Journal of the Inst. of Math. Jussieu (2005) **4**(4), 639–651 © Cambridge University Press 639
doi:10.1017/S1474748005000137 Printed in the United Kingdom

GEOMETRICAL VERSUS TOPOLOGICAL PROPERTIES OF MANIFOLDS

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(Received 29 January 2004; accepted 11 November 2004)

Abstract Given a compact n-dimensional immersed Riemannian manifold M^n in some Euclidean space we prove that if the Hausdorff dimension of the singular set of the Gauss map is small, then M^n is homeomorphic to the sphere S^n.

Also, we define a concept of finite geometrical type and prove that finite geometrical type hypersurfaces with a small set of points of zero Gauss–Kronecker curvature are topologically the sphere minus a finite number of points. A characterization of the $2n$-catenoid is obtained.

Keywords: Hausdorff dimension; finite geometrical type; immersions

AMS 2000 *Mathematics subject classification:* Primary 53-02

1. Introduction

Let $f : M^n \to N^m$ be a C^1 map. We denote by

$$\operatorname{rank}(f) := \min_{p \in M} \operatorname{rank}(D_p f).$$

If $n = \dim M = \dim N = m$, let $C := \{p \in M : \det D_p f = 0\}$ be the set of *critical points* of f and let $S := f(C)$ be the set of *critical values* of f.

Now, let M^n be a compact, connected, boundaryless, n-dimensional manifold. Denote by H_s the s-dimensional Hausdorff measure and by $\dim_H(A)$ the Hausdorff dimension of $A \subset M^n$. For definitions see Section 2 below. Let x be an immersion $x : M^n \to \mathbb{R}^{n+1}$. In this case, let $G : M^n \to S^n$ be the Gauss map associated with x, C the critical points of G and S the critical values of G. We denote by $\dim_H(x) := \dim_H(S)$. By Moreira's improvement of the Morse–Sard theorem (see [**8**]), since G is a smooth map, we have that $\dim_H(S) \leqslant n - 1$.

In other words, if

$$\mathrm{Imm} = \{x : M \to \mathbb{R}^{n+1} : x \text{ is an immersion}\},$$

then $\sup_{x\in \mathrm{Imm}} \dim_H(x) \leqslant n-1$. Clearly, this supremum could be equal to $n-1$, as some immersions of S^n in $\mathbb{R}^{n+1}$ show (e.g. immersions with 'cylindrical pieces'). Our interest here is the number $\inf \dim_H(x)$. Before we discuss this, we introduce some definitions.

Definition 1.1. Given an immersion $x : M^n \to \mathbb{R}^{n+1}$ we define $\mathrm{rank}(x) := \mathrm{rank}(G)$, where G is the Gauss map for x.

Definition 1.2. We denote by $\mathcal{R}(k)$ the set $\mathcal{R}(k) = \{x \in \mathrm{Imm} : \mathrm{rank}(x) \geqslant k\}$. Define by $\alpha_k(M)$ the numbers:

$$\alpha_k(M) = \inf_{x\in\mathcal{R}(k)} \dim_H(x), \quad k = 0, \ldots, n.$$

If $\mathcal{R}(k) = \emptyset$ we define $\alpha_k(M) = n-1$.

Now, we are in position to state our first result.

Theorem A. *If M^n is a compact manifold with $n \geqslant 3$ such that $\alpha_k(M^n) < k - [\frac{1}{2}n]$, for some integer k, then M^n is homeomorphic to S^n ($[r]$ is the integer part of r).*

The proof of this theorem in the cases $n = 3$ and $n \geqslant 4$ are quite different. For higher dimensions, we can use the generalized Poincaré conjecture (Smale and Freedman) to obtain that the given manifold is a sphere. Since the Poincaré conjecture is not available in three dimensions, the proof, in this case, is a little bit different. We use a characterization theorem due to Bing to compensate the loss of Poincaré conjecture, as commented before.

To prove this theorem in the case $n = 3$, we proceed as follows.

(i) By a theorem of Bing (see [**2**]), we just need to prove that every piecewise smooth simple curve γ in M^3 lies in a topological cube $\mathcal{R}$ of M^3.

(ii) In order to prove it, we shall show that it is enough to prove for $\gamma \subset M - G^{-1}(S)$ and that $G : M - G^{-1}(S) \to S^3 - S$ is a diffeomorphism.

(iii) Finally, we produce a cube $\tilde{\mathcal{R}} \supset G(\gamma)$ in $S^3 - S$ and we obtain $\mathcal{R}$ pulling back this cube by G.

Observe that, by [**3**], in three dimensions there are always Euclidean codimension 1 immersions. In particular, it is reasonable to consider the following consequence of Theorem A.

Corollary 1.3. *The following statement is equivalent to the Poincaré conjecture.*

> *Simply connected 3-manifolds admit Euclidean codimension 1 immersions with rank at least 2 and Hausdorff dimension of the singular set for their Gauss map less than 1.*

Our motivation behind proving this theorem are results by do Carmo and Elbert [**4**] and Barbosa, Fukuoka and Mercuri [**1**]. Roughly speaking, they obtain topological results about certain manifolds provided they admit special codimension 1 immersions. These results motivate the question: how does the space of immersions (extrinsic information) influence the topology of M (intrinsic information)? Theorems A and B below are a partial answer to this question. The proofs of the theorems depend on the concept of Hausdorff dimension. Essentially, Hausdorff dimension is a fractal dimension that measures how 'small' a given set is with respect to usual 'regular' sets (e.g. smooth submanifolds, that always have integer Hausdorff dimension).

In Section 6 of this paper we obtain the following generalizations of Theorems A and B.

Definition 1.4. Let $\bar{M}^n$ be a compact (oriented) manifold and $p_1, \dots, p_k \in \bar{M}^n$. Let $M = \bar{M}^n - \{p_1, \dots, p_k\}$. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is of *finite geometrical type* (in a weaker sense than that of [**1**]) if M^n is complete in the induced metric, the Gauss map $G : M^n \to S^n$ extends continuously to a function $\bar{G} : \bar{M}^n \to S^n$ and the set $G^{-1}(S)$ has $H_{n-1}(G^{-1}(S)) = 0$ (this last condition occurs if $\operatorname{rank}(x) \geqslant k$ and $H_{k-1}(S) = 0$).

The conditions in the previous definition are satisfied by complete hypersurfaces with finite total curvature whose Gauss–Kronecker curvature $H_n = k_1 \cdots k_n$ does not change sign and vanishes in a small set, as shown by [**4**]. Recall that a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ has total finite curvature if $\int_M |A|^n \, \mathrm{d}M < \infty$, $|A| = (\sum_i k_i^2)^{1/2}$, k_i are the principal curvatures. With these observations, one has the following theorem.

Theorem B. *If $x : M^n \to \mathbb{R}^{n+1}$ is a hypersurface with finite geometrical type and $H_{k-[n/2]}(S) = 0$, $\operatorname{rank}(x) \geqslant k$. Then M^n is topologically a sphere minus a finite number of points, i.e. $\bar{M}^n \simeq S^n$. In particular, this result holds for complete hypersurfaces with finite total curvature and $H_{k-[n/2]}(S) = 0$, $\operatorname{rank}(x) \geqslant k$.*

For even dimensions, we follow [**1**] and improve Theorem B. In particular, we obtain the following characterization of $2n$-catenoids, as the unique minimal hypersurfaces of finite geometrical type.

Theorem C. *Let $x : M^{2n} \to \mathbb{R}^{2n+1}$, $n \geqslant 2$, be an immersion of finite geometrical type with $H_{k-n}(S) = 0$, $\operatorname{rank}(x) \geqslant k$. Then M^{2n} is topologically a sphere minus two points. If M^{2n} is minimal, M^{2n} is a $2n$-catenoid.*

2. Notation and statements

Let M^n be a smooth manifold. Before stating the proofs of the statements we fix some notation and collect some (useful) standard propositions about Hausdorff dimension (and limit capacity, another fractal dimension). For the proofs of these propositions we refer to [**5**].

Let X be a compact metric space and $A \subset X$. We define the s-dimensional Hausdorff measure of A by

$$H_s(A) := \lim_{\varepsilon \to 0} \inf \left\{ \sum_i (\operatorname{diam} U_i)^s : A \subset \bigcup U_i, U_i \text{ is open and } \operatorname{diam} U_i \leqslant \varepsilon, \ \forall i \in \mathbb{N} \right\}.$$

The *Hausdorff dimension* of A is

$$\dim_H(A) := \sup\{d \geqslant 0 : H_d(A) = \infty\} = \inf\{d \geqslant 0 : H_d(A) = 0\}.$$

A remarkable fact is that H_n coincides with Lebesgue measure for a smooth manifold M^n.

A related notion are the lower and upper *limit capacity* (sometimes called box counting dimension) defined by

$$\underline{\dim_B}(A) := \liminf_{\varepsilon\to 0} \log n(A,\varepsilon)/(-\log\varepsilon),$$
$$\overline{\dim_B}(A) := \limsup_{\varepsilon\to 0} \log n(A,\varepsilon)/(-\log\varepsilon),$$

where $n(A,\varepsilon)$ is the minimum number of ε-balls that cover A. If $d(A) = \underline{\dim_B}(A) = \overline{\dim_B}(A)$, we say that the limit capacity of A is $\dim_B(A) = d(A)$.

These fractal dimensions satisfy the properties expected for 'natural' notions of dimensions. For instance, $\dim_H(A) = m$ if A is a smooth m-submanifold.

Proposition 2.1. *The properties listed below hold.*

(i) $\dim_H(E) \leqslant \dim_H(F)$ *if* $E \subset F$.

(ii) $\dim_H(E \cup F) = \max\{\dim_H(E), \dim_H(F)\}$.

(iii) *If f is a Lipschitz map with Lipschitz constant C, then $H_s(f(E)) \leqslant C \cdot H_s(E)$. As a consequence,* $\dim_H(f(E)) \leqslant \dim_H E$.

(iv) *If f is a bi-Lipschitz map (e.g. a diffeomorphism),* $\dim_H(f(E)) = \dim_H(E)$.

(v) $\dim_H(A) \leqslant \underline{\dim_B}(A)$.

Analogous properties hold for lower and upper limit capacity. If E is countable, $\dim_H(E) = 0$ (*although we may have* $\dim_B(E) > 0$).

When we are dealing with product spaces, the relationship between Hausdorff dimension and limit capacity are the product formulae that follow.

Proposition 2.2. $\dim_H(E) + \dim_H(F) \leqslant \dim_H(E \times F) \leqslant \dim_H(E) + \overline{\dim_B}(F)$. *Moreover, $c \cdot H_s(E) \cdot H_t(F) \leqslant H_{s+t}(E \times F) \leqslant C \cdot H_s(E)$, where c depends only on s and t, C depends only on s and* $\overline{\dim_B}(F)$.

Before starting the necessary lemmas to prove the central results, we observe that it follows from the above discussion that if M and N are diffeomorphic n-manifolds then $\alpha_k(M) = \alpha_k(N)$. This proves the following lemma.

Lemma 2.3. *The numbers*

$$\alpha_k(M) = \inf_{x\in\mathcal{R}(k)} \dim_H(x), \quad \textit{for } k = 0, \dots, n,$$

are smooth invariants of M.

In particular, if $n = 3$ we also have that α_k are topological invariants. It is a consequence of a theorem due to Moise [**7**], which states that if M and N are homeomorphic 3-manifolds then they are diffeomorphic. Then, the following conjecture arises from Theorem A.

Conjecture 2.4. *If M^3 is simply connected, then*

$$\alpha_2(M^3) = \inf_{x \in \mathcal{R}(2)} \dim_H(x) < 1.$$

Cohen's theorem [**3**] says that there are immersions of compact n-manifolds M^n in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of 1s in the binary expansion of n. This implies, for the case $n = 3$, that we always have that $\mathrm{Imm} \neq \emptyset$. In particular, the implicit hypothesis of existence of codimension 1 immersions in Theorem A is not too restrictive and our conjecture is reasonable. We point out that Conjecture 2.4 is true if the Poincaré conjecture holds and, in this case, $\sup_{x \in \mathrm{Imm}} \mathrm{rank}(x) = 3$ and $\inf_{x \in \mathcal{R}(k)} \dim_H(x) = 0$, for all $0 \leqslant k \leqslant 3$. A corollary of Theorem A and this observation is the following corollary.

Corollary 2.5. *The Poincaré conjecture is equivalent to Conjecture 2.4.*

From this, a natural approach to Conjecture 2.4 is a deformation and desingularization argument for metrics given by pull-back of immersions in Imm. We observe that Moreira's theorem give us $\alpha_2(M^3) \leqslant 2$. This motivates the following question, which is a kind of step toward the Poincaré conjecture. However, this question is of independent interest, since it can be true even if the Poincaré conjecture is false.

Question 2.6. *For simply connected* 3*-manifolds, is it true that* $\alpha_2(M^3) < 2$*?*

3. Some lemmas

In this section, we prove some useful facts on the way to establishing Theorems A and B. The first one relates the Hausdorff dimension of subsets of smooth manifolds and rank of smooth maps.

Proposition 3.1. *Let $f : M^m \to N^n$ be a C^1-map and $A \subset N$. Then $\dim_H f^{-1}(A) \leqslant \dim_H(A) + n - \mathrm{rank}(f)$.*

Proof. The computation of Hausdorff dimension is a local problem. So, we can consider $p \in f^{-1}(S)$, coordinate neighbourhoods $p \in U$, $f(p) \in V$ fixed and $f = (f_1, \dots, f_n) : U \to V$. Making a change of coordinates (which does not change Hausdorff dimensions), we can suppose that $\tilde{f} = (f_1, \dots, f_r)$ is a submersion, where $r = \mathrm{rank}(f)$. By the local form of submersions, there is a diffeomorphism φ such that $\tilde{f} \circ \varphi(y_1, \dots, y_m) = (y_1, \dots, y_r)$. This implies that

$$f \circ \varphi(y_1, \dots, y_m) = (y_1, \dots, y_r, g(\varphi(y_1, \dots, y_m))).$$

Then, if π denotes the projection in the r first variables, $x \in f^{-1}(S) \Rightarrow \pi\varphi^{-1}(x) \in \pi(S)$, i.e. $f^{-1}(S) \subset \varphi(\pi(S) \times \mathbb{R}^{n-r})$. By properties of Hausdorff dimension (see Section 2), we have

$$\dim_H f^{-1}(S) \leqslant \dim_H(\pi(S) \times \mathbb{R}^{n-r}) \leqslant \dim_H \pi(S) + \overline{\dim_B}(\mathbb{R}^{n-r}) \leqslant \dim_H(S) + n - r.$$

This concludes the proof. $\square$

The second proposition relates Hausdorff dimension with topological results.

Proposition 3.2. *Let $n \geqslant 2$ and let F be a closed subset of an n-dimensional connected (not necessarily compact) manifold M^n. If the Hausdorff dimension of F is strictly less than $n-1$ then $M^n - F$ is connected. If $M^n = \mathbb{R}^n$ or $M^n = S^n$, F is compact and the Hausdorff dimension of F is strictly less than $n-k-1$ then $M^n - F$ is k-connected (i.e. its homotopy groups π_i vanish for $i \leqslant k$).*

Proof. First, let $x, y \in M^n - F$ and $U_1, \ldots, U_k$ be coordinate neighbourhoods of M^n, with $x \in U_1$, $y \in U_k$ and $U_j \cap U_{j+1} \neq \emptyset$ for $j = 1, \ldots, k-1$. Choose points $x_j \in U_j \cap U_{j+1}$ for each j. Clearly, it suffices to find a path in U_j joining x_j to x_{j+1} that is disjoint form F. So assume x and y are in a ball U of $\mathbb{R}^n$.

F is closed so there are neighbourhoods $N(x)$, $N(y)$ of x and y, disjoint from F. Let D be a compact $(n-1)$ disc whose centre is on the midpoint of the segment J joining x to y and choose D orthogonal to J. Assume $N(x)$, $N(y)$ chosen small enough so that they are disjoint from D. Consider the truncated double cone C over D. The radial projection π (from x and y) to D gives a Lipschitz map $\pi : C \to D$. Since $\dim_H(\pi(C \cap F)) < n-1$, there is a point $\tilde{y} \in D - \pi(F)$. Then the segments joining x to $\tilde{y}$ and y to $\tilde{y}$ are disjoint from F.

Second, if F is a compact subset of $M^n = \mathbb{R}^n$, $\dim_H F < n-k-1$, let $[\Gamma] \in \pi_i(\mathbb{R}^n - F)$ be a homotopy class for $i \leqslant k$. Choose a smooth representative $\Gamma \in [\Gamma]$. Define $f : \Gamma \times F \to S^{n-1}$, $f(x,y) := (y-x)/\|y-x\|$. We will consider in $\Gamma \times F$ the sum norm, i.e. if $p, q \in \Gamma \times F$, $p = (x,y)$, $q = (z,w)$, then $\|p-q\| := \|x-z\| + \|y-w\|$. For this choice of norm we have

$$\|f(p) - f(q)\| = \frac{1}{\|y-x\|\|z-w\|}\|\{(y-x)\|z-w\| + \|y-x\|(z-w)\}\| \;\Rightarrow$$

$$\|f(p) - f(q)\| \leqslant \frac{\|(y-x)\|z-w\| - \|z-w\|(w-z)\|}{\|y-x\|\|z-w\|} + \frac{\|\,\|z-w\|(w-z) - \|y-x\|(w-z)\|}{\|y-x\|\|z-w\|} \;\Rightarrow$$

$$\|f(p) - f(q)\| \leqslant \frac{1}{\|y-x\|}\{\|(z-x) + (y-w)\|\} + \frac{1}{\|y-x\|}|\{\|(z-w)\| - \|(y-x)\|\}| \;\Rightarrow$$

$$\|f(p) - f(q)\| \leqslant 2C\|p-q\|,$$

where $C = 1/d(\Gamma, F)$. We have $d(\Gamma, F) > 0$ since these are compact disjoint sets. This computation shows that f is Lipschitz.

Then, we have (Proposition 2.1 and 2.2)

$$\dim_H f(\Gamma \times F) \leqslant \dim_H(\Gamma \times F) \leqslant \overline{\dim_B}(\Gamma) + \dim_H(F) < i + n - k - 1 \leqslant n - 1 \;\Rightarrow\; \exists v \notin f(\Gamma \times F).$$

Now, F is compact implies that there is a real N such that $F \subset B_N(0)$. Then, making a translation of Γ in the direction v, we can put, using this translation as homotopy, Γ outside B_N and the translated Γ remain disjoint from F. Since $\mathbb{R}^n - B_N$ is n-connected (for $n \geqslant 3$), $\pi_i(\mathbb{R}^n - F) = 0$. This concludes the proof. $\square$

Remark 3.3. We remark that the hypothesis F is closed in the previous proposition is necessary. For example, take $F = \mathbb{Q}^n$, $M^n = \mathbb{R}^n$. We have $\dim_H(F) = 0$ (F is a countable set) but $M^n - F$ is not connected.

We can think of Proposition 3.2 as a weak type of transversality. In fact, if F is a compact $(n-2)$-submanifold of M^n then $M - F$ is connected and if F is a compact $(n-3)$-submanifold of $\mathbb{R}^n$ (or S^n) then $\mathbb{R}^n - F$ is simply connected. This follows from basic transversality. However, our previous proposition does not assume regularity of F, but allows us to conclude the same results. It is natural these results are true because Hausdorff dimension translates the fact that F is, in some sense, 'smaller' than an $(n-1)$-submanifold N which has optimal dimension in order to disconnect M^n.

For later use, we generalize the first part of Proposition 3.2 as follows.

Lemma 3.4. *Suppose that $\Gamma \in \pi_i(M^n)$ is Lipschitz* (*e.g. if $i = 1$ and Γ is a piecewise smooth curve*) *and let $K \subset M^n$ be compact, $\dim_H K < n - i$. Then there are diffeomorphisms h of M, arbitrarily close to the identity map, such that $h(\Gamma) \cap K = \emptyset$. In particular, if $[\Gamma] \in \pi_i(M^n)$, $K \subset M^n$ a compact set, $\dim_H(K) < n - i$, there is a smooth representative $\Gamma \in [\Gamma]$ such that $\Gamma \cap K = \emptyset$, i.e. $\Gamma \in \pi_i(M^n - K)$.*

Proof. First, consider a parametrized neighbourhood $\phi : U \to B_3(0) \subset \mathbb{R}^n$ and suppose that Γ lies in $\bar{V}_1$, where $V_1 = \phi^{-1}(B_1(0))$. Let $K_1 = \phi(K) \subset \mathbb{R}^n$ and $\Gamma_1 = \phi(\Gamma) \subset \mathbb{R}^n$. Consider the map

$$F : \Gamma_1 \times K_1 \to \mathbb{R}^n \quad \text{defined by } F(x, y) = x - y.$$

Observe that, since Γ is Lipschitz and ϕ is a diffeomorphism, $\overline{\dim}_B \Gamma = \overline{\dim}_B \Gamma_1 \leqslant i$. This implies that $\dim_H(F(\Gamma_1 \times K_1)) < n$, since $\dim_H(K) < n - i$. This implies, in particular, that $\mathbb{R}^n - F(\Gamma_1 \times K_1)$ is an open and dense subset, since K is compact. Then, we may choose a vector $v \in \mathbb{R}^n - F(\Gamma_1 \times K_1)$ arbitrarily close to 0 such that $(\Gamma_1 + v) \subset B_2(0)$. Since, $v \in \mathbb{R}^n - F(\Gamma_1 \times K_1)$ we have that $(\Gamma_1 + v) \cap K_1 = \emptyset$.

To construct h we consider a bump function $\beta : \mathbb{R}^n \to [0, 1]$, such that $\beta(x) = 1$ if $x \in B_1(0)$ and $\beta(x) = 0$ for every $x \in \mathbb{R}^n - B_2(0)$. It is easy to see that h defined by

$$h(y) = y \quad \text{if } x \in M - U \quad \text{and} \quad h(y) = \phi^{-1}(\beta(\phi(y))v + \phi(y)),$$

is a diffeomorphism that satisfies $h(\Gamma) \cap K = \emptyset$, since $(\Gamma_1 + v) \cap K_1 = \emptyset$.

In the general case, we proceed as follows: first, considering a finite number of parametrized neighbourhoods $\phi_i : U_i \to B_3(0)$, $i \in \{1, \ldots, n\}$, and $V_i = \phi_i^{-1}(B_1(0))$ covering Γ, by the previous case, there exists h_1 arbitrarily close to the identity such that $h_1(\Gamma) \subset \bigcup_{i=1}^n V_i$ and such that $h_1(\Gamma \cap \bar{V}_1) \cap K = \emptyset$. Observe that, $d(h_1(\Gamma \cap \bar{V}_1), K) > \epsilon_1 > 0$, since $h_1(\Gamma \cap \bar{V}_1)$ is a compact set.

The next step is to repeat the previous argument considering h_2 arbitrarily close to the identity, such that $h_2(h_1(\Gamma) \cap V_2) \cap K = \emptyset$ and $h_2(h_1(\Gamma)) \subset \bigcup_{i=1}^n V_i$. If $d(h_2, id) < \frac{1}{2}\epsilon_1$ then $h_2(h_1(\Gamma) \cap V_1) \cap K = \emptyset$. Repeating this argument by induction, we obtain that $h = h_n \circ \cdots \circ h_1$ is a diffeomorphism such that $h(\Gamma) \cap K = \emptyset$. This concludes the proof. □

4. Proof of Theorem A in the case $n = 3$

Before giving a proof for Theorem A, we mention a lemma due to Bing [**2**].

Lemma 4.1 (Bing). *A compact, connected,* 3*-manifold* M *is topologically* S^3 *if and only if each piecewise smooth simple closed curve in* M *lies in a topological cube in* M*.*

A modern proof of this lemma can be found in [**9**]. In modern language, Bing's proof shows that the hypothesis above implies there is a *Heegaard* splitting of M into two balls. This implies M is a sphere.

In fact, Bing's theorem is not stated in [**2**,**9**] as above. But the lemma holds. Actually, to prove that M is homeomorphic to S^3, Bing uses only that, if a triangulation of M is fixed, every simple *polyhedral* closed curve lies in a topological cube. Observe that *polyhedral* curves are piecewise smooth curves, if we choose a smooth triangulation (smooth manifolds always admit smooth triangulation (see [**11**, p. 194] and also [**12**, p. 124]).

Proof of Theorem A in the case $n = 3$. If $\alpha_2(M) < 1$, there is an immersion $x : M^3 \to \mathbb{R}^4$ such that $\operatorname{rank}(x) \geqslant 2$, $\dim_H(x) < 1$. Let G be the Gauss map associated with x. By Propositions 3.2 and 3.1, since $\dim_H(S) < 1$, $M - G^{-1}(S)$, $S^3 - S$ are connected manifolds. Consider $G : M - G^{-1}(S) \to S^3 - S$. This is a proper map between connected manifolds whose Jacobian never vanishes. So it is a surjective covering map (see [**13**]). Since, moreover, $S^3 - S$ is simply connected (by Proposition 3.2), $G : M - G^{-1}(S) \to S^3 - S$ is a diffeomorphism. To prove that M^3 is homeomorphic to S^3, it is necessary and sufficient that every piecewise smooth simple closed curve $\gamma \subset M^3$ is contained in a topological cube $Q \subset M^3$ (by Lemma 4.1).

In order to prove that every piecewise smooth curve γ lies in a topological cube, observe that we may suppose that $\gamma \cap K = \emptyset$ (here $K = G^{-1}(S)$). Indeed, by Lemma 3.4 there exists a diffeomorphism h of M such that $h(\gamma) \cap K = \emptyset$. Then, if $h(\gamma)$ lies in a topological cube R, γ lies in the topological cube $h^{-1}(R)$, thus we can, in fact, make this assumption.

Now, since $\gamma \subset M - K$ and $M - K$ is diffeomorphic to $S^3 - S$, we may consider $\gamma \subset \mathbb{R}^3 - S$, S a compact subset of $\mathbb{R}^3$ with Hausdorff dimension less than 1 via identification by the diffeomorphism G and stereographic projection. In this case, we can follow the proof of Proposition 3.2 to conclude that $f : \gamma \times S \to S^2$, $f(x, y) = (x - y)/\|x - y\|$ is Lipschitz. Because $\overline{\dim_B} \gamma \leqslant 1$ $\dim_H S < 1$ (here we are using that γ is piecewise smooth), we obtain a direction $v \in S^2$ such that $F := \bigcup_{t \in \mathbb{R}}(L_t(\gamma))$ is disjoint from S, where $L_t(p) := p + t \cdot v$. By compactness of γ it is easy to see that F is a closed subset of $\mathbb{R}^n$. This implies that $3\epsilon = d(F, S) > 0$. Consider $F_\epsilon = \{x : d(x, F) \leqslant \epsilon\}$ and $S_\epsilon = \{x : d(x, S) \leqslant \epsilon\}$. By definition of $\epsilon > 0$, $F_\epsilon \cap S_\epsilon = \emptyset$, then we can choose $\varphi : \mathbb{R}^3 \to \mathbb{R}$ a smooth function such that $\varphi|_{F_\epsilon} = 1$, $\varphi|_{S_\epsilon} = 0$. Consider the vector field

$X(p) = \varphi(p) \cdot v$ and let X_t, $t \in \mathbb{R}$, be the X-flow. We have $X_t(p) = p + tv$ $\forall p \in \gamma$ and $X_t(p) = p$ $\forall p \in S$, for any $t \in \mathbb{R}$. Choosing N real such that $S \subset B_N(0)$ and T such that $t \geqslant T \Rightarrow L_t(\gamma) \cap B_N(0) = \emptyset$, we obtain a global homeomorphism X_t which sends γ outside $B_N(0)$ and keep fixed S, $\forall t \geqslant T$.

Observe that $X_t(\gamma)$ is contained in the interior of a topological cube $Q \subset \mathbb{R}^3 - B_N(0)$. Then, observing that X_t is a diffeomorphism and that $X_t(x) = x$ for every $x \in S$ and $t \in \mathbb{R}$, we have that $\gamma \subset X_{-t}(Q) \subset \mathbb{R}^3 - S$, $\forall t \geqslant T$. This concludes the proof. □

5. Proof of Theorem A in the case $n \geqslant 4$

We start this section with the statement of the generalized Poincaré conjecture.

Theorem 5.1. *A compact simply connected homological sphere M^n is homeomorphic to S^n, if $n \geqslant 4$ (diffeomorphic for $n = 5, 6$).*

The proof of the generalized Poincaré conjecture is due to Smale [**10**] for $n \geqslant 5$ and to Freedman [**6**] for $n = 4$. This theorem makes the proof of the Theorem B a little bit easier than the proof of Theorem A.

Proof of Theorem A in the case $n \geqslant 4$. If $k = n$, there is nothing to prove. Indeed, in this case, $G : M^n \to S^n$ is a diffeomorphism, by definition. Hence, we suppose $k \leqslant n-1$; $\alpha_k(M) < k - [\frac{1}{2}n]$. Then, there is an immersion $x : M^n \to \mathbb{R}^{n+1}$, $\operatorname{rank}(x) \geqslant k$, $\dim_H(x) < k - [\frac{1}{2}n]$. The hypothesis implies that $M - G^{-1}(S)$ is connected, $S^n - S$ is simply connected and G is a proper map whose Jacobian never vanishes. By [**13**], G is a surjective, covering map. So, we conclude that $G : M - G^{-1}(S) \to S^n - S$ is a diffeomorphism. But $S^n - S$ is $(n - 1 - k + [\frac{1}{2}n])$-connected, by Proposition 3.2. In particular, because $k \leqslant n-1$, $S^n - S$ is $[\frac{1}{2}n]$-connected and so, using the diffeomorphism G, $M - K$ is $[\frac{1}{2}n]$-connected, where $K = G^{-1}(S)$. It is sufficient to prove that M^n is a simply connected homological sphere, by Theorem 5.1. By Lemma 3.4, $M - K$ is $[\frac{1}{2}n]$-connected and $\dim_H(K) < n - [\frac{1}{2}n]$ (by Proposition3.2) implies that M itself is $[\frac{1}{2}n]$-connected. It is known that $H^i(M) = L(H_i(M)) \oplus T(H_{i-1}(M))$, L and T denote the free part and the torsion part of the group. By Poincaré duality, $H_{n-i}(M) \simeq H^i(M)$. The fact that M is $[\frac{1}{2}n]$-connected and the other facts give us $H_i(M) = 0$, for $0 < i < n$. This concludes the proof. □

6. Proof of Theorems B and C

In this section we make some comments on extensions of Theorem A. Although these extensions are quite easy, they were omitted so far to make the presentation of the paper more clear. Now, we are going to improve our previous results. First, all preceding arguments work with the assumption that $H_{k-[n/2]}(S) = 0$ and $\operatorname{rank}(x) \geqslant k$ in Theorems A and B (where H_s is the s-dimensional Hausdorff measure). We prefer to consider the hypothesis as its stands in these theorems because it is more interesting to define the invariants $\alpha_k(M)$. The reason this 'new' hypothesis works is that our proofs, essentially, depend on the existence of special directions $v \in S^{n-1}$. But these directions exist if the singular sets have Hausdorff measure 0. Secondly, M need not be compact. It is sufficient that M is of *finite geometric type* (here our definition of finite geometrical type is a little bit different [**1**]). We will make more precise these comments in the proof of Theorem B below, after recalling the following definition.

Definition 6.1. Let $\bar{M}^n$ be a compact (oriented) manifold and $q_1, \ldots, q_k \in \bar{M}^n$. Let $M^n = \bar{M}^n - \{q_1, \ldots, q_k\}$. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is of finite geometrical type if M^n is complete in the induced metric, the Gauss map $G : M^n \to S^n$ extends continuously to a function $\bar{G} : \bar{M}^n \to S^n$, and the set $G^{-1}(S)$ has $H_{n-1}(G^{-1}(S)) = 0$ (this last condition occurs if $\operatorname{rank}(x) \geqslant k$ and $\dim_H(x) < k - 1$, or, more generally, if $\operatorname{rank}(x) \geqslant k$ and $H_{k-1}(S) = 0$).

As pointed out in the introduction, the conditions in the previous definition are satisfied, for example, by complete hypersurfaces with finite total curvature whose Gauss–Kronecker curvature $H_n = k_1 \cdots k_n$ does not change sign and vanishes in a small set, as shown by [**4**]. Recall that a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ has total finite curvature if $\int_M |A|^n \,\mathrm{d}M < \infty$, $|A| = (\sum_i k_i^2)^{1/2}$, k_i are the principal curvatures. Thus, there are examples satisfying the definition. With these observations, we now prove our Theorem B.

Proof of Theorem B. To avoid unnecessary repetitions, we will only indicate the principal modifications needed in the proofs of Theorems A and B by stating 'new' propositions, which are analogous to the previous ones, and making a few comments in their proofs. The details are left to the reader.

Proposition 6.2 (Proposition 3.1′). *Let $f : M^m \to N^n$ be a C^1-map and $A \subset N$. If $H_s(A) = 0$, then $H_{s+n-\operatorname{rank}(f)}(f^{-1}(A)) = 0$.*

Proof. It suffices to show that for any $p \in f^{-1}(A)$, there is an open set $U = U(p) \ni p$ such that $H_{s+n-r}(f^{-1}(A) \cap U) = 0$. However, if U is chosen as in the proof of Proposition 3.1, we have $f^{-1}(A) \cap U \subset \varphi(\pi(A) \times \mathbb{R}^{n-r})$, where φ is a diffeomorphism, $r = \operatorname{rank}(f)$ and π is the projection in the first r variables. By Propositions 2.1 and 2.2,

$$H_{s+n-r}(f^{-1}(A) \cap U) \leqslant C_1 \cdot H_{s+n-r}(\pi(A) \times \mathbb{R}^{n-r}) \leqslant C_1 \cdot C_2 \cdot H_s(A) = 0,$$

where C_1 depends only on φ and C_2 depends only on $(n-r)$. This finishes the proof. □

Proposition 6.3 (Proposition 3.2′). *Let $n \geqslant 3$ and let F be a closed subset of M^n such that $H_{n-1}(F) = 0$, then $M - F$ is connected. If $M^n = \mathbb{R}^n$ or $M^n = S^n$, F is compact and $H_{n-k}(F) = 0$, then $M^n - F$ is k-connected.*

Proof. As in Proposition 3.2, let $x, y \in M^n - F$ and $U_1, \ldots, U_k$ be coordinate neighbourhoods of M^n, with $x \in U_1$, $y \in U_k$ and $U_j \cap U_{j+1} \neq \emptyset$ for $j = 1, \ldots, k-1$. Choose points $x_j \in U_j \cap U_{j+1}$ for each j. Clearly, it suffices to find a path in U_j joining x_j to x_{j+1} that is disjoint from F. So assume that x and y are in a ball U of $\mathbb{R}^n$.

F is closed so there are neighbourhoods $N(x)$, $N(y)$ of x and y, disjoint from F. Let D be a compact $(n-1)$ disc whose centre is on the midpoint of the segment J joining x to y and choose D orthogonal to J. Assume $N(x)$, $N(y)$ chosen small enough so that they are disjoint from D. We consider again the truncated double cone C over D. The radial projection π (from x and y) to D gives a Lipschitz map $\pi : C \to D$. Since $H^{n-1}(F) = 0$, we have by Proposition 2.1 that $H^{n-1}(\pi(F)) = 0$. Thus, there is a point $\tilde{y} \in D - \pi(F)$ and the segments joining x to $\tilde{y}$ and y to $\tilde{y}$ are disjoint from F.

Second, if $[\Gamma] \in \pi_i(\mathbb{R}^n - F)$, $i \leqslant k$, is a homotopy class and Γ is a smooth representative, define $f : \Gamma \times F \to S^{n-1}$, $f(x,y) = (x-y)/\|x-y\|$. Following the proof of Proposition 3.2, f is Lipschitz. Now, since $H_{n-k-1}(F) = 0$, we have, by Proposition 2.2, $H_{n-1}(\Gamma \times F) = 0$. Thus, Proposition 2.1 implies $H_{n-1}(f(\Gamma \times F)) = 0$. This concludes the proof. □

Lemma 6.4 (Lemma 3.4′). *Suppose that $\Gamma \in \pi_i(M^n)$ is Lipschitz and $K \subset M^n$ is compact, $H_{n-i}(K) = 0$. Then there are diffeomorphisms h of M, arbitrarily close to the identity map, such that $h(\Gamma) \cap K = \emptyset$.*

Proof. If Γ is Lipschitz and Γ lies in a parametrized neighbourhood, we can take $F : \Gamma \times K \to \mathbb{R}^n$, $F(x,y) = x - y$ a Lipschitz function. Because $H_n(\Gamma \times K) = 0$, then $H_n(F(\Gamma \times K)) = 0$. In the general case we proceed as in proof of Lemma 3.4. Take, by compactness, a finite number of parametrized neighbourhoods and apply the previous case. By finiteness of number of parametrized neighbourhoods and using that K is compact, an induction argument achieves the desired diffeomorphisms h. This concludes the proof. □

Returning to proof of Theorem B, observe that in Theorem A, we need $\bar{G} : \bar{M}^n - \bar{G}^{-1}(\tilde{S}) \to S^n - \tilde{S}$ to be a diffeomorphism, where $\tilde{S} = S \cup \{\bar{G}(q_i) : i = 1, \ldots, k\}$. This remains true because

$$H_{k-[n/2]}(S) = 0 \tag{$*$}$$

implies $S^n - \tilde{S}$ is $(n - 1 - k + [\frac{1}{2}n])$-connected. In fact, this is a consequence of $(*)$, Proposition 6.3 and $\{p_i : i = 1, \ldots, k\}$ is finite ($p_i := \bar{G}(q_i)$). Moreover, $\operatorname{rank}(x) \geqslant k$ imply, by Propositions 6.2 and 6.3, $\bar{M} - G^{-1}(\tilde{S})$ is connected. Indeed, these propositions say that $\operatorname{rank}(x) \geqslant k \Rightarrow H_{n-[n/2]}(G^{-1}(S)) = 0$ and $H_{n-1}(G^{-1}(S)) = 0 \Rightarrow M - G^{-1}(S)$ is connected. However, if $\bar{G}^{-1}(\tilde{S}) - (G^{-1}(S) \cup \{q_i : i = 1, \ldots, k\}) := A$, then, for all $x \in A$,

$$\det D_x G \neq 0. \tag{$**$}$$

In particular, since $G(A) \subset \{p_i : i = 1, \ldots, k\}$, $(**)$ implies $\dim_H(A) = 0$. Then, $H_{n-[n/2]}(\bar{G}^{-1}(\tilde{S})) = H_{n-[n/2]}(G^{-1}(S)) = 0$. Thus, by [**13**], G is a surjective and covering map (because it is proper and its Jacobian never vanishes). In particular, by simple connectivity, G is a diffeomorphism. At this point, using the previous lemma and propositions, it is sufficient to follow the proof of Theorem A, if $n = 3$, and the proof of Theorem B, if $n \geqslant 4$, to obtain $\bar{M}^n \simeq S^n$. This concludes the proof. □

For even dimensions, we can follow [**1**] and improve Theorem B.

Theorem 6.5 (Theorem C). *Let $x : M^{2n} \to \mathbb{R}^{2n+1}$, $n \geqslant 2$, be an immersion of finite geometrical type with $H_{k-n}(S) = 0$, $\operatorname{rank}(x) \geqslant k$. Then M^{2n} is topologically a sphere minus two points. If M^{2n} is minimal, M^{2n} is a $2n$-catenoid.*

For the sake of completeness we present an outline of the proof of Theorem C.

Outline of the proof of Theorem C. Barbosa, Fukuoka and Mercuri define to each *end* p of M a geometric index $I(p)$ that is related to the topology of M by the formula (see Theorem 2.3 of [**1**]):

$$\chi(\bar{M}^{2n}) = \sum_{i=1}^{k}(1 + I(p_i)) + 2\sigma m, \tag{6.1}$$

where σ is the sign of the Gauss–Kronecker curvature and m is the degree of $G : M^n \to S^n$. Now, the hypothesis $2n > 2$ implies (see [**1**]) $I(p_i) = 1$, $\forall i$. Since we know, by Theorem B, $\bar{M}^{2n}$ is a sphere, we have $2 = 2k + 2\sigma m$. But, it is easy to see that $m = \deg(G) = 1$ because G is a diffeomorphism outside the singular set. Then, $2 = 2k + 2\sigma \Rightarrow k = 2$, $\sigma = -1$. In particular, M is a sphere minus two points.

If M is minimal, we will use the following theorem of Schoen.

> *The only minimal immersions, which are regular at infinity and have two ends, are the catenoid and a pair of planes.*

The regularity at infinity in our case holds if the ends are embedded. However, $I(p) = 1$ means exactly this. So, we can use this theorem in the case of minimal hypersurfaces of finite geometric type. This concludes the outline of the proof. □

Remark 6.6. We can extend Theorem A in a different direction (without mention of $\operatorname{rank}(x)$). In fact, using only that G is Lipschitz, it suffices assume that $H_{n-[n/2]}(C) = 0$ (C is the set of points where the Gauss–Kronecker curvature vanishes). This is essentially the hypothesis of Barbosa, Fukuoka and Mercuri [**1**]. We prefer to state Theorems B and C as before since the classical theorems concerning estimates for Hausdorff dimension (Morse–Sard, Moreira) deal only with the critical values S and, in particular, our Corollary 2.5 will be more difficult if the hypothesis is changed to $H_1(C) = 0$ for some immersion $x : M^3 \to \mathbb{R}^4$ (although, in this assumption, we have no problems with $\operatorname{rank}(x)$, i.e. this assumption has some advantages).

Remark 6.7. It would be interesting to know if there are examples of codimension 1 immersions with a singular set which are not in the situation of Barbosa–Fukuoka–Mercuri and do Carmo–Elbert but instead satisfy our hypothesis. This question was posed to the second author by Walcy Santos during the ‘Differential Geometry’ seminar at IMPA. In fact, these immersions can be constructed with some extra work. Some examples will be presented in another work to appear elsewhere.

Acknowledgements. C.M. is supported by Faperj/Brazil. K.O. is supported by Pronex-CNPq/Brazil and Fapeal/Brazil.

The authors are thankful to Jairo Bochi, João Pedro Santos, Carlos Morales, Alexander Arbieto and Carlos Moreira for fruitful conversations. Also, the authors are indebted to Professor Manfredo do Carmo for his kind encouragement and to Professor Marcelo Viana for his suggestions and advice. The comments and suggestions of anonymous referees were useful to improve the presentation of this paper. Last, but not least, we are thankful to IMPA and its staff.

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