we obtain

$$\int_0^\delta E_3 \lesssim \left(\frac{N_1}{N}\right)^{\frac{3}{2}} \frac{N_4}{N_1 N_2 N_3} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- Exactly two frequencies are $\gtrsim N$. We consider the most difficult case $|n_4| \gtrsim N$, $|n_1| \sim |n_4|$ and $|n_2|, |n_3| \ll |n_1|, |n_4|$. The multiplier is estimated by

$$\frac{m(n_1 n_2 n_3) - m(n_1) m(n_2) m(n_3)}{m(n_1) m(n_2) m(n_3)} \lesssim \begin{cases} \langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \rangle \left(\frac{N_2}{N}\right)^{\frac{1}{2}}, & \text{if } |n_2| \geq N, \\ \langle \left(\frac{N_2}{N}\right)^{\frac{1}{2}} \rangle \left(\frac{N_3}{N}\right)^{\frac{1}{2}}, & \text{if } |n_3| \geq N, \\ \frac{N_2 + N_3}{N_1}, & \text{if } |n_2|, |n_3| \leq N. \end{cases}$$

Thus,

$$\int_0^\delta E_3 \lesssim N^{-1+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- Exactly three frequencies are $\gtrsim N$. The most difficult case is $|n_1| \sim |n_2| \sim |n_4| \gtrsim N$ and $|n_3| \ll |n_1|, |n_2|, |n_4|$. Here the multiplier is bounded by

$$\frac{m(n_1 n_2 n_3) - m(n_1) m(n_2) m(n_3)}{m(n_1) m(n_2) m(n_3)} \lesssim \left(\frac{N_1}{N} \frac{N_2}{N}\right)^{\frac{1}{2}} \left\langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \right\rangle.$$

Hence,

$$\int_0^\delta E_3 \lesssim \left(\frac{N_1}{N} \frac{N_2}{N}\right)^{\frac{1}{2}} \left\langle \left(\frac{N_3}{N}\right)^{\frac{1}{2}} \right\rangle \frac{N_4}{N_1 N_2 N_3} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-1+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

The contribution of $\int_0^\delta E_4$ is controlled if we are able to show that

$$(4.22) \quad \int_0^\delta \sum \frac{m(n_1 + n_2) - m(n_1) m(n_2)}{m(n_1) m(n_2)} \widehat{v}_1(n_1, t) |n_2| \widehat{v}_2(n_2, t) \widehat{u}_3(n_3, t) \widehat{u}_4(n_4, t) \lesssim N^{-1+} \delta^{\frac{7}{12}-} \prod_{j=1}^2 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}}.$$

We crudely bound the multiplier by

$$\left| \frac{m(n_1 + n_2) - m(n_1) m(n_2)}{m(n_1) m(n_2)} \right| \lesssim \left(\frac{N_{\max}}{N}\right)^{1-}.$$

The most difficult case is $|n_2| \geq N$. We have two possibilities:

- Exactly two frequencies are $\gtrsim N$. We can assume $N_3 \ll N_2$. In particular,

$$\int_0^\delta E_4 \lesssim \left(\frac{N_{\max}}{N}\right)^{1-} \frac{\delta^{\frac{7}{12}}}{N_1 N_3 N_4} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}} \lesssim N^{-1+} \delta^{\frac{7}{12}-} N_{\max}^{0-} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}}.$$

- At least three frequencies are $\gtrsim N$. In this case,

$$\int_0^\delta E_4 \lesssim N^{-2+\delta\frac{7}{12}} N_{\max}^{0-} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}}.$$

The expression $\int_0^\delta E_5$ is controlled if we are able to prove

$$(4.23) \quad \int_0^\delta \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, t) \widehat{u}_2(n_2, t) \widehat{v}_3(n_3, t) |n_4| \widehat{v}_4(n_4, t) \lesssim N^{-1+\delta\frac{7}{12}} \prod_{j=1}^2 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}}.$$

This follows directly from the previous analysis for (4.22).

For the term $\int_0^\delta E_6$, we apply the lemma 2.3 to obtain

$$\int_0^\delta E_6 \lesssim \|(Iv)_{xx}\|_{Y^{-1}} \|(|Iu|^2 - I(|u|^2))_x\|_{W^1} \lesssim \|Iv\|_{Y^1} \|(|Iu|^2 - I(|u|^2))_x\|_{W^1}.$$

So, the definition of the W^1 norm means that we have to prove

$$(4.24) \quad \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} + \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \lesssim \left\{ N^{-\frac{3}{2}+\delta\frac{1}{8}} + N^{-\frac{2}{3}\delta\frac{3}{8}} \right\} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}}.$$

Note that $\sum \tau_j = 0$ and $\sum n_j = 0$. In particular, we obtain the dispersion relation

$$\tau_3 - n_3^3 + \tau_2 + n_2^2 + \tau_1 + n_1^2 = -n_3^3 + n_1^2 + n_2^2.$$

- $|n_1| \gtrsim N$, $|n_2| \ll |n_1|$. Denoting by $L_1 := |\tau_1 + n_1^2|$, $L_2 := |\tau_2 + n_2^2|$ and $L_3 := |\tau_3 - n_3^3|$, the dispersion relation says that in the present situation $L_{\max} := \max\{L_j\} \gtrsim N^3$. Since the multiplier is bounded by

$$\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \begin{cases} \frac{\nabla m(n_1)n_2}{m(n_1)} \lesssim \frac{N_2}{N_1}, & \text{if } |n_2| \leq N, \\ \left(\frac{N_2}{N}\right)^{\frac{1}{2}}, & \text{if } |n_2| \geq N, \end{cases}$$

we deduce that

$$\begin{aligned} & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} + \\ & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \lesssim \\ & \frac{N_3^2}{N_3^{\frac{3}{2}-}} \frac{\delta^{\frac{1}{8}-}}{N N_1} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \lesssim N^{-\frac{3}{2}+} \delta^{\frac{1}{8}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}}. \end{aligned}$$

- $|n_1| \sim |n_2| \gtrsim N$, $|n_3|^3 \gg |n_2|^2$. In the present case the multiplier is bounded by $(\frac{N_1}{N})^{1-}$ and the dispersion relation says that $L_{\max} \gtrsim N_3^3$. Thus,

$$\begin{aligned} & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} + \\ & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \lesssim \\ & \frac{N_3^2}{N_3^{\frac{3}{2}-}} \left(\frac{N_1}{N} \right)^{1-} \frac{\delta^{\frac{1}{8}-}}{N_1 N_2} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \lesssim N^{-\frac{3}{2}+} \delta^{\frac{1}{8}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}}. \end{aligned}$$

- $|n_1| \sim |n_2| \gtrsim N$ and $|n_3|^3 \lesssim |n_2|^2$. Here the dispersion relation does not give useful information about L_{\max} . Since the multiplier is estimated by $(\frac{N_2}{N})^{\frac{1}{2}}$, we obtain the crude bound

$$\begin{aligned} & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3, \tau_3}^2} + \\ & \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3^3 \rangle^{\frac{1}{2}}} |n_3| \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{u}_1(n_1, \tau_1) \widehat{u}_2(n_2, \tau_2) \right\|_{L_{n_3}^2 L_{\tau_3}^1} \lesssim \\ & N_3^2 \left(\frac{N_2}{N} \right)^{\frac{1}{2}} \frac{\delta^{\frac{3}{8}-}}{N_1 N_2} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \lesssim N^{-\frac{2}{3}+} \delta^{\frac{3}{8}-} N_{\max}^{0-} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}}. \end{aligned}$$

Next, the desired bound related to $\int_0^\delta E_7$ follows from

$$(4.25) \quad \begin{aligned} & \int_0^\delta \sum \left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |n_1 + n_2| \widehat{u}_1(n_1, t) \widehat{v}_2(n_2, t) |n_3| \widehat{u}_3(n_3, t) \lesssim \\ & N^{-1+} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \end{aligned}$$

- $|n_1| \ll |n_2| \gtrsim N$. The multiplier is $\lesssim (|n_2|/N)^{1/2}$ so that

$$\int_0^\delta E_7 \lesssim \frac{1}{N^{1/2}} \int_0^\delta \sum |n_1 + n_2| \widehat{u}_1(n_1, t) |n_2|^{1/2} \widehat{v}_2(n_2, t) |n_3| \widehat{u}_3(n_3, t) \lesssim N^{-1} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.$$

- $|n_1| \sim |n_2| \gtrsim N$. The multiplier is $\lesssim |n_2|/N$. Hence,

$$\int_0^\delta E_7 \lesssim N^{-1} \delta^{\frac{19}{24}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.$$

- $|n_1| \gtrsim N$, $|n_2| \leq N$. The multiplier is again $\lesssim N_2/N$, so that it can be estimated as above.

Now we turn to the term $\int_0^\delta E_8$. The objective is to show that

$$(4.26) \quad \int_0^\delta \left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |n_1 + n_2| \prod_{j=1}^4 \widehat{u}_j(n_j, t) \lesssim N^{-1+\delta^{\frac{1}{2}-}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}$$

- At least three frequencies are $\gtrsim N$. We can assume $|n_1| \geq |n_2|$. The multiplier is bounded by N_{\max}/N so that

$$\int_0^\delta E_8 \lesssim \frac{N_{\max}}{N} \frac{\delta^{\frac{1}{2}-}}{N_2 N_3 N_4} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+\delta^{\frac{1}{2}-}} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- Exactly two frequencies are $\gtrsim N$. Without loss of generality, we suppose $|n_1| \sim |n_2| \gtrsim N$ and $|n_3|, |n_4| \ll N$. Since the multiplier satisfies

$$\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left(\frac{N_{\max}}{N} \right)^{1-},$$

we get the bound

$$\int_0^\delta E_8 \lesssim \left(\frac{N_{\max}}{N} \right)^{1-} \frac{\delta^{\frac{1}{2}-}}{N_2 N_3 N_4} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-1+\delta^{\frac{1}{2}-}} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

The contribution of $\int_0^\delta E_9$ is estimated if we prove that

$$(4.27) \quad \int_0^\delta \left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \widehat{u}_1(n_1, t) \widehat{v}_2(n_2, t) \widehat{u}_3(n_3, t) \widehat{v}_4(n_4, t) \lesssim N^{-2+\delta^{\frac{7}{12}-}} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}}.$$

This follows since at least two frequencies are $\gtrsim N$ and the multiplier is always bounded by $(N_{\max}/N)^{1-}$, so that

$$\begin{aligned} \int_0^\delta E_9 &\lesssim \left(\frac{N_{\max}}{N}\right)^{1-} \|u_1\|_{L^4} \|v_2\|_{L^4} \|u_3\|_{L^4} \|v_4\|_{L^4} \lesssim \\ &\left(\frac{N_{\max}}{N}\right)^{1-} \frac{\delta^{\frac{1}{4}+\frac{1}{3}-}}{N_1 N_2 N_3 N_4} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}} \lesssim \\ &N^{-2+} \delta^{\frac{7}{12}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}}. \end{aligned}$$

Now, we treat the term $\int_0^\delta E_{10}$. It is sufficient to prove

$$(4.28) \quad \int_0^\delta \sum \left| \frac{m(n_4 + n_5 + n_6) - m(n_4)m(n_5)m(n_6)}{m(n_4)m(n_5)m(n_6)} \right| \left| \prod_{j=1}^6 \widehat{u}_j(n_j, t) \right| \lesssim N^{-2+} \delta^{\frac{1}{2}-} \prod_{j=1}^6 \|u_j\|_{X^1}$$

However, this follows easily from the facts that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are $\gtrsim N$, say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$, the Strichartz bound $X^{0,3/8} \subset L^4$ and the inclusion¹ $X^{\frac{1}{2}+} \subset L_{xt}^\infty$. Indeed, if we combine these informations, it is not hard to get

$$\int_0^\delta E_{10} \lesssim \left(\frac{N_{\max}}{N}\right)^{\frac{3}{2}} \frac{1}{N_{i_1} N_{i_2} N_{i_3} N_{i_4}} \delta^{\frac{1}{2}-} \frac{1}{(N_{i_5} N_{i_6})^{1/2-}} \prod_{j=1}^6 \|u_j\|_{X^1} \lesssim N^{-2+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^6 \|u_j\|_{X^1}$$

For the expression $\int_0^\delta E_{11}$, we use again that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are $\gtrsim N$ (say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$), the Strichartz bounds in lemma 2.1 and the inclusions $X^{\frac{1}{2}+}, Y^{\frac{1}{2}+} \subset L_{xt}^\infty$ to obtain

$$(4.29) \quad \begin{aligned} &\int_0^\delta \sum \left| \frac{m(n_1 + n_2 + n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \right| \left| \prod_{j=1}^4 \widehat{u}_j(n_j, t) \widehat{v}_5(n_5, t) \right| \lesssim \\ &\left(\frac{N_{\max}}{N}\right)^{\frac{3}{2}} \frac{1}{N_{i_1} N_{i_2} N_{i_3} N_{i_4}} \frac{\delta^{\frac{1}{2}-}}{N_{i_5}^{1/2-}} \prod_{j=1}^4 \|u_j\|_{X^1} \|v_5\|_{Y^1} \lesssim \\ &N^{-2+} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^1} \|v_5\|_{Y^1}. \end{aligned}$$

The analysis of $\int_0^\delta E_{12}$ is similar to the $\int_0^\delta E_{11}$. This completes the proof of the proposition 4.2. \square

¹This inclusion is an easy consequence of Sobolev embedding.

5. GLOBAL WELL-POSEDNESS BELOW THE ENERGY SPACE

In this section we combine the variant local well-posedness result in proposition 3.1 with the two almost conservation results in the propositions 4.1 and 4.2 to prove the theorem 1.1.

Remark 5.1. *Note that the spatial mean $\int_{\mathbb{T}} v(t, x) dx$ is preserved during the evolution (1.1). Thus, we can assume that the initial data v_0 has zero-mean, since otherwise we make the change $w = v - \int_{\mathbb{T}} v_0 dx$ at the expense of two harmless linear terms (namely, $u \int_{\mathbb{T}} v_0 dx$ and $\partial_x v \int_{\mathbb{T}} v_0$).*

The definition of the I-operator implies that the initial data satisfies $\|Iu_0\|_{H^1}^2 + \|Iv_0\|_{H^1}^2 \lesssim N^{2(1-s)}$ and $\|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 \lesssim 1$. By the estimates (4.4) and (4.8), we get that $|L(Iu_0, Iv_0)| \lesssim N^{1-s}$ and $|E(Iu_0, Iv_0)| \lesssim N^{2(1-s)}$.

Also, any bound for $L(Iu, Iv)$ and $E(Iu, Iv)$ of the form $|L(Iu, Iv)| \lesssim N^{1-s}$ and $|E(Iu, Iv)| \lesssim N^{2(1-s)}$ implies that $\|Iu\|_{L^2}^2 \lesssim M$, $\|Iv\|_{L^2}^2 \lesssim N^{1-s}$ and $\|Iu\|_{H^1}^2 + \|Iv\|_{H^1}^2 \lesssim N^{2(1-s)}$.

Given a time T , if we can uniformly bound the H^1 -norms of the solution at times $t = \delta$, $t = 2\delta$, etc., the local existence result in proposition 3.1 says that the solution can be extended up to any time interval where such a uniform bound holds. On the other hand, given a time T , if we can interact $T\delta^{-1}$ times the local existence result, the solution exists in the time interval $[0, T]$. So, in view of the propositions 4.1 and 4.2, it suffices to show

$$(5.1) \quad (N^{-1+\delta^{\frac{19}{24}}-} N^{3(1-s)} + N^{-2+\delta^{\frac{1}{2}}-} N^{4(1-s)}) T \delta^{-1} \lesssim N^{1-s}$$

and

$$(5.2) \quad \left\{ (N^{-1+\delta^{\frac{1}{6}}-} + N^{-\frac{2}{3}+\delta^{\frac{3}{8}}-} + N^{-\frac{3}{2}+\delta^{\frac{1}{8}}-}) N^{3(1-s)} + N^{-1+\delta^{\frac{1}{2}}-} N^{4(1-s)} + N^{-2+\delta^{\frac{1}{2}}-} N^{6(1-s)} \right\} \frac{T}{\delta} \lesssim N^{2(1-s)}$$

At this point, we recall that the proposition 3.1 says that $\delta \sim N^{-\frac{16}{3}(1-s)-}$ if $\beta \neq 0$ and $\delta \sim N^{-8(1-s)-}$ if $\beta = 0$. Hence,

- $\beta \neq 0$. The condition (5.1) holds for

$$-1 + \frac{5}{24} \frac{16}{3} (1-s) + 3(1-s) < (1-s), \text{ i.e. , } s > 19/28$$

and

$$-2 + \frac{1}{2} \frac{16}{3} (1-s) + 4(1-s) < (1-s), \text{ i.e. , } s > 11/17;$$

Similarly, the condition (5.2) is satisfied if

$$-1 + \frac{5}{6} \frac{16}{3} (1-s) + 3(1-s) < 2(1-s), \text{ i.e. , } s > 40/49;$$

$$-\frac{2}{3} + \frac{5}{6} \frac{16}{3} (1-s) + 3(1-s) < 2(1-s), \text{ i.e. , } s > 11/13;$$

$$-\frac{3}{2} + \frac{7}{8} \frac{16}{3}(1-s) + 3(1-s) < 2(1-s), \text{ i.e. , } s > 25/34;$$

$$-1 + \frac{1}{2} \frac{16}{3}(1-s) + 4(1-s) < 2(1-s), \text{ i.e. , } s > 11/14$$

and

$$-2 + \frac{1}{2} \frac{16}{3}(1-s) + 6(1-s) < 2(1-s), \text{ i.e. , } s > 7/10.$$

Thus, we conclude that the non-resonant NLS-KdV system is globally well-posed for any $s > 11/13$.

- $\beta = 0$. The condition (5.1) is fulfilled when

$$-1 + \frac{5}{24} 8(1-s) + 3(1-s) < (1-s), \text{ i.e. , } s > 8/11$$

and

$$-2 + \frac{1}{2} 8(1-s) + 4(1-s) < (1-s), \text{ i.e. , } s > 5/7;$$

Similarly, the condition (5.2) is verified for

$$-1 + \frac{5}{6} 8(1-s) + 3(1-s) < 2(1-s), \text{ i.e. , } s > 20/23;$$

$$-\frac{2}{3} + \frac{5}{6} 8(1-s) + 3(1-s) < 2(1-s), \text{ i.e. , } s > 8/9;$$

$$-\frac{3}{2} + \frac{7}{8} 8(1-s) + 3(1-s) < 2(1-s), \text{ i.e. , } s > 13/16;$$

$$-1 + \frac{1}{2} 8(1-s) + 4(1-s) < 2(1-s), \text{ i.e. , } s > 5/6$$

and

$$-2 + \frac{1}{2} 8(1-s) + 6(1-s) < 2(1-s), \text{ i.e. , } s > 3/4.$$

Hence, we obtain that the resonant NLS-KdV system is globally well-posed for any $s > 8/9$.

REFERENCES

- [1] A. Arbieto, A. Corcho and C. Matheus, *Rough solutions for the periodic Schrödinger - Korteweg - deVries system*, Preprint (2005).
- [2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations*, Geometric and Functional Anal., **3** (1993), 107–156, 209–262.
- [3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Sharp global well-posedness for KdV and modified KdV on \mathbb{R} and \mathbb{T}* , J. Amer. Math. Soc., **16** (2003), 705–749.
- [4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Multilinear estimates for periodic KdV equations, and applications*, J. Funct. Analysis, **211** (2004), 173–218.
- [5] A. J. Corcho, and F. Linares, *Well-posedness for the Schrödinger - Kortweg-de Vries system*, Preprint (2005).

- [6] H. Pecher, *The Cauchy problem for a Schrödinger - Kortweg - de Vries system with rough data*, Preprint (2005).
- [7] M. Tsutsumi, *Well-posedness of the Cauchy problem for a coupled Schrödinger-KdV equation*, Math. Sciences Appl., **2** (1993), 513–528.

CARLOS MATHEUS

IMPA, ESTRADA DONA CASTORINA 110, RIO DE JANEIRO, 22460–320, BRAZIL.
E-mail address: `matheus@impa.br`