INTRODUCTION TO TEICHMÜLLER THEORY AND ITS APPLICATIONS TO DYNAMICS OF INTERVAL EXCHANGE TRANSFORMATIONS, FLOWS ON SURFACES AND BILLIARDS

GIOVANNI FORNI AND CARLOS MATHEUS

Abstract. This text is an expanded version of the lecture notes of a minicourse (with the same title of this text) delivered by the authors in the Bedlewo school “Modern Dynamics and its Interaction with Analysis, Geometry and Number Theory” (from 4 to 16 July, 2011).

In the first part of this text, i.e., from Sections 1 to 5, we discuss the Teichmüller and moduli space of translation surfaces, the Teichmüller flow and the $SL(2,\mathbb{R})$-action on these moduli spaces, the Kontsevich-Zorich cocycle over the Teichmüller geodesic flow, and we sketch two applications of the ergodic properties of the Teichmüller flow and Kontsevich-Zorich cocycle with respect to Masur-Veech measures to the unique ergodicity and weak mixing properties of typical interval exchange transformations and translation flows based on the fundamental fact that the Teichmüller flow and the Kontsevich-Zorich cocycle work as renormalization dynamics for interval exchange transformations and translation flows.

In the second part of this text, i.e., from Sections 6 to 9, we start by pointing out that it is interesting to study the ergodic properties of the Kontsevich-Zorich cocycle with respect to other invariant measures than the Masur-Veech ones in view of potential applications to the investigation of billiards in rational polygons (for instance). Then, we study some examples of measures such that the ergodic properties of the Kontsevich-Zorich cocycle with respect to these measures is very different from the case of Masur-Veech measures. Finally, we close these notes by constructing some examples closed $SL(2,\mathbb{R})$-orbits such that the restriction of the Teichmüller flow to them has arbitrary small rate of exponential mixing, or, equivalently, a naturally associated unitary $SL(2,\mathbb{R})$-representation has arbitrarily small spectral gap (and in particular it has complementary series).

Contents

1. Quick review of basic elements of Teichmüller theory 3
   1.1. Deformation of Riemann surfaces: moduli and Teichmüller spaces of curves 3
   1.2. Beltrami differentials and Teichmüller metric 4
   1.3. Quadratic differentials and the cotangent bundle of the moduli space of curves 6

Date: June 22, 2012.

Key words and phrases. Moduli spaces, Abelian differentials, translation surfaces, Teichmüller flow, $SL(2,\mathbb{R})$-action on moduli spaces, Kontsevich-Zorich cocycle, Lyapunov exponents.

The authors are thankful to their coauthors A. Avila, J.-C. Yoccoz and A. Zorich for the pleasure of working with them on some topics discussed in these notes. C.M. is grateful to his wife Aline Cerqueira for the immense help with several figures in this text. C.M. was partially supported by the Balzan Research Project of J. Palis.
1.4. An example: Teichmüller and moduli spaces of elliptic curves (torii) 9

2. Some structures on the set of Abelian differentials 11
  2.1. Abelian differentials and translation structures 12
  2.2. Stratification 19
  2.3. Period map and local coordinates 20
  2.4. Connectedness of strata 23

3. Dynamics on the moduli space of Abelian differentials 24
  3.1. $GL^+(2, \mathbb{R})$-action on $\mathcal{H}_g$ 25
  3.2. $SL(2, \mathbb{R})$-action on $\mathcal{H}_g^{(1)}$ and Teichmüller geodesic flow 26
  3.3. Teichmüller flow and Kontsevich-Zorich cocycle on the Hodge bundle over $\mathcal{H}_g^{(1)}$ 30
  3.4. Hodge norm on the Hodge bundle over $\mathcal{H}_g^{(1)}$ 34

3.5. First variation of Hodge norm and hyperbolic features of Teichmüller flow 35

4. Ergodic theory of Teichmüller flow with respect to Masur-Veech measure 38
  4.1. Finiteness of Masur-Veech measure and unique ergodicity of interval exchange maps 38
  4.2. Ergodicity of Teichmüller flow 43
  4.3. Exponential mixing (and spectral gap of $SL(2, \mathbb{R})$ representations) 44

5. Ergodic Theory of KZ cocycle with respect to Masur-Veech measures 47
  5.1. Kontsevich-Zorich conjecture (after G. Forni, and A. Ávila & M. Viana) 47
  5.2. Second Lyapunov exponent of KZ cocycle with respect to Masur-Veech measures 51
  5.3. Weak mixing property for i.e.t.’s and translation flows 59

6. Veech’s question 61
  6.1. *Eierlegende Wollmilchsau* 62
  6.2. *Ornithorynque* 66
  6.3. M. Möller’s work on Shimura and Teichmüller curves 68
  6.4. Sums of Lyapunov exponents (after A. Eskin, M. Kontsevich & A. Zorich) 69

7. Explicit computation of Kontsevich-Zorich cocycle over two totally degenerate examples 76
  7.1. Affine diffeomorphisms, automorphisms and Veech groups 76
  7.2. Affine diffeomorphisms of square-tiled surfaces and Kontsevich-Zorich cocycle 77
  7.3. Combinatorics of square-tiled cyclic covers 78
  7.4. *Eierlegende Wollmilchsau* and the quaternion group 80
  7.5. The action of the affine diffeomorphisms of the *Eierlegende Wollmilchsau* 84
  7.6. The action of the affine diffeomorphisms of the *Ornithorynque* 86

8. Cyclic covers 88
  8.1. Hodge theory and the Lyapunov exponents of square-tiled cyclic covers 88
  8.2. Other cyclic covers 92

9. Arithmetic Teichmüller curves with complementary series 103
  9.1. J. Ellenberg and D. McReynolds theorem 103
  9.2. Teichmüller curves with complementary series 104
1. Quick review of basic elements of Teichmüller theory

The long-term goal of these lecture notes is the study of the so-called Teichmüller geodesic flow and its noble cousin Kontsevich-Zorich cocycle, and some of its applications to interval exchange transformations, translation flows and billiards. As any respectable geodesic flow, Teichmüller flow acts naturally in a certain unit cotangent bundle. More precisely, the phase space of the Teichmüller geodesic flow is the unit cotangent bundle of the moduli space of Riemann surfaces.

For this initial section, we’ll briefly recall some basic results of Teichmüller theory leading to the conclusion that the unit cotangent bundle of the moduli space of Riemann surfaces (i.e., the phase space of the Teichmüller geodesic flow) is naturally identified to the moduli space of quadratic differentials. As we’ll see later in this text, the relevance of this identification resides in the fact that it makes apparent the existence of a natural $SL(2,\mathbb{R})$ action on the moduli space of quadratic differentials such that the Teichmüller flow corresponds to the action of the diagonal subgroup $g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ of $SL(2,\mathbb{R})$. In any event, the basic reference for this section is J. Hubbard’s book [40].

1.1. Deformation of Riemann surfaces: moduli and Teichmüller spaces of curves. Let us consider two Riemann surface structures $S_0$ and $S_1$ on a fixed (topological) compact surface of genus $g \geq 1$. If $S_0$ and $S_1$ are not biholomorphic (i.e., they are “distinct”), there is no way to produce a conformal map (i.e., holomorphic map with non-vanishing derivative) $f : S_0 \to S_1$. However, we can try to produce maps $f : S_0 \to S_1$ which are as “nearly conformal” as possible. To do so, we need a reasonable way to “measure” the amount of “non-conformality” of $f$. A fairly standard procedure is the following one. Given a point $x \in S_0$ and some local coordinates around $x$ and $f(x)$, we write the derivative $Df(x)$ of $f$ at $x$ as $Df(x)u = \frac{\partial f}{\partial x}(x)u + \frac{\partial f}{\partial \bar{x}}(x)\bar{u}$, so that $Df(x)$
sends infinitesimal circles into infinitesimal ellipses of eccentricity

$$\left| \frac{\partial f}{\partial z}(x) \right| + \left| \frac{\partial f}{\partial z}(x) \right| = \frac{1 + k(f, x)}{1 - k(f, x)} := K(f, x),$$

where $k(f, x) := \frac{\left| \frac{\partial f}{\partial z}(x) \right|}{\left| \frac{\partial f}{\partial z}(x) \right|}$. This is illustrated in the figure below:

![Diagram illustrating infinitesimal circles and ellipses]

We say that $K(f, x)$ is the eccentricity coefficient of $f$ at $x$, while

$$K(f) := \sup_{x \in S_0} K(f, x)$$

is the eccentricity coefficient of $f$. Note that, by definition, $K(f) \geq 1$ and $f$ is a conformal map if and only if $K(f) = 1$ (or equivalently $k(f, x) = 0$ for every $x \in S_0$). Hence, $K(f)$ accomplishes the task of measuring the amount of non-conformality of $f$. We call $f : S_0 \to S_1$ quasiconformal whenever $K = K(f) < \infty$.

In the next subsection, we’ll see that quasiconformal maps are useful to compare distinct Riemann structures on a given topological compact surface $S$. In a more advanced language, we consider the moduli space $\mathcal{M}(S)$ of Riemann surface structures on $S$ modulo conformal maps and the Teichmüller space $\mathcal{T}(S)$ of Riemann surface structures on $S$ modulo conformal maps isotopic to the identity. It follows that $\mathcal{M}(S)$ is the quotient of $\mathcal{T}(S)$ by the so-called modular group (or mapping class group) $\Gamma(S) := \Gamma_g := \text{Diff}^+(S)/\text{Diff}^+_0(S)$ of isotopy classes of diffeomorphisms of $S$ (here $\text{Diff}^+(S)$ is the set of orientation-preserving diffeomorphisms and $\text{Diff}^+_0(S)$ is the set of orientation-preserving diffeomorphisms isotopic to the identity). Therefore, the problem of studying deformations of Riemann surface structures corresponds to the study of the nature of the moduli space $\mathcal{M}(S)$ (and the Teichmüller space $\mathcal{T}(S)$).

1.2. Beltrami differentials and Teichmüller metric. Let’s come back to the definition of $K(f)$ in order to investigate the nature of the quantities $k(f, x) := \frac{\left| \frac{\partial f}{\partial z}(x) \right|}{\left| \frac{\partial f}{\partial z}(x) \right|}$. Since we are dealing with Riemann surfaces (and we used local charts to perform calculations), $k(f, x)$ doesn’t provide a globally defined function on $S_0$. Instead, by looking at how $k(f, x)$ transforms under changes of
coordinates, one can check that the quantities \( k(f, x) \) can be collected to globally define a tensor \( \mu(x) \) (of type \((-1, 1)\)) via the formula:

\[
\mu(x) = \frac{\partial f}{\partial z}(x) dz - \frac{\partial f}{\partial \overline{z}}(x) d\overline{z}.
\]

In the literature, \( \mu(x) \) is called a Beltrami differential. Note that \( \|\mu\|_{L^\infty} < 1 \) when \( f \) is an orientation-preserving quasiconformal map. The intimate relationship between quasiconformal maps and Beltrami differentials is revealed by the following profound theorem of L. Ahlfors and L. Bers:

**Theorem 1** (Measurable Riemann mapping theorem). Let \( U \subset \mathbb{C} \) be an open subset and consider \( \mu \in L^\infty(U) \) verifying \( \|\mu\|_{L^\infty} < 1 \). Then, there exists a quasiconformal mapping \( f : U \to \mathbb{C} \) such that the Beltrami equation

\[
\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}
\]

is satisfied in the sense of distributions. Furthermore, \( f \) is unique modulo composition with conformal maps: if \( g \) is another solution of Beltrami equation above, then there exists an injective conformal map \( \phi : f(U) \to \mathbb{C} \) such that \( g = \phi \circ f \).

A direct consequence of this striking result to the deformation of Riemann surface structures is the following proposition (whose proof is left as an exercise to the reader):

**Proposition 2.** Let \( X \) be a Riemann surface and \( \mu \) a Beltrami differential on \( X \). Given an atlas \( \phi_i : U_i \to \mathbb{C} \) of \( X \), denote by \( \mu_i \) the function on \( V_i := \phi_i(U_i) \subset \mathbb{C} \) defined by

\[
\mu_i|_{U_i} = \phi_i^* \left( \mu \frac{dz}{\overline{z}} \right).
\]

Then, there is a family of mappings \( \psi_i(\mu) : V_i \to \mathbb{C} \) solving the Beltrami equations

\[
\frac{\partial \psi_i(\mu)}{\partial \overline{z}} = \mu_i \frac{\partial \psi_i(\mu)}{\partial z}
\]

such that \( \psi_i(\mu) \) are homeomorphisms from \( V_i \) to \( \psi_i(\mu)(V_i) \).

Moreover, \( \psi_i \circ \phi_i : U_i \to \mathbb{C} \) form an atlas giving a well-defined Riemann surface structure \( X_\mu \) in the sense that it is independent of the initial choice of the atlas \( \phi_i : U_i \to \mathbb{C} \) and the choice of \( \phi_i \) verifying the corresponding Beltrami equations.

In other words, the measurable Riemann mapping theorem of Alhfors and Bers implies that one can use Beltrami differentials to naturally deform Riemann surfaces through quasiconformal mappings. Of course, we can ask to what extend this is a general phenomena: namely, given two Riemann surface structures \( S_0 \) and \( S_1 \), can we relate them by quasiconformal mappings? The answer to this question is provided by the remarkable theorem of O. Teichmüller:

**Theorem 3** (O. Teichmüller). Given two Riemann surfaces structures \( S_0 \) and \( S_1 \) on a compact topological surface \( S \) of genus \( g \geq 1 \), there exists a quasiconformal mapping \( f : S_0 \to S_1 \) minimizing the eccentricity coefficient \( K(g) \) among all quasiconformal maps \( g : S_0 \to S_1 \) isotopic to the
identity map $\text{id}: S \to S$. Furthermore, whenever a quasiconformal map $f: S_0 \to S_1$ minimizes the eccentricity coefficient in the isotopy class of a given orientation-preserving diffeomorphism $h: S \to S$, we have that the eccentricity coefficient of $f$ at any point $x \in S_0$ is constant, i.e.,

$$K(f, x) = K(f)$$

except for a finite number of points $x_1, \ldots, x_n \in S_0$. Also, quasiconformal mappings minimizing the eccentricity coefficient in a given isotopy class are unique modulo (pre and post) composition with conformal maps.

In the literature, any such minimizing quasiconformal map in a given isotopy class is called an extremal map. Using the extremal quasiconformal mappings, we can naturally introduce a distance between two Riemann surface structures $S_0$ and $S_1$ by

$$d(S_0, S_1) = \frac{1}{2} \ln K(f)$$

where $f: S_0 \to S_1$ is an extremal map isotopic to the identity. The metric $d$ is called Teichmüller metric. The major concern of these notes is the study of the geodesic flow associated to Teichmüller metric on the moduli space of Riemann surfaces. As we advanced in the introduction, it is quite convenient to regard a geodesic flow living on the cotangent bundle of the underlying space. The discussion of the cotangent bundle of $T(S)$ is the subject of the next subsection.

1.3. Quadratic differentials and the cotangent bundle of the moduli space of curves.

The results of the previous subsection show that the Teichmüller space is modeled by the space of Beltrami differentials. Recall that Beltrami differentials are measurable tensors $\mu$ of type $(-1,1)$ such that $\|\mu\|_{L^\infty} < 1$. It follows that the tangent bundle to $T(S)$ is modeled by the space of measurable and essentially bounded ($L^\infty$) tensors of type $(-1,1)$ (because Beltrami differentials form the unit ball of this Banach space). Hence, the cotangent bundle to $T(S)$ can be identified with the space $\mathcal{Q}(S)$ of integrable quadratic differentials on $S$, i.e., the space of (integrable) tensors $q$ of type $(2,0)$ (that is, $q$ is written as $q(z)dz^2$ in a local coordinate $z$). In fact, we can determine the cotangent bundle once one can find an object (a tensor of some type) such that the pairing

$$\langle \mu, q \rangle = \int_S q\mu$$

is well-defined; when $\mu$ is a tensor of type $(-1,1)$ and $q$ is a tensor of type $(2,0)$, we can write $q\mu = q(z)\mu(z)dz^2 \frac{\partial^2}{\partial z^2} = q(z)\mu(z)dz \, d\bar{z} = q(z)\mu(z)|dz|^2$ in local coordinates, i.e., we obtain a tensor of type $(1,1)$, that is, an area form. Therefore, since $\mu$ is essentially bounded, we see that the requirement that this pairing makes sense is equivalent to ask that the tensor $q$ of type $(2,0)$ is integrable.

Next, let’s see how the geodesic flow associated to the Teichmüller metric looks like after the identification of the cotangent bundle of $T(S)$ with the space $\mathcal{Q}(S)$ of integrable quadratic differentials. Firstly, we need to investigate more closely the geometry of extremal quasiconformal
maps between two Riemann surface structures. To do so, we recall another notable theorem of O. Teichmüller:

**Theorem 4** (O. Teichmüller). *Given an extremal map \( f : S_0 \to S_1 \), there is an atlas \( \phi_i \) on \( S_0 \) compatible with the underlying complex structure such that*

- the changes of coordinates are all of the form \( z \mapsto \pm z + c \), \( c \in \mathbb{C} \) outside the neighborhoods of a finite number of points,
- the horizontal (resp., vertical) foliation \( \{ \Im \phi_i = 0 \} \) (resp., \( \{ \Re \phi_i = 0 \} \)) is tangent to the major (resp.minor) axis of the infinitesimal ellipses obtained as the images of infinitesimal circles under the derivative \( Df \), and
- in terms of these coordinates, \( f \) expands the horizontal direction by the constant factor of \( \sqrt{K} \) and \( f \) contracts the vertical direction by the constant factor of \( 1/\sqrt{K} \).

An atlas \( \phi_i \) satisfying the property of the first item of Teichmüller theorem above is called a *half-translation structure*. In this language, Teichmüller’s theorem says that extremal maps \( f : S_0 \to S_1 \) (i.e., deformations of Riemann surface structures) can be easily understood in terms of half-translation structures: it suffices to expand (resp., contract) the corresponding horizontal (resp., vertical) foliation on \( S_0 \) by a constant factor equal to \( e^{d(S_0,S_1)} \) in order to get a horizontal (resp., vertical) foliation of a half-translation structure compatible with the Riemann surface structure of \( S_1 \). This provides a simple way to describe the Teichmüller geodesic flow in terms of half-translation structures. Thus, it remains to relate half-translation structures with quadratic differentials to get a pleasant formulation of this geodesic flow. While we could do this job right now, we’ll postpone this discussion to the third section of these notes for two reasons:

- Teichmüller geodesic flow is naturally embedded into a \( SL(2,\mathbb{R}) \)-action (as a consequence of this relationship between half-translation structures and quadratic differentials), so that it is preferable to give a unified treatment of this fact later;
- for pedagogical motivations, once we know that quadratic differentials is the correct object to study, it seems more reasonable to introduce the fine structures of the space \( Q(S) \) before the dynamics on this space (than the other way around).

In particular, we’ll proceed as follows: for the remainder of this subsection, we’ll briefly sketch the *bijective correspondence* between half-translation structures and quadratic differentials; after that, we make some remarks on the Teichmüller metric (and other metric structures on \( Q(S) \)) and we pass to the next subsection where we work out the particular (but important) case of genus 1 surfaces; then, in the spirit of the two items above, we devote Section 2 to the fine structures of \( Q(S) \), and Section 3 to the dynamics on \( Q(S) \).

Given a half-translation structure \( \phi_i : U_i \to \mathbb{C} \) on a Riemann surface \( S \), one can easily construct a quadratic differential \( q \) on \( S \) by pulling back the quadratic differential \( dz^2 \) on \( \mathbb{C} \) through \( \phi_i \); indeed, this procedure leads to a well-defined global quadratic differential on \( S \) because we are assuming that the changes of coordinates (outside the neighborhoods of finitely many points) have
the form $z \mapsto \pm z + c$. Conversely, given a quadratic differential $q$ on a Riemann surface $S$, we take an atlas $\phi_i : U_i \to \mathbb{C}$ such that $q|_{U_i} = \phi_i^*(dz^2)$ outside the neighborhoods of finitely many singularities of $q$. Note that the fact that $q$ is obtained by pulling back the quadratic differential $dz^2$ on $\mathbb{C}$ means that the associated changes of coordinates $z \mapsto z'$ send the quadratic differential $dz^2$ to $(dz')^2$. Thus, our changes of coordinates outside the neighborhoods of the singularities of $q$ have the form $z \mapsto z' = \pm z + c$, i.e., $\phi_i$ is a half-translation structure.

**Remark 5.** Generally speaking, a quadratic differential on a Riemann surface is either **orientable** or **non-orientable**. More precisely, given a quadratic differential $q$, consider the underlying half-translation structure $\phi_i$ and define two foliations by $\{ \Im \phi_i = c \}$ and $\{ \Re \phi_i = c \}$ (these are called the horizontal and vertical foliations associated to $q$). We say that $q$ is orientable if these foliations are orientable and $q$ is non-orientable otherwise. Alternatively, we say that $q$ is orientable if the changes of coordinates of the underlying half-translation structure $\phi_i$ outside the singularities of $q$ have the form $z \mapsto z + c$. Equivalently, $q$ is orientable if it is the global square of a holomorphic 1-form, i.e., $q = \omega^2$, where $\omega$ is a holomorphic 1-form, that is, an **Abelian differential**. For the sake of simplicity of the exposition, from now on, we’ll deal **exclusively** with orientable quadratic differentials $q$, or, more precisely, we’ll restrict our attention to **Abelian differentials**. The reason to doing so is two-fold: firstly, most of our computations below become more easy and clear in the orientable setting, and secondly, **usually** (but not always) some results about Abelian differentials can be extended to the non-orientable setting by a double cover construction, that is, one consider a (canonical) double cover of the initial Riemann surface equipped with a non-orientable quadratic differential $q$ such that a global square of the lift of $q$ is well-defined. In the sequel, we denote the space of Abelian differentials on a compact surface $S$ of genus $g$ by $\mathcal{H}(S)$ or $\mathcal{H}_g$.

**Remark 6.** In Subsection 2.1 we will come back to the correspondence between quadratic differentials and half-translation structures in the context of Abelian differentials: more precisely, in this subsection we will see that **Abelian differentials** bijectively correspond to the so-called translation structures.

We close this subsection with the following comments.

**Remark 7.** The Teichmüller metric is induced by the family of norms on the cotangent bundle $Q(S)$ of Teichmüller space $T(S)$ given by the $L^1$ norm of quadratic differentials (see Theorem 6.6.5 of [40]). However, this family of norms doesn’t depend smoothly on the base point in general, so that it doesn’t originate from a Riemannian metric. In fact, this family of norms defines only a Finsler metric, i.e., it is a family of norms depending continuously on the base point.

**Remark 8.** The Teichmüller space $T(S)$ of a compact surface $S$ of genus $g \geq 2$ is a nice complex-analytic manifold of complex dimension $3g - 3$ and it is homeomorphic to the unit open ball of $\mathbb{C}^{3g-3}$, while the moduli space $\mathcal{M}(S)$ is an orbifold in general. In fact, we are going to face this phenomenon in the next subsection (when we review the particular important case of genus 1 curves).
Remark 9. Another important metric on Teichmüller spaces whose geometrical and dynamical properties are the subject of several recent works is the so-called Weil-Petersson metric. It is the metric coming from the Hermitian inner product $\langle q_1, q_2 \rangle_{WP} := \int_S \frac{\text{tr} q_1 q_2}{\rho_S^2} \text{d}Q(S)$, where $\rho_S$ is the hyperbolic metric of the Riemann surface $S$ and $\rho_S^2$ is the associated area form. A profound result says that Weil-Petersson metric is a Kähler metric, i.e., the 2-form $\mathfrak{Im}\langle \cdot, \cdot \rangle_{WP}$ gives $\mathfrak{Im}\langle \cdot, \cdot \rangle_{WP}$ given by the imaginary part of the Weil-Petersson metric is closed. Furthermore, a beautiful theorem of S. Wolpert says that this 2-form admits a simple expression in terms of the Fenchel-Nielsen coordinates on Teichmüller space. Other important facts about the Weil-Petersson geodesic flow (i.e., the geodesic flow associated to $\langle \cdot, \cdot \rangle_{WP}$) are:

- it is a negatively curved incomplete metric with unbounded sectional curvatures (i.e., the sectional curvatures can approach 0 and/or $-\infty$ in general), so that the Weil-Petersson geodesic flow is a natural example of singular hyperbolic dynamics;
- S. Wolpert showed that this geodesic flow is defined for all times on a set of full measure of $Q(S)$;
- J. Brock, H. Masur and Y. Minsky showed that this geodesic flow is transitive, its set of periodic orbits is dense and it has infinite topological entropy;
- based on important previous works of S. Wolpert and C. McMullen, K. Burns, H. Masur and A. Wilkinson proved that this geodesic flow is ergodic with respect to Weil-Petersson volume form.

We refer to the excellent introduction of the paper [12] (and references therein) of K. Burns, H. Masur and A. Wilkinson for a nice account on the Weil-Petersson metric. Ending this remark, we note that the basic difference between the Teichmüller metric and the Weil-Petersson metric is the following: as we already indicated, the Teichmüller metric is related to flat (half-translation) structures, while the Weil-Petersson metric can be better understood in terms of hyperbolic structures.

1.4. An example: Teichmüller and moduli spaces of elliptic curves (torii). The goal of this subsection is the illustration of the role of the several objects introduced previously in the concrete case of genus 1 surfaces (elliptic curves). Indeed, we’ll see that, in this particular case, one can do “everything” by hand.

We begin by recalling that an elliptic curve, i.e., a Riemann surface of genus 1, is uniformized by the complex plane. In other words, any elliptic curve is biholomorphic to a quotient $\mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{C}$ is a lattice. Given a lattice $\Lambda \subset \mathbb{C}$ generated by two elements $w_1$ and $w_2$, that is, $\Lambda = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, we see that the multiplication by $1/w_1$ or $1/w_2$ provides a biholomorphism isotopic to the identity between $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda(w)$, where $\Lambda(w) := \mathbb{Z} \oplus \mathbb{Z}w$ is the lattice generated by 1 and $w \in \mathbb{H} \subset \mathbb{C}$ (the upper-half plane of the complex plane). In fact, $w = w_2/w_1$ or $w = w_1/w_2$ here. Next, we observe that any biholomorphism $f$ between $\mathbb{C}/\Lambda(w')$ and $\mathbb{C}/\Lambda(w)$ can be lifted to an automorphism $F$ of the complex plane $\mathbb{C}$. This implies that $F$ has the form $F(z) = Az + B$ for
some $A, B \in \mathbb{C}$. On the other hand, since $F$ is a lift of $f$, we can find $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that

$$
\begin{align*}
F(z + 1) - F(z) &= \delta + \gamma w \\
F(z + w') - F(z) &= \beta + \alpha w
\end{align*}
$$

Expanding these equations using the fact that $F(z) = az + b$, we get

$$
w' = \frac{\alpha w + \beta}{\gamma w + \delta}
$$

Also, since we’re dealing with invertible objects ($f$ and $F$), it is not hard to check that $\alpha \delta - \beta \gamma = 1$ (because it is an integer number whose inverse is also an integer). In other words, recalling that $SL(2, \mathbb{R})$ acts on $H$ via

$$
\begin{pmatrix} a & b \\
    c & d \end{pmatrix} \in SL(2, \mathbb{R}) \longleftrightarrow z \mapsto \frac{az + b}{cz + d},
$$

we see that the torii $\mathbb{C}/\Lambda(w)$ and $\mathbb{C}/\Lambda(w')$ are biholomorphic if and only if $w' \in SL(2, \mathbb{Z}) \cdot w$.

For example, we show below the torii $\mathbb{C}/\Lambda(i)$ (on the left) and $\mathbb{C}/\Lambda(1 + i)$ (in the middle).

Since $1 + i$ is deduced from $i$ via the action of $T = \begin{pmatrix} 1 & 1 \\
    0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$ on $\mathbb{H}$, we see that these torii are biholomorphic and hence they represent the same point in the moduli space $\mathcal{M}_1$ (see the right hand side part of the figure above). On the other hand, they represent distinct points in the Teichmüller space $T_1$ (because $T \neq id$).

Our discussion so far implies that the Teichmüller space $T_1$ of elliptic curves is naturally identified with the upper-half plane $\mathbb{H}$ and the moduli space $\mathcal{M}_1$ of elliptic curves is naturally identified with $\mathbb{H}/SL(2, \mathbb{Z})$. Furthermore, it is possible to show that, under this identification, the Teichmüller metric on $T_1$ corresponds to the hyperbolic metric (of constant curvature $-1$) on $\mathbb{H}$, so that the Teichmüller geodesic flow on $T_1$ and $\mathcal{M}_1$ are the geodesic flows of the hyperbolic metric on $\mathbb{H}$ and $\mathbb{H}/SL(2, \mathbb{Z})$. In order to better understand the moduli space $\mathcal{M}_1$, we’ll make the geometry of the quotient $\mathbb{H}/SL(2, \mathbb{Z})$ (called modular curve in the literature) more clear by presenting a fundamental domain of the $SL(2, \mathbb{Z})$-action on $\mathbb{H}$. It is a classical result (see Proposition 3.9.14 of [10]) that the region $F_0 := \{ z \in \mathbb{H} : -1/2 \leq \Re z \leq 1/2 \text{ and } |z| \geq 1 \}$ is a fundamental domain of this action in the sense that every $SL(2, \mathbb{Z})$-orbit intersects $F_0$, but it can intersect the interior $\text{int}(F_0)$ of $F_0$ at most once. In the specific case at hand, $SL(2, \mathbb{Z})$ acts on the boundary $\partial F_0$ of $F_0$ is $\partial F_0 = \{|z| \geq 1 \text{ and } \Re z = \pm 1/2\} \cup \{|z| = 1 \text{ and } |\Re z| \leq 1/2\}$ by sending
• \( \{ |z| \geq 1 \text{ and } \Re z = -1/2 \} \) to \( \{ |z| \geq 1 \text{ and } \Re z = 1/2 \} \) through the translation \( z \mapsto z + 1 \) or equivalently the parabolic matrix \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), and

• \( \{-1/2 \leq \Re z \leq 0 \text{ and } |z| = 1 \} \) to \( \{ 0 \leq \Re z \leq 1/2 \text{ and } |z| = 1 \} \) through the “inversion” \( z \mapsto -1/z \) or equivalently the elliptic (rotation) matrix \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

See the figure below for an illustration of the fundamental domain \( F_0 \):

![Diagram of fundamental domain F0]

This explicit description of the genus 1 case allows to clarify the role of the several objects introduced above. From the dynamical point of view, it is more interesting to consider the Teichmüller flow on moduli spaces than Teichmüller spaces: indeed, the Teichmüller flow on Teichmüller space is somewhat boring (for instance, it is not recurrent), while it is very interesting on moduli space (for instance, in the genus 1 case [i.e., the geodesic flow on the modular curve], it exhibits all nice features [recurrence, exponential mixing, ...] of hyperbolic systems [these properties are usually derived from the connection with continued fractions]). However, from the point of nice analytic structures, Teichmüller spaces are better than moduli spaces because Teichmüller spaces are complex-analytic manifolds while moduli spaces are orbifolds.\(^1\) In any case, it is natural to consider both spaces from the topological point of view because Teichmüller spaces are simply connected so that they are the universal covers of moduli spaces. Finally, closing this section, we note that our discussion above also shows that, in the genus 1 case, the mapping class group \( \Gamma_1 \) is \( SL(2, \mathbb{Z}) \).

2. SOME STRUCTURES ON THE SET OF ABELIAN DIFFERENTIALS

We denote by \( L_g \) the set of Abelian differentials on a Riemann surface of genus \( g \geq 1 \), or more precisely, the set of pairs (Riemann surface structure on \( M, \omega \) where \( M \) is a compact topological surface of genus \( g \) and \( \omega \) is a 1-form which is holomorphic with respect to the underlying Riemann surface structure. In this notation, the Teichmüller space of Abelian differentials is the quotient \( TH_g := L_g/\text{Diff}_0^+ (M) \) and the moduli space of Abelian differentials is the quotient \( H_g := L_g/\Gamma_g \).

\(^1\)In general, the mapping class group doesn’t act properly discontinuously on Teichmüller space because some Riemann surfaces are “more symmetric” (i.e., they have larger automorphisms group) than others. In fact, we already saw this in the case of genus 1: the modular curve \( \mathbb{H}/SL(2, \mathbb{Z}) \) isn’t smooth near the points \( w = i \) and \( w = e^{\pi i/3} \) because the (square and hexagonal) torii corresponding to these points have larger automorphisms groups when compared with a typical torus \( \mathbb{C}/\Lambda(w) \).
Here $\text{Diff}^+_0(M)$ and $\Gamma_g := \text{Diff}^+_0(M)/\text{Diff}^+_0(M)$ (the set of diffeomorphisms isotopic to the identity and the mapping class group resp.) act on the set of Riemann surface structure in the usual manner, while they act on Abelian differentials by pull-back.

In order to equip $L_g$, $TH_g$ and $H_g$ with nice structures, we need a more “concrete” presentation of Abelian differentials. In the next subsection, we will see that the notion of translation structures provide such a description of Abelian differentials.

2.1. Abelian differentials and translation structures. Given any point $p \in M - \Sigma$, let’s select $U(p)$ a small path-connected neighborhood of $p$ such that $U(p) \cap \Sigma = \emptyset$. In this setting, the “period” map $\phi_p : U(p) \to \mathbb{C}$, $\phi_p(x) := \int_x^p \omega$ obtained by integration along a path inside $U(p)$ connecting $p$ and $x$ is well-defined: indeed, this follows from the fact that any Abelian differential is closed (because they are holomorphic) and the neighborhood $U(p)$ doesn’t contain any zeroes of $\omega$ (so that the integral $\int_x^p \omega$ doesn’t depend on the choice of the path inside $U(p)$ connecting $p$ and $x$). Furthermore, since $p \in M - \Sigma$ (so that $\omega(p) \neq 0$), we see that, by reducing $U(p)$ if necessary, this “period” map is a biholomorphism.

In other words, the collection of all such “period” maps $\phi_p$ provides an atlas of $M - \Sigma$ compatible with the Riemann surface structure. Also, by definition, the push-forward of the Abelian differential $\omega$ by any such $\phi_p$ is precisely the canonical Abelian differential $(\phi_p)_*(\omega) = dz$ on the complex plane $\mathbb{C}$. Moreover, the “local” equality $\int_{p}^{x} \omega = \int_{q}^{x} \omega + \int_{p}^{q} \omega$ implies that all changes of coordinates have the form $\phi_q \circ \phi_p^{-1}(z) = z + c$ where $c = \int_{p}^{q} \omega \in \mathbb{C}$ is a constant (since it doesn’t depend on $z$). Furthermore, since $\omega$ has finite order at its zeroes, it is easy to deduce from Riemann’s theorem on removal singularities that this atlas of $M - \Sigma$ can be extended to $M$ in such a way that the push-forward of $\omega$ by a local chart around a zero $p \in \Sigma$ of order $k$ is the holomorphic form $z^k dz$.

In the literature, a compact surface $M$ of genus $g \geq 1$ equipped with an atlas whose changes of coordinates are given by translations $z \mapsto z + c$ of the complex plane outside a finite set of points $\Sigma \subset M$ is called a translation surface structure on $M$. In this language, our previous discussion simply says that any non-trivial Abelian differential $\omega$ on a compact Riemann surface $M$ gives rise to a translation surface structure on $M$ such that $\omega$ is the pull-back of the canonical holomorphic form $dz$ on $\mathbb{C}$. On the other hand, it is clear that every translation surface structure determines a Riemann surface (since translations are a very particular case of local biholomorphism) and an Abelian differential (by pulling back $dz$ on $\mathbb{C}$: this pull-back $\omega$ is well-defined because $dz$ is translation-invariant).

In resume, we just saw the proof of the following proposition:

**Proposition 10.** The set $L_g$ of Abelian differentials on Riemann surfaces of genus $g \geq 1$ is canonically identified to the set of translation structures on a compact (topological) surface $M$ of genus $g \geq 1$.

**Example 11.** During Riemann surfaces courses, a complex torus is quite often presented through a translation surface structure: indeed, by giving a lattice $\Lambda = Zw_1 \oplus Zw_2 \subset \mathbb{C}$, we are saying
that the complex torus $\mathbb{C}/\Lambda$ equipped with the (non-vanishing) Abelian differential $dz$ is canonically identified with the translation surface structure represented in the picture below (it truly represents a translation structure since we’re gluing opposite parallel sides of the parallelogram determined by $w_1$ and $w_2$ through the translations $z \mapsto z + w_1$ and $z \mapsto z + w_2$).

![Figure 1. Complex torus as a translation surface](image)

**Example 12.** Consider the polygon $P$ of Figure 2 below.

![Figure 2. A genus 2 translation surface](image)

In this picture, we are gluing parallel opposite sides $v_j$, $j = 1, \ldots, 4$, of $P$, so that this is again a valid presentation of a translation surface structure. Let’s denote by $(M, \omega)$ the corresponding Riemann surface and Abelian differential. Observe that, by following the sides identifications as indicated in this figure, we see that the vertices of $P$ are all identified to a single point $p$. Moreover, we see that $p$ is a special point when compared with any point of $P - \{p\}$ because, by turning around $p$, we note that the “total angle” around it is $6\pi$ while the total angle around any point of $P - \{p\}$ is $2\pi$, that is, a neighborhood of $p$ inside $M$ resembles “3 copies” of the flat complex plane while a neighborhood of any other point $q \neq p$ resembles only 1 copy of the flat complex plane. In other words, a natural local coordinate around $p$ is $\zeta = z^3$, so that $\omega = d\zeta = d(z^3) = 3z^2dz$, i.e., the Abelian differential $\omega$ has a unique zero of order 2 at $p$. From this, we can infer that $M$ is a compact Riemann surface of genus 2: indeed, by Riemann-Hurwitz theorem, the sum of orders of zeroes of an Abelian differential equals $2g - 2$ (where $g$ is the genus); in the present case, this means $2 = 2g - 2$, i.e., $g = 2$; alternatively, one can use Poincaré-Hopf index theorem to the vector field
given by the vertical direction on $M - \{ p \}$ (this is well-defined because these points correspond to regular points of the polygon $P$) and vanishing at $p$ (where a choice of “vertical direction” doesn’t make sense since we have multiple copies of the plane glued together).

**Example 13 (Rational billiards).** Let $P$ be a rational polygon, that is, a polygon whose angles are all rational multiples of $\pi$. Consider the billiard on $P$: the trajectory of a point in $P$ in a certain direction consists of straight lines until we hit the boundary $\partial P$ of the polygon; at this moment, we apply the usual reflection laws (saying that the angle between the outgoing ray with $\partial P$ is the same as the angle between the incoming ray and $\partial P$) to prolongate the trajectory. See the figure below for an illustration of such an trajectory.

In the literature, the study of general billiards (where $P$ is not necessarily a polygon) is a classical subject with physical origins (e.g., mechanics and thermodynamics of Lorenz gases). In the particular case of billiards in rational polygons, an unfolding construction (due to R. Fox and R. Keshner [35], and A. Katok and A. Zemlyakov [46]) allows to think of billiards on rational polygons as translation flows on translation surfaces. Roughly speaking, the idea is that each time the trajectory hits the boundary $\partial P$, instead of reflecting the trajectory, we reflect the table itself so that the trajectory keeps in straight line:

The group $G$ generated by the reflections with respect to the edges of $P$ is finite when $P$ is a rational polygon, so that the natural surface $X$ obtained by this unfolding procedure is compact. Furthermore, the surface $X$ comes equipped with a natural translation structure, and the billiard dynamics on $P$ becomes the translation (straight line) flow on $X$. In the picture below we drew the translation surface (Swiss cross) obtained by unfolding a L-shaped polygon,
and in the picture below we drew the translation surface (regular octagon) obtained by unfolding a triangle with angles $\pi/8$, $\pi/2$ and $3\pi/8$.

In general, a rational polygon $P$ with $k$ edges and angles $\pi m_i/n_i$, $i = 1, \ldots, N$ has a group of reflections $G$ of order $2N$ and, by unfolding $P$, we obtain a translation surface $X$ of genus $g$ where

$$2 - 2g = N(2 - k + \sum (1/n_i))$$

In particular, it is possible to show that the only genus 1 translation surfaces obtained by the unfolding procedure come from the following polygons: a square, an equilateral triangle, a triangle with angles $\pi/3$, $\pi/2$, $\pi/6$, and a triangle with angles $\pi/4$, $\pi/2$, $\pi/4$ (see the figure below).
For more informations about translation surfaces coming from billiards on rational polygons, see this survey of H. Masur and S. Tabachnikov [54].

**Example 14** (Suspensions of interval exchange transformations). An interval exchange transformation (i.e.t. for short) is a map $T : D_T \to D_{T^{-1}}$ where $D_T, D_{T^{-1}} \subset I$ are subsets of an open bounded interval $I$ such that $\#(I - D_T) = \#(I - D_{T^{-1}}) = d < \infty$ and the restriction of $T$ to each connected component of $I - D_T$ is a translation onto some connected component of $D_{T^{-1}}$. For concrete examples, see Figure 3 below.

![Figure 3. Three examples of interval exchange transformations.](image)

Usually, we obtain an i.e.t. as a return map of a translation flow on a translation surface. Conversely, given an i.e.t. $T$, it is possible to “suspend” it (in several ways) to construct translation flows on translation surfaces such that $T$ is the first return map to an adequate transversal to the translation flow. For instance, we illustrated in the figure below a suspension construction due to H. Masur [51] applied to the third i.e.t. of Figure 3.

![suspension construction](image)

Here, the idea is that:

- the vectors $\zeta_i$ have the form $\zeta_i = \lambda_i + \sqrt{-1} \tau_i \in \mathbb{C} \cong \mathbb{R}^2$ where $\lambda_i$ are the lengths of the intervals permuted by $T$;
then, we organize the vectors $\zeta_i$ on the plane $\mathbb{R}^2$ in order to get a polygon $P$ so that by going upstairs we meet the vectors $\zeta_i$ in the usual order (i.e., $\zeta_1, \zeta_2,$ etc.) while by going downstairs we meet the vectors $\zeta_i$ in the order determined by $T$, i.e., by following the combinatorial receipt (say a permutation $\pi$ of $d$ elements) used by $T$ to permute intervals;

- by gluing by translations the pairs of sides label by vectors $\zeta_i$, we obtain a translation surface whose vertical flow has the i.e.t. $T$ as first return map to the horizontal axis $\mathbb{R} \times \{0\}$ (e.g., in the picture we drew a trajectory of the vertical flow starting at the interval $B$ on the “top part” of $\mathbb{R} \times \{0\}$ and coming back at the interval $B$ on the bottom part of $\mathbb{R} \times \{0\}$);

- finally, the suspension data $\tau_i$ is chosen “arbitrarily” as soon as the planar figure $P$ is not degenerate: formally, one imposes the condition $\sum_{j<i} \tau_j > 0$ and $\sum_{\pi(j)<i} \tau_j < 0$ for every $1 < i \leq d$.

Of course, this is not the sole way of suspending i.e.t.’s to get translation surfaces: for instance, in this survey [72] of J.-C. Yoccoz, one can find a detailed description of an alternative suspension procedure due to W. Veech (and nowadays called Veech’s zippered rectangles construction).

**Example 15** (Square-tiled surfaces). Consider a finite collection of unit squares on the plane such that the leftmost side of each square is glued (by translation) to the rightmost side of another (maybe the same) square, and the bottom side of each square is glued (by translation) to the top side of another (maybe the same) square. Here, we assume that, after performing the identifications, the resulting surface is connected. Again, since our identifications are given by translations, this procedure gives at the end of the day a translation surface structure, that is, a Riemann surface $M$ equipped with an Abelian differential (obtained by pulling back $dz$ on each square). For obvious reasons, these surfaces are called square-tiled surfaces and/or origamis in the literature. For sake of concreteness, we drew in Figure 4 below a L-shaped square-tiled surface derived from 3 unit squares identified as in the picture (i.e., pairs of sides with the same marks are glued together by translation).

![Figure 4. A L-shaped square-tiled surface.](image)

By following the same arguments used in the previous example, the reader can easily verify that this L-shaped square-tiled surface with 3 squares corresponds to an Abelian differential with a single zero of order 2 in a Riemann surface of genus 2.

**Remark 16.** So far we produced examples of translation surfaces/Abelian differentials from identifications by translation of pairs of parallel sides of a finite collection of polygons. The curious
reader maybe asking whether all translation surface structures can be recovered by this procedure. In fact, it is possible to prove that any translation surface admits a triangulation such that the zeros of the Abelian differential appear only in the vertices of the triangles (so that the sides of the triangles are saddle connections in the sense that they connect zeroes of the Abelian differential), so that the translation surface can be recovered from this finite collection of triangles. However, if we are “ambitious” and try to represent translation surfaces by side identifications of a single polygon (like in Example 12) instead of using a finite collection of polygons, then we’ll fail: indeed, there are examples where the saddles connections are badly placed so that one polygon never suffices. However, it is possible to prove (with the aid of Veech’s zippered rectangle construction) that all translation surfaces outside a set formed by a countable union of codimension 2 real-analytic suborbifolds can be represented by a sole polygon whose sides are conveniently identified. See [72] for further details.

Despite its intrinsic beauty, a great advantage of talking about translation structures instead of Abelian differentials is the fact that several additional structures come for free due to the translation invariance of the corresponding structures on the complex plane $\mathbb{C}$:

- **flat metric**: since the usual (flat) Euclidean metric $dx^2 + dy^2$ on the complex plane $\mathbb{C}$ is translation-invariant, its pullback by the local charts provided by the translation structure gives a well-defined flat metric on $M - \Sigma$;
- **area form**: again, since the usual Euclidean area form $dx \cdot dy$ on the complex plane $\mathbb{C}$ is also translation-invariant, we get a well-defined area form $dA$ on $M$;
- **canonical choice of a vertical vector-field**: as we implicitly mentioned by the end of Example 12, the vertical vertical vector field $\partial/\partial y$ on $\mathbb{C}$ can be pulled back to $M - \Sigma$ in a coherent way to define a canonical choice of north direction;
- **pair of transverse measured foliations**: the pullback to $M - \Sigma$ of the horizontal $\{x = \text{constant}\}$ and vertical $\{y = \text{constant}\}$ foliations of the complex plane $\mathbb{C}$ are well-defined and produce a pair of transverse foliations $\mathcal{F}_h$ and $\mathcal{F}_v$ which are *measured* in the sense on Thurston: the leaves of these foliations come with a canonical notion of length measures $|dy|$ and $|dx|$ transversely to them.

Remark 17. It is important to observe that the flat metric introduced above is a *singular* metric: indeed, although it is a smooth Riemannian metric on $M - \Sigma$, it degenerates when we approach a zero $p \in \Sigma$ of the Abelian differential. Of course, we know from Gauss-Bonnet theorem that no compact surface of genus $g \geq 2$ admit a completely flat metric, so that, in some sense, if we wish to have a flat metric in a large portion of the surface, we’re obliged to “concentrate” the curvature at tiny places. From this point of view, the fact that the flat metric obtained from translation structures are degenerate at a finite number of points reflects the fact that the “sole” way to produce a “almost completely” flat model of our genus $g \geq 2$ surface is by concentrating all curvature at a finite number of points.
Once we know that Abelian differentials and translation structures are essentially the same object, we can put some structures on $L_g$, $TH_g$ and $H_g$.

2.2. Stratification. Given a non-trivial Abelian differential $\omega$ on a Riemann surface of genus $g \geq 1$, we can form a list $\kappa = (k_1, \ldots, k_\sigma)$ by collecting the multiplicities of the (finite set of) zeroes of $\omega$. Observe that any such list $\kappa = (k_1, \ldots, k_\sigma)$ verifies the constraint $\sum_{l=1}^{\sigma} k_l = 2g - 2$ in view of Poincaré-Hopf theorem (or alternatively Gauss-Bonnet theorem). Given an unordered list $\kappa = (k_1, \ldots, k_\sigma)$ with $\sum_{l=1}^{\sigma} k_l = 2g - 2$, the set $\mathcal{L}(\kappa)$ of Abelian differentials whose list of multiplicities of its zeroes coincide with $\kappa$ is called a stratum of $L_g$. Since the actions of $\text{Diff}_g^+(M)$ and $\Gamma_g$ respect the multiplicities of zeroes, the quotients $TH(\kappa) := \mathcal{L}(\kappa)/\text{Diff}_g^+(M)$ and $H(\kappa) := \mathcal{L}(\kappa)/\Gamma_g$ are well-defined. By obvious reasons, $TH(\kappa)$ and $H(\kappa)$ are called stratum of $TH_g$ and $H_g$ (resp.). Notice that, by definition,

$$L_g = \bigsqcup_{\kappa=(k_1,\ldots,k_\sigma),\sum_{l=1}^{\sigma} k_l = 2g-2} \mathcal{L}(\kappa), \quad TH_g = \bigsqcup_{\kappa=(k_1,\ldots,k_\sigma),\sum_{l=1}^{\sigma} k_l = 2g-2} TH(\kappa),$$

and

$$H_g = \bigsqcup_{\kappa=(k_1,\ldots,k_\sigma),\sum_{l=1}^{\sigma} k_l = 2g-2} H(\kappa).$$

In other words, the sets $L_g$, $TH_g$ and $H_g$ are naturally “decomposed” into the subsets (strata) $\mathcal{L}(\kappa)$, $TH(\kappa)$ and $H(\kappa)$. However, at this stage, we can’t promote this decomposition into disjoint subsets to a stratification because, in the literature, a stratification of a set $X$ is a decomposition $X = \bigcup_{i \in I} X_i$ where $I$ is a finite set of indices and the strata $X_i$ are disjoint manifolds/orbifolds of distinct dimensions. Thus, one can’t call stratification our decomposition of $L_g$, $TH_g$ and $H_g$ until we put nice manifold/orbifold structures (of different dimension) on the corresponding strata. The introduction of nice complex-analytic manifold/orbifold structures on $TH(\kappa)$ and $H(\kappa)$ are the topic of our next subsection.

Remark 18. The curious reader might be asking at this point whether the strata $H(\kappa)$ are non-empty. In fact, this is a natural question because, while the condition $\sum_{s=1}^{\sigma} k_s = 2g-2$ is a necessary condition (in view of Poincaré-Hopf theorem say), it is not completely obvious that this condition is also sufficient. In any case, it is possible to show that the strata $H(\kappa)$ of Abelian differentials are non-empty whenever $\sum_{s=1}^{\sigma} k_s = 2g-2$. For comparison, we note that exact analog result for strata of non-orientable quadratic differentials is false. Indeed, if we denote by $Q(d_1,\ldots,d_m,-1^p)$ the stratum of non-orientable quadratic differentials with $m$ zeroes of orders $d_1,\ldots,d_m \geq 1$ and $p$ simple poles, the Poincaré-Hopf theorem says that a necessary condition is $\sum_{l=1}^{m} d_l - p = 4g - 4$, and a theorem of H. Masur and J. Smillie says that this condition is almost sufficient: except for the empty strata $Q(\emptyset)$, $Q(1,-1)$, $Q(1,3)$, $Q(4)$ in genera 1 and 2, any strata verifying the previous condition is non-empty. See [53] for more details.
2.3. Period map and local coordinates. Let $\mathcal{TH}(\kappa)$ be a stratum, say $\kappa = (k_1, \ldots, k_g)$, $\sum k_l = 2g - 2$. Given an Abelian differential $\omega \in \mathcal{TH}(\kappa)$, we denote by $\Sigma(\omega)$ the set of zeroes of $\omega$. It is possible to prove that, for every $\omega_0 \in \mathcal{TH}(\kappa)$, there is an open set $\omega_0 \in U_0 \subset \mathcal{TH}(\kappa)$ such that, after identifying, for all $\omega \in U_0$, the cohomology $H^1(M, \Sigma(\omega), \mathbb{C})$ with the fixed complex vector space $H^1(M, \Sigma(\omega_0), \mathbb{C})$ via the Gauss-Manin connection (i.e., through identification of the integer lattices $H^1(M, \Sigma(\omega), \mathbb{Z} \oplus i\mathbb{Z})$ and $H^1(M, \Sigma(\omega_0), \mathbb{Z} \oplus i\mathbb{Z})$), the period map $\Theta : U_0 \to H^1(M, \Sigma(\omega_0), \mathbb{C})$ given by

$$\Theta(\omega) := \left( \gamma \to \int_{\gamma} \omega \right) \in \text{Hom}(H_1(M, \Sigma(\omega), \mathbb{Z}), \mathbb{C}) \simeq H^1(M, \Sigma(\omega), \mathbb{C}) \simeq H^1(M, \Sigma(\omega_0), \mathbb{C})$$

is a local homeomorphism. A sketch of proof of this fact (along the lines given in this article of A. Katok) goes as follows. We need to prove that two closed 1-forms $\eta_0$ and $\eta_1$ with the same relative periods, i.e., $\Theta(\eta_0) = \Theta(\eta_1)$, are isotopic as far as they are close enough to each other. The idea to construct such an isotopy is to apply a variant of the so-called Moser’s homotopy trick. More precisely, one considers $\eta_t = (1 - t)\eta_0 + t\eta_1$, and one tries to find the desired isotopy $\phi_t$ by solving

$$\phi_t^*(\eta_t) = \eta_0.$$

In this direction, let’s see what are the properties satisfied by a solution $\phi_t$. By taking the derivative, we find

$$0 = \frac{d}{dt} \phi_t^*(\eta_t).$$

Assuming that $\phi_t$ is the flow of a (non-autonomous) vector field $X_t$, we get

$$0 = \frac{d}{dt} (\phi_t^*(\eta_t)) = \phi_t^*(\dot{\eta}_t) + \mathcal{L}_{X_t} \eta_t = \phi_t^*(\eta_1 - \eta_0) + \mathcal{L}_{X_t} \eta_t,$$

where $\mathcal{L}_{X_t}$ is the Lie derivative along the direction of $X_t$.

By hypothesis, $\Theta(\eta_0) = \Theta(\eta_1)$. In particular, $\eta_0$ and $\eta_1$ have the same absolute periods, so that $[\eta_1] = [\eta_1 - \eta_0] = 0$ in $H^1(M, \mathbb{R})$. In other words, we can find a smooth family $U_t$ of functions with $dU_t = \eta_t$. By inserting this into the previous equation, it follows that

$$0 = dU_t + \mathcal{L}_{X_t} \eta_t.$$

On the other hand, by Cartan’s magic formula, $\mathcal{L}_{X_t} \eta_t = i_{X_t} (d\eta_t) + d(i_{X_t} \eta_t)$. Since $\eta_t$ is closed, $d\eta_t = 0$, so that $\mathcal{L}_{X_t} \eta_t = d(i_{X_t} \eta_t)$. By inserting this into the previous equation, we get

$$0 = d(U_t + i_{X_t} \eta_t).$$

\textsuperscript{2}Here we’re considering the natural topology on strata $\mathcal{TH}(\kappa)$ induced by the developing map. More precisely, given $\omega \in \mathcal{L}(\kappa)$, fix $p_1 \in \Sigma(\omega)$, an universal cover $p : \tilde{M} \to M$ and a point $P_1 \in \tilde{M}$ over $p_1$. By integration of $p^* \omega$ from $P_1$ to a point $Q \in \tilde{M}$, we get, by definition, a developing map $D_\omega : (\tilde{M}, P_1) \to (\mathbb{C}, 0)$ completely determining the translation surface $(M, \omega)$. In this way, the injective map $\omega \mapsto D_\omega$ allows us to see $\mathcal{L}(\kappa)$ as a subset of the space $C^0(\tilde{M}, \mathbb{C})$ of complex-valued continuous functions of $\tilde{M}$. By equipping $C^0(\tilde{M}, \mathbb{C})$ with the compact-open topology, we get natural topologies for $\mathcal{L}(\kappa)$ and $\mathcal{TH}(\kappa)$.
At this point, we see how one can hope to solve the original equation \( \phi_t^* \eta_t = \eta_0 \); firstly we fix a smooth family \( U_t \) with \( dU_t = \eta_t = \eta_1 - \eta_0 \) (unique up to additive constant), secondly we define a vector field \( X_t \) such that \( i_{X_t} \eta_t = -U_t \), so that \( U_t + i_{X_t} \eta_t = 0 \) and a fortiori \( d(U_t + i_{X_t} \eta_t) = 0 \), and finally we let \( \phi_t \) be the isometry associated to \( X_t \). Of course, one must check that this is well-defined: for instance, since \( \eta_t \) have singularities (zeroes) at the finite set \( \Sigma = \Sigma(\eta_t) = \Sigma(\eta_1) \), we need to know that one can take \( U_t = 0 \) at \( \Sigma \), and this is possible because \( U_t(p_i) - U_t(p_i) = \int_{p_i}^{p_i} \eta_t = \int_{p_i}^{p_i} (\eta_1 - \eta_0) \), and \( \eta_0 \) and \( \eta_1 \) have the same relative periods. We leave the verification of the details of the definition of \( \phi_t \) as an exercise to the reader (whose solution is presented in [15]).

For an alternative proof of the fact that the period maps are local coordinates based on W. Veech’s zippered rectangles, see e.g. J.-C. Yoccoz’s survey [72].

Remark 19. Concerning the possibility of using Veech’s zippered rectangles to show that the period maps are local homeomorphisms, we vaguely mentioned this particular strategy because the zippered rectangles are a fundamental tool: indeed, besides its usage to put nice structures on \( \mathcal{H}(\kappa) \), it can be applied to understand the connected components of \( \mathcal{H}(\kappa) \), make volume estimates for \( \mu_\kappa \), study the dynamics of Teichmüller flow on \( \mathcal{H}(\kappa) \) through combinatorial methods (Markov partitions), etc. In other words, Veech’s zippered rectangles is a powerful tool to study global geometry and dynamics on \( \mathcal{H}(\kappa) \). Furthermore, it allows to connect the dynamics on \( \mathcal{H}(\kappa) \) to the interval exchange transformations and translation flows, so that it is also relevant in applications of the dynamics of Teichmüller flow. However, taking into account the usual limitations of space and time, and the fact that this tool is largely discussed in Yoccoz’s survey [72], we will not give here more details on Veech’s zippered rectangles and applications.

Recall that \( H^1(M, \Sigma(\omega_0), \mathbb{C}) \) is a complex vector space isomorphic to \( \mathbb{C}^{2g+\sigma-1} \). An explicit way to see this fact is by taking \( \{\alpha_j, \beta_j\}_{j=1}^g \) a basis of the absolute homology group \( H_1(M, \mathbb{Z}) \), and relative cycles \( \gamma_1, \ldots, \gamma_{\sigma-1} \) joining an arbitrarily chosen point \( p_0 \in \Sigma(\omega_0) \) to the other points \( p_1, \ldots, p_{\sigma-1} \in \Sigma(\omega) \). Then, the map
\[
\omega \in H^1(M, \Sigma(\omega_0), \mathbb{C}) \mapsto \left( \int_{\alpha_1} \omega, \int_{\beta_1} \omega, \ldots, \int_{\alpha_g} \omega, \int_{\beta_g} \omega, \int_{\gamma_1} \omega, \ldots, \int_{\gamma_{\sigma-1}} \omega \right) \in \mathbb{C}^{2g+\sigma-1}
\]
gives the desired isomorphism. Moreover, by composing the period maps with such isomorphisms, we see that all changes of coordinates are given by affine maps of \( \mathbb{C}^{2g+\sigma-1} \). In particular, we can also pullback Lebesgue measure on \( \mathbb{C}^{2g+\sigma-1} \) to get a natural measure \( \lambda_\kappa \) on \( \mathcal{T}(\kappa) \): indeed, \( \lambda_\kappa \) is well-defined modulo normalization because the changes of coordinates are affine maps; to remedy the normalization problem, we ask that the integral lattice \( H^1(M, \Sigma(\omega_0), \mathbb{Z} \oplus i\mathbb{Z}) \) has covolume 1, i.e., the volume of the torus \( H^1(M, \Sigma(\omega_0), \mathbb{C})/H^1(M, \Sigma(\omega_0), \mathbb{Z} \oplus i\mathbb{Z}) \) is 1.

In resume, by using the period maps as local coordinates, \( \mathcal{T}(\kappa) \) has a structure of affine complex manifold of complex dimension \( 2g + \sigma - 1 \) and a natural (Lebesgue) measure \( \lambda_\kappa \). Also, it is not hard to check that the action of the modular group \( \Gamma_g := \text{Diff}^+(M)/\text{Diff}_0^+(M) \) is compatible with the affine complex manifold structure on \( \mathcal{T}(\kappa) \), so that, by passing to the quotient, one gets
that the stratum $\mathcal{H}(\kappa)$ of the moduli space of Abelian differentials has the structure of a affine complex orbifold and a natural Lebesgue measure $\mu_\kappa$.

**Remark 20.** Note that, as we emphasized above, after passing to the quotient, $\mathcal{H}(\kappa)$ is an affine complex orbifold at best. In fact, we can’t expect $\mathcal{H}(\kappa)$ to be a manifold since, as we saw in Subsection 1.4 even in the most simple example of genus 1, $\mathcal{H}(\emptyset)$ is an orbifold but not a manifold. In particular, the fact that the period maps are local homeomorphisms is intimately related to the fact that we were talking specifically about the Teichmüller space of Abelian differentials $T\mathcal{H}(\kappa)$ (and not of its cousin $\mathcal{H}(\kappa)$).

The following example shows a concrete way to geometrically interpret the role of period maps as local coordinates of the strata $T\mathcal{H}(\kappa)$.

**Example 21.** Let $Q$ be a polygon and $(M, \omega_0)$ a translation surface like in Example 12. As we saw in this example, $M$ is a genus 2 Riemann surface and $\omega_0$ has an unique zero (of order 2) at the point $p \in M$ coming from the vertices of $Q$ (because the total angle around $p$ is $6\pi$), that is, $\omega_0 \in T\mathcal{H}(2)$. Also, it can be checked that the four closed loops $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $M$ obtained by projecting to $M$ the four sides $v_1, v_2, v_3, v_4$ from $Q$ are a basis of the absolute homology group $H_1(M, \mathbb{Z})$. It follows that, in this case, the period map in a small neighborhood $U_0$ of $\omega_0$ is

$$\omega \in U_0 \subset T\mathcal{H}(2) \mapsto \left(\int_{\alpha_1} \omega, \int_{\alpha_2} \omega, \int_{\alpha_3} \omega, \int_{\alpha_4} \omega\right) \in V_0 \subset \mathbb{C}^4,$$

where $V_0$ is a small neighborhood of $(v_1, v_2, v_3, v_4)$. Consequently, we see that any Abelian differential $\omega \in \mathcal{H}(2)$ sufficiently close to $\omega_0$ can be obtained geometrically by small arbitrary perturbations (indicated by dashed red lines in the figure below) of the sides $v_1, v_2, v_3, v_4$ of our initial polygon $Q$ (indicated by blue lines in the figure below).

![Diagram of a polygon with vertices and sides labeled](image)

**Remark 22.** The introduction of nice affine complex structures in the case of non-orientable quadratic differentials is slightly different from the case of Abelian differentials. Given a non-orientable quadratic differential $q \in Q(\kappa)$ on a genus $g$ Riemann surface $M$, there is a canonical (orienting) double-cover $\pi_\kappa : \hat{M} \to M$ such that $\pi_\kappa^*(q) = \hat{\omega}^2$, where $\hat{\omega}$ is an Abelian differential on $\hat{M}$. Here, $\hat{M}$ is a connected Riemann surface because $q$ is non-orientable (otherwise it would be two disjoint copies of $M$). Also, by writing $\kappa = (\alpha_1, \ldots, \alpha_r, e_1, \ldots, e_\nu)$ where $\alpha_j$ are odd integers and $e_j$ are even...
connected components of strata, but this is a hard combinatorial problem: for instance, when trying to compute extended Rauzy classes associated to connected components.

\(^3\)A slightly modified version of Rauzy classes, a combinatorial invariant (composed of pair of permutations with \(d\) symbols with \(d = 2g + s - 1\), where \(g\) is the genus and \(s\) is the number of zeroes) introduced by G. Rauzy \[65\] in his study of interval exchange transformations.
of strata of Abelian differentials of genus \( g \), one should perform several combinatorial operations with pairs of permutations on an alphabet of \( d \geq 2g \) letters\(^4\). Nevertheless, M. Kontsevich and A. Zorich [48] managed to classify completely the connected components of strata of Abelian differentials with the aid of some invariants from algebraic and geometrical nature: technically speaking, there are exactly three types of connected components of strata – hyperelliptic, even spin and odd spin. The outcome of Kontsevich-Zorich classification are the following results:

**Theorem 23** (M. Kontsevich and A. Zorich). Fix \( g \geq 4 \).

- the minimal stratum \( \mathcal{H}(2g-2) \) has 3 connected components;
- \( \mathcal{H}(2l, 2l) \), \( l \geq 2 \), has 3 connected components;
- any \( \mathcal{H}(2l_1, \ldots, 2l_n) \neq \mathcal{H}(2l, 2l) \), \( l_i \geq 1 \), has 2 connected components;
- \( \mathcal{H}(2l-1, 2l-1) \), \( l \geq 2 \), has 2 connected components;
- all other strata of Abelian differentials of genus \( g \) are connected.

The classification of the connected components of strata of Abelian differentials of genus \( g = 2 \) and 3 are slightly different:

**Theorem 24** (M. Kontsevich and A. Zorich). In genus 2, the strata \( \mathcal{H}(2) \) and \( \mathcal{H}(1, 1) \) are connected. In genus 3, both of the strata \( \mathcal{H}(4) \) and \( \mathcal{H}(2, 2) \) have two connected components, while the other strata in genus 3 are connected.

For more details, we strongly recommend the original article by M. Kontsevich and A. Zorich.

**Remark 25.** The classification of connected components of strata of non-orientable quadratic differentials was performed by E. Lanneau [49]. For genus \( g \geq 3 \), each of the four strata \( Q(9, -1) \), \( Q(6, 3, -1) \), \( Q(3, 3, 3, -1) \), \( Q(12) \) have exactly two connected components, each of the following strata \( Q(2j-1, 2j-1, 2k-1) \) \((j, k \geq 0, j + k = g)\), \( Q(2j-1, 2j-1, 4k+2) \) \((j, k \geq 0, j+k = g-1)\), \( Q(4j+2, 4k+2) \) \((j, k \geq 0, j+k = g-2)\) also have exactly two connected components, and all other strata are connected. For genus \( 0 \leq g \leq 2 \), we have that any strata in genus 0 and 1 are connected, and, in genus 2, \( Q(6, -1^2) \), \( Q(3, 3, -1^2) \) have exactly two connected components each, while the other strata are connected.

### 3. Dynamics on the moduli space of Abelian differentials

Let \((M, \omega)\) be a compact Riemann surface \( M \) of genus \( g \geq 1 \) equipped with a non-trivial Abelian differential (that is, a holomorphic 1-form) \( \omega \neq 0 \). In the sequel, we denote by \( \Sigma \) the (finite) set of zeroes of \( \omega \).

\(^4\)Just to give an idea of how fast the size of Rauzy classes does grow, let’s mention that the cardinality of the largest Rauzy classes in genera 2, 3, 4 and 5 are 15, 2177, 617401 and 300296573 resp. (can you guess the next number for genus 6? :)) For more informations on how these numbers can be computed see V. Delecroix’s work [15].
3.1. $GL^+(2, \mathbb{R})$-action on $\mathcal{H}_g$. The canonical identification between Abelian differentials and translation structures makes transparent the existence of a natural action of $GL^+(2, \mathbb{R})$ on the set of Abelian differentials $L_g$. Indeed, given an Abelian differential $\omega$, let’s denote by $\{\phi_\alpha(\omega)\}_{\alpha \in I}$ the maximal atlas on $M - \Sigma$ giving the translation structure corresponding to $\omega$ (so that $\phi_\alpha(\omega) \ast dz = \omega$ for every index $\alpha$). Here, the local charts $\phi_\alpha(\omega)$ map some open set of $M - \Sigma$ to $\mathbb{C}$.

Since any matrix $A \in GL^+(2, \mathbb{R})$ acts on $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, we can post-compose the local charts $\phi_\alpha(\omega)$ with $A$, so that we obtain a new atlas $\{A \circ \phi_\alpha(\omega)\}$ on $M - \Sigma$. Observe that the changes of coordinates of this new atlas are also solely by translations, as a quick computation reveals:

$$(A \circ \phi_\beta(\omega)) \circ (A \circ \phi_\alpha)^{-1}(z) = A \circ (\phi_\beta(\omega) \circ \phi_\alpha(\omega)^{-1}) \circ A^{-1}(z) = A(A^{-1}(z) + c) = z + A(c).$$

In other words, $\{A \circ \phi_\alpha(\omega)\}$ is a new translation structure. The corresponding Abelian differential is, by definition, $A \cdot \omega$. Observe that the complex structure of the plane is not preserved by the action of a (typical) $GL^+(2, \mathbb{R})$ matrix. Therefore, the Riemann surface structure with respect to which the 1-form $A \cdot \omega$ is holomorphic is usually distinct (not biholomorphic) to the Riemann surface structure related to $\omega$. Here, a notable exception is the group of rotations $SO(2, \mathbb{R}) \subset GL^+(2, \mathbb{R})$ who may change the Abelian differential without touching the Riemann surface structure (because any rotation preserves the complex structure of the plane).

By definition, it is utterly trivial to see this $GL^+(2, \mathbb{R})$-action on Abelian differentials given by sides identifications of collections of polygons (as in the previous examples): in fact, given $A \in GL^+(2, \mathbb{R})$ and an Abelian differential $\omega$ related to a finite collection of polygons $\mathcal{P}$ with parallel sides glued by translations, the Abelian differential $A \cdot \omega$ corresponds, by definition, to the finite collection of polygons $A \cdot \mathcal{P}$ obtained by letting $A$ act on the polygons forming $\mathcal{P}$ as a planar subset and keeping the sides identifications of parallel sides by translations. Notice that the linear action of $A$ on the plane evidently respects the notion of parallelism, so that this procedure is well-defined. For sake of concreteness, we drew below some illustrative examples (see Figures 5, 6 and 7) of the actions of the matrices $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on a L-shaped square-tiled surface.

![Figure 5. Action of $g_t$](image-url)
Concerning the structures on $\mathcal{H}_g$ introduced in the previous section, we notice that this $GL^+(2, \mathbb{R})$-action preserves each of its strata $\mathcal{H}(\kappa)$ ($\kappa = (k_1, \ldots, k_\sigma)$, $\sum_{l=1}^\sigma k_l = 2g - 2$) and the natural (Lebesgue) measures $\lambda_\kappa$ on them since the underlying affine structures of strata are respected. Of course, this observation opens up the possibility of studying this action via ergodic-theoretical methods applied to $\lambda_\kappa$. However, it turns out that the strata $\mathcal{H}(\kappa)$ are a somewhat big: for instance, they are ruled in the sense that the complex lines $\mathbb{C} \cdot \omega$ foliate them. As a result, it is possible to prove that each $\lambda_\kappa$ has infinite mass, so that the use of standard ergodic-theoretical methods is not possible, and although there are some ergodic theorems for systems preserving a measure of infinite mass, they don’t seem to lead us as far as the usual Ergodic Theory.

Anyway, this difficulty can be bypassed by normalizing the area form $A$ associated to the Abelian differential. This should be compared with the case of the Euclidean space $\mathbb{R}^n$: indeed, while the Lebesgue measure on $\mathbb{R}^n$ has infinite mass, after “killing” the scaling factor and restricting ourselves to the unit sphere $S^{n-1} \subset \mathbb{R}^n$, we end up with a finite measure. The details of this procedure are the content of the next subsection.

3.2. $SL(2, \mathbb{R})$-action on $\mathcal{H}_g^{(1)}$ and Teichmüller geodesic flow. We denote by $\mathcal{H}_g^{(1)}$ the set of Abelian differentials $\omega$ on a genus $g$ Riemann surface $M$ whose induced area form $dA(\omega)$ on $M$ has total area $A(\omega) := \int_M dA(\omega) = 1$. At first sight, one is tempted to say that $\mathcal{H}_g^{(1)}$ is some sort of “unit sphere” of $\mathcal{H}_g$. However, since the area form $A(\omega)$ of an arbitrary Abelian differential $\omega$
can be expressed as
\[ A(\omega) = \frac{i}{2} \int_M \omega \wedge \bar{\omega} = \frac{i}{2} \sum_{j=1}^g (A_j B_j - \bar{A}_j B_j) \]
where \(A_j, B_j\) form a canonical basis of absolute periods of \(\omega\), i.e., \(A_j = \int_{\alpha_j} \omega, B_j = \int_{\beta_j} \omega\) and \(\{\alpha_j, \beta_j\}_{j=1}^g\) is a symplectic basis of \(H_1(M, \mathbb{R})\) (with respect to the intersection form), we see that \(\mathcal{H}_g^{(1)}\) resembles more a “unit hyperboloid”.

Again, we can stratify \(\mathcal{H}_g^{(1)}\) by considering \(\mathcal{H}^{(1)}(\kappa) := \mathcal{H}(\kappa) \cap \mathcal{H}_g^{(1)}\) and, from the definition of the \(GL^+(2, \mathbb{R})\)-action on the plane \(\mathbb{C} \simeq \mathbb{R}^2\), we see that \(\mathcal{H}_g^{(1)}\) and its strata \(\mathcal{H}^{(1)}(\kappa)\) come equipped with a natural \(SL(2, \mathbb{R})\)-action. Moreover, by disintegrating the natural Lebesgue measure on \(\lambda_\kappa\) with respect to the level sets of the total area function \(A : \mathcal{H}_g \to \mathbb{R}_+\), \(\omega \mapsto A(\omega)\), we can write
\[ d\lambda_\kappa = dA \cdot d\lambda_\kappa^{(1)} \]
where \(\lambda_\kappa^{(1)}\) is a natural “Lebesgue” measure on \(\mathcal{H}^{(1)}(\kappa)\). We encourage the reader to compare this with the analogous procedure to get the Lebesgue measure on the unit sphere \(S^{n-1}\) by disintegration of the Lebesgue measure of the Euclidean space \(\mathbb{R}^n\).

Of course, from the “naturality” of the construction, it follows that \(\lambda_\kappa^{(1)}\) is a \(SL(2, \mathbb{R})\)-invariant measure on \(\mathcal{H}(\kappa)\). The following fundamental result was proved by H. Masur [51] and W. Veech [68]:

**Theorem 26** (H. Masur/W. Veech). The total volume (mass) of \(\lambda_\kappa^{(1)}\) is finite.

**Remark 27.** The computation of the actual values of these volumes took essentially 20 years to be performed and it is due to A. Eskin and A. Okounkov [25]. We will make some comments on this later (in Section 1.2).

For a presentation (from scratch) of the proof of Theorem 26 based on Veech’s zippered rectangle construction, see e.g. J.-C. Yoccoz survey [72].

For the sake of reader’s convenience, we present here the following intuitive argument of H. Masur [52] explaining why \(\lambda_\kappa^{(1)}\) has finite mass. In the genus 1 case (of torii), we saw in Subsection 1.4 that the moduli space \(\mathcal{H}^{(1)}_1 \simeq SL(2, \mathbb{R})/SL(2, \mathbb{Z})\) and the Masur-Veech measure \(\lambda_\kappa^{(1)}\) comes from the Haar measure \(\lambda_{SL(2, \mathbb{R})}\) of \(SL(2, \mathbb{R})\). In this situation, the fact \(\lambda_\kappa^{(1)}\) has finite mass is a reformulation of the fact that \(SL(2, \mathbb{Z})\) is a lattice of \(SL(2, \mathbb{R})\). More concretely, \(\mathcal{H}^{(1)}_1\) correspond to pairs of vectors \(v_1, v_2 \in \mathbb{R}^2\) with \(|v_1 \wedge v_2| = 1\), so that, by direct calculation\(^5\)
\[ \lambda_\kappa^{(1)}(\mathcal{H}^{(1)}_1) \leq \text{Leb}(\{(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : |v_1 \wedge v_2| \leq 1\}) < \infty \]

For the discussion of the general case, we will need the notion of maximal cylinders of translation surfaces. In simple terms, given a closed regular geodesic \(\gamma\) in a translation surface \((M, \omega)\), we can

\(^5\)Actually, since \(\mathcal{H}^{(1)}_1\) is the unit cotangent bundle of \(H/SL(2, \mathbb{Z})\) and the fundamental domain \(\mathcal{F}_0\) in Subsection 1.4 has hyperbolic area \(\pi/12\) (for the hyperbolic metric of curvature \(-4\) compatible with our time normalization of the geodesic flow), one has \(\lambda_\kappa^{(1)}(\mathcal{H}^{(1)}_1) = \pi^2/6\). However, we will not insist on this explicit computation because it is not easy to generalize it for moduli spaces of higher genera Abelian differentials.
form a maximal cylinder \( C \) by collecting all closed geodesics of \((M, \omega)\) parallel to \( \gamma \) not meeting any zero of \( \omega \). In particular, the boundary of \( C \) contains zeroes of \( \omega \). Given a maximal cylinder \( C \), we denote by \( w(C) \) its width (i.e., the length of its waist curve \( \gamma \)) and by \( h(C) \) its height (i.e., the distance across \( C \)). For example, in the figure below we illustrate two closed geodesics \( \gamma_1 \) and \( \gamma_2 \) (in the horizontal direction) and the two corresponding maximal cylinders \( C_1 \) and \( C_2 \) of the L-shaped square-tiled surface of Example 15. In this picture, we see that \( C_1 \) has width 2, \( C_2 \) has width 1, and both \( C_1 \) and \( C_2 \) have height 1.

Continuing with the argument for the finiteness of the mass of \( \lambda^{(1)}_\kappa \), one shows that, for each \( g \geq 2 \), there exists an universal constant \( C(g) \) such that any translation surface \((M, \omega)\) of genus \( g \) and diameter \( \text{diam}(M, \omega) \geq C(g) \) has a maximal cylinder of height \( h \sim \text{diam}(M, \omega) \) in some direction (see the figure below)

Next, we recall that \( \lambda^{(1)}_\kappa \) was defined via the relative periods. In particular, the set of translation surfaces \((M, \omega) \in \mathcal{H}^{(1)}(\kappa)\) of genus \( g \geq 2 \) with diameter \( \leq C(g) \) has finite \( \lambda^{(1)}_\kappa \)-measure. Hence,
it remains to estimate the $\lambda_{\kappa}^{(1)}$-measure of the set of translation surfaces $(M, \omega)$ with diameter $\geq C(g)$. Here, we recall that there is a maximal cylinder $C \subset (M, \omega)$ with height $h \sim \text{diam}(M, \omega)$. Since $(M, \omega)$ has area one, this forces the width $w$ of $C$, i.e., the length of a closed geodesic (waist curve) $\gamma$ in $C$ to be small. By taking $\gamma$ and a curve $\rho$ across $C$ as a part of the basis of the relative homology of $(M, \omega)$, we get two vectors $v = \int_{\rho} \omega$ and $u = \int_{\rho} \omega$ with $|v \wedge u| \leq 1$ and $v$ small. In other words, we can think of the cusp of $H^{(1)}(\kappa)$ corresponding to translation surfaces $(M, \omega)$ with $v$ small as a subset of the set $\{(v, u) \in \mathbb{R}^2 \times \mathbb{R}^2 : |v \wedge u| \leq 1\}$. This ends the sketch of proof of Theorem 26 because the $\lambda_{\kappa}^{(1)}$-measure of cusps is then bounded by

$$\text{Leb}(\{(v, u) \in \mathbb{R}^2 \times \mathbb{R}^2 : |v \wedge u| \leq 1\}) < \infty.$$  

In what follows, given any connected component $C$ of some stratum $H^{(1)}(\kappa)$, we call the $SL(2, \mathbb{R})$-invariant probability measure $\mu_C$ obtained from the normalization of the (restriction to $C$) of $\lambda_{\kappa}^{(1)}$ the Masur-Veech measure of $C$.

In this language, the global picture is the following: we dispose of a $SL(2, \mathbb{R})$-action on connected components $C$ of strata $H^{(1)}(\kappa)$ of the moduli space of Abelian differentials with unit area and a naturally invariant probability $\mu_C$ (Masur-Veech measure).

Of course, it is tempting to start the study of the statistics of $SL(2, \mathbb{R})$-orbits of this action with respect to Masur-Veech measure, but we’ll momentarily refrain ourselves from doing so (instead we postpone to the next section this discussion) because this is the appropriate place to introduce the so-called Teichmüller (geodesic) flow.

The Teichmüller flow $g_t$ on $H^{(1)}_g$ is simply the action of the diagonal subgroup $g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ of $SL(2, \mathbb{R})$. The discussions we had so far imply that $g_t$ is the geodesic flow of the Teichmüller metric (introduced in Section 1). Indeed, from Teichmüller’s theorem 1, it follows that the path $\{(M_t, \omega_t) : t \in \mathbb{R}\}$, where $\omega_t = g_t(\omega_0)$ and $M_t$ is the underlying Riemann surface structure such that $\omega_t$ is holomorphic, is a geodesic of Teichmüller metric $d$, and $d((M_0, \omega_0), (M_t, \omega_t)) = t$ for all $t \in \mathbb{R}$ (i.e., $t$ is the arc-length parameter).

In Figure 5 above, we saw the action of Teichmüller geodesic flow $g_t$ on an Abelian differential $\omega \in H(2)$ associated to a L-shaped square-tiled surface derived from 3 squares. At a first glance, if the reader forgot the discussion by the end of Section 1 he/she will find (again) the dynamics of $g_t$ very uninteresting: the initial L-shaped square-tiled surface gets indefinitely squeezed in the vertical direction and stretched in the horizontal direction, so that we don’t have any hope of finding a surface whose shape is somehow “close” to the initial shape (that is, $g_t$ doesn’t seem to have any interesting dynamical feature such as recurrence). However, as we already mentioned in by the end of Section 1 (in the genus 1 case), while this is true in Teichmüller spaces $TH(\kappa)$, this is not exactly true in moduli spaces $H(\kappa)$: in fact, while in Teichmüller spaces we can only identify “points” by diffeomorphisms isotopic to the identity, one can profit of the (orientation-preserving) diffeomorphisms not isotopic to identity in the case of moduli spaces to eventually bring deformed shapes close to a given one. In other words, the very fact that we deal with the modular group
Γ_g = Diff^+(M)/Diff_0^+(M) (i.e., diffeomorphisms not necessarily isotopic to identity) in the case of moduli spaces allows to change names of homology classes of the surfaces as we wish, that is, geometrically we can cut our surface along any closed loop to extract a piece of it, glue back by translation (!) this piece at some other part of the surface, and, by definition, the resulting surface will represent the same point in moduli space as our initial surface. Below, we included a picture illustrating this:

![Figure 8. Actions of g_t and an element ρ of Γ_g](image)

To further analyze the dynamics of Teichmüller flow g_t (and/or SL(2,R)-action) on \(H_g^{(1)}\), it is surely important to know its derivative \(Dg_t\). In the next subsection, we will follow M. Kontsevich and A. Zorich to show that the dynamically relevant information about \(Dg_t\) is captured by the so-called Kontsevich-Zorich cocycle.

3.3. Teichmüller flow and Kontsevich-Zorich cocycle on the Hodge bundle over \(H_g^{(1)}\).

We start with the following trivial bundle over Teichmüller space of Abelian differentials \(T \mathcal{H}_g^{(1)}\):

\[
\widehat{H}_g^1 := T \mathcal{H}_g^{(1)} \times H^1(M, \mathbb{R})
\]

and the trivial (dynamical) cocycle over Teichmüller flow \(g_t\):

\[
\widehat{G}_t^{KZ} : \widehat{H}_g^1 \to \widehat{H}_g^1, \quad \widehat{G}_t^{KZ}(\omega, [c]) = (g_t(\omega), [c]).
\]

Of course, there is not much to say here: we act through Teichmüller flow in the first component and we’re not acting (or acting trivially if you wish) in the second component of the trivial bundle \(\widehat{H}_g^1\).

Now, we observe that the modular group \(\Gamma_g\) acts on both components of \(\widehat{H}_g^1\) by pull-back, and, as we already saw, the action of Teichmüller flow \(g_t\) commutes with the action of \(\Gamma_g\) (since \(g_t\) acts by post-composition on the local charts of a translation structure while \(\Gamma_g\) acts by pre-composition on them). Therefore, it makes sense to take the quotients

\[
H_g^1 := (T \mathcal{H}_g^{(1)} \times H^1(M, \mathbb{R}))/\Gamma_g
\]
and $G^{KZ}_t := \mathcal{G}^{KZ}_t / \Gamma_g$. In the literature, $H^1_0$ is the (real) Hodge bundle over the moduli space of Abelian differentials $\mathcal{H}_g = \mathcal{T} \mathcal{H}_g / \Gamma_g$ and $G^{KZ}_t$ is the Kontsevich-Zorich cocycle (KZ cocycle for short) over Teichmüller flow $g_t$.

We begin by pointing out that the Kontsevich-Zorich cocycle $G^{KZ}_t$ (unlike its “parent” $\mathcal{G}^{KZ}_t$) is very far from being trivial. Indeed, since we identify $(\omega, [c])$ with $(\rho^*\omega, \rho^*([c]))$ for any $\rho \in \Gamma_g$ to construct the Hodge bundle and $G^{KZ}_t$, it follows that the fibers of $\mathcal{H}^1_g$ over $\omega$ and $\rho^*\omega$ are identified in a non-trivial way if the (standard cohomological) action of $\rho$ on $H^1(M, \mathbb{R})$ is non-trivial.

Alternatively, suppose we fix a fundamental domain $D$ of $\Gamma_g$ on $\mathcal{T} \mathcal{H}_g$ (e.g., through Veech’s zippered rectangle construction) and let’s say we start with some point $\omega$ at the boundary of $D$, a cohomology class $[c] \in H^1(M, \mathbb{R})$ and assume that the Teichmüller geodesic through $\omega$ points towards the interior of $D$. Now, we run Teichmüller flow for some (long) time $t_0$ until we hit again the boundary of $D$ and our geodesic is pointing outwards $D$. At this stage, from the definition of Kontsevich-Zorich cocycle, we have the “option” to apply an element $\rho$ of the modular group $\Gamma_g$ so that Teichmüller flow through $\rho^*(g_{t_0}\omega)$ points towards $D$ “at the cost” of replacing the cohomology class $[c]$ by $\rho^*([c])$. In this way, we see that $G^{KZ}_{t_0}(\omega, [c]) = (\rho^*(\omega), \rho^*([c]))$ is non-trivial in general.

Below we illustrate (see Figures 9 and 10) this “fundamental domain”-based discussion in both genus 1 and $g \geq 2$ cases (the picture in the higher-genus case being idealized, of course, since the moduli space is higher-dimensional).

![Figure 9. Kontsevich-Zorich cocycle in the moduli space of torii.](image)

---

\[^6\]In fact, this doesn’t lead to a linear cocycle in the usual sense of Dynamical Systems because the Hodge bundle is an orbifold bundle. Indeed, $G^{KZ}_t$ is well-defined along $g_t$-orbits of translation surfaces without non-trivial automorphisms, but there is an ambiguity when the translation surface $(M, \omega)$ has a non-trivial group of automorphisms $\text{Aut}(M, \omega)$. In simple terms, this ambiguity comes from the fact that the fiber of the Hodge bundle over $(M, \omega)$ is the quotient of $H^1(M, \mathbb{R})$ by the group $\text{Aut}(M, \omega)$, so that, if $\text{Aut}(M, \omega) \neq \{\text{id}\}$, the KZ cocycle induces only linear maps on $H^1(M, \mathbb{R})$ modulo the cohomological action of $\text{Aut}(M, \mathbb{R})$. Notice that $\text{Aut}(M, \omega) = \{\text{id}\}$ for almost every $(M, \omega)$ (wrt Masur-Veech measures), so that this ambiguity problem doesn’t concern generic orbits. In any event, as far as Lyapunov exponents are concerned, this ambiguity is not hard to solve. By Hurwitz theorem, $\#\text{Aut}(M, \omega) \leq 84(g-1) < \infty$, so that one can get rid of the ambiguity by taking adequate finite covers of KZ cocycle (e.g., by marking horizontal separatrices of the translation surfaces). See, e.g., [58] for more details and comments on this.
Here we are projecting the picture from the unit cotangent bundle $H(\emptyset) = SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ to the moduli space of torii $M_1 = \mathbb{H}/SL(2,\mathbb{Z})$, so that the evolution of the Abelian differentials $g_t(\omega)$ are designed by the tangent vectors to the hyperbolic geodesics, while the evolution of cohomology classes is designed by the transversal vectors to these geodesics.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Kontsevich-Zorich cocycle on the Hodge bundle $H^1_g$, $g \geq 2$.}
\end{figure}

Next, we observe that the $G^KZ_t$ is a symplectic cocycle because the action by pull-back of the elements of $\Gamma_g$ on $H^1(M,\mathbb{R})$ preserves the intersection form $([c],[c']) = \int_M [c] \wedge [c']$, a symplectic form on the $2g$-dimensional real vector space $H^1(M,\mathbb{R})$. This has the following consequence for the Ergodic Theory of KZ cocycle. Given any ergodic Teichmüller flow invariant probability $\mu$, we know from Oseledets theorem that there are real numbers (Lyapunov exponents) $\lambda^\mu_1 > \cdots > \lambda^\mu_k$ and a Teichmüller flow equivariant decomposition $H^1(M,\mathbb{R}) = E^1(\omega) \oplus \cdots \oplus E^k(\omega)$ at $\mu$-almost every point $\omega$ such that $E^i(\omega)$ depends measurably on $\omega$ and

$$\lim_{t \to \pm \infty} \frac{1}{t} \log(\|G^KZ_t(\omega,v)\|/\|v\|) = \lambda^\mu_i$$

for every $v \in E^i(\omega) - \{0\}$ and any choice of $\|\cdot\|$ such that $\int \log^+ \|G^KZ_t\|d\mu < \infty$. If we allow ourselves to repeat each $\lambda^\mu_i$ accordingly to its multiplicity $\dim E^i(\omega)$, we get a list of $2g$ Lyapunov exponents

$$\lambda^\mu_1 \geq \cdots \geq \lambda^\mu_{2g}.$$  

Such a list is commonly called Lyapunov spectrum (of KZ cocycle with respect to $\mu$). The fact that KZ cocycle is symplectic means that the Lyapunov spectrum is always symmetric with respect to $\cdots$

\footnote{We will see in Remark 31 below that one can choose the so-called Hodge norm here.}
the origin:

\[ \lambda_1^\mu \geq \cdots \geq \lambda_g^\mu \geq 0 \geq -\lambda_g^\mu \geq \cdots - \lambda_1^\mu \]

that is, \( \lambda_{2g-i}^\mu = -\lambda_{i+1}^\mu \) for every \( i = 0, \ldots, g - 1 \). Roughly speaking, this symmetry correspond to
the fact that whenever \( \theta \) appears as an eigenvalue of a symplectic matrix \( A \), \( \theta^{-1} \) is also an eigenvalue of \( A \) (so that, by taking logarithms, we “see” that the appearance of a Lyapunov exponent \( \lambda \) forces the appearance of a Lyapunov exponent \(-\lambda\)). Thus, it suffices to study the non-negative Lyapunov exponents of KZ cocycle to determine its entire Lyapunov spectrum.

Also, in the specific case of KZ cocycle, it is not hard to deduce that \( \pm 1 \) belong to the Lyapunov spectrum of any ergodic probability \( \mu \). Indeed, by the definition, the family of symplectic planes \( E(\omega) = \mathbb{R} \cdot [\text{Re}(\omega)] \oplus \mathbb{R} \cdot [\text{Im}(\omega)] \subset H^1(M, \mathbb{R}) \) generated by the cohomology classes of the real and imaginary parts of \( \omega \) are Teichmüller flow (and even \( SL(2, \mathbb{R}) \)) equivariant. Also, the action of Teichmüller flow restricted to these planes is, by definition, isomorphic to the action of the matrices \( g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \) on the usual plane \( \mathbb{R}^2 \) if we identify \( [\text{Re}(\omega)] \) with the canonical vector \( e_1 = (1, 0) \in \mathbb{R}^2 \) and \( [\text{Im}(\omega)] \) with the canonical vector \( e_2 = (0, 1) \in \mathbb{R}^2 \). Actually, the same is true for the entire \( SL(2, \mathbb{R}) \)-action restricted to these planes (where we replace \( g_t \) by the corresponding matrices). Since the Lyapunov exponents of the \( g_t \) action on \( \mathbb{R}^2 \) are \( \pm 1 \), we get that \( \pm 1 \) belong to the Lyapunov spectrum of KZ cocycle.

Actually, it is possible to prove that \( \lambda_1^\mu = 1 \) (i.e., 1 is always the top exponent), and \( \lambda_1^\mu = 1 > \lambda_2^\mu \), i.e., the top exponent has always multiplicity 1, or, in other words, the Lyapunov exponent \( \lambda_1^\mu \) is always simple. However, since this requires some machinery (variational formulas for the Hodge norm on the Hodge bundle), we postpone this discussion to Subsection 3.5 below, and we close this subsection by relating this cocycle with the derivative of Teichmüller flow \( g_t \) (which is one of the interests of KZ cocycle).

By writing \( H^1(M, \Sigma, \mathbb{C}) = \mathbb{C} \otimes H^1(M, \Sigma, \mathbb{R}) = \mathbb{R}^2 \otimes H^1(M, \Sigma, \mathbb{R}) \) and considering the action of \( Dg_t \) on each factor of this tensor product, one can check (from the fact that local coordinates of connected components of strata \( H^{(1)}(\kappa) \) are given by period maps) that \( Dg_t \) acts through the usual action of the matrices \( g_t = \text{diag}(e^t, e^{-t}) \) on the first factor \( \mathbb{R}^2 \) and it acts through the natural generalization \( \tilde{G}_t^{KZ} \) of KZ cocycle \( G_t^{KZ} \) on the second factor \( H^1(M, \Sigma, \mathbb{R}) \). In particular, the Lyapunov exponents of \( Dg_t \) have the form \( \pm 1 + \lambda \) where \( \lambda \) are Lyapunov exponents of \( \tilde{G}_t^{KZ} \).

Now, we observe that the relative part don’t contribute with interesting exponents of \( \tilde{G}_t^{KZ} \), so that it suffices to understand the absolute part. More precisely, we affirm that the natural action of \( \tilde{G}_t^{KZ} \) on the quotient \( \mathbb{H}^1(M, \Sigma, \mathbb{R})/H^1(M, \mathbb{R}) \) (“relative part” of dimension \( \sigma - 1 \)) is through bounded linear transformations, so that this part contributes with \( \sigma - 1 \) zero exponents of \( \tilde{G}_t^{KZ} \) (where \( \sigma = \#\Sigma \)). Indeed, this affirmation is true because there are no long relative cycles in \( H_1(M, \Sigma, \mathbb{R})/H_1(M, \mathbb{R}) \): for instance, in the figure below we see that any attempt to

\footnote{This argument would be easier to perform if one disposes of equivariant relative parts (i.e., equivariant supplements of \( H^1(M, \mathbb{R}) \) in \( H^1(M, \Sigma, \mathbb{R}) \)). However, as we will see in Remark 28 below, this is not true in general.}
produce a long relative cycle $c_1 \in H_1(M, \Sigma, \mathbb{R})$ between two points $p_1, p_2 \in \Sigma$ by applying $\tilde{G}_t^{KZ}$, say $c_1 = \tilde{G}_t^{KZ}(\omega, c_0)$ for a large $t > 0$, can be “counter-reacted” by taking a bounded cycle $c_2$ between $p_1$ and $p_2$: in this way, $c_1$ differs from $c_2$ by an absolute cycle $\gamma = c_1 - c_2 \in H_1(M, \mathbb{R})$, i.e., $c_1$ and $c_2$ represent the same element of $H_1(M, \Sigma, \mathbb{R})/H_1(M, \mathbb{R})$, so that the claim that $\tilde{G}_t^{KZ}$ acts on $H_1(M, \Sigma, \mathbb{R})/H_1(M, \mathbb{R})$ via bounded linear transformations follows.

In other words, the interesting Lyapunov exponents of $\tilde{G}_t^{KZ}$ come from the absolute part $H^1(M, \mathbb{R})$, i.e., it is the KZ cocycle $G_t^{KZ} := \tilde{G}_t^{KZ}|_{H^1(M, \mathbb{R})}$ who describes the most exciting Lyapunov exponents. Equivalently, the Lyapunov spectrum of $\tilde{G}_t^{KZ}$ consists of $\sigma - 1$ zero exponents and the $2g$ Lyapunov exponents $\pm \lambda_i^\mu$, $1 \leq i \leq g$ of the KZ cocycle.

Thus, in resume, the Lyapunov spectrum of Teichmüller flow $g_t$ with respect to an ergodic probability measure $\mu$ supported on a stratum $\mathcal{H}(\kappa)$ (with $\kappa = (k_1, \ldots, k_\sigma)$) has the form

$$2 \geq 1 + \lambda_2^\mu \geq \cdots \geq 1 + \lambda_g^\mu \geq 1 \geq \cdots = 1 = 1 - \lambda_g^\mu \geq \cdots \geq 1 - \lambda_2^\mu \geq 0$$

$$\geq -1 + \lambda_2^\mu \geq \cdots \geq -1 + \lambda_g^\mu \geq -1 = \cdots = -1 \geq -1 - \lambda_g^\mu \geq \cdots \geq -1 - \lambda_2^\mu \geq -2$$

where $1 \geq \lambda_2^\mu \geq \cdots \lambda_g^\mu$ are the non-negative exponents of KZ cocycle $G_t^{KZ}$ with respect to $\mu$.

**Remark 28.** Concerning the computation of exponents $\pm 1$ in the relative part of cohomology (i.e., before passing to absolute cohomology), our job would be easier if $H^1(M, \mathbb{C})$ admitted a equivariant supplement inside $H^1(M, \Sigma, \mathbb{C})$. However, it is possible to construct examples to show that this doesn’t happen in general. See Appendix B of [56] for more details.

Therefore, the KZ cocycle captures the “main part” of the derivative cocycle $Dg_t$, so that, since we’re interested in the Ergodic Theory of Teichmüller flow, we will spend sometime in the next sections to analyze KZ cocycle (without much reference to $Dg_t$).

3.4. **Hodge norm on the Hodge bundle over $\mathcal{H}_{g_1}$.** By definition, the task of studying Lyapunov exponents consists precisely in understanding the growth of norm of vectors. Of course, the particular choice of norm doesn’t affect the values of Lyapunov exponents (essentially because two
norms on a finite-dimensional vector space are equivalent), but for the sake of our discussion it will be convenient to work with the so-called Hodge norm.

Let $M$ be a Riemann surface. The Hodge (intersection) form $(\cdot,\cdot)$ on $H^1(M,\mathbb{C})$ is given

$$<(\alpha,\beta) := \frac{i}{2} \int_M \alpha \wedge \overline{\beta}$$

for each $\alpha,\beta \in H^1(M,\mathbb{C})$.

The Hodge form is positive-definite on the space $H^{1,0}(M)$ of holomorphic 1-forms on $M$, and negative-definite on the space $H^{0,1}(M)$ of anti-holomorphic 1-forms on $M$. For instance, given a holomorphic 1-form $\alpha \neq 0$, we can locally write $\alpha(z) = f(z)dz$, so that

$$\alpha(z) \wedge \overline{\alpha(z)} = |f(z)|^2 dz \wedge \overline{dz} = -2i|f(z)|^2 dx \wedge dy.$$

Since $dx \wedge dy$ is an area form on $M$ and $|f(z)|^2 \geq 0$, we get that $(\alpha,\alpha) > 0$.

In particular, since $H^1(M,\mathbb{C}) = H^{1,0}(M) \oplus H^{0,1}(M)$, and $H^{1,0}(M)$ and $H^{0,1}(M)$ are $g$-dimensional complex vector spaces, one has that the Hodge form is an Hermitian form of signature $(g,g)$ on $H^1(M,\mathbb{C})$.

The Hodge form is equivariant with respect to the natural action of the mapping-class group $\Gamma_g$. In particular, it induces an Hermitian form (also called Hodge form and denoted by $(\cdot,\cdot)$) on the complex Hodge bundle $H^1_g(C) = (T \mathcal{H}^{(1)}_g \times H^1(M,\mathbb{C}))/\Gamma_g$ over $\mathcal{H}^{(1)}_g$.

The so-called Hodge representation theorem says that any real cohomology class $c \in H^1(M,\mathbb{R})$ is the real part of an unique holomorphic 1-form $h(c) \in H^{1,0}(M)$, i.e., $c = \Re h(c)$. In particular, one can use the Hodge form $(\cdot,\cdot)$ to induce an inner product on $H^1(M,\mathbb{R})$ via:

$$(c_1,c_2) := (\Re h(c_1),\Re h(c_2))$$

for each $c_1, c_2 \in H^1(M,\mathbb{R})$.

Again, this induces an inner product $(\cdot,\cdot)$ and a norm $\|\cdot\|$ on the real Hodge bundle $H^1_g(\mathbb{R}) = H^1_g = (T \mathcal{H}^{(1)}_g \times H^1(M,\mathbb{R}))/\Gamma_g$ over $\mathcal{H}^{(1)}_g$. In the literature, $(\cdot,\cdot)$ is the Hodge inner product and $\|\cdot\|$ is the Hodge norm on the real Hodge bundle.

Observe that, in general, the subspaces $H^{1,0}$ and $H^{0,1}$ are not equivariant with respect to the (natural complex version of the) KZ cocycle (on $H^1_g(C)$), and this is one of the reasons why the Hodge norm $\|\cdot\|$ is not preserved by the KZ cocycle in general. In the next subsection, we will study first variation formulas for the Hodge norm along the KZ cocycle and its applications to the Teichmüller flow.

3.5. First variation of Hodge norm and hyperbolic features of Teichmüller flow. Let $c \in H^1(M,\mathbb{R})$ a vector in the fiber of the real Hodge bundle over $\omega \in \mathcal{H}^{(1)}_g$. Denote by $\alpha_0$ the holomorphic 1-forms with $c = \Re \alpha_0$. By applying the Teichmüller flow $g_t$ to $\omega$, we endow $M$ with a new Riemann surface structure such that $\omega_t = g_t(\omega)$ is an Abelian differential. In particular, $c = \Re \alpha_t$ where $\alpha_t$ is a holomorphic 1-form with respect to the new Riemann surface structure associated to $\omega_t$. 
Of course, by definition, KZ cocycle acts by parallel transport on the Hodge bundle, so that
the cohomology classes $c$ are not “changing”. However, since the representatives $\alpha_t$ we use to
“measure” the “size” (Hodge norm) of $c$ are changing, it is an interesting (and natural) problem
to know how fast the Hodge norm changes along KZ cocycle, or, equivalently, to compute the first
variation of the Hodge norm along KZ cocycle:

$$\frac{d}{dt} \| \pi_2(G^K_\omega, c) \|^2_{\omega(t)} |_{t=0} := \frac{d}{dt} \| c \|^2_{\omega(t)} := \frac{d}{dt} (\alpha_t, \alpha_t) |_{t=0}$$

where $\pi_2 : H^1_\omega \to H^1_\omega$ is the projection in the second factor of the Hodge bundle and $\| . \|_{\omega(t)}$
is the Hodge norm with respect to the Riemann surface structure induced by $\omega_t$.

In this subsection we will calculate this quantity by following the original article [27]. By working
locally outside the zeroes of $\omega_t$, we can choose local holomorphic coordinates $z_t$ with $\omega_t = dz_t$. Now,
we note that, by definition of the Teichmüller flow, $d\omega_t = \omega_t := e^{t}d\bar{z} + i e^{-t}dy$, where $dz = dx + i dy$,
so that

$$\omega_t = \frac{d}{dt} \omega_t = e^{t}[\Re \omega] - ie^{-t}[\Im \omega] = \frac{d}{dt} \bar{\omega}.$$  

Next, we write $0 = c - c = [\Re \alpha_t] - [\Re \alpha_0]$, so that we find smooth family $u_t$ with $du_t = \Re \alpha_t - \Re \alpha_0$.
By writing $\alpha_t = f_t \omega_t$, and by taking derivatives, we have locally

$$du_t = \bar{f}_t \omega_t + f_t \bar{\omega}_t + \bar{f}_t \bar{\omega}_t = \bar{f}_t dz_t + f_t d\bar{z}_t + \bar{f}_t d\bar{z}_t + \bar{f}_t dz_t$$

In particular, since $(\partial u_t/\partial z_t)dz_t + (\partial u_t/\partial \bar{z}_t)d\bar{z}_t := du_t$, we find that $\partial u_t/\partial z_t = \bar{f}_t + \bar{f}_t$.

Finally, we can (locally) compute

$$\frac{d}{dt} (\alpha_t, \alpha_t) = \Re \frac{i}{2} \frac{d}{dt} \int \alpha_t \wedge \bar{\alpha}_t = \Re \frac{i}{2} \frac{d}{dt} \int f_t \bar{f}_t (\omega_t \wedge \bar{\omega}_t) = \Re \frac{i}{2} \frac{d}{dt} \int f_t \bar{f}_t (dz_t \wedge d\bar{z}_t)$$

$$= 2\Re \frac{i}{2} \int f_t \bar{f}_t (dz_t \wedge d\bar{z}_t) = 2\Re \frac{i}{2} \int f_t \left( -\bar{f}_t + \frac{\partial \bar{u}_t}{\partial \bar{z}_t} \right) dz_t \wedge d\bar{z}_t$$

$$= -2\Re \frac{i}{2} \int f_t f_t (dz_t \wedge d\bar{z}_t) = -2\Re \frac{i}{2} \int \frac{\alpha_t \cdot \alpha_t}{\omega_t} \omega_t \wedge \bar{\omega}_t$$

In resume, we proved the following formula (originally from Lemma 2.1’ of [27]):

**Theorem 29** (G. Forni). Let $\omega$ be an Abelian differential and $c \in H^1(M, \mathbb{R})$. Denote by $\alpha_0$ the
holomorphic (with respect to $(M, \omega)$) 1-form with $c = [\Re \alpha_0]$. Then,

$$\frac{d}{dt} \| \pi_2(G^K_\omega, c) \|^2_{\omega(t)} |_{t=0} = \frac{d}{dt} \| c \|^2_{\omega(t)} |_{t=0} = -2\Re B_\omega(\alpha_0, \alpha_0)$$

where $B_\omega(\alpha, \beta) := \frac{1}{2} \int_M (\alpha/\omega)(\beta/\omega) \omega \wedge \bar{\omega}$.

In order to alleviate the notation, we put $B^\Re_\omega(c, c) := B_\omega(\alpha_0, \alpha_0)$ where $\alpha_0$ is the unique $(M, \omega)$-holomorphic 1-form with $c = [\Re \alpha_0]$. Observe that $B^\Re_\omega$ is a complex-valued bilinear form.

**Corollary 30.** One has

$$\frac{d}{dt} \log \| c \|_{\omega(t)} |_{t=0} = -\frac{\Re B^\Re_\omega(c, c)}{\| c \|^2_{\omega}}$$
In particular,
\[
\frac{d}{dt} \log \|c\|_{\omega, t=0} \leq 1
\]

Proof. The first statement of this corollary follows from the main formula in Theorem \[29\] while the second statement follows from an application of Cauchy-Schwarz inequality:
\[
|B_{\omega}(\alpha, \beta)| \leq \int |(\alpha/\omega)(\beta/\omega)| \omega \wedge \omega \leq \left( \int |\alpha/\omega|^2 \omega \wedge \omega \right)^{1/2} \left( \int |\beta/\omega|^2 \omega \wedge \omega \right)^{1/2} = \|\alpha\|_{\omega} \|\beta\|_{\omega}
\]

Remark 31. This corollary implies that the KZ cocycle is log-bounded with respect to the Hodge norm, that is, \( \log \|G^{KZ}_{\pm 1}(\omega, c)\|_{g_\pm(\omega)} \leq 1 \) for all \( c \in H^1(M, \mathbb{R}) \) with \( \|c\|_{\omega} = 1 \). Hence, given any finite mass measure \( \mu \) on \( H^{(1)}_g \), we have that
\[
\int \log^+ \|G^{KZ}_{\pm 1}(\omega)\|d\mu < \infty
\]

Corollary 32. Let \( \mu \) be any \( g_t \)-invariant ergodic probability on \( H^{(1)}_g \). Then, \( \lambda_2^\mu < 1 = \lambda_1^\mu \).

Proof. By Corollary \[30\] we have that \( \lambda_1^\mu \leq 1 \). Moreover, since the Teichmüller flow \( g_t(\omega) := e^{t\Re \omega} + ie^{-t\Im \omega} = \omega_1 \), we have that the \( G^{KZ}_t \)-invariant 2-plane \( H^{(1)}_{1,1}(M, \mathbb{R}) := \mathbb{R} \cdot [\Re \omega] \oplus \mathbb{R} \cdot [\Im \omega] \) contributes with Lyapunov exponents \( \pm 1 \). In particular, \( \lambda_1^\mu = 1 \).

Now, we note that \( H^1_{(0)}(M, \mathbb{R}) := \{ c \in H^1(M, \mathbb{R}) : c \wedge \omega = 0 \} \) is \( G^{KZ}_t \)-invariant because the KZ cocycle is symplectic with respect to the intersection form on \( H^1(M, \mathbb{R}) \) and \( H^1_{(0)}(M, \mathbb{R}) \) is the symplectic orthogonal of the (symplectic) 2-plane \( H^{(1)}_{1,1}(M, \mathbb{R}) \). Therefore, \( \lambda_2^\mu \) is the largest Lyapunov exponent of the restriction of KZ cocycle to \( H^1_{(0)}(M, \mathbb{R}) \).

In order to estimate \( \lambda_2^\mu \), we observe that, for any \( c \in H^1_{(0)}(M, \mathbb{R}) - \{0\} \),
\[
\frac{d}{dt} \log \|c\|_{\omega, t=0} = -\frac{\Re B_{\omega}(h, c)}{\|c\|^2_{\omega}} \leq \Lambda^+(\omega) := \max \left\{ \frac{|B_{\omega}(h, h)|}{\|h\|_{\omega}^2} : h \in H^1_{(0)}(M, \mathbb{R}) - \{0\} \right\}
\]
by Corollary \[30\] Hence, by integration,
\[
\frac{1}{T} \left( \log \|c\|_{g_T(\omega)} - \log \|c\|_{\omega} \right) \leq \frac{1}{T} \int_0^T \Lambda^+(g_t(\omega))dt
\]
By Oseledets theorem and Birkhoff’s theorem, for \( \mu \)-almost every \( \omega \in H^{(1)}_g \), we obtain that
\[
\lambda_2^\mu = \lim_{T \to \infty} \frac{1}{T} \log \|c\|_{g_T(\omega)} \leq \lim_{T \to \infty} \frac{1}{T} \int_0^T \Lambda^+(g_t(\omega))dt = \int_{H^{(1)}_g} \Lambda^+(\omega)d\mu(\omega)
\]
This reduces the task of proving that \( \lambda_2^\mu < 1 \) to show that \( \Lambda^+(\omega) < 1 \) for every \( \omega \in H^{(1)}_g \).
Here, we proceed by contradiction. Assume that \( \Lambda^+(\omega) = 1 \) for some Abelian differential \( \omega \). By definition, this means that
\[
|B_{\omega}(h, h)| = \|h\|_{\omega}^2
\]
for some \( h \in H^1_{(0)}(M, \mathbb{R}) - \{0\} \). In other words, by looking at the proof of Corollary \[30\] we have a case of equality in an estimate derived from Cauchy-Schwarz inequality. It follows that, by
denoting \( \alpha(h) \neq 0 \) the \((M, \omega)\)-holomorphic 1-form with \( h = \text{Re}(\alpha(h)) \), the functions \( u(h) := \alpha(h)/\omega \) and \( \overline{u(h)} = \overline{\alpha(h)/\omega} \) differ by a multiplicative constant \( a \in \mathbb{C} \), i.e.,

\[
\overline{u(h)} = a \cdot u(h)
\]

Since \( u(h) \) is a meromorphic function and, \textit{a fortiori}, \( \overline{u(h)} \) is an anti-meromorphic function, this is only possible when \( u(h) \) is a \textit{constant} function, that is, \( \alpha(h) \in \mathbb{C} \cdot \omega - \{0\} \). In particular, \( h \wedge \omega \neq 0 \), a contradiction with the fact that \( h \in H^1_{(0)}(M, \mathbb{R}) \). \( \square \)

At this stage, one could work more to derive further applications of the Hodge norm to Teichmüller dynamics: for instance, using the Hodge norm it is possible to show some uniform hyperbolicity and quantitative recurrence estimates for the Teichmüller flow \( g_t \) with respect to any compact set \( K \subset \mathcal{H}_{g}^{(1)} \), and this information was used by J. Athreya and G. Forni \cite{1} to study deviations of ergodic averages for billiards on rational polygons. However, we will refrain ourselves from doing so because we prefer to give in the next section some interesting applications of the facts derived in this subsection.

4. **Ergodic theory of Teichmüller flow with respect to Masur-Veech measure**

Let \( C \) be a connected component of a stratum \( \mathcal{H}_{\kappa}^{(1)} \) of Abelian differentials with unit area, and denote by \( \mu_C \) the corresponding Masur-Veech probability measure.

4.1. **Finiteness of Masur-Veech measure and unique ergodicity of interval exchange maps.** One important application of the fact that the Teichmüller flow preserves a natural (Masur-Veech) probability measure is the \textit{unique ergodicity} of “almost every” \textit{interval exchange transformation} (i.e.t. for short). Recall that an \textit{i.e.t.} is a map \( T : D_T \to D_T^{-1} \) where \( D_T, D_T^{-1} \subset I \) are subsets of an open bounded interval \( I \) such that \( I - D_T \) and \( I - D_T^{-1} \) are finite sets and the restriction of \( T \) to each connected component of \( I - D_T \) is a translation onto some connected component of \( D_T^{-1} \). For a concrete example, see the picture below.

It is not hard to see that an i.e.t. \( T \) is determined by a \textit{metric data}, i.e., lengths of the connected components of \( I - D_T \), and \textit{combinatorial data}, i.e., a permutation \( \pi \) indicating how the connected components of \( I - D_T \) are “rearranged” after applying \( T \) to them. For instance, in the example of picture above where 4 intervals are exchanged, the combinatorial data is the permutation \( \pi : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\} \) with \( (\pi(1), \pi(2), \pi(3), \pi(4)) = (4, 3, 2, 1) \).

In particular, it makes sense to talk about “almost every” i.e.t.: it means that a certain property holds for almost every choice of metric data with respect to the Lebesgue measure.
Remark 33. In the sequel, we will always assume that the combinatorial data $\pi$ is irreducible, i.e., if $\pi$ is a permutation of $d$ elements $\{1, \ldots, d\}$, we require that, for every $k < d$, $\pi(\{1, \ldots, k\}) \neq \{1, \ldots, k\}$. The meaning of this condition is very simple: if $\pi$ is not irreducible, there is $k < d$ such that $\pi(\{1, \ldots, k\}) = \{1, \ldots, k\}$ and hence we can study any i.e.t. $T$ with combinatorial data $\pi$ by juxtaposing two i.e.t.’s (one with $k$ intervals and another with $d - k$ intervals).

By applying their result (Theorem 26), H. Masur \[51\] and W. Veech \[68\] deduced that:

**Theorem 34** (H. Masur, W. Veech). *Almost every i.e.t. is uniquely ergodic.*

Philosophically speaking, the derivation of this result from Theorem 26 is part of a long tradition (in Dynamical Systems) of “plough in parameter space, and harvest in phase space” (as it was said by Adrien Douady about complex quadratic polynomials and Mandelbrot set [cf. http://en.wikipedia.org/wiki/Mandelbrot]). In broad terms, the idea is that given a parameter family of dynamical systems and an appropriate renormalization procedure (defined at least for a significant part of the parameters), one can often infer properties of the dynamical system for “typical parameters” by studying the dynamics of the renormalization.

For the case at hand, we can describe this idea in a nutshell as follows. An i.e.t. $T$ can be “suspended” (in several ways) to a translation surface $(M, \omega)$: the two most “popular” ways are Masur’s suspension construction and Veech’s zippered rectangles construction (cf. Example 14 above). For example, in Figure 11 below, we see a genus 2 surface (obtained by gluing the opposites sides of the polygon marked with the same letter $A$, $B$, $C$ or $D$ by translation) presented as a (Masur’s) suspension of an i.e.t. with combinatorial data $(\pi(1), \pi(2), \pi(3), \pi(4)) = (4, 3, 2, 1)$. To see that this is the combinatorial data of the i.e.t., it suffices to “compute” the return map of vertical translation flow to the special segment in the “middle” of the polygon.

By definition, $T$ is the first return time map of the vertical translation flow of the Abelian differential $\omega$ to an appropriate horizontal separatrix associated to some singularity of $\omega$. Here, the vertical translation flow $\phi_{t}^{\omega}$ associated to a translation surface $(M, \omega)$ is the flow obtained by following (with unit speed) vertical geodesics of the flat metric corresponding to $\omega$. In particular,
since the flat metric has singularities (in general), $\phi_t^\omega$ is defined \textit{almost} everywhere (as vertical trajectories are “forced” to stop when they hit singular points [zeroes] of $\omega$)! See Figure 11 below for an illustration of these objects. There one can see an orbit through a point $q$ hitting a singularity in finite time (and hence stopping by then) and an orbit through a point $p$ whose orbit never hits a singularity (and hence it can be prolonged forever).

![Figure 11](image.png)

\textbf{Figure 11.} Two pieces of orbits of a vertical translation flow: the orbit through $q$ (in red) hits a singularity in finite time (and then stops), while the orbit through $p$ (in blue) winds around the surface without hitting singularities (and thus can be continued indefinitely).

In particular, we can study orbits of $T$ by looking at orbits of the vertical flow on $(M, \omega)$. Here, the idea is that long orbits of the vertical flow can wrap around a lot on $(M, \omega)$, so that a natural procedure is to use Teichmüller flow $g_t = \text{diag}(e^t, e^{-t})$ to make the long vertical orbit shorter and shorter (so that it wraps less and less), thus making it reasonably easier to analyse. I.e., one uses Teichmüller flow to \textit{renormalize} the dynamics of the vertical flow on translation surfaces (and/or i.e.t.’s). Of course, the price we pay is that this procedure changes the shape of $(M, \omega)$ (into $(M, g_t(\omega))$). But, if the Teichmüller flow $g_t$ has nice \textit{recurrence} properties (so that the shape $(M, g_t(\omega))$ is very close to $(M, \omega)$ for appropriate choice of large $t$), one can hope to bypass the difficulty imposed by the change of shape.

In the case of showing unique ergodicity of almost every i.e.t., H. Masur and W. Veech observed that this can be derived from Poincaré’s recurrence theorem applied to Teichmüller flow endowed with Masur-Veech measure. Of course, for this application of Poincaré recurrence theorem, it is utterly important to know that Masur-Veech measure is a probability (i.e., it has finite mass), a fact ensured by Theorem 26.

Evidently, this is a very rough sketch of the proof of Theorem 34. For more details, see J.-C. Yoccoz survey [72] for a complete proof using Rauzy-Veech induction.

Notice that the same kind of reasoning as above indicates that the unique ergodicity property must also be true for “almost every” translation flow in the sense that the vertical translation
flow on $\mu_C$ almost every translation surface structure $(M, \omega) \in \mathcal{C}$ is uniquely ergodic. Indeed, the following theorem (again by H. Masur [51] and W. Veech [68]) says that this is the case:

**Theorem 35** (H. Masur, W. Veech). *Almost every translation flow is uniquely ergodic.*

**Remark 36.** During his original proof, H. Masur showed the following result: if the vertical translation flow on the translation surface $(M, \omega)$ is minimal but not uniquely ergodic, then the trajectory $g_t(\omega)$ of $(M, \omega)$ under the Teichmüller flow is not recurrent. Equivalently, H. Masur gave a sufficient criterion (nowadays known as *Masur’s criterion*) for the unique ergodicity of a minimal vertical translation flows: it suffices to check that the Teichmüller trajectory of the underlying translation surface is recurrent. Note that the converse is not true in general as it was shown by Y. Cheung and A. Eskin [14].

In the sequel, we will present a sketch of proof of this result based on the recurrence of Teichmüller flow, and the simplicity of the top exponent $1 = \lambda_1^{\mu_C} > \lambda_2^{\mu_C}$ (see Corollary [32] above). We start by assuming that the vertical translation flow $\phi_t^\omega$ of our translation surface $(M, \omega)$ is minimal, that is, every orbit defined for all times $t \geq 0$ are dense: this condition is well-known to be related to the absence of saddle connections (see, e.g., J.-C. Yoccoz survey [72]), and the last property has full measure (since the presence of saddle connections for $(M, e^{i\theta} \omega)$ corresponds to a countable set of directions $\theta \in \mathbb{R}$, and the Masur-Veech measure $\mu_C$ is natural).

Now, given an ergodic $\phi_t^\omega$-invariant probability $\mu$, consider $x \in M$ a $\mu$-typical point, and $T \geq 0$. Let $\gamma_T(x) \in H_1(M, \mathbb{R})$ be the homology class obtained by “closing” the piece of (vertical) trajectory $[x, \phi_t^\omega(x)] := \{\phi_t^\omega(x) : t \in [0, T]\}$ with a bounded (usually small) segment connecting $x$ to $\phi_T^\omega(x)$.

A well-known theorem of Schwartzman [66] says that

$$\lim_{T \to \infty} \frac{\gamma_T(x)}{T} = \rho(\mu) \in H_1(M, \mathbb{R}) - \{0\}$$

In the literature, $\rho(\mu)$ is called *Schwartzman asymptotic cycle*. By Poincaré duality, the Poincaré dual of $\rho(\mu)$ gives us a class $c(\mu) \in H^1(M, \mathbb{R}) - \{0\}$. Geometrically, $c(\mu)$ is related to the flux of $\phi_t^\omega$ through transverse closed curves $\gamma$ with respect to $\mu$. More precisely, given $\gamma$ a closed curve transverse to $\phi_t^\omega$, the flux is

$$\langle c(\mu), \gamma \rangle := \lim_{t \to 0} \frac{\mu}{t} \left( \bigcup_{s \in [0, t]} \phi_t^\omega(\gamma) \right)$$

For the sake of the subsequent discussion, we recall that any $\phi_t^\omega$-invariant probability $\mu$ induces a transverse measure $\hat{\mu}$ on pieces of segments $\delta$ transverse to $\phi_t^\omega$: indeed, we define $\hat{\mu}(\delta)$ by the flux through $\delta$, i.e., $\lim_{t \to 0} \mu(\bigcup_{s \in [0, t]} \phi_t^\omega(\delta))/t$. Since $\phi_t^\omega$ is simply a translation along the leaves of the vertical foliation of $\omega$, we see that $\mu$ can be locally written as $\text{Leb} \times \hat{\mu}$ in any “product” open set of the form $\bigcup_{s \in [0, t]} \phi_t^\omega(\delta)$ not meeting singularities of $\phi_t^\omega$ (where $\delta$ is a transverse segment).
We claim that the map $\mu \to c(\mu)$ is injective. Indeed, given two ergodic $\phi_\gamma^t$-invariant probabilities $\mu_1$ and $\mu_2$ with $c(\mu_1) = c(\mu_2)$, we observe that the transverse measures $\hat{\mu}_1$ and $\hat{\mu}_2$ induced by them on a closed curve $\gamma$ transverse to $\phi_\gamma^t$ differ by the derivative of a continuous function $U$ on $\gamma$. Indeed, $U$ can be obtained by integration: by fixing an “origin” $0 \in \gamma$ and an orientation on $\gamma$, we declare $U(0) = 0$ and $U(x) := \hat{\mu}_1(\gamma([0,x])) - \hat{\mu}_2(\gamma([0,x])) + U(0)$, where $\gamma([0,x])$ is the segment of $\gamma$ going from 0 to $x$ in the positive sense (with respect to the fixed orientation). Of course, the fact that $U$ is well-defined$^9$ is guaranteed by the assumption $c(\mu_1) = c(\mu_2)$. Now, we note that $U$ is invariant under the return map induced by $\phi_\gamma^t$, so that, by minimality of $\phi_\gamma^t$, we conclude that the continuous function $U$ must be constant. Therefore, $\hat{\mu}_1 = \hat{\mu}_2$, i.e., $\mu_1$ and $\mu_2$ have the same transverse measures. Since $\mu_1$ and $\mu_2$ are the Lebesgue measure along the flow direction, we obtain that $\mu_1 = \mu_2$, so that the claim is proved.

Next, we affirm that $c(\mu)$ (or equivalently $\rho(\mu)$) decays exponentially fast like $e^{-t}$ under KZ cocycle whenever the Teichmüller flow orbit $g_t(\omega)$ of $\omega$ is recurrent. Indeed, let us fix $t \geq 0$ such that $g_t(\omega)$ is very close to $\omega$, and we consider the action of KZ cocycle $G_t^{KZ}$ on $\rho(\mu)$. Since, by definition, $\rho(\mu)$ is approximated by $\gamma_T(x)/T$ as $T \to \infty$, we have that

$$G_t^{KZ}(\rho(\mu)) = \lim_{T \to \infty} \frac{1}{T} G_t^{KZ}(\gamma_T(x)).$$

On the other hand, since $g_t$ contracts the vertical direction by a factor of $e^{-t}$ and $\gamma_T(x)$ is essentially a vertical trajectory (except for a bounded piece of segment connecting $x$ to $\phi_\gamma^t(x)$), we get

$$\|G_t^{KZ}(\rho(\mu))\|_{g_t(\omega)} = \lim_{T \to \infty} \frac{1}{T} \|G_t^{KZ}(\gamma_T(x))\|_{g_t(\omega)} = e^{-t} \lim_{T \to \infty} \frac{1}{T} \|\gamma_T(x)\|_\omega = e^{-t} \|\rho(\mu)\|_\omega$$

where $\|\cdot\|_\theta$ is the stable norm on $H_1(M, \mathbb{R})$ with respect to $\theta$ (obtained by measuring the length of [primitive] closed curves [i.e., elements of $H_1(M, \mathbb{Z})$] using the flat structure induced by $\theta$ and extending this “by linearity”). In the previous calculation, we implicitly used the fact that $g_t(\omega)$ is very close to $\omega$, so that the stable norms $\|\cdot\|_{g_t(\omega)}$ and $\|\cdot\|_\omega$ are comparable by definite factors, and thus the factor of $1/T$ can “kill” eventual (bounded) error terms coming from the “closing” procedure used to define $\gamma_T(x)$. Therefore, our affirmation is proved.

Finally, we note that the fact $1 = \lambda_1^{\phi_\gamma^t} > \lambda_2^{\phi_\gamma^t}$ (i.e., simplicity of the top KZ cocycle exponent, see Corollary 32 above) means that there is only one direction in $H_1(M, \mathbb{R})$ which is contracted like $e^{-t}$ (namely, $\mathbb{R} \cdot [\text{Im}(\omega)]$). Therefore, given $\omega$ with minimal vertical translation flow and recurrent Teichmüller flow orbit, any $\phi_\gamma^t$-invariant ergodic probability $\mu$ satisfies $c(\mu) \in \mathbb{R} \cdot [\text{Im}(\omega)]$. Since $\phi_\gamma^t$ preserves the Lebesgue measure $\text{Leb}$ (flat area induced by $\omega$), we obtain that any $\phi_\gamma^t$-invariant ergodic probability $\mu$ is a multiple of $\text{Leb}$, and, a fortiori, $\mu = \text{Leb}$. Thus, $\phi_\gamma^t$ is uniquely ergodic for such $\omega$’s. Since we already saw that $\mu_C$ almost everywhere the vertical translation flow is minimal, we have only to show that $\mu_C$-almost every $\omega$ is recurrent under Teichmüller flow to complete the proof of Theorem 35, but this is immediate from Poincaré’s recurrence theorem (since Teichmüller flow preserves the Masur-Veech measure $\mu_C$, a finite mass measure).

$^9$I.e., it produces the same value for $U(0)$ when we go around $\gamma$. 
4.2. Ergodicity of Teichmüller flow. In the fundamental papers [51] and [69], H. Masur and W. Veech independently showed the following result:

**Theorem 37** (H. Masur, W. Veech). The Teichmüller geodesic flow $g_t$ is ergodic, and actually mixing, with respect to $\mu_C$.

For a complete proof of this result using Rauzy-Veech induction, see (again) J.-C. Yoccoz survey [72].

Concerning the first part of the statement, we observe that the ergodicity of the Teichmüller flow $g_t$ is essentially a consequence of the simplicity of the top exponent $1 = \lambda_{1C} > \lambda_{2C}$ and the existence of nice (“long”) stable and unstable manifolds for $g_t$. Indeed, as we already know, the simplicity of the top exponent $\lambda_{1C}$ implies that, except for the zero Lyapunov exponent coming from the flow direction, the Teichmüller flow $g_t$ has no other zero exponent (since $1 - \lambda_{2C} > 0$ is the second smallest non-negative exponent). In other words, the Teichmüller flow is non-uniformly hyperbolic in the sense of the Pesin theory. This indicates that Hopf’s argument may apply in our context. Recall that Hopf’s argument starts by observing that ergodic averages are constant along stable and unstable manifolds: more precisely, given a point $x$ such that the ergodic average

$$\varphi(x) := \lim_{t \to +\infty} \frac{1}{t} \int_0^t \varphi(g_s(x))ds$$

exists for a (uniformly) continuous observable $\varphi : C \to \mathbb{R}$, then the ergodic averages

$$\varphi(y) := \lim_{t \to +\infty} \frac{1}{t} \int_0^t \varphi(g_s(y))ds$$

exists and $\varphi(y) = \varphi(x)$ for any $y$ in the stable manifold $W^s(x)$ of $x$. Actually, since $y \in W^s(x)$, we have $\lim_{s \to +\infty} d(g_s(y), g_s(x)) = 0$, so that, by the uniform continuity of $\varphi$, the desired claim follows.

Of course, a similar result for ergodic averages along unstable manifolds holds if we replace $t \to +\infty$ by $t \to -\infty$ in the definition of $\varphi$. Now, the fact that we consider “future” ($t \to +\infty$) ergodic averages along stable manifolds and “past” ($t \to -\infty$) ergodic averages along unstable manifolds is not a major problem since Birkhoff’s ergodic theorem ensures that these two “types” of ergodic averages coincide at $\mu_C$ almost every point.

In particular, since the ergodicity of $\mu_C$ is equivalent to the fact that $\varphi$ is constant at $\mu_C$ almost every point, if one could access any point $y$ starting from any point $x$ using pieces of stable and unstable manifolds like in Figure 12 below, we would be in good shape (here, we’re skipping some details because Hopf’s argument needs that the intersection points appearing in Figure 12 to satisfy Birkhoff’s ergodic theorem; in general, this is issue is strongly related to the so-called absolute continuity property of the stable and unstable manifolds, but this is not a problem in our context since Pesin’s theory ensures absolute continuity of $W^s$ and $W^u$).

However, it is a general fact that Pesin theory of non-uniformly hyperbolic systems only provides the existence of short stable and unstable manifolds. Even worse, the function associating to a typical point the size of its stable/unstable manifolds is only measurable. In particular, the nice
scenario drew below may not happen in general (and actually the best Hopf’s argument [alone]
can do is to ensure the presence of a countable number of ergodic components [at most]).

Fortunately, in the specific case of Teichmüller flow, one can determine explicitly
the stable and unstable manifolds: since $g_t$ acts on $\omega$ by multiplying $[\Re(\omega)]$ by $e^t$
and $[\Im(\omega)]$ by $e^{-t}$, we infer that

$$W^s(\omega_0) = \{\omega \in \mathbb{C} : [\Re(\omega)] = [\Re(\omega_0)]\} \quad \text{and} \quad W^u(\omega_0) = \{\omega \in \mathbb{C} : [\Im(\omega)] = [\Im(\omega_0)]\}.$$ 

In particular, we see that these invariant manifolds are “large” subsets corresponding to affine
subspaces in period coordinates. Therefore, the potential problem pointed out in the previous
paragraph doesn’t exist, and one can proceed with Hopf’s argument to eventually derive the
ergodicity of Teichmüller flow with respect to Masur-Veech measure $\mu_C$.

Concerning the second part of the statement of this theorem, we should say that the mixing
property of Teichmüller flow is a consequence of its ergodicity and the mere existence of the
SL(2, $\mathbb{R}$)-action: indeed, while ergodicity alone doesn’t imply mixing in general (e.g., irrational
rotations of the circle are ergodic but not mixing), the fact that Teichmüller flow is part of a whole
SL(2, $\mathbb{R}$)-action permits to derive mixing from ergodicity in view of the nice representation theory
of SL(2, $\mathbb{R}$). We discuss this together with the exponential mixing property of Teichmüller flow in
the next subsection.

4.3. Exponential mixing (and spectral gap of SL(2, $\mathbb{R}$) representations). Generally speak-
ing, we say that a flow $(\phi_t)_{t \in \mathbb{R}}$ on a space $X$ is mixing with respect to an invariant probability $\mu$
when the correlation function $C_t(f,g) := \int_X (f \circ \phi_t \cdot g) d\mu - \int_X f d\mu \cdot \int_X g d\mu$ satisfies

$$\lim_{t \to \infty} |C_t(f,g)| = 0$$

for every $f, g \in L^2(X, \mu)$. Of course, the mixing property always implies ergodicity of $(\phi_t)_{t \in \mathbb{R}}$ but
the converse is not always true (e.g., irrational translation flows on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ are
ergodic but not mixing). However, as we’re going to see in a moment, when the flow $(\phi_t)_{t \in \mathbb{R}}$ is
part of a larger $SL(2, \mathbb{R})$ action, it is possible to show that ergodicity implies mixing.
More precisely, suppose that we have a $SL(2, \mathbb{R})$ action on a space $X$ preserving a probability measure $\mu$, and let $(\phi_t)_{t \in \mathbb{R}}$ be the flow on $X$ corresponding to the action of the diagonal subgroup $\text{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$. In this setting, one has:

**Proposition 38.** Assume that $(\phi_t)_{t \in \mathbb{R}}$ is $\mu$-ergodic. Then, $(\phi_t)_{t \in \mathbb{R}}$ is $\mu$-mixing.

Of course, the Teichmüller flow $g_t$ on a connected component $C$ of a stratum of the moduli space of Abelian differentials equipped with its natural (Masur-Veech) probability measure $\mu_C$ is a prototype example of flow verifying the assumptions of the previous proposition.

As we pointed out above, the proof of this result uses knowledge of the representation theory of $SL(2, \mathbb{R})$. We strongly recommend reading Livio Flaminio’s notes in this volume for a nice discussion of this subject. For sake of convenience, we quickly reviewed some results on this topic in Appendix A below. In particular, we will borrow the notations from this Appendix.

We begin by observing that the $SL(2, \mathbb{R})$ action on $(X, \mu)$ induces an unitary representation of $SL(2, \mathbb{R})$ on $\mathcal{H} = L^2_0(X, \mu)$. Here, $L^2_0(X, \mu)$ is the Hilbert space of $L^2$ functions of $(X, \mu)$ with zero mean. In particular, from the semisimplicity of $SL(2, \mathbb{R})$, we can write $\mathcal{H}$ as a integral of irreducible unitary $SL(2, \mathbb{R})$ representations $\mathcal{H}_\xi$:

$$\mathcal{H} = \int \mathcal{H}_\xi d\lambda(\xi)$$

The fact that $(\phi_t)_{t \in \mathbb{R}}$ is $\mu$-ergodic implies that the $SL(2, \mathbb{R})$ action is $\mu$-ergodic, that is, the trivial representation doesn’t appear in the previous integral decomposition. By Bargmann’s classification, every nontrivial unitary irreducible $SL(2, \mathbb{R})$ representation belongs to one of the following three classes (or series): **principal series**, **discrete series** and **complementary series**. See Livio Flaminio’s notes in this volume and/or Appendix A below for more discussion.

By M. Ratner’s work [63], we know that, for every $t \geq 1$ and for every $v, w \in \mathcal{H}_\xi$ with $\mathcal{H}_\xi$ in the **principal or discrete series**, 

$$|C_\xi(v, w)| := \int (v \circ \phi_t) \cdot w \, d\mu \leq C \cdot t \cdot e^{-t} \cdot \|v\|_{L^2(X, \mu)} \cdot \|w\|_{L^2(X, \mu)}$$

where $C > 0$ is an universal constant. Of course, we’re implicitly using the fact that, by hypothesis, $\phi_t$ is exactly the action of the diagonal subgroup $\text{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$ on $X$. Also, for every $t \geq 1$ and for every $v, w \in \mathcal{H}_\xi C^3$ vectors (see Appendix A for more details) with $\mathcal{H}_\xi$ in the **complementary series**, one can find a parameter $s = s(\mathcal{H}_\xi) \in (0, 1)$ (related to the eigenvalue $-1/4 < \lambda(\mathcal{H}_\xi) < 0$ of the Casimir operator of $\mathcal{H}_\xi$) such that

$$|C_\xi(v, w)| := \int (v \circ \phi_t) \cdot w \, d\mu \leq C_s \cdot e^{(-1+s)t} \cdot \|v\|_{C^3} \cdot \|w\|_{C^3}$$

where $C_s > 0$ is a constant depending only on $s$ and $\|u\|_{C^3}$ is the $C^3$ norm of a $C^3$ vector $u$ along the $SO(2, \mathbb{R})$ direction. In the notation of Appendix A, $\|u\|_{C^3} := \|u\|_{L^2(X, \mu)} + \|L_W u\|_{L^2(X, \mu)} + \|L^3_W u\|_{L^2(X, \mu)}$. Furthermore, $C_s$ can be taken uniform on intervals of the form $s \in [1 - s_0, s_0]$ with $1/2 < s_0 < 1$. 


Putting these informations together (and using the classical fact that $C^3$ vectors are dense), one obtains that $|C_t(v, w)| \to 0$ as $t \to \infty$ (actually, it goes “exponentially fast” to zero in the sense explained above) for:

- all vectors $v, w \in H_\xi$ when $H_\xi$ belongs to the principal or discrete series;
- a dense subset (e.g., $C^3$ vectors) of vectors $v, w \in H_\xi$ when $H_\xi$ belongs to the complementary series.

Using this (and the integral decomposition $H = \int H_\xi d\lambda(\xi)$), we conclude that $|C_t(v, w)| \to 0$ as $t \to \infty$ for a dense subset of vectors $v, w \in H = L_0^2(X, \mu)$. Now, an easy approximation argument shows that $|C_t(v, w)| \to 0$ as $t \to \infty$ for all $v, w \in L_0^2(X, \mu)$. Hence, $(\phi_t)_{t \in \mathbb{R}}$ is $\mu$-mixing and the proof of Proposition 38 is complete.

Once the Proposition 38 is proved, a natural question concerns the “speed”/“rate” of convergence of $C_t(f, g)$ to zero (as $t \to \infty$). In a certain sense, this question was already answered during the proof of Proposition 38 using Ratner’s results [63], one can show that $C_t(f, g)$ converges exponentially fast to zero for all $f, g$ in a dense subset of $L_0^2(X, \mu)$ (e.g., $f, g$ $C^3$ vectors) if and only if the unitary $SL(2, \mathbb{R})$ representation $H = L_0^2(X, \mu)$ has spectral gap, i.e., there exists $s_0 \in (0, 1)$ such that, when writing $H$ as an integral $H = \int H_\xi d\lambda(\xi)$ of unitary irreducible $SL(2, \mathbb{R})$ representations, no $H_\xi$ in the complementary series has parameter $s = s(H_\xi) \in (s_0, 1)$. Actually, it is possible to show that the spectral gap property is equivalent to the nonexistence of almost invariant vectors: recall that a representation of a Lie group $G$ on a Hilbert space $H$ has almost vector when, for all compact subsets $K$ and for all $\varepsilon > 0$, there exists an unit vector $v \in H$ such that $\|gv - v\| < \varepsilon$ for all $g \in K$.

In general, it is a hard task to prove the spectral gap property for a given unitary $SL(2, \mathbb{R})$ representation. For the case of the unitary $SL(2, \mathbb{R})$ representation $L_0^2(C, \mu_C)$ obtained from the $SL(2, \mathbb{R})$ action on a connected component $C$ of a stratum of the moduli space of Abelian differentials equipped with the natural Masur-Veech measure $\mu_C$, A. Ávila, S. Gouëzel and J.-C. Yoccoz showed the following theorem:

**Theorem 39** (A. Ávila, S. Gouëzel, J.-C. Yoccoz). The Teichmüller flow $g_t$ on $C$ is exponentially mixing with respect to $\mu_C$ (in the sense that $C_t(f, g) \to 0$ exponentially as $t \to \infty$ for “sufficiently smooth” $f, g$), and the unitary $SL(2, \mathbb{R})$ representation $L_0^2(C, \mu_C)$ has spectral gap.

In the proof of this result, Ávila, Gouëzel and Yoccoz [5] proves firstly that the Teichmüller geodesic flow (i.e., the action of the diagonal subgroup $A = \{a(t) : t \in \mathbb{R}\}$ on the moduli space $Q_g$ of Abelian differentials) is exponentially mixing with respect to Masur-Veech measure (indeed this is the main result of their paper) and they use a reverse Ratner estimate to derive the spectral gap property from the exponential mixing (and not the other way around!). Here, the proof of the exponential mixing property with respect to Masur-Veech measure is obtained by delicate (mostly combinatorial) estimates on the so-called Rauzy-Veech induction.

---

10This is essentially due to the fact that $SL(2, \mathbb{R})$ doesn’t have the so-called Kazhdan’s property $T$. 

---
More recently, Ávila and Gouëzel [4] developed a more geometrical (and less combinatorial) approach to the exponential mixing of algebraic $SL(2, \mathbb{R})$-invariant probabilities.

Roughly speaking, an algebraic $SL(2, \mathbb{R})$-invariant measure $\mu$ is a probability measure supported on an affine suborbifold $\text{supp}(\mu)$ of $\mathbb{C}$ (in the sense that $\text{supp}(\mu)$ corresponds, in local period coordinates, to affine subspaces in relative homology) such that $\mu$ is absolutely continuous (wrt the Lebesgue measure on the affine subspaces corresponding to $\text{supp}(\mu)$ in period charts) and its density is locally constant in period coordinates. The class of algebraic $SL(2, \mathbb{R})$-invariant probabilities contains all “known” examples (e.g., Masur-Veech measures $\mu_C$ and the probabilities supported on the $SL(2, \mathbb{R})$-orbits of Veech surfaces [in particular, square-tiled surfaces]). Actually, an important conjecture in Teichmüller dynamics claims that all $SL(2, \mathbb{R})$-invariant probabilities are algebraic. If it is true, this conjecture would provide a non-homogenous counterpart to Ratner’s theorems [64] on unipotent actions in homogenous spaces.

After the celebrated works of K. Calta [13] and C. McMullen [61], there is a complete classification of $SL(2, \mathbb{R})$-invariant measures in genus 2 (i.e., $\mathcal{C} = \mathcal{H}(2)$ or $\mathcal{H}(1, 1)$). In particular, it follows that such measures are always algebraic (in genus 2). Furthermore, it was recently announced by A. Eskin and M. Mirzakhani [24] that the full conjecture is true.

In any case, the result obtained by Ávila and Gouëzel [4] is:

**Theorem 40** (A. Ávila and S. Gouëzel). Let $\mu$ be an algebraic $SL(2, \mathbb{R})$-invariant probability, and consider the integral decomposition $L^2_0(\mathcal{C}, \mu) = \int \mathcal{H}_\xi d\lambda(\xi)$ of the unitary $SL(2, \mathbb{R})$ representation $L^2_0(\mathcal{C}, \mu)$ into irreducible factors $\mathcal{H}_\xi$. Then, for any $\delta > 0$, the representations $\mathcal{H}_\xi$ of the complementary series with parameter $s(\mathcal{H}_\xi) \in [\delta, 1]$ appear only discretely (i.e., $\{s \in [\delta, 1] : s = s(\mathcal{H}_\xi) \text{ for some } \xi\} \text{ is finite}$) and with finite multiplicity (i.e., for each $s \in [\delta, 1], \{\xi : s(\mathcal{H}_\xi) = s\} \text{ is finite}$). In particular, the Teichmüller geodesic flow $g_t$ is exponentially mixing with respect to $\mu$.

This completes the discussion of this section on the first ergodic properties of the Teichmüller flow with respect to Masur-Veech measures and its applications to the unique ergodicity of i.e.t.’s and translation flows. In the next section, we will study more ergodic properties of the Teichmüller flow and KZ cocycle with respect to Masur-Veech measures and its applications to the weak mixing property of i.e.t.’s and translation flows.

5. Ergodic Theory of KZ cocycle with respect to Masur-Veech measures

Again, let $\mathcal{C}$ be a connected component of a stratum $\mathcal{H}^{(1)}(\kappa)$ of Abelian differentials with unit area, and denote by $\mu_C$ the corresponding Masur-Veech probability measure.

5.1. Kontsevich-Zorich conjecture (after G. Forni, and A. Ávila & M. Viana). Around 1996, A. Zorich and M. Kontsevich [75], [47] performed several numerical experiments leading them to conjecture that the Lyapunov spectra of the Kontsevich-Zorich cocycle with respect to Masur-Veech measures $\mu_C$ are simple, i.e., the multiplicity of each Lyapunov exponent $\lambda^{(i)}_C$, $i = 1, \ldots, 2g$

\[\text{Cf. Subsection 6.3 below for more details on the construction of these probabilities.}\]
is 1:
\[ 1 = \lambda_1^{\mu C} > \lambda_2^{\mu C} > \cdots > \lambda_g^{\mu C} > \lambda_{g+1}^{\mu C} > \cdots > \lambda_{2g}^{\mu C} = -1 \]

As we discussed in the previous section, the Kontsevich-Zorich cocycle \( G_{t}^{KZ} \) is symplectic, so that its Lyapunov exponents (with respect to any invariant ergodic probability \( \mu \)) are symmetric with respect to the origin: \( \lambda_i^{\mu} = -\lambda_{i+1}^{\mu} \). Also, the top Lyapunov exponent \( 1 = \lambda_1^{\mu} \) is always simple (i.e., \( \lambda_1^{\mu} > \lambda_2^{\mu} \)). Therefore, the Kontsevich-Zorich conjecture is equivalent to

\[ \lambda_2^{\mu C} > \cdots > \lambda_g^{\mu C} > 0 \]

In 2002, G. Forni [27] was able to show that \( \lambda_g^{\mu C} > 0 \) via second variational formulas for the Hodge norm and certain formulas for the sum of the Lyapunov exponents of the KZ cocycle (inspired by M. Kontsevich’s work). In Subsection 5.2 below, we’ll illustrate some of G. Forni’s techniques by showing the positivity of the second Lyapunov exponent \( \lambda_2^{\mu C} \) of the KZ cocycle with respect to Masur-Veech measure \( \mu_C \). While the fact \( \lambda_2^{\mu C} > 0 \) is certainly a weaker statement than Forni’s theorem \( \lambda_g^{\mu C} > 0 \), it turns out that it is sufficient to some interesting applications to interval exchange transformations and vertical translation flows. Indeed, using a technical machinery of parameter exclusion strongly based on the fact that \( \lambda_g^{\mu C} > 0 \), A. Ávila and G. Forni [3] were able to show that almost every i.e.t. (not corresponding to “rotations”) and almost every vertical translation flow (on genus \( g \geq 2 \) translation surfaces) are weakly mixing. Here, we say that an i.e.t. corresponds to a rotation if its combinatorial data \( \pi : \{1, \ldots, d\} \to \{1, \ldots, d\} \) has the form \( \pi(i) = i + 1 \) (mod \( d \)). In this case, one can see that the corresponding i.e.t. can be conjugated to a rotation of the circle, and hence it is never weak-mixing. Observe that, in general, weak-mixing property is the “best” dynamical property we can expect: indeed, as it was shown by A. Katok [44], interval exchange transformations and suspension flows over i.e.t.’s with a roof function of bounded variation (e.g., translation flows) are never mixing. We will come back to this point later in this section.

In 2007, A. Ávila and M. Viana [7] proved the full Kontsevich-Zorich conjecture by studying a discrete-time analog of Kontsevich-Zorich cocycle over the Rauzy-Veech induction. In few words, Ávila and Viana showed that the symplectic monoid associated to Rauzy-Veech induction is pinching (“it contains matrices with simple spectrum”) and twisting (“any subspace can be put into generic position by using some matrix of the monoid”), and they used the pinching and twisting properties to ensure simplicity of Lyapunov spectra. In a certain sense, these conditions (pinching and twisting) are analogues (for deterministic chaotic dynamical systems) of the strong irreducibility and proximality conditions (sometimes derived from a stronger Zariski density property) used by Y. Guivarch and A. Raugi [37], and I. Goldsheid and G. Margulis [36] to derive simplicity of Lyapunov exponents for random products of matrices.

**Remark 41.** More recently, G. Forni extended some techniques of his article [27] to prove in [29] a geometric criterion for the non-uniform hyperbolicity of KZ cocycle (i.e., \( \lambda_g^{\mu} > 0 \)) of “general” \( SL(2, \mathbb{R}) \)-invariant ergodic probability measures \( \mu \) (see Remark 58 below). As a matter of fact,
this general recent criterion strictly includes Masur-Veech measures, but it doesn’t allow to derive simplicity of the Lyapunov spectrum in general (see the appendix to [29] for more details). Also, it was recently shown by V. Delecroix and the second author [18] that there is no converse to G. Forni’s criterion. Here, the arguments of V. Delecroix and the second author are based on a recent criterion for the simplicity of the Lyapunov exponents of KZ cocycle with respect to $SL(2, \mathbb{R})$-invariant ergodic probabilities supported on the $SL(2, \mathbb{R})$-orbits of square-tiled surfaces due to M. Möller, J.-C. Yoccoz and the second author [57]. For more comments on this, see Section D below.

As the reader can imagine, the Kontsevich-Zorich conjecture has applications to the study of deviations of ergodic averages along trajectories of vertical translation flows and interval exchanges transformations. Actually, this was the initial motivation for the introduction of the Kontsevich-Zorich cocycle by A. Zorich and M. Kontsevich.

For the case of vertical translation flows, we begin with a typical vertical translation flow $\phi^\omega$ on a translation surface $(M, \omega)$ (so that it is uniquely ergodic) and we choose a typical point $p$ (so that $\phi^\omega$ is defined for every time $t$), e.g., as in Figure 11 above. For all $T > 0$ large enough, let us denote by $\gamma_T(x) \in H_1(M, \mathbb{R})$ the homology class obtained by “closing” the piece of (vertical) trajectory $[x, \phi_T^\omega(x)] := \{\phi_t^\omega(x) : t \in [0, T]\}$ with a bounded (usually small) segment connecting $x$ to $\phi_T^\omega$. Recall that Schwartzman theorem [66] says that

$$\lim_{T \to \infty} \frac{\gamma_T(x)}{T} = c \in H_1(M, \mathbb{R}) - \{0\}.$$

For genus $g = 1$ translation surfaces (i.e., flat torii), this is very good and fairly complete result: indeed, it is not hard to see that the deviation of $\gamma_T(x)$ from the line $E_1 := \mathbb{R} \cdot c$ spanned by the Schwartzman asymptotic cycle is bounded.

For genus $g = 2$ translation surfaces, the global scenario gets richer: by doing numerical experiments, what one sees is that the deviation of $\gamma_T(x)$ from the line $E_1$ has amplitude $T^{\lambda_2}$ with $\lambda_2 < 1$ around a certain line. In other words, the deviation of $\gamma_T(x)$ from the Schwartzman asymptotic cycle is not completely random: it occurs along an isotropic 2-dimensional plane $E_2 \subset H_1(M, \mathbb{R})$ containing $E_1$. Again, in genus $g = 2$, this is a “complete” picture in the sense that numerical experiments indicate that the deviation of $\gamma_T(x)$ from $E_2$ is again bounded.

More generally, for arbitrary genus $g$, the numerical experiments indicate that existence of an asymptotic Lagrangian flag, i.e., a sequence of isotropic subspaces $E_1 \subset E_2 \subset \cdots \subset E_g \subset H_1(M, \mathbb{R})$ with $\dim(E_i) = i$ and a deviation spectrum $1 = \lambda_1 > \lambda_2 > \cdots > \lambda_g > 0$ such that

$$\lim_{T \to \infty} \frac{\log \text{dist}(\gamma_T(x), E_i)}{\log T} = \lambda_{i+1}$$

for every $i = 1, \ldots, g - 1$, and

$$\sup_{T \in [0, \infty)} \text{dist}(\gamma_T(x), E_g) < \infty.$$

\[\text{I.e., the conditions of Forni’s criterion are sufficient but not necessary for non-uniform hyperbolicity}\]
For instance, the reader can see below two pictures (Figures 13 and 14) extracted from A. Zorich’s survey [74] and showing numerical experiments related to the deviation phenomenon or Zorich phenomenon discussed above in a genus 3 translation surface. There, we have a slightly different notation for the involved objects: $c_n$ denotes $\gamma_{T_n}(x)$ for a convenient choice of $T_n$, the subspaces $V_i$ correspond to the subspaces $E_i$, and the numbers $\nu_i$ correspond to the numbers $\lambda_i$.

**Figure 13.** Projection of a broken line joining $c_1, c_2, \ldots, c_{100000}$ to a plane orthogonal to Schwartzmann cycle (in a genus 3 case).

**Figure 14.** Deviation from Schwartzmann asymptotic cycle.

This scenario supported by numerical experimental was made rigorous by A. Zorich [75] using the Kontsevich-Zorich cocycle: more precisely, he proved that the previous statement is true with $E_i$ corresponding to the sum of the Oseledets subspaces associated to the first $i$ non-negative exponents of KZ cocycle, and $\lambda_i$ corresponding to the $i$-th Lyapunov exponent of the KZ cocycle.
with respect to Masur-Veech measure $\mu_C$. Of course, to get the complete description of the deviation phenomenon (i.e., the fact that $\dim E_i = i$, that is, the asymptotic flat $E_1 \subset \cdots \subset E_g$ is Lagrangian and complete), one needs to know that Kontsevich-Zorich conjecture is true. So, in this sense, A. Zorich’s theorem is a conditional statement depending on Kontsevich-Zorich conjecture.

Closing this subsection, let us mention that a similar scenario of deviations of ergodic averages for i.e.t.’s is true (as proved by A. Zorich in [75]), but its precise statement is somewhat technical because we need to talk first about special Birkhoff sums (which are Birkhoff sums along trajectories of our initial i.e.t. from a point $x$ until its return to special intervals [determined by Rauzy-Veech algorithm]), and then decompose general Birkhoff sums into a sum of relatively few special Birkhoff sums. In particular, we’ll not comment on this here, and we refer the curious reader to A. Zorich’s original paper [75] and J.-C. Yoccoz survey [72].

5.2. Second Lyapunov exponent of KZ cocycle with respect to Masur-Veech measures. We dedicate this subsection to give a sketch of proof of the following result:

**Theorem 42.** Let $C$ be a connected component of some stratum of $\mathcal{H}_g$ and denote by $\mu_C$ the corresponding Masur-Veech measure. Then, $\lambda_{2C}^\mu > 0$.

As we already mentioned, this result is part of one of the main results of [27] showing that $\lambda_g^{\mu_C} > 0$. However, we’ll not discuss the proof of the more general result $\lambda_g^{\mu_C} > 0$ because

- one already finds several of the ideas used to show $\lambda_g^{\mu_C} > 0$ during the sketch of proof of $\lambda_2^{\mu_C} > 0$, and
- by sticking to the study of $\lambda_2^{\mu_C} > 0$ we avoid the (rather technical) discussion of characterizing Oseledets unstable subspaces of KZ cocycle via basic currents.

In any event, we start the sketch of proof of Theorem 42 by recalling (from Subsection 3.5, Theorem 29) that the form $B_\omega(\alpha, \beta) := \frac{1}{2} \int \frac{\alpha \beta}{\omega} \omega$, $\alpha, \beta \in H^1(M, \mathbb{R})$, is relevant in the study of first variation of the Hodge norm in view of the formula:

$$\frac{d}{dt} \|c\|_{\omega_t} \big|_{t=0} = -2 \Re B_\omega(\alpha_0, \alpha_0)$$

where $c = [\Re \alpha_0]$, $\alpha_0 \in H^{1,0}$.

Also, recall that $H^1(M, \mathbb{R}) = H^1_{st}(M, \mathbb{R}) \oplus H^1_{(0)}(M, \mathbb{R})$ where $H^1_{st}(M, \mathbb{R}) = \Re \{\Re \omega\} \oplus \Re \{\Im \omega\}$ and $H^1_{(0)}(M, \mathbb{R}) := \{c \in H^1(M, \mathbb{R}) : c \wedge \omega = 0\}$. Moreover, $H^1_{st}(M, \mathbb{R})$ is KZ cocycle invariant and it contributes with the $\pm 1 = \pm \lambda_i^\mu$ Lyapunov exponents. Hence, since KZ cocycle preserves the symplectic intersection form, we get that the Lyapunov exponents $\lambda_i^\mu$, $2 \leq i \leq g$ come from the restriction of KZ cocycle to $H^1_{(0)}(M, \mathbb{R})$.

Denoting by $B_\omega^R(c_1, c_2) = B_\omega(\alpha_1, \alpha_2)$ where $c_i = [\Re \alpha_i]$, $i = 1, 2$, the complex-valued bilinear form on $H^1(M, \mathbb{R})$ induced by $B_\omega$ (on $H^{1,0}(M)$) via Hodge representation theorem (cf. Subsection 3.4), we obtain the following nice immediate consequence of this discussion:

---

13Note that it depends real-analytically (in particular continuously) on $\omega$. 
Corollary 43. Let μ be an ergodic \( g_t \)-invariant probability and suppose that \( \text{rank}(B^\omega_\mu|\mathcal{H}^1_{\omega}(\mathcal{M},\mathbb{R})) = 0 \) for all \( \omega \in \text{supp}(\mu) \). Then, \( \lambda^\mu_0 = \cdots = \lambda^\mu_g = 0 \).

Geometrically, \( B_\omega \) is essentially the second fundamental form (or Kodaira-Spencer map) of the holomorphic subbundle \( H^{1,0} \) of the complex Hodge bundle \( H^1_\mathbb{C} \) equipped with the Gauss-Manin connection. Roughly speaking, recall that the second fundamental form \( II_\omega : H^{1,0} \rightarrow H^{0,1} \) is associated to \( \mu \)-invariant probability and suppose that \( \text{rank}(B^\omega_\mu|\mathcal{H}^1_{\omega}(\mathcal{M},\mathbb{R})) = 0 \) for all \( \omega \in \text{supp}(\mu) \). Then, \( \lambda^\mu_0 = \cdots = \lambda^\mu_g = 0 \).

Theorem 44 (M. Kontsevich, G. Forni). Let μ be a \( SL(2,\mathbb{R}) \)-invariant probability on a connected component \( \mathcal{C} \) of some stratum of \( \mathcal{H}_g \). Then, one has the following formula for the sum of non-negative Lyapunov exponents of KZ cocycle with respect to μ:

\[
\lambda^\mu_1 + \cdots + \lambda^\mu_g = \int_\mathcal{C} (\Lambda_1(\omega) + \cdots + \Lambda_g(\omega)) \, d\mu(\omega)
\]

Remark 45. Since \( B_\omega(\omega,\omega) := 1 \), one can use the argument (Cauchy-Schwarz inequality) of the proof of Corollary 30 to see that \( \Lambda_1(\omega) \equiv 1 \) for all \( \omega \). In particular, since \( \lambda^\mu_1 = 1 \), one can rewrite
the formula above as
\[ \lambda_2^g + \cdots + \lambda_g^g = \int_C (\Lambda_2(\omega) + \cdots + \Lambda_g(\omega)) \, d\mu(\omega) \]

Remark 46. Note that there is an important difference in the hypothesis of Theorem 29 and Theorem 44 is: in the former \( \mu \) is any \( g_t \)-invariant while in the latter \( \mu \) is \( SL(2, \mathbb{R}) \)-invariant!

Before giving a sketch of proof of Theorem 44 we observe that from it (and Remark 45) one can immediately deduced the following “converse” to Corollary 43:

Corollary 47. Let \( \mu \) be a \( SL(2, \mathbb{R}) \)-invariant \( g_t \)-ergodic probability on a connected component \( C \) of some stratum of \( \mathcal{H}_g \). Suppose that \( \lambda_2^\mu = \cdots = \lambda_g^\mu = 0 \). Then, \( \text{rank}(B^\mathbb{R}_{\omega}|_{H^1_0(M, \mathbb{R})}) = 0 \) for all \( \omega \in \text{supp}(\mu) \).

Evidently, this corollary shows how one can prove Theorem 42: since Masur-Veech measures \( \mu_C \) are fully supported, it suffices to check that \( \text{rank}(B^\mathbb{R}_{\omega}|_{H^1_0(M, \mathbb{R})}) > 0 \) for some \( \omega \in C = \text{supp}(\mu_C) \). In other words, by assuming Theorem 44 we just saw that:

Corollary 48. If \( \text{rank}(B^\mathbb{R}_{\omega}|_{H^1_0(M, \mathbb{R})}) > 0 \) for some \( \omega \in C \) then \( \lambda_2^{\mu_C} > 0 \).

Now, before trying to use this corollary, let’s give an outline of proof of Theorem 44.

Sketch of proof of Theorem 44. Given \( 1 \leq k \leq g \), let
\[ \Phi_k(\omega, I_k) := 2 \sum_{i=1}^k H_\omega(c_i, c_i) - \sum_{j,m=1}^k |B^\mathbb{R}_{\omega}(c_j, c_m)|^2 \]
where \( I_k \) is a \( k \)-dimensional isotropic subspace of the real Hodge bundle \( H^1_\mathbb{R} \) and \( \{c_1, \ldots, c_k\} \) is any Hodge-orthonormal basis of \( I_k \).

In the sequel, we will use the following three lemmas (see [27] or [33] for proofs and more details).

Lemma 49 (Lemma 5.2’ of [27]). Let \( \{c_1, \ldots, c_k, c_{k+1}, \ldots, c_g\} \) be any Hodge-orthonormal completion of \( \{c_1, \ldots, c_k\} \) into basis of a Lagrangian subspace of \( H^1(M, \mathbb{R}) \). Then,
\[ \Phi_k(\omega, I_k) = \sum_{i=1}^g \Lambda_i(\omega) - \sum_{j,m=k+1}^g |B^\mathbb{R}_{\omega}(c_j, c_m)|^2 \]

Remark 50 (M. Kontsevich’s fundamental remark). In the extremal case \( k = g \), the right-hand side of the previous equality doesn’t depend on the Lagrangian subspace \( I_g \):
\[ \Phi_g(\omega, I_g) = \sum_{i=1}^g \Lambda_i(\omega) = \text{tr}(H_\omega) \]

This fundamental observation of Maxim Kontsevich lies at the heart of the main formula of Theorem 44.

\[ \text{Of course, it is implicit here that the expression } 2 \sum_{i=1}^k H_\omega(c_i, c_i) - \sum_{j,m=1}^k |B^\mathbb{R}_{\omega}(c_j, c_m)|^2 \text{ doesn’t depend on the choice of Hodge-orthonormal basis } \{c_1, \ldots, c_k\} \text{ but only on the isotropic subspace } I_k \subset H^1(M, \mathbb{R}). \]
It is not hard to see that the notion of Hodge norm $\|\cdot\|_\omega$ on vectors $c \in H^1(M, \mathbb{R})$ can be extended to any polyvector $c_1 \wedge \cdots \wedge c_k$ coming from a (Hodge-orthonormal) basis $\{c_1, \ldots, c_k\}$ of an isotropic subspace $I_k$. By slightly abusing of the notation, we will denote by $\|c_1 \wedge \cdots \wedge c_k\|_\omega$ the Hodge norm of such a polyvector.

Note that the Hodge norm $\|\cdot\|_\omega$ depends only on the complex structure, so that $\|\cdot\|_\omega = \|\cdot\|_{\omega'}$ whenever $\omega' = \text{constant} \cdot \omega$. In particular, it makes sense to consider the Hodge norm $\|\cdot\|_h$ over the Teichmüller disk $h \in SO(2, \mathbb{R}) \setminus SL(2, \mathbb{R}) \cdot \omega$. For subsequent use, we denote by $\Delta_{\text{hyp}}$ the hyperbolic (leafwise) Laplacian on $SO(2, \mathbb{R}) \setminus SL(2, \mathbb{R}) \cdot \omega$ (here, we’re taking advantage of the fact that $SO(2, \mathbb{R}) \setminus SL(2, \mathbb{R})$ is isomorphic to Poincaré’s hyperbolic disk $D$).

**Lemma 51** (Lemma 5.2 of [27]). One has $\Delta_{\text{hyp}} \log \|c_1 \wedge \cdots \wedge c_k\|_\omega = 2\Phi_k(\omega, I_k)$.

Finally, in order to connect the previous two lemmas with Oseledets theorem (and Lyapunov exponents), one needs the following fact about hyperbolic geometry:

**Lemma 52** (Lemma 3.1 of [27]). Let $L : D \to \mathbb{R}$ be a smooth function. Then,
\[
\frac{1}{2\pi} \int_0^{2\pi} L(t, \theta) \frac{d\theta}{\text{area}(D_t)} \int_{D_t} \Delta_{\text{hyp}} L(t, \theta) \frac{d\text{area}_P}{\text{area}(D_t)} = \frac{1}{2} \tanh(t) \int_0^T \Phi_k(t, \theta) \frac{d\text{area}_P}{\text{area}(D_t)}
\]

where $\Lambda := \Delta_{\text{hyp}} L$, $(t, \theta)$ are polar coordinates on Poincaré’s disk, $D_t$ is the disk of radius $t$ centered at the origin $0 \in D$ and area$_P$ is Poincaré’s area form on $D$.

Next, the idea to derive Theorem 44 from the previous three lemmas is the following. Denote by $R_\theta = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right)$, and, given $\omega \in H_g$, for $SO(2, \mathbb{R}) \setminus SL(2, \mathbb{R}) \cdot \omega \ni h = g_t R_\theta \omega = (t, \theta)$, let $L(h) := \|c_1 \wedge \cdots \wedge c_k\|_h$. In plain terms, $L$ is measuring how the (Hodge norm) size of the polyvector $c_1 \wedge \cdots \wedge c_k$ changes along the Teichmüller disk of $\omega$. In particular, as we’re going to see in a moment, it is not surprising that $L$ has “something to do” with Lyapunov exponents.

By Lemma 52, one has
\[
\frac{1}{2\pi} \int_0^{2\pi} L(t, \theta) \frac{d\theta}{\text{area}(D_t)} \int_{D_t} \Delta_{\text{hyp}} L(t, \theta) \frac{d\text{area}_P}{\text{area}(D_t)} = \frac{1}{2} \tanh(t) \int_0^T \Phi_k(t, \theta) \frac{d\text{area}_P}{\text{area}(D_t)}
\]

Then, by integrating with respect to the $t$-variable in the interval $[0, T]$ and by using Lemma 51 for the computation of $\Delta_{\text{hyp}} L$, one deduces
\[
\frac{1}{2\pi} \int_0^{2\pi} (L(T, \theta) - L(0, \theta)) \frac{d\theta}{\text{area}(D_t)} = \frac{1}{T} \int_0^T \Phi_k(t, \theta) \frac{d\text{area}_P}{\text{area}(D_t)} \frac{d\text{area}_P}{\text{area}(D_t)}
\]

At this point, by taking an average with respect to $\mu$ and using the $SL(2, \mathbb{R})$-invariance of $\mu$ to get rid of the integration with respect to $\theta$, we deduce that
\[
\frac{1}{T} \int_C (L(g_T(\omega)) - L(\omega)) d\mu(\omega) = \frac{1}{T} \int_C \int_0^T \frac{\tanh(t)}{\text{area}(D_t)} \int_{D_t} \Phi_k(g_t R_\theta \omega, I_k) \frac{d\text{area}_P}{\text{area}(D_t)} \frac{d\text{area}_P}{\text{area}(D_t)} dt
d\mu(\omega)
\]

Now, we observe that:
• by Oseledets theorem, for a “generic” isotropic subspace $I_k$ and $\mu$-almost every $\omega$, one has that $\frac{1}{T} L(g_T(\omega))$ converges\(^{17}\) to $\lambda_1^\mu + \cdots + \lambda_k^\mu$ as $T \to \infty$, and

• by Remark 50 for $k = g$, $\Phi_g(\omega, I_g) = \Phi_g(\omega) = \Lambda_1(\omega) + \cdots + \Lambda_g(\omega)$ is independent on $I_g$.

So, for $k = g$, this discussion\(^{18}\) allows to show that

$$\lambda_1^\mu + \cdots + \lambda_g^\mu = \int_C (\Lambda_1(\omega) + \cdots + \Lambda_g(\omega)) \, d\mu(\omega)$$

This completes the sketch of proof of Theorem 44. \qed

Remark 53. Essentially the same argument above allows to derive formulas for partial sums of Lyapunov exponents. More precisely, given $\mu$ a $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probability with $\lambda_k^\mu > \lambda_{k+1}^\mu$ (for some $1 \leq k \leq g - 1$), one has

$$\lambda_1^\mu + \cdots + \lambda_k^\mu = \int_C \Phi_k(\omega, E_k^+(\omega)) \, d\mu(\omega)$$

where $E_k^+(\omega)$ is the Oseledets subspace associated to the $k$ top Lyapunov exponents.

In general, this formula is harder to use than Theorem 44 because the right-hand side of the former implicitly assumes some \textit{a priori} control of $E_k^+(\omega)$ while the right-hand side of the latter is independent of Lagrangian subspaces (as noticed by M. Kontsevich).

After obtaining Theorem 44, we’re ready to use Corollary 47 to reduce the proof of Theorem 42 to the following theorem:

**Theorem 54.** In any connected component $C$ of a stratum of $H_g$ one can find some $\omega \in C$ with

$$\text{rank}(B_{\omega}|_{H^1_{\text{top}}(M, \mathbb{R})}) = 2g - 2.$$  

Roughly speaking, the basic idea (somehow \textit{recurrent} in Teichmüller dynamics) to show this result is to look for $\omega$ near the boundary of $C$ after passing to an appropriate compactification. More precisely, one shows that, by considering the so-called Deligne-Mumford compactification $\overline{C} = C \cap \partial C$, there exists an open set $U \subset C$ near some boundary point $\omega_\infty \in \partial C$ such that

$$\text{rank}(B_{\omega}|_{H^1_{\text{top}}(M, \mathbb{R})}) = 2g - 2$$

for any $\omega \in U$ “simply” because the “same” is true for $\omega_\infty$.

A complete formalization of this idea is out of the scope of these notes as it would lead us to a serious discussion of Deligne-Mumford compactification, some variational formulas of J. Fay and A. Yamada, etc. Instead, we offer below a very rough sketch of proof of Theorem 54 based on some “intuitive” properties of Deligne-Mumford compactification (while providing adequate references for the omitted details).

The first step towards finding the boundary point $\omega_\infty$ is to start with the notion of Abelian differentials with \textit{periodic Lagrangian horizontal foliation:}

\(^{17}\)Recall that, by definition, the function $t \mapsto L(g_t(\omega))$ is measuring the growth (in Hodge norm) of the polyvector $c_1 \wedge \cdots \wedge c_k$ along the Teichmüller orbit $g_t(\omega)$.

\(^{18}\)Combined with an application of Lebesgue dominated convergence theorem and the fact that $\tanh(t)/\text{area}(D_t) \to 1$ as $t \to \infty$. See [27] and [33] for more details.
**Definition 55.** Let \( \omega \) be an Abelian differential on a Riemann surface \( M \). We say that the horizontal foliation \( F_{\text{hor}}(\omega) := \{ 3\omega = \text{constant} \} \) is periodic whenever all regular leaves of \( F_{\text{hor}}(\omega) \) are closed, i.e., the translation surface \( (M, \omega) = \bigcup_i C_i \) can be completely decomposed into maximal cylinders \( C_i \) corresponding to some closed regular geodesics \( \gamma_i \) in the horizontal direction.

The **homological dimension** of \( \omega \) with periodic horizontal foliation is the dimension of the (isotropic) subspace of \( H_1(M, \mathbb{R}) \) generated by the waist curves \( \gamma_i \) of the horizontal maximal cylinders \( C_i \) decomposing \( (M, \omega) \).

We say that \( \omega \) has periodic Lagrangian horizontal foliation whenever its homological dimension is maximal (i.e., \( g \)).

In general, it is not hard to find Abelian differentials with periodic horizontal foliation: for instance, any **square-tiled surface** (see Example 15) verifies this property and the class of square-tiled surfaces\(^{19}\) is *dense* on \( \mathcal{C} \).

Next, we claim that:

**Lemma 56.** \( \mathcal{C} \) contains Abelian differentials with periodic Lagrangian horizontal foliation.

**Proof.** Of course, the lemma follows once we can show that given \( \omega \in \mathcal{C} \) with homological dimension \( k < g \), one can produce an Abelian differential \( \tilde{\omega} \) with homological dimension \( k+1 \). In this direction, given such an \( \omega \), we can select a closed curve \( \gamma \) disjoint from (i.e., zero algebraic intersection with) the waist curves \( \gamma_i \) of horizontal maximal cylinders \( C_i \) of \( (M, \omega) \) and \( \gamma \neq 0 \) in \( H_1(M, \mathbb{R}) \).

Then, let’s denote by \([df] \in H^1(M, \mathbb{Z})\) the Poincaré dual of \( \gamma \) given by taking a small tubular neighborhoods \( V \subset U \) of \( \gamma \) and taking a smooth function \( f \) on \( M - \gamma \) such that

\[
    f(p) = \begin{cases} 
        1 & \text{for } x \in V_- \\
        0 & \text{for } x \in M - U_-
    \end{cases}
\]

where \( U^\pm \) (resp. \( V^\pm \)) is the connected component of \( U - \gamma \) (resp. \( V - \gamma \)) to the right/left of \( \gamma \) with respect to its orientation of \( \gamma \) (see the figure below)

\(^{19}\)As square-tiled surfaces \( (M, \omega) \) are characterized by the rationality of their periods (i.e., \( \int_\gamma \omega \in \mathbb{Q} \oplus i\mathbb{Q} \) for any \( \gamma \in H_1(M, \Sigma, \mathbb{Z}) \)). See [38] for more details.
and let

\[ [df] := \begin{cases} 
  df & \text{on } U - \gamma \\
  0 & \text{on } (M - U) \cup \gamma
\end{cases} \]

In this setting, since the waist curves \( \gamma_i \) of maximal cylinders of \( C_i \) of \( \omega \) generate a \( k \)-dimensional isotropic subspace \( I_k \subset H_1(M, \mathbb{R}) \) (as \( \omega \) has homological dimension \( k \)) and \( \gamma \) is disjoint from \( \gamma_i \)'s, it is possible to check (see the proof of Lemma 4.4 of [27]) that the Abelian differential \( \bar{\omega} = \omega + r[df] \) has homological dimension \( k + 1 \) whenever \( r \in \mathbb{Q} - \{0\} \) is sufficiently small.

This completes the proof of the lemma. \( \Box \)

Now, let’s fix \( \omega \in \mathcal{C} \) with periodic Lagrangian horizontal foliation and let’s try to use \( \omega \) to reach some nice boundary point \( \omega_{\infty} \) on the Deligne-Mumford compactification of \( \mathcal{C} \) (whatever this means...). Intuitively, we note that horizontal maximal cylinders \( C_i \) of \( \omega \) and their waist curves \( \gamma_i \) looks like this

\[ \gamma_3 \]
\[ \gamma_2 \]
\[ \gamma_1 \]

In particular, by applying Teichmüller flow \( g_t = \text{diag}(e^t, e^{-t}) \) and letting \( t \to -\infty \), we start to pinching off the waist curves \( \gamma_i \). As it was observed by H. Masur (see Section 4 of [27] and references
therein), by an appropriate scaling process on $\omega_t = g_t(\omega)$, one can makes sense of a limiting object $\omega_\infty$ in the Deligne-Mumford compactification of $C$ looking like this:

Roughly speaking, this picture is intended to say that $\omega_\infty$ lives in a stable curve $M_\infty$, i.e., a Riemann surface with nodes at the punctures $p_i$ obtained after pinching $\gamma_i$’s off, and it is a meromorphic quadratic differential with double poles (and strictly positive residues) at the punctures and the same zeroes of $\omega_t$.

If $\omega$ has homological dimension $g$, it is possible to check that $\omega_\infty$ lives in a sphere with $2g$ paired punctures and $\omega$ has strictly positive residues on each of them. In this situation, certain variational formulas of J. Fay and A. Yamada allowing to show that as $\omega_t$ approaches $\omega_\infty$, one has

$$B_{\omega_t}(c_i^t, c_j^t) \to -\delta_{ij}$$

whenever $\{c_1^t, \ldots, c_g^t\}$ is a Hodge-orthonormal basis of the dual of the ($g$-dimensional) subspace of $H_1(M, \mathbb{R})$ generated by the waist curves $\gamma_i$’s of $\omega$. In other words, up to orthogonal matrices, the matrix of the form $B_{\omega_t}$ approaches $-\text{Id}_{g\times g}$ as $t \to -\infty$. Hence, $\text{rank}(B_{\omega_t}^\mathbb{R}) := 2 \cdot \text{rank}(B_{\omega_t}) = 2g$ as $t \to -\infty$, and, a fortiori, the rank of $B_{\omega_t}^\mathbb{R}|_{H_1^{(0)}(M, \mathbb{R})}$ is $2g - 2$ as $t \to -\infty$. Thus, this completes the sketch of proof of Theorem 54.

**Remark 57.** Actually, the fact that $B_{\omega_t}$ “approaches” $-\text{Id}_{g\times g}$ can be used to show that

$$\sup_{\omega \in C} \Lambda_i(\omega) = 1 \quad \text{for all } 1 \leq i \leq g.$$

In a nutshell, the previous discussion around Theorem 54 can be resumed as follows: firstly, we searched (in $C$) some $\omega$ with periodic Lagrangian horizontal foliation; then, by using the Teichmüller flow orbit $\omega_t = g_t(\omega)$ of $\omega$ and by letting $t \to -\infty$, we spotted an open region $U$ of $C$ (near a certain “boundary” point $\omega_\infty$) where the form $B$ becomes an “almost” diagonal matrix with non-vanishing diagonal terms, so that the rank of $B$ is maximal. Here, we “insist” that the inspiration for spotting $U$ near the boundary of $C$ comes from the fact that $B$ is a sort of derivative of the so-called period matrix $\Pi$, and one knows since the works of J. Fay and A. Yamada that the period matrix $\Pi$ (and therefore $B$) has nice asymptotic expansions near the boundary of $C$. The following picture is a résumé of the discussion of this paragraph:

---

20Here, it is implicit the fundamental fact that $B$ can be interpreted as the “derivative of the period matrix”. See Section 4 of [27] for more comments.
Obviously, the proof of the main result of this subsection (namely Theorem 42) is now complete in view of Theorem 54 and Corollary 47.

Remark 58. These arguments (concerning exclusively Masur-Veech measures) were extended by G. Forni [29] to give the following far-reaching criterion for the non-uniform hyperbolicity of KZ cocycle with respect to a $SL(2,\mathbb{R})$-invariant $g_t$-ergodic probabilities $\mu$ (satisfying a certain local product structure property): if one can find $\omega$ in the support of $\mu$ with periodic Lagrangian horizontal foliation (i.e., there is some $\omega \in \text{supp}(\mu)$ with homological dimension $g$), then $\lambda^\mu_g > 0$.

5.3. Weak mixing property for i.e.t.’s and translation flows. The plan for this subsection is to vaguely sketch how the knowledge of the positivity of the second Lyapunov exponent $\lambda^\mu_C$ of KZ cocycle with respect to Masur-Veech measures $\mu_C$ on connected components $C$ of strata was used by A. Avila and G. Forni [3] to show weak mixing property for i.e.t.’s and translation flows. The basic references for the subsection are the original article [3] and the survey [30].

Recall that a dynamical system $T : X \to X$ preserving a probability $\mu$ is weak mixing whenever
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0
\]
for any measurable subsets $A,B \subset X$. Equivalently, $(T,\mu)$ is weak mixing if given measurable subsets $A,B \subset X$, there exists a subset $E \subset \mathbb{N}$ of density one\footnote{I.e., $\lim_{N \to \infty} \frac{1}{N} \cdot \#(E \cap \{1, \ldots, N\}) = 1.$} such that
\[
\lim_{n \to \infty} \frac{1}{|E|} \sum_{n \in E} \mu(T^{-n}(A) \cap B) = \mu(A) \cdot \mu(B)
\]

For the case of i.e.t.’s and translation flows, it is particularly interesting to consider the following spectral characterization of weak mixing.
An i.e.t. \( T : D_T \to D_{T-1} \) is weak mixing if for any \( t \in \mathbb{R} \) there is no non-constant measurable function \( f : D_T \to \mathbb{C} \) such that

\[
f(T(x)) = e^{2\pi it} f(x)
\]

for every \( x \in D_T \).

Similarly, a (vertical) translation flow \( \phi^\omega_s \) on a translation surface \((M,\omega)\) represented by the suspension of an i.e.t. \( T : D_T \to D_{T-1} \), say \( D_T = \bigcup_{\alpha \in \Lambda} I_\alpha \) with a (piecewise constant) roof function \( h(x) = h_\alpha \) for \( x \in I_\alpha \) (see the picture below)

is weak mixing if for any \( t \in \mathbb{R} \) there is no non-constant measurable function \( f : D_T \to \mathbb{C} \) such that

\[
f(T(x)) = e^{2\pi ith_\alpha} f(x)
\]

for every \( x \in I_\alpha, \alpha \in \Lambda \).

This spectral characterization of weak mixing allowed W. Veech to setup a criterion of weak mixing for i.e.t.'s and translation flows: roughly speaking, in the case of translation flows \( \phi^\omega_s \), it says that if \( \phi^\omega_s \) is not weak mixing, say the equation

\[
f(T(x)) = e^{2\pi ith_\alpha} f(x)
\]

has a non-constant measurable solution \( f \) for some \( t \in \mathbb{R} \), then, by considering the times when the Teichmüller orbit of the translation surface comes back near itself, i.e., the times \( t_n \) such that the \( g_{t_n}(\omega) \) is close \( \omega \), the KZ cocycle \( G^{KZ}_{t_n}(\omega) \) sends \( t \cdot h \), where \( h := (h_\alpha)_{\alpha \in \Lambda} \in \mathbb{R}^\Lambda \) is thought as an element of the (relative) homology of \((M,\omega)\), i.e., \( h \in H_1(M,\Sigma,\mathbb{R}) \), near the integer lattice \( \mathbb{Z}^\Lambda \simeq H_1(M,\Sigma,\mathbb{Z}) \) in (relative) homology:

\[
\lim_{n \to \infty} \text{dist}_{\mathbb{R}^\Lambda}(G^{KZ}_{t_n}(\omega)(t \cdot h), \mathbb{Z}^\Lambda) = 0
\]  

(5.1)

Actually, this is a very crude approximation of Veech’s criterion: the formal statement depends on the relationship between Teichmüller flow/KZ cocycle and Rauzy-Veech-Zorich algorithm, and we will not try to recall it here.
Instead, we will close this subsection by saying that the idea to deduce weak mixing for “almost every” i.e.t.’s and translation flows is to carefully analyze the KZ cocycle in order to prove that $G^t_{KZ}$ “tends” to keep “typical” lines $t \cdot h \in \mathbb{R} \cdot h \subset H_1(M, \Sigma, \mathbb{R})$ in homology sufficiently “far away” from the integral lattice $H_1(M, \Sigma, \mathbb{Z})$ when the second Lyapunov exponent $\lambda^{\mu_{C}}_2$ (with respect to Masur-Veech measures $\mu_{C}$) is positive. In other words, one of the (many) key ideas in [3] is to show that Equation 5.1 can be contradicted for “almost every” i.e.t.’s and translation flows when $\lambda^{\mu_{C}}_2 > 0$, so that Veech’s criterion implies weak mixing property for “almost every” i.e.t.’s and translation flows.

6. Veech’s question

As we saw in the previous chapter, after the works of G. Forni [27], and A. Avila and M. Viana [7] on the Kontsevich-Zorich conjecture, the Lyapunov exponents of the Kontsevich-Zorich cocycle with respect to Masur-Veech measures $\mu_{C}$ are “well-understood”: their multiplicities are 1 (i.e., they are simple) and 0 doesn’t belong to the Lyapunov spectrum. Moreover, a part of this result (namely, the positivity of the second top Lyapunov exponent) was used by Avila and Forni [3] to derive the weak mixing property for “typical” translation flows (and i.e.t.’s) with respect to $\mu_{C}$. In other words, one can say that we reasonably understand the Ergodic Theory of the Kontsevich-Zorich cocycle with respect to Masur-Veech measures (and its consequences to “typical” i.e.t.’s and translation flows).

On the other hand, by working with Masur-Veech measures, one rules out the possibility of using the KZ cocycle to the study of particular but physically interesting translations flows (such as the ones associated to billiards in rational polygon). Indeed, this is so because these particular translation flows are usually associated to Abelian differentials in closed SL(2, $\mathbb{R}$)-invariant sets of zero Masur-Veech measure (and hence the tools from Ergodic Theory can’t be applied directly).

Therefore, it is natural to ask how much of discussion of the previous chapter still applies to other (Teichmüller and/or SL(2, $\mathbb{R}$)) invariant measures. In this direction, after the completion of the work [27], W. Veech asked G. Forni whether the positivity of the $g$th top exponent (i.e., $\lambda^{\mu_{G}}_g > 0$) and/or the “simplicity scenario” from the Kontsevich-Zorich conjecture can be extended to arbitrary SL(2, $\mathbb{R}$)-invariant probability measures $\mu$.

The reader maybe wondering why W. Veech doesn’t also include all Teichmüller invariant probabilities in his question. As it turns out, there are at least two good reasons to do so:

- the Teichmüller flow is non-uniformly hyperbolic with respect to Masur-Veech measure, and hence it has a lot of invariant measures and a complete study of the Lyapunov spectrum of all such measures seems a very hard task. More concretely, W. Veech [68] (see also Appendix [B] below) constructed (with the aid of the so-called Rauzy-Veech diagrams) a periodic orbit of the Teichmüller flow (i.e., a pseudo-Anosov element of the mapping class group) in the stratum $\mathcal{H}(2)$ of genus 2 Abelian differentials with a single double zero such

\[\text{Cf. Example 13 of Section 2.}\]
that the Teichmüller invariant probability supported in this periodic orbit has a \textit{vanishing} second Lyapunov exponent for the KZ cocycle. This is in sharp contrast with the fact that the second Lyapunov exponent of KZ cocycle w.r.t. the Masur-Veech measure $\mu_{\mathcal{H}(2)}$ of the (connected) stratum $\mathcal{H}(2)$ is non-zero (as it follows from the work of G. Forni [27], and Avila and Viana [7]) and it shows that the description of Lyapunov spectra of KZ cocycle with respect to arbitrary Teichmüller invariant measures can be difficult \textit{even} at the level of periodic orbits (pseudo-Anosov elements).

- As we mentioned in the previous chapter, in analogy to Ratner’s work on unipotent flows, it is conjectured that any $SL(2, \mathbb{R})$-invariant probability in the moduli space of Abelian differentials is “algebraic” (and a recent progress on this conjecture was announced by A. Eskin and M. Mirzakhani [24]), and this is actually the case for all known examples of $SL(2, \mathbb{R})$-invariant measures. In particular, at least conjecturally or in known examples, $SL(2, \mathbb{R})$-invariant probabilities are much more well-behaved than general Teichmüller invariant measures (which can be supported in fractal-like objects), and thus they constitute a natural family of measures to begin the study of the possible Lyapunov spectra of KZ cocycle.

During this section, we’ll describe two examples answering Veech’s question.

6.1. \textbf{Eierlegende Wollmilchsau}. Given $x_1, \ldots, x_4 \in \mathbb{C}$ four distinct points in the Riemann sphere $\mathbb{C} = \mathbb{C} \cup \{\infty\}$, let’s consider the Riemann surface $M_3(x_1, \ldots, x_4)$ defined by (the solutions of) the algebraic equation

$$\{(x, y) : y^4 = (x - x_1) \cdots (x - x_4)\}.$$ 

This Riemann surface is a \textit{cyclic cover} of the Riemann sphere branched at 4 points in the sense that we have the covering map $p : M_3(x_1, \ldots, x_4) \to \overline{\mathbb{C}}$, $p(x, y) = x$, is ramified precisely over $x_1, \ldots, x_4$, and the automorphism $T(x, y) = (x, iy)$, $i = \sqrt{-1}$, of $M_3(x_1, \ldots, x_4)$ is a generator of the Galois group $\mathbb{Z}/4\mathbb{Z}$ (cyclic group of order 4) of the covering $p$.

\textit{Remark 59}. By Galois theory, a Riemann surface $M$ coming from a normal cover $p : M \to \overline{\mathbb{C}}$ branched at 4 points $x_1, \ldots, x_4 \in \overline{\mathbb{C}}$ and with cyclic Galois group (of deck transformations) is given by an algebraic equation of the form $y^N = (x - x_1)^{a_1} \cdots (x - x_4)^{a_4}$.

As the reader can check, $p$ is \textit{not} ramified at $\infty$, and, by Riemann-Hurwitz formula, $M_3(x_1, \ldots, x_4)$ has genus 3.

\textit{Remark 60}. Since the group of Möbius transformations (automorphisms of $\overline{\mathbb{C}}$) acts (sharply) 3-transitively on $\overline{\mathbb{C}}$, we get that $M_3(x_1, \ldots, x_4)$ is isomorphic to $M_3(0, 1, \infty, \lambda)$ where

$$\lambda = \lambda(x_1, \ldots, x_4) := \frac{(x_4 - x_1)(x_2 - x_3)}{(x_4 - x_3)(x_2 - x_1)} \in \mathbb{C} - \{0, 1, \infty\}$$

is the \textit{cross-ratio} of $x_1, \ldots, x_4$. In other words, the complex structures of the family $M_3(x_1, \ldots, x_4)$ are parametrized by a single complex parameter $\lambda \in \mathbb{C} - \{0, 1, \infty\}$.
Next, we consider $dx/y^2$ on $M_3(x_1,\ldots,x_4)$.

**Lemma 61.** $dx/y^2$ is an Abelian differential with 4 simple zeroes at $x_1,\ldots,x_4$. In particular, $(M_3(x_1,\ldots,x_4), dx/y^2) \in \mathcal{H}(1,1,1,1)$.

**Proof.** The lemma follows by studying $dx/y^2$ near the points $x_1,\ldots,x_4$ and $\infty$:

- near $x_i$, i.e., $x \sim x_i$, the natural coordinate is $y$ and one has $y^4 \sim (x - x_i)$, so that $y^3 dy \sim dx$; hence, near $x_i$, $\frac{dx}{y^2} \sim \frac{y^3}{y^2} = ydy$; i.e., $dx/y^2$ has simple zeroes at $x_i$'s.
- near $\infty$, the natural coordinate is $\zeta = 1/x$ and one has $y^4 = x^4(1-x_1/x)\ldots(1-x_4/x) \sim \zeta^{-4}$ (i.e., $y\zeta \sim 1$) and $d\zeta \sim dx/x^2 = \zeta^2 dx$, so that $dx/y^2 \sim \zeta^{-2} d\zeta/\zeta^{-2} = d\zeta$, that is, $dx/y^2$ is holomorphic and non-vanishing near $\infty$.

Thus, one has that $(M_3(x_1,\ldots,x_4), dx/y^2) \in \mathcal{H}(1,1,1,1)$.

Now, we define $\omega_{EW} = c(x_1,\ldots,x_4)dx/y^2$ on $M_3(x_1,\ldots,x_4)$ where $c(x_1,\ldots,x_4) \in \mathbb{R}$ is the unique positive real number such that the translation surface $(M_3(x_1,\ldots,x_4), \omega_{EW})$ has unit area.

**Lemma 62.** $\mathcal{E}W := \{(M_3(x_1,\ldots,x_4), \omega_{EW}) : x_1,\ldots,x_4 \in \mathbb{C} \text{ distinct} \}$ is the $SL(2,\mathbb{R})$-orbit of a square-tiled surface. In particular, $\mathcal{E}W$ is a closed $SL(2,\mathbb{R})$-invariant locus of $\mathcal{H}(1)(1,1,1,1)$.

**Proof.** Let's show that $\mathcal{E}W \subset \mathcal{H}(1)(1,1,1,1)$ is a closed $SL(2,\mathbb{R})$-invariant locus.

Note that $\omega_{EW}$ is anti-invariant with respect to the action $T^*$ (by pull-back) of the automorphism $T(x,y) = (x, iy)$ of $M_3(x_1,\ldots,x_4)$, i.e., $T^*(\omega_{EW}) = -\omega_{EW}$. In fact,

$$T^*(\omega_{EW}) := c(x_1,\ldots,x_4)dx/(iy)^2 = -c(x_1,\ldots,x_4)dx/y^2 = -\omega_{EW}.$$ 

Therefore, the quadratic differential $q_{EW} = \omega_{EW}^2$ is $T^*$-invariant. Since $T$ generates the Galois group of deck transformations of $p$, this means that $q$ projects under $p$ to a quadratic differential $q_0$ on $\mathbb{C}$ with 4 simple poles.

One can see this directly: since $q_{EW} = \omega_{EW}^2 = c(x_1,\ldots,x_4)^2dx^2/y^4$, it projects to $q_0 = q_0(x_1,\ldots,x_4) := c(x_1,\ldots,x_4)^2dx^2/(x-x_1)\ldots(x-x_4)$ under $p(x,y) = x$.

Therefore, the elements of $\mathcal{E}W$ are obtained by appropriate cyclic covers of elements

$$(\mathbb{C}, q_0(x_1,\ldots,x_4)) \in \mathcal{Q}(-1,-1,-1,-1)$$

Actually, since $\omega_{EW}$ has unit area and $p$ has degree 4, one has that $q_0$ has area $1/8$, i.e.,

$$(\mathbb{C}, q_0(x_1,\ldots,x_4)) \in \mathcal{Q}^{(1/8)}(-1,-1,-1,-1) \simeq \mathcal{Q}^{(1)}(-1,-1,-1,-1)$$

In other words, the locus $\mathcal{E}W$ is a copy of $\mathcal{Q}^{(1)}(-1,-1,-1,-1)$ inside $\mathcal{H}^{(1)}(1,1,1,1)$.

---

23Note that this is coherent with Riemann-Hurwitz formula $2g - 2 = \sum k_i$ where $g$ is the genus of $M$ and $k_i$ are the orders of zeroes of an Abelian differential $\omega$ on $M$.

24Cf. Example 15 of Section 2 for the definition of square-tiled surface

25Essentially this is the fact that $M_3(x_1,\ldots,x_4)$ is given by the equation $y^4 = (x-x_1)\ldots(x-x_4)$. 


Since any (ramified) cover is defined by pre-composition with charts and $SL(2, \mathbb{R})$ acts by post-composition with charts, the operations of taking covers and letting $SL(2, \mathbb{R})$ act commute. Hence, it follows that $\mathcal{E}W$ is a closed $SL(2, \mathbb{R})$-invariant locus of $H^{(1)}(1,1,1,1)$ simply because the elements of $\mathcal{E}W$ are obtained by appropriate cyclic covers of elements of the closed $SL(2, \mathbb{R})$-invariant locus $Q^{(1)}(-1, -1, -1, -1)$.

Finally, $\mathcal{E}W$ is the $SL(2, \mathbb{R})$-orbit of a square-tiled surface because $h(x, y) = (x, y^2)$ is a covering map from $M_3(x_1, \ldots, x_4)$ to the elliptic curve (genus 1 Riemann surface)

$$E(x_1, \ldots, x_4) = \{ w^2 = (x - x_1) \ldots (x - x_4) \}$$

such that $h_*(\omega_{EW}) = c(x_1, \ldots, x_4)dx/w$, and the locus $\{ E(x_1, \ldots, x_4), c(x_1, \ldots, x_4)dx/w \}$ is precisely the moduli space $H^{(1/4)}(\emptyset)$ of genus 1 Abelian differentials with area 1/4 and we know that $H^{(1/4)}(\emptyset) \simeq SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ is (isomorphic to) the $SL(2, \mathbb{R})$-orbit of the square-tiled surface $(\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}), (1/4)dx)$.

More concretely, by taking $(x_1, x_2, x_3, x_4) = (-1, 0, 1, \infty)$, since the following integrals

$$\int_{-\infty}^{-1} \frac{dx}{\sqrt{x^3 - x}} = \int_{-1}^{0} \frac{dx}{\sqrt{x^3 - x}} = \int_{0}^{1} \frac{dx}{\sqrt{x^3 - x}} = \int_{1}^{\infty} \frac{dx}{\sqrt{x^3 - x}} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}$$

representing several periods coincide, one has that $(E(-1, 0, 1, \infty), dx/w)$ is isomorphic (up to isogeny, i.e., scaling factor) to $(\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}), dz)$, so that $(M_3(-1, 0, 1, \infty), \omega_{EW})$ is a square-tiled surface and its $SL(2, \mathbb{R})$-orbit is contained in $\mathcal{E}W$.

Because $\mathcal{E}W$ is a closed connected locus of real dimension 3 (as it is a copy of the stratum $H^{(1)}(\emptyset) \simeq SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ and $\dim(SL(2, \mathbb{R})) = 3$), one gets that $\mathcal{E}W$ must coincide with the $SL(2, \mathbb{R})$-orbit of the square-tiled surface $(M_3(-1, 0, 1, \infty), \omega_{EW})$.

The $SL(2, \mathbb{R})$-orbits of square-tiled surfaces $(M, \omega)$ are “well-behaved” objects in moduli spaces of Abelian differentials. For instance, since a square-tiled surface $(M, \omega)$ is naturally a finite cover of the flat torus $T^3 = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}), dz)$ ramified only at $0 \in T^2$, one can see that the stabilizer $SL(M, \omega)$ of $(M, \omega)$ in $SL(2, \mathbb{R})$ is a finite-index subgroup of $SL(2, \mathbb{Z})$ (when the periods of $(M, \omega)$ generate the lattice $\mathbb{Z} \oplus i\mathbb{Z}$). Because $SL(2, \mathbb{Z})$ is a lattice of $SL(2, \mathbb{R})$, we have that $SL(2, \mathbb{R}) \cdot (M, \omega) \simeq SL(2, \mathbb{R})/SL(M, \omega)$ supports an unique $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probability $\mu$. In particular, it makes sense to talk about Lyapunov exponents of square-tiled surfaces $(M, \omega)$: they’re the Lyapunov exponents of the KZ cocycle with respect to the unique $SL(2, \mathbb{R})$-invariant measure $\mu$ supported on $SL(2, \mathbb{R}) \cdot (M, \omega)$.

By combining this discussion with Lemma 6.2 we have that $\mathcal{E}W$ supports an unique $SL(2, \mathbb{R})$-invariant probability $\mu_{\mathcal{E}W}$. 

---

20In particular, the translation atlas of $(M_3, \omega_{EW})$ is obtained by pre-composing (half) translation charts of $(\mathbb{C}, q_0)$ with the covering map $p$.

21This is true because any stratum of quadratic/Abelian differentials is closed and $SL(2, \mathbb{R})$-invariant.


23We will see a concrete model of this translation surface in next section.
Theorem 63 (Forni [28]). The Lyapunov spectrum of KZ cocycle with respect to $\mu_{EW}$ is totally degenerate in the sense that

$$\lambda_2^{\mu_{EW}} = \lambda_3^{\mu_{EW}} = 0$$

Proof. As the reader can check, the set

$$\{ \theta_1 = \omega_{EW} = c \cdot \frac{dx}{y^2}, \quad \theta_2 := \frac{dx}{y^2}, \quad \theta_3 := \frac{x \, dx}{y^3} \}$$

is a basis of the space $H^{1,0}(M_3)$ of holomorphic 1-forms on $M_3 = M_3(x_1, \ldots, x_4)$. Note that it diagonalizes the cohomological action $T^*$ of the automorphism $T(x, y) = (x, iy)$: indeed,

$$T^*(\omega_{EW}) = -\omega_{EW}, \quad T^*(\theta_2) = i\theta_2, \quad T^*(\theta_3) = i\theta_3.$$ 

Let’s denote by $\lambda(n)$ the $T^*$-eigenvalue of $\theta_n$, i.e., $T^*(\theta_n) = \lambda(n)\theta_n$.

We can compute the (symmetric, complex-valued) form $B_{\omega_{EW}}$ on $H^{1,0}$ in this basis as follows. Firstly, we recall that $B_{\omega_{EW}}(\omega_{EW}, \omega_{EW}) = 1$. Secondly, by using the automorphism $T$ to perform a change of variables, we get that

$$B_{\omega_{EW}}(\theta_n, \theta_m) := \frac{i}{2} \int \frac{\theta_n \theta_m}{\theta_1} \overline{\theta_1} = \frac{i}{2} \int \frac{T^*(\theta_n)T^*(\theta_m)}{T^*(\theta_1)} \overline{T^*(\theta_1)} = \lambda(n)\lambda(m) \frac{i}{2} \int \frac{\theta_n \theta_m}{\theta_1} \overline{\theta_1} = \lambda(n)\lambda(m) B_{\omega_{EW}}(\theta_n, \theta_m)$$

In particular, $B_{\omega_{EW}}(\theta_n, \theta_n) = 0$ whenever $\lambda(n)\lambda(m) \neq 1$. Since $\lambda(n)\lambda(m) \neq 1$ for $(n, m) \neq (1, 1)$, we obtain that the matrix of $B$ in the basis $\{\theta_1, \theta_2, \theta_3\}$ is

$$B_{\omega_{EW}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $B_{\omega_{EW}}$ has rank 1 and, by Theorem 44 we conclude that the $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probability $\mu_{EW}$ has Lyapunov exponents $\lambda_2^{\mu_{EW}} = \lambda_3^{\mu_{EW}} = 0$. \qed

Evidently, this theorem (of G. Forni in 2006) answers Veech’s question in a definitive way: there is no non-uniform hyperbolicity and/or simplicity statement for KZ cocycle with respect to general $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probabilities!

In the literature, the square-tiled surface $(M_3(-1, 0, 1, \infty), \omega_{EW})$ in the support of $\mu_{EW}$ was coined Eierlegende Wollmilchsau by F. Herrlich, M. Möller, G. Schmithüsen [39] because it has marvelous algebro-geometrical properties (see [39]) in addition to its totally degenerate Lyapunov spectrum. In fact, the German term Eierlegende Wollmilchsau literally is “egg-laying wool-milk-sow” in English and it means “a tool for several purposes” (after “Wiktionary”). The picture below (found on internet) resumes the meaning of this German expression:

Remark 64. In fact, by analyzing the argument above, one realizes that it was shown that $B_{\omega_{EW}}^g$ vanishes on $H^1_{(0)}(M_3, \mathbb{R})$. In particular, by combining this with Theorem 29 one concludes that KZ cocycle $G_t^{KZ}$ acts by isometries (of the Hodge norm) on $H^1_{(0)}(M_3, \mathbb{R})$. Notice that this is stronger...
than simply saying that Lyapunov exponents vanish: indeed, zero Lyapunov exponents in general allow to conclude only subexponential growth (at most) of norms of vectors, while in this example we observe no growth at all! In particular, this illustrates the usefulness of variational formulas (such as Theorem 29) for the Hodge norm in the particular context of KZ cocycle.

6.2. Ornithorynque. Let’s now construct another $SL(2,\mathbb{R})$-invariant $g_t$-invariant probability $\mu_O$ with totally degenerate Lyapunov spectrum in the sense that KZ cocycle has vanishing second Lyapunov exponent with respect to $\mu_O$.

Consider the Riemann surface $M_4(x_1,\ldots,x_4)$ determined by the desingularization of the algebraic equation

$$y^6 = (x - x_1)^3(x - x_2)(x - x_3)(x - x_4)$$

where $x_1,\ldots,x_4 \in \overline{\mathbb{C}}$ are four distinct points.

The map $p : M_4(x_1,\ldots,x_4) \to \overline{\mathbb{C}}$, $p(x,y) = x$, is a covering branched (precisely) at $x_1,\ldots,x_4$, and the Galois group of its deck transformations is generated by the automorphism $T(x,y) = (x,\varepsilon_6 y)$, $\varepsilon_6 = \exp(2\pi i/6)$ (and hence it is isomorphic to $\mathbb{Z}/6\mathbb{Z}$).

In resume, $M_4(x_1,\ldots,x_4)$ is (also) a cyclic cover of the Riemann sphere branched at 4 points. By Riemann-Hurwitz formula applied to $p$, the reader can check that $M_4(x_1,\ldots,x_4)$ has genus 4.

By reasoning similarly to the proofs of Lemmas 61 and 62 one can show that

**Lemma 65.** $(x - x_1)dx/y^3$ is an Abelian differential with 3 double zeroes at $x_2,x_3,x_4$, i.e., $(M_4(x_1,\ldots,x_4),(x - x_1)dx/y^3) \in \mathcal{H}(2,2,2)$.

Moreover, by letting $\omega_O = c(x_1,\ldots,x_4)(x - x_1)dx/y^3$ where $c(x_1,\ldots,x_4) \in \mathbb{R}$ is the unique positive real number such that the translation surface $(M_4(x_1,\ldots,x_4),\omega_O)$ has unit area, one has that $O := \{(M_4(x_1,\ldots,x_4),\omega_O) : x_1,\ldots,x_4 \in \overline{\mathbb{C}} \text{ distinct}\}$ is the $SL(2,\mathbb{R})$-orbit of a square-tiled surface. In particular, $O$ is a closed $SL(2,\mathbb{R})$-invariant locus of $\mathcal{H}^{(1)}(2,2,2)$. 

Remark 66. By the classification of connected components of strata of Kontsevich and Zorich [45], the stratum $\mathcal{H}(2,2,2)$ is not connected. Indeed, it has 2 connected components distinguished by the so-called parity of the spin structure. In [56], it is shown that $\mathcal{O}$ is contained in the even spin connected component of $\mathcal{H}(2,2,2)$. See these articles for more details. On the other hand, notice that this issue wasn’t raised in the case of $\mathcal{E}W$ because the stratum $\mathcal{H}(1,1,1,1)$ is connected.

By Lemma 65, we have an unique $SL(2,\mathbb{R})$-invariant probability $\mu_{\mathcal{O}}$ supported on $\mathcal{O}$.

**Theorem 67.** The Lyapunov spectrum of KZ cocycle with respect to $\mu_{\mathcal{O}}$ is (also) totally degenerate in the sense that

$$\lambda_2^{\mu_{\mathcal{O}}} = \lambda_3^{\mu_{\mathcal{O}}} = \lambda_4^{\mu_{\mathcal{O}}} = 0$$

Furthermore, KZ cocycle acts isometrically (with respect to the Hodge norm) on $H^1_{(0)}(M_4,\mathbb{R})$.

**Proof.** The argument is similar to the proof of Theorem 63. One starts by noticing that the set

$$\left\{ \theta_1 = \omega_{\mathcal{O}} = c \cdot \frac{(x-x_1)dx}{y^3}, \quad \theta_2 := \frac{(x-x_1)dx}{y^4}, \quad \theta_3 := \frac{(x-x_1)^2dx}{y^3}, \quad \theta_4 := \frac{(x-x_1)^3dx}{y^5} \right\}$$

is a basis of the space $H^{1,0}(M_4)$ of holomorphic 1-forms on $M_4 = M_3(x_1,\ldots,x_4)$ diagonalizing the cohomological action $T^*$ of the automorphism $T(x,y) = (x,\varepsilon_6y)$:

$$T^*(\omega_{\mathcal{O}}) = -\omega_{\mathcal{O}}, \quad T^*(\theta_2) = \varepsilon_6^2\theta_2, \quad T^*(\theta_3) = \varepsilon_6\theta_3, \quad T^*(\theta_4) = \varepsilon_6\theta_4$$

Again, let’s denote by $\lambda(n)$ the $T^*$-eigenvalue of $\theta_n$, i.e., $T^*(\theta_n) = \lambda(n)\theta_n$, and let’s use the automorphism $T$ to perform a change of variables to compute $B_{\omega_{\mathcal{O}}}$:

$$B_{\omega_{\mathcal{O}}} = \frac{i}{2} \int \frac{\theta_n \theta_m}{\theta_1} \overline{\theta_1} = \frac{i}{2} \int \frac{T^*(\theta_n) T^*(\theta_m)}{T^*(\theta_1)} T^* \overline{\theta_1} = \lambda(n)\lambda(m) \frac{i}{2} \int \frac{\theta_n \theta_m}{\theta_1} \overline{\theta_1} = \lambda(n)\lambda(m) B_{\omega_{\mathcal{O}}}$$

As before, $B_{\omega_{\mathcal{O}}} = 0$ whenever $\lambda(n)\lambda(m) \neq 1$, and $\lambda(n)\lambda(m) \neq 1$ for $(n,m) \neq (1,1)$. Since $B_{\omega_{\mathcal{O}}} = 1$, we conclude that the matrix of $B$ in the basis $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ is

$$B_{\omega_{\mathcal{O}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $B_{\omega_{\mathcal{O}}}$ has rank 1 and, by Theorem 44, one has $\lambda_2^{\mu_{\mathcal{O}}} = \lambda_3^{\mu_{\mathcal{O}}} = \lambda_4^{\mu_{\mathcal{O}}} = 0$. Finally, the last statement of the theorem follows from Theorem 29 because $B_{\omega_{\mathcal{O}}}^R$ vanishes on $H^1_{(0)}(M,\mathbb{R})$. $\square$

In resume, $\mu_{\mathcal{O}}$ is another example answering (negatively) Veech’s question. This example was announced in [31] and it appeared later in [32]. After a suggestion of Vincent Delecroix and Barak Weiss, the square-tiled surface $(M_4(-1,0,1,\infty),\omega_{\mathcal{O}}) \in \text{supp}(\mu_{\mathcal{O}})$ was coined *Ornithorynque* (Platypus in French):
In fact, as we’ll see later (in Section 8), *Eierlegende Wollmilchsau* and *Ornithorynque* are members of a class of examples called *square-tiled cyclic covers*, but let’s not insist on this for now. Instead, let’s ask:

*What about other examples of $\text{SL}(2, \mathbb{R})$-invariant $g_t$-ergodic probabilities with totally degenerate Lyapunov spectrum?*

In the rest of this section, we’ll see an *almost* complete answer to this question.

6.3. M. Möller’s work on Shimura and Teichmüller curves. In the sequel, we will need the following notion:

**Definition 68.** A *Teichmüller curve* in $\mathcal{H}_g$ is a closed $\text{SL}(2, \mathbb{R})$-orbit.

By a theorem of J. Smillie [67], a $\text{SL}(2, \mathbb{R})$-orbit $\text{SL}(2, \mathbb{R}) \cdot (M, \omega)$ is closed if and only if $(M, \omega)$ is a *Veech surface*:

**Definition 69.** The *Veech group* $\text{SL}(M, \omega)$ of a translation surface $(M, \omega) \in \mathcal{H}_g$ is the stabilizer of $(M, \omega)$ with respect to the natural action of $\text{SL}(2, \mathbb{R})$ on $\mathcal{H}_g$.

In this language, a *Veech surface* is a translation surface $(M, \omega)$ whose Veech group $\text{SL}(M, \omega)$ is a *lattice* in $\text{SL}(2, \mathbb{R})$, i.e., $\text{SL}(M, \omega)$ has finite covolume in $\text{SL}(2, \mathbb{R})$.

For example, as we already mentioned, square-tiled surfaces are Veech surfaces because their Veech groups are finite-index subgroups of $\text{SL}(2, \mathbb{Z})$.

The motivation for the nomenclature *Teichmüller curve* comes from the following facts:

- let $\pi : \mathcal{H}_g \to \mathcal{M}_g$, $\pi(M, \omega) = M$ be the natural projection from the moduli space of Abelian differentials $\mathcal{H}_g$ to the moduli space of curves $\mathcal{M}_g$; then, the image of closed $\text{SL}(2, \mathbb{R})$-orbits under $\pi$ are *complex geodesics* of $\mathcal{M}_g$, i.e., algebraic curves (Riemann surfaces) immersed in $\mathcal{M}_g$ in an *isometric* way with respect to Teichmüller metric (cf. Section 1):
• Conversely, all totally geodesic algebraic curves in $\mathcal{M}_g$ are projections of closed $SL(2, \mathbb{R})$-orbits in $\mathcal{H}_g$.

The characterization of closed $SL(2, \mathbb{R})$-orbits via Veech surfaces shows that they support $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probabilities: indeed, since $SL(2, \mathbb{R}) \cdot (M, \omega) \simeq SL(2, \mathbb{R})/SL(M, \omega)$, one has that $SL(2, \mathbb{R}) \cdot (M, \omega)$ carries an unique $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probability when $(M, \omega)$ is a Veech surface. In particular, it makes sense to talk about Lyapunov exponents of Veech surfaces: this means simply the Lyapunov exponents of KZ cocycle with respect to the $SL(2, \mathbb{R})$-invariant probability supported in the corresponding closed $SL(2, \mathbb{R})$-orbit.

In the work [59], M. Möller studied the question of classifying Shimura-Teichmüller curves, i.e., Veech surfaces with totally degenerate Lyapunov spectrum. The name Shimura-Teichmüller curve is motivated by the fact that Teichmüller curves (Veech surfaces) with totally degenerate Lyapunov spectrum have the following algebro-geometrical characterization. Let $\tilde{\pi} : \mathcal{H}_g \rightarrow \mathcal{A}_g$ the natural map obtained by composition of $\pi : \mathcal{H}_g \rightarrow \mathcal{M}_g$ with the natural (Jacobian) inclusion $\mathcal{M}_g \rightarrow \mathcal{A}_g$ of $\mathcal{M}_g$ into the moduli space of principally polarized Abelian varieties of dimension $g$. Then, a Teichmüller curve $SL(2, \mathbb{R}) \cdot (M, \omega)$ has totally degenerate Lyapunov spectrum if and only if $\tilde{\pi}(SL(2, \mathbb{R}) \cdot (M, \omega))$ is isometric with respect to the Hodge norm or equivalently, the family of Jacobians $\tilde{\pi}(SL(2, \mathbb{R}) \cdot (M, \omega))$ has a fixed part of (maximal) dimension $g - 1$. See the original article [59] for more comments and references.

In this setting, M. Möller [59] proved that:

**Theorem 70** (M. Möller). There are no Shimura-Teichmüller curves in genera $g = 2$ and $g \geq 6$, while in genera $g = 3$ and $g = 4$ (resp.), the sole Shimura-Teichmüller curves are Eierlegende Wollmilchsau and Ornithorynque (resp.).

In other words, this theorem says that essentially we know all Shimura-Teichmüller curves: there are no other Shimura-Teichmüller curves besides Eierlegende Wollmilchsau and Ornithorynque except possibly for some new examples of genus $g = 5$.

Actually, M. Möller [59] showed that any candidate for Shimura-Teichmüller curve in genus $g = 5$ must satisfy several constraints (e.g., they must belong to specific strata, etc.). In particular, he conjectures that there are no Shimura-Teichmüller curves in genus 5.

In resume, we have a fairly satisfactory understanding of $SL(2, \mathbb{R})$-invariant probabilities with totally degenerate Lyapunov exponents coming from Veech surfaces. Next, let’s pass to the study of the analogous question for more general classes of $SL(2, \mathbb{R})$-invariant probabilities.

6.4. **Sums of Lyapunov exponents (after A. Eskin, M. Kontsevich & A. Zorich).** In a recent work, A. Eskin, M. Kontsevich and A. Zorich [21] proved a formula (announced 15 years ago) for the sum of Lyapunov exponents of certain $SL(2, \mathbb{R})$-invariant probabilities. In order to state their theorem, we’ll need a couple of definitions.

\[\text{This should be compared with Remark 64}\]
6.4.1. Regular affine measures on moduli spaces. A $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probability $\mu$ on a stratum $H^{(1)}(\kappa)$ is called affine whenever

- $\mathbb{R} \cdot \text{supp}(\mu) = \{ \omega \in H(\kappa) : \frac{1}{\text{area}(\omega)} \omega \in \text{supp}(\mu) \}$ is an affine suborbifold of $H(\kappa)$ in the sense that it is described by affine subspaces of relative cohomology in local period coordinates (cf. Subsection 2.3 of Section 2 for the definitions);
- the measure $\nu$ on $H(\kappa)$ given by $d\nu = da \cdot d\mu$ (where $a(\omega) = \text{area}(\omega)$ is the total area function) is equivalent to the Lebesgue measure on the affine suborbifold $\mathbb{R} \cdot \text{supp}(\mu)$ (or equivalently, the Lebesgue measure of the affine subspaces representing $\mathbb{R} \cdot \text{supp}(\mu)$ in local period coordinates).

It was recently announced\(^{31}\) by A. Eskin and M. Mirzakhani\(^{20}\) that all $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probability are affine.

In any event, one of the main goals in\(^{21}\) was the development of a formula for the sums of Lyapunov exponents of KZ cocycle with respect to affine measures. However, for technical reasons (related to a certain “integration by parts” argument), A. Eskin, M. Kontsevich and A. Zorich need a “regularity” condition. More precisely, we say that an affine $\mu$ on $H^{(1)}(\kappa)$ is regular if there exists a constant $K > 0$ such that

$$\lim_{\varepsilon \to 0} \frac{\mu(C_2(K, \varepsilon))}{\varepsilon^2} = 0.$$ 

Here, $C_2(K, \varepsilon)$ is the set of (unit area) translation surfaces $(M, \omega) \in H^{(1)}(\kappa)$ possessing two non-parallel (flat) maximal cylinders\(^{32}\) $C_1, C_2$ of widths $w(C_i) < \varepsilon$ and heights $h(C_i) > Kw(C_i)$ (i.e., $\text{moduli} \mod(C_i) = h(C_i)/w(C_i) > K$).

In plain terms, $\mu$ is regular if the probability of seeing non-parallel “very thin and high” cylinders in translation surfaces in the support of $\mu$ is “very small”.

As a matter of fact, all known examples of affine measures are regular and it is conjectured in\(^{21}\) that any affine measure is regular.

6.4.2. Siegel-Veech constants. The idea of Eskin-Kontsevich-Zorich formula is to express the sum of Lyapunov exponents of a regular affine measures $\mu$ in terms of its Siegel-Veech constant, a geometrical quantity that we discuss below.

**Definition 71.** Let $(M, \omega)$ be a translation surface. Given $L > 0$, we define

$$N_{\text{area}}(\omega, L) = \sum_{C \text{ maximal horizontal cylinder of width } w(C) < L} \frac{\text{area}(C)}{\text{area}(\omega)}.$$ 

Informally, $N_{\text{area}}(\omega, L)$ counts the fraction of the area of the translation surface $(M, \omega)$ occupied by maximal horizontal cylinders of width bounded by $L$.

---

\(^{31}\)This result can be thought of a version of Ratner’s theorem in the non-homogenous context of moduli spaces of Abelian differentials.

\(^{32}\)See Subsection 3.2 of Section 3 for the definitions and some comments.
Remark 72. Our choice of normalization of the quantity \( \frac{1}{L^2} \int N_{\text{area}}(\omega, L) d\mu(\omega) \) leading to the Siegel-Veech constant here is not the same of \([21]\). Indeed, what \([21]\) call Siegel-Veech constant is \(3c(\mu)/\pi^2\) in our notation. Of course, there is no conceptual different between these normalizations, but we prefer to take a different convention from \([21]\) because \(c(\mu)\) appears more “naturally” in the statement of Eskin-Kontsevich-Zorich formula.

Remark 73. It is not hard to see from the definition that Siegel-Veech constants \(c(\mu)\) are always positive, i.e., \(c(\mu) > 0\) for any \(SL(2, \mathbb{R})\)-invariant \(g_t\)-ergodic \(\mu\).

The Siegel-Veech constants of Masur-Veech measures were computed by A. Eskin, H. Masur and A. Zorich \([23]\) and they are intimately related to volumes \(3A_k^{(1)}(H(1)(\kappa))\) of strata calculated by A. Eskin and A. Okounkov \([24]\). However, we’ll not further discuss this interesting topic in these notes because it would lead us too far from the study of Lyapunov exponents of KZ cocycle. Instead, we’ll conclude our considerations on Siegel-Veech constants by showing the following result (of A. Eskin, M. Kontsevich and A. Zorich \([21]\)) allowing to compute Siegel-Veech constants of measures coming from square-tiled surfaces.

Let \(S_0 = (M_0, \omega_0)\) be a square-tiled surface, i.e., \(S_0\) comes from a finite covering \((M_0, \omega_0) \to \mathbb{T}^2 = (\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}), dz)\) branched only at 0. Since \(SL(2, \mathbb{Z})\) is the stabilizer of \(\mathbb{T}^2\) in \(SL(2, \mathbb{R})\) (when the periods of \((M_0, \omega_0)\) generate the lattice \(\mathbb{Z} \oplus i\mathbb{Z}\)), the \(SL(2, \mathbb{Z})\)-orbit of \((M_0, \omega_0)\) give all square-tiled surfaces in the \(SL(2, \mathbb{R})\)-orbit of \((M_0, \omega_0)\). Moreover, since the Veech group \(SL(2, \mathbb{R})\) is a finite-index subgroup of \(SL(2, \mathbb{Z})\), one has

\[
SL(2, \mathbb{Z}) \cdot (M_0, \omega_0) = \{S_0, S_1, \ldots, S_{k-1}\},
\]

where \(k = |SL(2, \mathbb{Z}) : SL(M_0, \omega_0)| = #SL(2, \mathbb{Z}) \cdot (M_0, \omega_0)\).

In this context, for each \(S_j \in SL(2, \mathbb{Z}) \cdot (M_0, \omega_0)\), we write \(S_j = \bigcup C_{ij}\) where \(C_{ij}\) are the maximal horizontal cylinders of \(S_j\), and we denote the width and height of \(C_{ij}\) by \(w_{ij}\) and \(h_{ij}\).

**Theorem 74** (Theorem 4 of \([21]\)). The Siegel-Veech constant of the \(SL(2, \mathbb{R})\)-invariant \(g_t\)-ergodic probability supported on the \(SL(2, \mathbb{R})\)-orbit of the square-tiled surface \((M_0, \omega_0)\) is

\[
\frac{1}{#SL(2, \mathbb{Z}) \cdot (M_0, \omega_0)} \sum_{S_j \in SL(2, \mathbb{Z}) \cdot (M_0, \omega_0)} \sum_{S_j = \bigcup C_{ij}} \frac{h_{ij}}{w_{ij}}
\]

For example, the picture below illustrates the computation of the \(SL(2, \mathbb{Z})\)-orbit of a \(L\)-shaped square-tiled surface \((M_0, \omega_0)\) with 3 squares (shown in the middle of the picture):

\[^{33}\text{Cf. Subsection 3.2 of Section 3 and Appendix C.}\]
Here, we’re using the fact that the group \( SL(2, \mathbb{Z}) \) is generated by the matrices
\[
S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]
and
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
so that \( SL(2, \mathbb{Z}) \)-orbits of square-tiled surfaces can be determined by successive applications of \( S \) and \( T \).

From the picture we infer that \( \# SL(2, \mathbb{Z}) \cdot (M_0, \omega_0) = 3 \) and

- \((M_0, \omega_0) = C_{10} \cup C_{20} \) where \( C_{10}, C_{20} \) are horizontal maximal cylinders with \( h(C_{10}) = h(C_{20}) = 1 \) and \( w(C_{10}) = 1, w(C_{20}) = 2 \);
- \((M_1, \omega_1) := S \cdot (M, \omega_0) = C_{11} \) where \( C_{11} \) is a horizontal cylinder of heigth 1 and width 3;
- \((M_2, \omega_2) := T \cdot (M_0, \omega_0) = C_{12} \cup C_{22} \) where \( C_{12}, C_{22} \) are horizontal maximal cylinders with \( h(C_{12}) = h(C_{22}) = 1 \) and \( w(C_{12}) = 1, w(C_{22}) = 2 \).

By plugging this into Theorem 74, we get that the Siegel-Veech constant of the \( SL(2, \mathbb{R}) \)-invariant probability supported on \( SL(2, \mathbb{R}) \cdot (M_0, \omega_0) \) is
\[
\frac{1}{3} \left\{ \frac{1}{1 + \frac{1}{2}} + \frac{1}{3} + \frac{1}{1 + \frac{1}{2}} \right\} = \frac{10}{9}
\]

6.4.3. Statement of Eskin-Kontsevich-Zorich formula and some of its consequences. At this point, we dispose of all elements to state:

**Theorem 75** (Eskin-Kontsevich-Zorich formula [21]). Let \( \mu \) be a regular affine probability measure supported on a stratum \( H^{(1)}(k_1, \ldots, k_s) \) of the moduli space of Abelian differentials of genus \( g \geq 2 \). Then, the sum of the top \( g \) Lyapunov exponents of KZ cocycle with respect to \( \mu \) is
\[
\lambda_1^\mu + \cdots + \lambda_g^\mu = \frac{1}{12} \sum_{j=1}^s \frac{k_j(k_j + 2)}{(k_j + 1)} + c(\mu)
\]
The proof of this fundamental theorem is long and sophisticated, and hence a complete discussion is out of the scope of these notes. Instead, we offer only a very rough idea on how the argument goes on. Firstly, one uses the formula (Theorem 44) of M. Kontsevich and G. Forni for sums of Lyapunov exponents to think of \( \lambda^1 + \cdots + \lambda^g \) as a certain integral over the stratum \( \mathcal{H}^{(1)}(k_1, \ldots, k_s) \). Then, by studying the integral, one can apply an integration by parts (“Stokes”) argument to express it as a main term and a boundary term. At this point, the so-called Riemann-Roch-Hirzebruch-Grothendieck theorem allows to compute the main term and the outcome (depending only on the stratum) is precisely

\[
\frac{1}{12} \sum_{j=1}^{s} k_j(k_j + 2) \frac{1}{(k_j + 1)}
\]

If the (strata of) moduli spaces of Abelian differentials were compact, there would be no contribution from the boundary term and the deduction of the formula would be complete. Of course, \( \mathcal{H}^{(1)}(k_1, \ldots, k_s) \) is never compact, and the contribution of the boundary term is not negligible. Here, the study of the geometry of translation surfaces near the boundary of the moduli spaces (and the regularity assumption on \( \mu \)) plays a role to show that the boundary term is given by the Siegel-Veech constant \( c(\mu) \) and this completes this crude sketch of the arguments in [21].

Coming back to the question of studying \( SL(2, \mathbb{R}) \)-invariant measures with totally degenerate Lyapunov spectrum, let’s now apply Eskin-Kontsevich-Zorich formula to rule out the existence of regular affine measures with totally degenerate spectrum in high genus:

**Proposition 76** (Corollary 5 of [21]). Let \( \mu \) be a regular affine probability measure on a stratum \( \mathcal{H}^{(1)}(k_1, \ldots, k_s) \) of Abelian differentials of genus \( g \geq 7 \). Then,

\[
\lambda_2^\mu > 0
\]

(and, actually, \( \lambda_{(g-1)g/(6g-3)+1}^\mu > 0 \)).

**Proof.** Since \( \lambda_1^\mu = 1 \), it suffices to show that the right-hand side of Eskin-Kontsevich-Zorich formula is \( > 1 \) to get that \( \lambda_2^\mu > 0 \), and this follows from the computation

\[
\lambda_1^\mu + \cdots + \lambda_g^\mu = \frac{1}{12} \sum_{j=1}^{s} k_j(k_j + 2) \frac{1}{(k_j + 1)} + c(\mu) \geq \frac{1}{12} \sum_{j=1}^{s} k_j(k_j + 2) \frac{1}{(k_j + 1)} > \frac{1}{12} \sum_{j=1}^{s} k_j = \frac{2g - 2}{12} \geq 1
\]

based on the non-negativity of the Siegel-Veech constant \( c(\mu) \) and the assumption \( g \geq 7 \). \( \square \)

At this stage, we can resume this section as follows. The Eierlegende Wollmilchsau and Ornithorhyncque are two examples answering Veech’s question negatively because their Lyapunov spectra are totally degenerate, but these examples are rare among regular affine measures.

We close this section with the following remarks.

**Remark 77.** A. Eskin, M. Kontsevich and A. Zorich also showed in [21] a version of their formula for quadratic differentials, and they used it to compute Siegel-Veech constants of \( SL(2, \mathbb{R}) \)-invariant

\[34\text{The article [21] has 106 pages!}\]
$g_t$-ergodic probabilities $\mu$ supported in the hyperelliptic connected components $H_{\text{hyp}}(2g-2)$ and $H_{\text{hyp}}(g-1, g-1)$ of the strata $H(2g-2)$ and $H(g-1, g-1)$. The outcome of their computation is the fact that Siegel-Veech constant of any such $SL(2, \mathbb{R})$-invariant $g_t$-ergodic $\mu$ is

$$c(\mu) = \begin{cases} \frac{g(2g+1)}{6(2g-1)} & \text{if } \text{supp}(\mu) \subset H_{\text{hyp}}(2g-2) \\ \frac{2g^4+3g^2+1}{6g} & \text{if } \text{supp}(\mu) \subset H_{\text{hyp}}(g-1, g-1) \end{cases}$$

and, hence (by Theorem 75), in this case, the sum of Lyapunov exponents is

$$\lambda_1^{\mu} + \cdots + \lambda_2^{\mu} = \begin{cases} \frac{g^2}{2g-1} & \text{if } \text{supp}(\mu) \subset H_{\text{hyp}}(2g-2) \\ 1/2 & \text{if } \text{supp}(\mu) \subset H(1,1), \text{ or } \text{supp}(\mu) \subset H(g-1, g-1) \end{cases}$$

In particular, since the sole two strata $H(2)$ and $H(1,1)$ in genus 2 are hyperelliptic connected components, one has that, for any $SL(2, \mathbb{R})$-invariant $g_t$-ergodic $\mu$,

$$\lambda_2^{\mu} = \begin{cases} 1/3 & \text{if } \text{supp}(\mu) \subset H(2) \\ 1/2 & \text{if } \text{supp}(\mu) \subset H(1,1) \end{cases}$$

because $\lambda_2^{\mu} = 1$. This fact was conjectured by M. Kontsevich and A. Zorich, and it was firstly demonstrated by M. Bainbridge [9] a few years before the article [21] was available.

Remark 78. In a very recent work, D. Aulicino [2] further studied the problem of classifying $SL(2, \mathbb{R})$-invariant measures with totally degenerate spectrum from the point of view of the Teichmüller disks contained in the rank-one locus. More precisely, following [27] and [28], we define the rank-$k$ locus of the moduli space $H_g$ of Abelian differentials of genus $g$ is $D_g(k) := \{ \omega \in H_g : \text{rank}(B_\omega) \leq k \}$. Note that $D_g(1) \subset \cdots \subset D_g(g-1)$. In the literature, the locus $D_g(g-1)$ is sometimes called determinant locus (because $D_g(g-1) = \{ \omega \in H_g : \text{det} B_\omega = 0 \}$). Observe that these loci are naturally related to the study of Lyapunov exponents of KZ cocycle: for instance, by Theorem 44 any $SL(2, \mathbb{R})$-invariant probability $\mu$ with $\text{supp}(\mu) \subset D_g(1)$ has totally degenerate spectrum. In his work [2], D. Aulicino showed that there are no Teichmüller disks $SL(2, \mathbb{R}) \cdot (M, \omega)$ contained in $D_g(1)$ for $g = 2$ or $g \geq 13$, the Eierlegende Wollmilchsau and Ornithorynque are the sole Teichmüller disks contained in $D_3(1)$ and $D_4(1)$, and, furthermore, if there are no Teichmüller curves contained in $D_5(1)$, then there are no Teichmüller disks contained in $D_9(1)$ for $g \geq 5$. It is worth to point out that Teichmüller disks are more general objects than regular affine measures, so that Proposition 76 doesn’t allow to recover the results of D. Aulicino.

Remark 79. In the next section, we will show that

- Eierlegende Wollmilchsau $(M_3(-1, 0, 1, \infty), \omega_{EW}) \in H(1, 1, 1, 1)$ is decomposed in two maximal horizontal cylinders $C_1, C_2$ with $h(C_1) = h(C_2) = 1$ and $w(C_1) = w(C_2) = 4$, and its Veech group is $SL(2, \mathbb{Z})$;

\footnote{I.e., $SL(2, \mathbb{R})$-orbits. Note that Teichmüller disks are more general objects than Teichmüller curves because we don’t require $SL(2, \mathbb{R})$-orbits to be closed in the definition of the former.}
• Ornithorynque \((M_4(-1,0,1,\infty),\omega_0)\) \(\in \mathcal{H}(2,2,2)\) is decomposed in two maximal horizontal cylinders \(C_1, C_2\) with \(h(C_1) = h(C_2) = 1\) and \(w(C_1) = w(C_2) = 6\) and its Veech group is \(SL(2,\mathbb{Z})\).

See Remark 83 below for more details. By plugging these facts into Theorem 74 one can compute the Siegel-Veech constants of the measures \(\mu_{\mathcal{E}W}\) and \(\mu_{\mathcal{O}}\), and then, by Theorem 75 one can calculate the sum of their Lyapunov exponents. By doing so, one finds:

\[
\lambda_1^{\mu_{\mathcal{E}W}} + \lambda_2^{\mu_{\mathcal{E}W}} + \lambda_3^{\mu_{\mathcal{E}W}} = \frac{1}{12} \cdot 4 \cdot \frac{1}{2} + \frac{1}{1} \cdot \left( \frac{1}{4} + \frac{1}{4} \right) = 1
\]

and

\[
\lambda_1^{\mu_{\mathcal{O}}} + \lambda_2^{\mu_{\mathcal{O}}} + \lambda_3^{\mu_{\mathcal{O}}} + \lambda_4^{\mu_{\mathcal{O}}} = \frac{1}{12} \cdot 3 \cdot \frac{2}{3} + \frac{1}{1} \cdot \left( \frac{1}{6} + \frac{1}{6} \right) = 1
\]

Since \(\lambda^0_i = 1\) for any \(g_i\)-invariant ergodic \(\mu\), one concludes that \(\lambda_2^{\mu_{\mathcal{E}W}} = \lambda_2^{\mu_{\mathcal{O}}} = 0\) and \(\lambda_2^{\mu_{\mathcal{O}}} = \lambda_3^{\mu_{\mathcal{O}}} = \lambda_4^{\mu_{\mathcal{O}}} = 0\), a fact that we already knew from Theorems 63 and 67.

Remark 80. In the case of \(\mu\) coming from square-tiled surfaces \((M_0,\omega_0)\), the formula in Theorem 74 for \(c(\mu)\) combined with Theorem 75 suggests that one can write down computer programs to calculate the sum of Lyapunov exponents.

Indeed, as we’ll see in Appendix C \((M_0,\omega_0)\) determines\(^{36}\) by a pair of permutations \(h, v \in S_N\) modulo simultaneous conjugations, the heights and widths of its horizontal cylinders are determined by the cycles of the permutation \(h\) and, under this correspondence, the matrices \(S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\) and \(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) act on pairs of permutations as \(S(h,v) = (hv^{-1},v)\) and \(T(h,v) = (h,vh^{-1})\).

So, by recalling that \(S\) and \(T\) generate \(SL(2,\mathbb{Z})\), we can use the right-hand side of the formula in Theorem 74 to convert the computation of the Siegel-Veech constant of the \(SL(2,\mathbb{R})\)-invariant probability supported on \((M_0,\omega_0)\) into a combinatorial calculation with pairs of permutations that an adequate computer program can perform.

In fact, such computer programs for \(Mathematica\) and \(SAGE\) were written by, e.g., A. Zorich and V. Delecroix, and it is likely that they will be publicly available soon.

\(^{36}\)A pair of permutations \(h,v \in S_N\) gives rise to a square-tiled surface with \(N\) squares by taking \(N\) unit squares \(Q_i, i = 1, \ldots, N\), and by gluing (by translations) the rightmost vertical side of \(Q_i\) to the leftmost vertical side of \(Q_{h(i)}\) and the topmost horizontal side of \(Q_i\) to the bottommost horizontal side of \(Q_{v(i)}\). Of course, by renumbering the squares of a given square-tiled surface we may end up with different pairs of permutations, so that a square-tiled surface determines a pair \(h,v \in S_N\) modulo simultaneous conjugation, i.e., modulo the equivalence relation \((h',v') \sim (h,v)\) if and only if \(h' = \phi h \phi^{-1}\) and \(v' = \phi v \phi^{-1}\) for some \(\phi \in S_N\). Finally, it is possible to check that the action of the matrices \(S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\) and \(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) on square-tiled surfaces translate into the action \(S(h,v) = (hv^{-1},v)\) and \(T(h,v) = (h,vh^{-1})\) on pairs of permutations. Cf. Appendix C for more comments on this construction.
7. Explicit computation of Kontsevich-Zorich cocycle over two totally degenerate examples

We saw in the previous section that examples of $SL(2, \mathbb{R})$-invariant $g_t$-ergodic probabilities with totally degenerate spectrum are rare and it is likely that there are only two of them coming from two square-tiled surfaces, namely, the Eierlegende Wollmilchsau and the Ornithorynque.

In this section, we will investigate more closely the Kontsevich-Zorich cocycle over the $SL(2, \mathbb{R})$-orbit of these special examples of square-tiled surfaces.

7.1. Affine diffeomorphisms, automorphisms and Veech groups. Let $(M, \omega)$ be a translation surface. We denote by $\text{Aff}(M, \omega)$ its group of affine diffeomorphisms, i.e., the group of (orientation-preserving) homeomorphisms $f$ of $M$ preserving the set $\Sigma$ of zeroes of $\omega$ that are affine in the translation charts (local primitives of $\omega$) of $M - \Sigma$. In translation charts, the linear part (differential) of $f \in \text{Aff}(M, \omega)$ is a well-defined matrix $Df \in SL(2, \mathbb{R})$. One obtains in this way a homomorphism:

$$D : \text{Aff}(M, \omega) \rightarrow SL(2, \mathbb{R})$$

The kernel of $D$ is, by definition, the group $\text{Aut}(M, \omega)$ of automorphisms of $(M, \omega)$, the image of $D$ is, by definition, the Veech group $SL(M, \omega)$ of $(M, \omega)$, and it is possible to show that

$$1 \rightarrow \text{Aut}(M, \omega) \rightarrow \text{Aff}(M, \omega) \rightarrow SL(M, \omega) \rightarrow 1$$

is an exact sequence.

Remark 81. In fact, we introduced the Veech group in Definition 69 above as the stabilizer of $(M, \omega)$ with respect to the action of $SL(2, \mathbb{R})$ on $\mathcal{H}_g$. As it turns out, it is possible to show that these definitions coincide. See the survey [41] of P. Hubert and T. Schmidt for more details.

It is possible to show that, in genus $g \geq 2$, the affine group $\text{Aff}(M, \omega)$ injects in the modular group $\Gamma_g$, and the stabilizer of the $SL(2, \mathbb{R})$-orbit of $(M, \omega)$ in $\Gamma_g$ is precisely $\text{Aff}(M, \omega)$ (see [70]). In particular, since KZ cocycle is the quotient of the trivial cocycle over Teichmüller flow

$$\text{diag}(e^t, e^{-t}) \times \text{id} : \mathcal{T}\mathcal{H}_g \times H_1(M, \mathbb{R}) \rightarrow \mathcal{T}\mathcal{H}_g \times H_1(M, \mathbb{R})$$

by the natural action of $\Gamma_g$, we conclude that KZ cocycle on $SL(2, \mathbb{R}) \cdot (M, \omega)$ is the quotient of the trivial cocycle

$$\text{diag}(e^t, e^{-t}) \times \text{id} : SL(2, \mathbb{R}) \cdot (M, \omega) \times H_1(M, \mathbb{R}) \rightarrow SL(2, \mathbb{R}) \cdot (M, \omega) \times H_1(M, \mathbb{R})$$

by the natural action of the affine group $\text{Aff}(M, \omega)$.

Remark 82. Our original definition of KZ cocycle used cohomology groups $H^1(M, \mathbb{R})$ in the fibers instead of homology groups $H_1(M, \mathbb{R})$. However, since cohomology groups are dual to homology groups, there is no harm in replacing cohomology by homology in our considerations (as far as, e.g., Lyapunov exponents are concerned).
7.2. Affine diffeomorphisms of square-tiled surfaces and Kontsevich-Zorich cocycle. Let $(M, \omega)$ be a square-tiled surface, i.e., a finite covering $p : M \rightarrow \mathbb{T}^2$ of the torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified only at $0 \in \mathbb{T}^2$ with $\omega = p^*(dz)$.

Using the action of $p$ on homology groups, we define $H_1^{(0)}(M, \mathbb{Q})$ the kernel of $p$ and $H_1^{sy}(M, \mathbb{Q}) := p^{-1}(H_1(\mathbb{T}^2, \mathbb{Q}))$. It is not hard to show that

$$H_1(M, \mathbb{Q}) = H_1^{(0)}(M, \mathbb{Q}) \oplus H_1^{sy}(M, \mathbb{Q})$$

The standard subspace $H_1^{sy}(M, \mathbb{Q})$ and $H_1^{(0)}(M, \mathbb{Q})$ can be alternatively described as follows. Let $\text{Sq}(M, \omega)$ be the set of squares constituting $(M, \omega)$, i.e., $\text{Sq}(M, \omega)$ is the set of connected components of $p^{-1}((0,1)^2)$ (where $(0,1)^2 \subset \mathbb{R}^2$ is the open unit square inside $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$). Put $\Sigma' = p^{-1}(\{0\})$, so that $\Sigma'$ contains the set $\Sigma$ of zeroes of $\omega$, and, for each square $\alpha \in \text{Sq}(M, \omega)$, let $\sigma_\alpha \in H_1(M, \Sigma', \mathbb{Z})$ be the cycle going from the bottom left corner of $\alpha$ to the bottom right corner of $\alpha$, and $\zeta_\alpha \in H_1(M, \Sigma', \mathbb{Z})$ be the cycle going from the bottom left corner of $\alpha$ to the top left corner of $\alpha$.

In this notation, we can form the absolute cycles

$$\sigma = \sum_{\alpha \in \text{Sq}(M, \omega)} \sigma_\alpha \quad \text{and} \quad \zeta = \sum_{\alpha \in \text{Sq}(M, \omega)} \zeta_\alpha,$$

and one can show that $H_1^{sy}(M, \mathbb{Q}) = \mathbb{Q}\sigma \oplus \mathbb{Q}\zeta$, and $H_1^{(0)}(M, \mathbb{Q})$ is the symplectic orthogonal (with respect to the intersection form) to $H_1^{sy}(M, \mathbb{Q})$.

Note that, from this description, we see several facts. Firstly, $H_1^{sy}$ is a 2-dimensional (symplectic) subspace and $H_1^{(0)}$ is a $2g - 2$-dimensional (symplectic) subspace. Secondly, the homological action of $\text{Aff}(M, \omega)$ on $H_1^{sy}$ occurs via the standard action of the Veech group $D(\text{Aff}(M, \omega)) := \text{SL}(M, \omega) \subset \text{SL}(2, \mathbb{R})$ on the plane $\mathbb{R}^2 \simeq \mathbb{R}\sigma \oplus \mathbb{R}\zeta$. Finally, the homological action of $\text{Aff}(M, \omega)$ preserves the decomposition

$$H_1(M, \mathbb{Q}) = H_1^{(0)}(M, \mathbb{Q}) \oplus H_1^{sy}(M, \mathbb{Q})$$

because this action is symplectic with respect to the intersection form on homology.

Consider now the restriction of KZ cocycle to the closed orbit of the square-tiled surface $(M, \omega)$ and denote by $\mu$ the $\text{SL}(2, \mathbb{R})$-invariant probability supported on $\text{SL}(2, \mathbb{R}) \cdot (M, \omega)$. By combining the discussion of the previous paragraph with the fact that KZ cocycle acts on $\text{SL}(2, \mathbb{R}) \cdot (M, \omega)$ through $\text{Aff}(M, \omega)$, we see that (in the present context) the tautological Lyapunov exponents $\lambda_1^\mu = 1$ and $\lambda_2^{2g} = -1$ of KZ cocycle (with respect to $\mu$) come from the restriction of KZ cocycle to the
2-dimensional symplectic subspace $H^1_\mu$. Therefore, the interesting part $\lambda^g_2 \geq \cdots \geq \lambda^g_0$ of the Lyapunov spectrum of KZ cocycle with respect to $\mu$ comes from its restriction to $H^1_\mu$.

In particular, we “reduced” the study of KZ cocycle over the Eierlegende Wollmilchsau and Ornithorynque to the computation of the homological action of their affine diffeomorphisms on $H^1_\mu$. Evidently, it is convenient to get concrete models of these square-tiled surfaces because they allow to write down explicit basis of $H^1_\mu$, so that the action of affine diffeomorphisms can be encoded by concrete matrices.

Keeping this in mind, we start now the description of concrete models of the Eierlegende Wollmilchsau and Ornithorynque.

7.3. **Combinatorics of square-tiled cyclic covers.** The Eierlegende Wollmilchsau and Ornithorynque are naturally included into the following family. Let $N$, $0 < a_1, \ldots, a_4 < N$ be non-negative integers such that

- (a) $\gcd(N, a_1, \ldots, a_4) = 1$;
- (b) $a_1 + \cdots + a_4$ is a multiple of $N$, i.e., $a_1 + \cdots + a_4 \in \{N, 2N, 3N\}$;
- (c) $N$ is even and $a_1, \ldots, a_4$ are odd.

Using these parameters, one can form a family $M_N(a_1, \ldots, a_4)$ of pairs (Riemann surface, Abelian differential) by taking $x_1, \ldots, x_4 \in \mathbb{C}$ distinct and considering the algebraic equations

$$y^N = (x-x_1)^{a_1} \cdots (x-x_4)^{a_4}$$

equipped with the Abelian differential $\omega = (x-x_1)^{b_1} \cdots (x-x_4)^{b_4} dx/y^{N/2}$, where $2b_j := a_j - 1$.

Here, (a) ensures that the Riemann surfaces are connected, (b) ensures that they are cyclic covers of $\mathbb{C}$ branched at $x_1, \ldots, x_4$ but not at $\infty$, and (c) ensures that $\omega$ is a well-defined, holomorphic 1-form.

In this notation, the Eierlegende Wollmilchsau is $M_4(1, 1, 1, 1)$ and Ornithorynque is $M_6(3, 1, 1, 1)$.

The family $M_N(a_1, \ldots, a_4)$ is called square-tiled cyclic cover. In the sequel, our discussion will follow closely [22].

Note that the square of the Abelian differential $\omega$ is the pull-back of the quadratic differential

$$q_0 = \frac{(dx)^2}{(x-x_1) \cdots (x-x_4)}$$

on $\mathbb{C}$ under the natural projection $p(x, y) = x$. By choosing $(x_1, \ldots, x_4) = (-1, 0, 1, \infty)$, we know that the flat structure associated to $q_0$ is given by two flat unit squares glued by their boundaries:

Now, we can use this concrete description of the flat structure of $q_0$ to obtain a concrete model for $(M_N(a_1, \ldots, a_4), \omega)$ as follows. We have 2 squares tiling the flat model of $q_0$, a white and

---

37This is better appreciated by noticing that the square of the Abelian differential $dx/z$ on the elliptic curve (torus) $z^2 = (x-x_1) \cdots (x-x_4)$ is the pull-back of $q_0$ under $h(x, z) = x$, and $(E(-1, 0, 1, \infty), dx/z)$ is the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ (up to scaling) because the periods $\int_{-1}^{-i} \frac{dz}{\sqrt{z^2 - x}} = \int_0^1 \frac{dx}{\sqrt{x^2 - z}} = \int_0^1 \frac{dz}{\sqrt{z^2 - x}} = \int_0^\infty \frac{dx}{\sqrt{x^2 - z}} = \frac{\Gamma(1/4)^2}{2^{1/2}}$. 

a black. Since the covering \( p : M_N(a_1, \ldots, a_4) \to \mathbb{C}, p(x, y) = x \), has degree \( N \), we have that \((M_N(a_1, \ldots, a_4), \omega)\) has \(2N\) squares naturally colored white or black.

Let’s take arbitrarily a white square \( S_0 \) of \( M_N(a_1, \ldots, a_4) \) and let’s number it 0. Then, we consider the black square \( S_1 \) adjacent to 0 via the side \([CD]\) (see the figure above) and we number it 1. Then, by denoting by \( T(x, y) = (x, \varepsilon y) \), \( \varepsilon = \exp(2\pi i k/N) \), the automorphism of \( M_N(a_1, \ldots, a_4) \) generating the Galois group of the covering \( p \), we number \( 2k \) the white square \( S_{2k} = T^k(S_0) \) and \( 2k+1 \) the black square \( S_{2k+1} = T^k(S_1) \). Here, we take \( k \) modulo \( N \) (so that one may always think \( 0 \leq k < N \)).

The endpoint of the lift of the path \( \tau_h \) (see the figure above) to \( M_N(a_1, \ldots, a_4) \) is deduced from the starting point of the lifted path by applying \( T^{a_1+a_4} = T^{-a_2-a_3} \) (here item (b) above was used). In this way, by moving to two squares to the right, we go from the square number \( j \) to the square number \( j + 2(a_1 + a_3)(\text{mod} \ N) \). In particular, by successively applying \( T^{a_1+a_3} \) we can deduce all horizontal cylinders of \((M_N(a_1, \ldots, a_4), \omega)\).

Similarly, we can deduce neighbors in the vertical direction by using small (positively oriented) paths \( \sigma_i \) encircling \( z_i \) (see the picture above for \( \sigma_1 \)). Indeed, since the extremal points of the lift of \( \sigma_i \) to \( M_N(a_1, \ldots, a_4) \) differ by \( T^{a_i} \), by going around a corner (in the counterclockwise sense) of the square numbered \( j \), we end up in the square numbered \( j + 2a_i(\text{mod} \ N) \).

These “local moves” obtained by lifting \( \tau_h \) and \( \sigma_1, \sigma_1^{-1}, \sigma_2, \ldots \) are described in a nutshell in the following picture:

Of course, the picture above makes clear that we can extract concrete square-tiled models for \( M_N(a_1, \ldots, a_4) \).

For example, it is an instructive exercise to the reader to apply this procedure to Eierlegende Wollmilchsau \( M_4(1,1,1,1) \) and Ornithorynque \( M_6(3,1,1,1) \), and check that the pictures one get for them are the following:

Remark 83. From Figure 15 we see that the Eierlegende Wollmilchsau can be decomposed into two maximal horizontal cylinders, both of height 1 and width 4. Similarly, from Figure 16 we see that the Ornithorynque can be decomposed into two maximal horizontal cylinders, both of height 1 and width 6. Moreover, by applying the matrices \( S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) and \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) to the two figures above, and by using adequate elements of the modular group to cut and paste the resulting objects, the reader can verify that \( S \) and \( T \) stabilize both the Eierlegende Wollmilchsau
and Ornithorynque, i.e., $S$ and $T$ belong to the Veech group of them. Because $S$ and $T$ generate $SL(2,\mathbb{Z})$, this shows that the Veech group of both Eierlegende Wollmilchsau and Ornithorynque is $SL(2,\mathbb{Z})$.

Once we dispose of these concrete models for Eierlegende Wollmilchsau and Ornithorynque (and more generally square-tiled cyclic covers), it is time use them to produce nice basis of their homology groups.

7.4. Eierlegende Wollmilchsau and the quaternion group. By carefully looking at Figure 15 the reader can verify that Eierlegende Wollmilchsau admits the following presentation:

Here, the squares of Eierlegende Wollmilchsau are labelled via the elements of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$.

A great advantage of this presentation is the fact that one can easily deduce the neighbors of squares by right multiplication by $\pm i$ or $\pm j$: indeed, given a square $g \in Q$, its neighbor to the right is the square $g \cdot i$ and its neighbor on the top is $g \cdot j$. In this way, we can identify the group of automorphisms $\text{Aut}(M_{EW}, \omega_{EW})$ of Eierlegende Wollmilchsau $(M_{EW}, \omega_{EW})$ with the quaternion group $Q$ by associated to $h \in Q$ the automorphism sending the square $g$ to the square $h \cdot g$ deduced from $g$ by left multiplication by $h$.

\[ i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = k. \]
Also, the zeroes of $\omega_{EW}$ are located at the left bottom corners of the squares $g$. Since the left bottom corners of the squares $g$ and $-g$ are identified together, the set $\Sigma_{EW}$ of zeroes of $\omega$
is naturally identified with the group $\overline{Q} = Q/\{\pm 1\} = \{1, i, j, k\}$. Note that $\overline{Q}$ is isomorphic to Klein’s group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In what follows, we’ll follow closely [56] to compute the homological action of affine group of Eierlegende Wollmilchsau using the cycles $\sigma_g, \zeta_g \in H_1(M_{EW}, \Sigma_{EW}, \omega_{EW})$ introduced in Subsection 7.2 above, i.e.,

\[
\begin{array}{ccc}
\sigma_{ij} & \zeta_{ij} \\
\zeta_{ij} & g & \sigma_{ij} \\
\sigma_{ij} & \\
\end{array}
\]

**Remark 84.** Note that, from this picture, we have that $\sigma_g + \zeta_{qi} - \sigma_{gj} - \zeta_g = 0$ in homology. We’ll systematically use this relation in the sequel.

However, before rushing to the study of the whole action of the affine group, let’s first investigate the action of the group of automorphisms. The automorphism group of Eierlegende Wollmilchsau is isomorphic to $Q$. In particular, one can select a nice basis on the homology of Eierlegende Wollmilchsau using the representation theory of $Q$. More precisely, we know $Q$ has 4 irreducible 1-dimensional representations $\chi_1, \chi_i, \chi_j, \chi_k$ and 1 irreducible 2-dimensional representation $\chi_2$. They can be seen inside the regular representation of $Q$ in $\mathbb{Z}(Q)$ via the submodules generated by:

- $\chi_1 : [1] + [-1] + [i] + [-i] + [j] + [-j] + [k] + [-k]$
- $\chi_i : [1] + [-1] + [i] + [-i] - [j] - [-j] - [k] - [-k]$
- $\chi_k : [1] + [-1] - [i] - [-i] - [j] - [-j] + [k] + [-k]$
- $\chi_2 : [1] - [-1], [i] - [-i], [j] - [-j], [k] - [-k]$

Thus, we see that the character table of $Q$ is

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-1</th>
<th>$\pm i$</th>
<th>$\pm j$</th>
<th>$\pm k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_i$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_j$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_k$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\text{tr } \chi_2$</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
This motivates the introduction of the following relative cycles:

\[ w_i = \zeta_1 + \zeta_{-1} + \zeta_i + \zeta_{-i} - \zeta_j - \zeta_{-j} - \zeta_k - \zeta_{-k} \]
\[ w_j = \sigma_1 + \sigma_{-1} + \sigma_j + \sigma_{-j} - \sigma_i - \sigma_{-i} - \sigma_k - \sigma_{-k} \]
\[ w_k = \zeta_1 + \zeta_{-1} + \zeta_k + \zeta_{-k} - \zeta_i - \zeta_{-i} - \zeta_j - \zeta_{-j} \]
\[ = \sigma_1 + \sigma_{-1} + \sigma_k + \sigma_{-k} - \sigma_i - \sigma_{-i} - \sigma_j - \sigma_{-j} \]

**Remark 85.** Note that

\[ \zeta_1 + \zeta_{-1} + \zeta_j + \zeta_{-j} + \zeta_k - \zeta_{-k} = \sigma_1 + \sigma_{-1} + \sigma_i - \sigma_{-i} - \sigma_j - \sigma_{-j} - \sigma_k - \sigma_{-k} = 0 \]

in homology.

Indeed, we have that the cycle \( w_i \) is relative because its boundary is \( \partial w_i = 4(\bar{j} + \bar{k} - \bar{i} - \bar{t}) \) and the action of an automorphism \( g \in Q \) is \( g \cdot w_i = \chi_i(g) \cdot w_i \) (see Figure 17). Also, one has similar formulas for \( w_j \) and \( w_k \), so that the action of \( Q = \text{Aut}(M_{EW}, \omega_{EW}) \) on the subspace \( H_{rel} := Qw_i \oplus Qw_j \oplus Qk \) is fairly well-understood.

Observe that \( H_{rel} \) is a relative subspace in the sense that it *complements* the absolute homology group \( H_1(M_{EW}, Q) \) in \( H_1(M_{EW}, \Sigma_{EW}, Q) \).

Next, we consider the following absolute cycles:

\[ \sigma := \sum_{g \in Q} \sigma_g, \quad \zeta := \sum_{g \in Q} \zeta_g, \]
\[ \tilde{\sigma}_g := \sigma_g - \sigma_{-g}, \quad \tilde{\zeta}_g := \zeta_g - \zeta_{-g}, \]
\[ \varepsilon_g := \tilde{\sigma}_g - \tilde{\sigma}_{gj} = \tilde{\zeta}_g - \tilde{\zeta}_{gj} \]

Using the notations introduced in Subsection 7.2, one can check that \( H_1^{st} = Q\sigma \oplus Q\zeta \) and \( H_1^{(0)}(M_{EW}, Q) \) is spanned by the cycles \( \tilde{\sigma}_g, \tilde{\zeta}_g \). Here, we notice that, since Eierlegende Wollmilchsau has genus \( g = 3 \), i.e., the dimension of \( H_1(M_{EW}, Q) \) is \( 2g - 2 = 6 \), \( H_1^{st} \) has dimension 2 and

\[ H_1(M_{EW}, Q) = H_1^{st}(M_{EW}, Q) \oplus H_1^{(0)}(M_{EW}, Q), \]

one has that \( H_1^{(0)}(M_{EW}, Q) \) has dimension 4.

For later use, we observe that \( \tilde{\sigma}_{-g} = -\tilde{\sigma}_g, \quad \tilde{\zeta}_{-g} = -\tilde{\zeta}_g, \quad \varepsilon_{-g} = -\varepsilon_g \) and

\[ \tilde{\sigma}_g = \frac{1}{2}(\varepsilon_g + \varepsilon_{gj}), \quad \tilde{\zeta}_g = \frac{1}{2}(\varepsilon_g + \varepsilon_{gj}) \]

Therefore, one can write \( H_1^{(0)}(M_{EW}, Q) \) (in several ways) as a sum of two copies of the 2-dimensional irreducible \( Q \)-representation \( \chi_2 \).

Finally, we observe that, for any \( g \in Q \simeq \text{Aut}(M_{EW}, \omega_{EW}) \),

\[ g \cdot \sigma = \sigma, \quad g \cdot \zeta = \zeta \]

and

\[ g \cdot \tilde{\sigma}_h = \tilde{\sigma}_{gh}, \quad g \cdot \tilde{\zeta}_h = \tilde{\zeta}_{gh}, \quad g \cdot \varepsilon_h = \varepsilon_{gh} \]
See Figure 17. Hence, the action of \( Q = \text{Aut}(M_{\text{EW}}, \omega_{\text{EW}}) \) on the absolute homology \( H_1(M_{\text{EW}}, \mathbb{Q}) = H_1^{st} \oplus H_1^{(0)} \) is also fairly well-understood.

In resume, by looking at the representation theory of the (finite) group of automorphisms \( Q \) of Eierlegende Wollmilchsau, we selected a nice generating set of the relative cycles \( \sigma, \zeta, \hat{\sigma}, \hat{\zeta} \) such that the action of \( Q \) is easily computed.

After this first (preparatory) step of studying the homological action of \( \text{Aut}(M_{\text{EW}}, \omega_{\text{EW}}) \simeq Q \), we are ready to face the homological action of \( \text{Aff}(M_{\text{EW}}, \omega_{\text{EW}}) \).

### 7.5. The action of the affine diffeomorphisms of the Eierlegende Wollmilchsau

Denote by \( \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \) the subgroup of \( \text{Aff}(M_{\text{EW}}, \omega_{\text{EW}}) \) consisting of affine diffeomorphisms fixing \( \tilde{T} \in \Sigma_{\text{EW}} \) (and, \( \text{a fortiori} \), each \( \tilde{\gamma} \in \Sigma_{\text{EW}} \)). Because \( \text{Aut}(M_{\text{EW}}, \omega_{\text{EW}}) \simeq Q \) acts transitively on \( \Sigma_{\text{EW}} \), we have that \( \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \) has index 4 inside \( \text{Aff}(M_{\text{EW}}, \omega_{\text{EW}}) \).

Since the elements of \( \text{Aff}(M_{\text{EW}}, \omega_{\text{EW}}) \) differ from those \( \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \) by composition with some element in \( \text{Aut}(M_{\text{EW}}, \omega_{\text{EW}}) \simeq Q \) and we completely understand the homological action of \( Q \), our task is reduced to compute the action of \( \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \).

At this point, we introduce the elements \( \tilde{S} \in \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \), resp. \( \tilde{T} \in \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \), obtained by the lifting \( S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \), resp. \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \), in such a way that the square 1 intersects its image under \( \tilde{S} \), resp. \( \tilde{T} \). It can be checked that
\[
(\tilde{S}\tilde{T}^{-1}\tilde{S})^4 = -1
\]
so that \( \tilde{S} \) and \( \tilde{T} \) generate \( \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \). In other words, it suffices to compute the action of \( \tilde{S} \) and \( \tilde{T} \) to get our hands in the action of \( \text{Aff}(1)(M_{\text{EW}}, \omega_{\text{EW}}) \).

A direct inspection of Figure 17 reveals that the actions of \( \tilde{S} \) and \( \tilde{T} \) on the cycles \( \sigma_g \) and \( \zeta_g \) are

\[
\tilde{S}(\zeta_g) = \begin{cases} \zeta_g & \text{if } g \in \{\pm 1, \pm j\}, \\ \zeta_{gj} & \text{if } g \in \{\pm i, \pm k\}, \end{cases} \quad \tilde{S}(\sigma_g) = \begin{cases} \sigma_g + \zeta_{gi} & \text{if } g \in \{\pm 1, \pm j\}, \\ \sigma_{gj} + \zeta_g & \text{if } g \in \{\pm i, \pm k\}, \end{cases}
\]

and

\[
\tilde{T}(\sigma_g) = \begin{cases} \sigma_g & \text{if } g \in \{\pm 1, \pm i\}, \\ \sigma_{sg} & \text{if } g \in \{\pm j, \pm k\}, \end{cases} \quad \tilde{T}(\zeta_g) = \begin{cases} \zeta_g + \sigma_{gi} & \text{if } g \in \{\pm 1, \pm i\}, \\ \zeta_{gj} + \sigma_g & \text{if } g \in \{\pm j, \pm k\}. \end{cases}
\]

From these formulas, one deduces that \( \tilde{S} \) and \( \tilde{T} \) act on \( H_1^{st} = \mathbb{Q}\sigma \oplus \mathbb{Q}\zeta \) in the standard way (cf. Subsection 7.2)

\[
\tilde{S}(\sigma) = \sigma + \zeta, \quad \tilde{S}(\zeta) = \zeta,
\]

\[
\tilde{T}(\zeta) = \sigma + \zeta, \quad \tilde{T}(\sigma) = \sigma,
\]

while they act on the relative part \( H_{rel} = \mathbb{Q}w_i \oplus \mathbb{Q}w_j \oplus \mathbb{Q}w_k \) via the symmetry group of a tetrahedron:

\[
\tilde{S}(w_i) = w_k, \quad \tilde{S}(w_j) = w_j, \quad \tilde{S}(w_k) = w_i,
\]

\[
\tilde{T}(w_i) = w_i, \quad \tilde{T}(w_j) = w_k, \quad \tilde{T}(w_k) = w_j.
\]
Last, but not least, we have $\tilde{S}$ and $\tilde{T}$ act on $H_1^{(0)}$ (that is, the subspace containing the non-tautological exponents of KZ cocycle) as:

$$\tilde{S}(\varepsilon_g) = \begin{cases} 
\frac{1}{2}(\varepsilon_g + \varepsilon_{gi} + \varepsilon_{gj} + \varepsilon_{gk}) & \text{if } g \in \{\pm 1, \pm j\}, \\
\frac{1}{2}(\varepsilon_g - \varepsilon_{gi} - \varepsilon_{gj} + \varepsilon_{gk}) & \text{if } g \in \{\pm i, \pm k\}, 
\end{cases}$$

and

$$\tilde{T}(\varepsilon_g) = \begin{cases} 
\frac{1}{2}(\varepsilon_g + \varepsilon_{gi} + \varepsilon_{gj} - \varepsilon_{gk}) & \text{if } g \in \{\pm 1, \pm i\}, \\
\frac{1}{2}(\varepsilon_g - \varepsilon_{gi} - \varepsilon_{gj} - \varepsilon_{gk}) & \text{if } g \in \{\pm j, \pm k\}.
\end{cases}$$

Here, our choice of computing $\tilde{S}$ and $\tilde{T}$ in terms of $\varepsilon_g$ was not arbitrary: indeed, a closer inspection of these formulas shows that $\tilde{S}$ and $\tilde{T}$ are acting on the 4-dimensional subspace $H_1^{(0)}$ via the automorphism group of the root system. Of course, $\varepsilon_g + \varepsilon_h : g \neq \pm h$

of type $D_4$, that is, by equipping $H_1^{(0)}$ with the inner product such that $\{\varepsilon_1, \varepsilon_i, \varepsilon_j, \varepsilon_k\}$ is orthonormal, one has that $\tilde{S}$ and $\tilde{T}$ act on $H_1^{(0)}$ via the finite group $O(R)$ of orthogonal linear transformations of $H_1^{(0)}$ preserving $R$.

In particular, this discussion shows that $\text{Aff}(1)(M_{EW}, \omega_{EW})$ acts on $H_1^{(0)}$ via a certain finite subgroup of orthogonal $4 \times 4$ matrices.

Actually, one can follow [58] to develop these calculations (using the knowledge of the structure of the automorphism and Weyl groups of root systems of type $D_4$) to prove that the affine group $\text{Aff}(M_{EW}, \omega_{EW})$ acts on $H_1^{(0)}$ via a subgroup of order $96$ of orthogonal $4 \times 4$ matrices. However, we will not insist more on this point. Instead, we take the opportunity to observe that this allows to re-derive the total degeneracy of the Lyapunov spectrum of Eierlegende Wollmilchsau. Indeed, since $\text{Aff}(M_{EW}, \omega_{EW})$ acts on $H_1^{(0)}$ via a finite group of matrices, it preserves some norm (actually we already determined it), that is, $\text{Aff}(M_{EW}, \omega_{EW})$ acts isometrically on $H_1^{(0)}$. Hence, by our discussion in Subsection 7.2, this means that the restriction of KZ cocycle to $H_1^{(0)}(M_{EW}, \omega_{EW})$ acts isometrically over the $SL(2, \mathbb{R})$-orbit of Eierlegende Wollmilchsau, so that all (non-tautological) Lyapunov exponents of Eierlegende Wollmilchsau must vanish.

Remark 86. Of course, a priori the fact that the restriction of KZ cocycle to $H_1^{(0)}(M_{EW}, \omega_{EW})$ acts via a finite group is stronger than simply knowing that it acts isometrically. But, as it turns out, it is possible to show that in the case of square-tiled surfaces, if KZ cocycle acts isometrically on $H_1^{(0)}$, then it must act through a finite group: indeed, in the square-tiled surface case, KZ cocycle acts on $H_1^{(0)}$ via the discrete group $Sp(2g - 2, \mathbb{Z}[1/N])$ for some $N \in \mathbb{N}$; so, if KZ cocycle also acts on $H_1^{(0)}$ isometrically, one conclude that KZ cocycle acts on $H_1^{(0)}$ through $Sp(2g - 2, \mathbb{Z}[1/N]) \cap O(2g - 2)$, a finite group. See [59] for more details. However, we avoided using this fact during this section to convince the reader that the homological action of the affine group is so concrete that one

$^{39}$The order of $O(R)$ is order 1152, so that the affine group of Eierlegende Wollmilchsau acts via an index 12 subgroup of $O(R)$. 


can actually determine (with bare hands) the matrices involved in such action (at least if one is sufficiently patient).

7.6. The action of the affine diffeomorphisms of the Ornithorynque. As the reader can imagine, the calculations of the previous two subsections can be mimicked in the context of Ornithorynque. Evidently, the required modifications are somewhat straightforward, so let’s content ourselves with a mere outline of the computations (referring to the original article [54] for details).

We start by considering a “better” presentation of Ornithorynque \((M_O, \omega_O)\) (compare with Figure 10):

\[
\begin{array}{c}
\sigma_{i+2} & \sigma'_{i+2} \\
\zeta_{i+1} & \zeta'_{i+1} \\
\zeta_{i} & \zeta'_{i} \\
\sigma_{i} & \sigma'_{i} \\
\end{array}
\]

The indication \(i = 0, 1, 2 \pmod{3}\) means that we’re considering three copies of the “basic” pattern, and we identify sides with the same “name” \(\sigma_i, \sigma'_i, \zeta_i\) or \(\zeta'_i\) by taking into account that the subindices \(i\) are thought modulo 3. The (three) black dots are regular points while the other “special” points indicated the (three) double zeroes of the Abelian differential \(\omega_O\). Also, it is clear from this picture that the cycles \(\sigma_i, \sigma'_i, \zeta_i\) and \(\zeta'_i\) are relative, and they verify the relation

\[
\sigma_i + \sigma'_i + \zeta_{i-1} + \zeta'_{i+1} - \sigma'_{i-1} - \sigma_{i+1} - \zeta'_i - \zeta_i = 0
\]

for each \(i \in \mathbb{Z}/3\mathbb{Z}\).

Let’s now focus on the action of \(\text{Aff}(M_O, \omega_O)\) on \(H_1^{(0)}(M_O, \omega_O)\) (the interesting part of the homology containing the non-tautological exponents of KZ cocycle). The natural \(\mathbb{Z}/3\mathbb{Z}\) symmetry (i.e., group of automorphisms) of the figure above motivates the following choice of cycles in \(H_1^{(0)}\):

\[
a_i := \sigma_i - \sigma_{i+1}, \quad a'_i := \sigma'_i - \sigma'_{i+1}
\]

and

\[
b_i := \zeta_i - \zeta_{i+1}, \quad b'_i := \zeta'_i - \zeta'_{i+1}
\]

The relation between the cycles \(\sigma_i, \sigma'_i, \zeta_i\) and \(\zeta'_i\) written above implies that

\[
a_i - a'_{i-1} + b_{i-1} - b'_i = 0
\]
This suggests the introduction of the following cycles in $H^{(0)}_1$:

$$
\tau_i := a_i - a'_{i-1} = b'_i - b_{i-1}, \quad \bar{\sigma}_i := a_i + a'_{i-1}, \quad \hat{\zeta}_i = b'_i + b_{i-1}
$$

Note that $\sum_{i \in \mathbb{Z}/3\mathbb{Z}} \tau_i = \sum_{i \in \mathbb{Z}/3\mathbb{Z}} \bar{\sigma}_i = \sum_{i \in \mathbb{Z}/3\mathbb{Z}} \hat{\zeta}_i = 0$. Actually, this is the sole relation satisfied by them because it is possible to show that

$$
\{\tau_i, \bar{\sigma}_i, \hat{\zeta}_i : i \in \mathbb{Z}/3\mathbb{Z} - \{0\}\}
$$

is a basis of $H^{(0)}_1(M_O, \mathbb{Q})$. Observe that this is coherent with the fact that $H^{(0)}_1$ has dimension 6 ($= 2g - 2$ for $g = g(M_O) = 4$).

Let’s denote by $H_\tau$ the 2-dimensional subspace of $H^{(0)}_1$ spanned by $\tau_i$’s and $\hat{H}$ the 4-dimensional subspace of $H^{(0)}_1$ spanned by $\bar{\sigma}_i$’s and $\hat{\zeta}_i$’s, so that

$$H^{(0)}_1(M_O, \mathbb{Q}) = H_\tau \oplus \hat{H}$$

This splitting is natural in our context because it is preserved by the homological action of the affine group. Indeed, it is easy to check that this splitting is preserved by the group of automorphisms $\mathbb{Z}/3\mathbb{Z}$, so that we can restrict our attention to the subgroup $\text{Aff}_{(1)}(M_O, \omega_O)$ of affine diffeomorphisms fixing each zero of $\omega_O$.

Denote by $\tilde{S} \in \text{Aff}_{(1)}(M_O, \omega_O)$, resp. $\tilde{T} \in \text{Aff}_{(1)}(M_O, \omega_O)$, the elements with linear parts $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$, resp. $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$. Again, one can check that $\tilde{S}$ and $\tilde{T}$ generate $\text{Aff}_{(1)}(M_O, \omega_O)$, and their action on the subspaces $H_\tau$ and $\hat{H}$ are given by the formulas

$$
\tilde{S}(\tau_i) = -\tau_{i+1}, \quad \tilde{S}(\bar{\sigma}_i) = \bar{\sigma}_i + \hat{\zeta}_{i-1}, \quad \tilde{S}(\hat{\zeta}_i) = \hat{\zeta}_{i+1}
$$

and

$$
\tilde{T}(\tau_i) = -\tau_{i-1}, \quad \tilde{T}(\bar{\sigma}_i) = \bar{\sigma}_{i-1}, \quad \tilde{T}(\hat{\zeta}_i) = \hat{\zeta}_{i+1}
$$

Therefore, $\text{Aff}_{(1)}(M_O, \omega_O)$ preserves the decomposition $H^{(0)}_1 = H_\tau \oplus \hat{H}$ and it acts on $H_\tau$ via the cyclic group $\mathbb{Z}/3\mathbb{Z}$.

Finally, a careful inspection of the formulas above shows that the action of $\text{Aff}_{(1)}(M_O, \omega_O)$ on $\hat{H}$ preserves the root system $R = \{\pm \bar{\sigma}_i, \pm \hat{\zeta}_i, \pm (\bar{\sigma}_i + \hat{\zeta}_{i-1}), \pm (\bar{\sigma}_i - \hat{\zeta}_{i+1})\}$ of type $D_4$. Hence, $\text{Aff}_{(1)}(M_O, \omega_O)$ acts on $\hat{H}$ via a subgroup of the finite group of automorphisms of the root system $R$. Actually, one can perform this computation jusqu’au but to check that the (whole) affine group $\text{Aff}(M_O, \omega_O)$, or equivalently, the KZ cocycle over the $SL(2, \mathbb{R})$-orbit of Ornithorynque, acts via an explicit subgroup of order 72 of the group of automorphisms of $R$. 

8. Cyclic covers

In last section we studied combinatorial models of Eierlegende Wollmilchsau and Ornithorynque by taking advantage of the fact that they belong to the class of square-tiled cyclic covers. Then, we used these combinatorial models to “put our hands” on the KZ cocycle over their $SL(2, \mathbb{R})$-orbits via the homological action of the group of affine diffeomorphisms.

In this section, we’ll be “less concrete but more conceptual” in order to systematically treat the Lyapunov spectrum of KZ cocycle over square-tiled cyclic covers in an unified way. From this framework we will prove that all zero Lyapunov exponents in the class of square-tiled cyclic covers have a nice geometrical explanation: they come from the annihilator of the second fundamental form $B_{\omega}$. A striking consequence of this fact is the continuous (actually, real-analytic) dependence of the neutral Oseledets subspace on the base point. However, by the end of this section, we will see that this beautiful scenario is not true in general: indeed, we’ll construct other (not square-tiled) cyclic covers leading to a merely measurable neutral Oseledets subspace.

8.1. Hodge theory and the Lyapunov exponents of square-tiled cyclic covers. Consider a square-tiled cyclic cover

$$M = M_N(a_1, \ldots, a_4) = \{ y^N = (x - x_1)^{a_1} \cdots (x - x_4)^{a_4} \}, \quad \omega := (x - x_1)^{b_1} \cdots (x - x_4)^{b_4} dx / y^{N/2}$$

where $\gcd(N, a_1, \ldots, a_4) = 1$, $2b_j = a_j - 1$, $N$ is even and $0 < a_j < N$ are odd. Cf. Subsection 7.3 for more comments.

The arguments in the previous two sections readily show that the locus of $(M_N(a_1, \ldots, a_4), \omega)$ is the $SL(2, \mathbb{R})$-orbit of a square-tiled surface. Thus, it makes sense to discuss the Lyapunov exponents of KZ cocycle with respect to the unique $SL(2, \mathbb{R})$-invariant probability supported on the locus of $(M_N(a_1, \ldots, a_4), \omega)$.

For “linear algebra reasons”, it is better to work with complex version of the KZ cocycle on the complex Hodge bundle: indeed, as we shall see in a moment, it is easier to diagonalize by blocks the complex KZ cocycle, while we keep the same Lyapunov exponents of usual (real) KZ cocycle.

We can diagonalize by blocks the complex KZ cocycle over $(M_N(a_1, \ldots, a_4), \omega)$ using the automorphism $T(x, y) = (x, \varepsilon y)$, $\varepsilon = \exp(2\pi i / N)$. More precisely, since $T^N = \text{Id}$, one can write

$$H^1(M, \mathbb{C}) = \bigoplus_{j=1}^{N-1} H^1(\varepsilon^j)$$

where $H^1(\varepsilon^j)$ is the eigenspace of the cohomological action $T^*$ of the automorphism $T$.

Remark 87. Here, the eigenspace $H^1(\varepsilon^0)$ is not present because any element of $H^1(\varepsilon^0)$ is $T^*$-invariant and hence it projects to $H^1(\overline{\mathbb{C}}) = \{0\}$ under $p : M_N(a_1, \ldots, a_4) \to \overline{\mathbb{C}}$, $p(x, y) = x$.

\[40\text{Furthermore, the Veech group has a simple dependence on } N, a_1, \ldots, a_4 \text{ and it has index 1, 2, 3 or 6 inside } SL(2, \mathbb{Z}). \text{ See [32].} \]
We affirm that these blocks are invariant under the complex KZ cocycle. Indeed, since the automorphism $T$ acts by pre-composition with the translation charts of $(M_N(a_1, \ldots, a_4), \omega)$ and $\text{SL}(2, \mathbb{R})$ acts by post-composition with translation charts, it follows that $T^*$ commutes with the complex KZ cocycle and, a fortiori, the eigenspaces of $T^*$ serve to diagonalize by blocks the complex KZ cocycle.

Next, we recall that the complex KZ cocycle preserves the Hodge form $(\alpha, \beta) = \frac{1}{2} \int \alpha \wedge \beta$, a positive definite form on $H^{1,0}$ and negative definite form on $H^{0,1}$ (cf. Subsection 3.4). Since $H^1(M, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$, we have that $H^1(\varepsilon^j) = H^{1,0}(\varepsilon^j) \oplus H^{0,1}(\varepsilon^j)$, and, thus, the restriction of the complex KZ cocycle to $H^1(\varepsilon^j)$ acts via some elements of the group $U(p_j, q_j)$ of complex matrices preserving a non-degenerate (pseudo-)Hermitian form of signature

$$p_j := \dim \mathbb{C}H^{1,0}(\varepsilon^j), \quad q_j := \dim \mathbb{C}H^{0,1}(\varepsilon^j)$$

In the setting of square-tiled cyclic covers, these signatures are easy to compute in terms of $N, a_1, \ldots, a_4$ in view of the next lemma:

**Lemma 88** (I. Bouw). Let $[x]$ be the integer part of $x$. One has

$$p_j = \sum_{n=1}^{4} \left\lfloor \frac{a_n j}{N} \right\rfloor - 1$$

and

$$q_j = \sum_{n=1}^{4} \left\lfloor \frac{a_n (N - j)}{N} \right\rfloor - 1$$

In particular, $p_j, q_j \in \{0, 1, 2\}$ and $p_j + q_j \in \{0, 1, 2\}$.

**Proof.** A sketch of proof of this result goes as follows. Since $H^{0,1}(\varepsilon^j) = \overline{H^{1,0}(\varepsilon^N-j)}$, we have that $q_j = p_{N-j}$ and, therefore, it suffices to compute $p_j$.

For this, it suffices to study whether the meromorphic form

$$\alpha_j(b_1, b_2, b_3, b_4) := (x - x_1)^{b_1}(x - x_2)^{b_2}(x - x_3)^{b_3}(x - x_4)^{b_4}dx/y^j$$

is holomorphic near $x_1, \ldots, x_4$ and $\infty$.

By performing the necessary calculations, one verifies that there exists a choice of $b_1, \ldots, b_4$ with $\alpha_j(b_1, \ldots, b_4)$ holomorphic if and only if

$$\sum_{n=1}^{4} \left\lfloor \frac{a_n j}{N} \right\rfloor \geq 2$$

Moreover, if this inequality is satisfied then $\alpha_j(b_1, \ldots, b_4)$ is holomorphic for $b_n := \left\lfloor \frac{a_n j}{N} \right\rfloor$. Furthermore, if $\sum_{n=1}^{4} \left\lfloor \frac{a_n j}{N} \right\rfloor = 3$, one can also check that $H^{1,0}(\varepsilon^j)$ is spanned by $\alpha_j(b_1, \ldots, b_4)$ and $x \cdot \alpha_j(b_1, \ldots, b_4)$ for $b_n := \left\lfloor \frac{a_n j}{N} \right\rfloor$. For the details of this computation, see, e.g., [22].
Note that this completes the sketch of proof of lemma: indeed, since $0 < a_n < N$ and $a_1 + \cdots + a_4$ is a multiple of $N$, we have that
\[ 1 \leq \sum_{n=1}^{4} \left[ \frac{a_n}{N} \right] \leq 3 \]
because $a_1 + \cdots + a_4 \in \{N, 2N, 3N\}$. So, our discussion above covers all cases. $\square$

This lemma suggests the following “separation” between the blocks $H^1(\varepsilon^j)$:
\[ \mathcal{N} := \{0 < j < N - 1 : p_j \text{ or } q_j = 0\} \]
and
\[ \mathcal{P} := \{0 < j < N - 1 : p_j = q_j = 1\} \]
In fact, by Lemma 88, we have that $\mathcal{N} \cup \mathcal{P} = \{1, \ldots, N - 1\}$, $N/2 \in \mathcal{P}$, and the restriction of complex KZ cocycle to $H^1(\varepsilon^j)$ acts via matrices in the group:
- $U(p_j, 0)$, $0 \leq p_j \leq 2$, or $U(0, q_j)$, $0 \leq q_j \leq 2$, whenever $j \in \mathcal{N}$;
- $U(1, 1)$ whenever $j \in \mathcal{P}$.
From the point of view of Lyapunov exponents, this “separation” is natural because the groups $U(p_j, 0)$ or $U(0, q_j)$ are compact while the group $U(1, 1) \simeq SL(2, \mathbb{R})$ is not compact. More concretely, an immediate consequence of the compactness of the groups $U(p_j, 0)$ or $U(0, q_j)$ is:

**Corollary 89.** If $j \in \mathcal{N}$, then all Lyapunov exponents of the restriction of the complex KZ cocycle to $H^1(\varepsilon^j)$ vanish.

Alternatively, this corollary can be derived by analyzing the second fundament form $B_\omega$. More precisely, let $H^1_0(M, \mathbb{R}) := (H^1(\varepsilon^j) \oplus H^1(\varepsilon^{N-j})) \cap H^1(M, \mathbb{R})$. It is possible to show (by adapting the arguments of the proof of Theorem 44) that the sum of non-negative Lyapunov exponents of KZ cocycle in $H^1_0(M, \mathbb{R})$ coincide with the average of the sum of eigenvalues of the restriction of $H_\omega = B_\omega \cdot B^*_\omega$ to $H^{1,0}(\varepsilon^j) \oplus H^{1,0}(\varepsilon^{N-j})$. See, e.g., [33] for more details. So, we can re-obtain this corollary by showing that $B_\omega$ vanishes on $H^{1,0}(\varepsilon^j) \oplus H^{1,0}(\varepsilon^{N-j})$ when $j \in \mathcal{N}$. Then, one realizes that this is true since $j \in \mathcal{N}$ implies that $H^{1,0}(\varepsilon^j) \oplus H^{1,0}(\varepsilon^{N-j}) = H^{1,0}(\varepsilon^j)$ or $H^{1,0}(\varepsilon^{N-j})$, and the restriction of $B_\omega$ to $H^{1,0}(\varepsilon^k)$ vanishes for every $k \neq N/2$ because of the following computation:

\[ B_\omega(\alpha, \beta) = B_{T^*(\omega)}(T^*(\alpha), T^*(\beta)) = \varepsilon^{2k} B_\omega(\alpha, \beta) \]

Actually, one can further play with the form $B_\omega$ to show a “converse” to this corollary, that is, the Lyapunov exponents of the restriction of the complex KZ cocycle to $H^1(\varepsilon^j)$ are non-zero whenever $j \in \mathcal{P}$. In fact, if $j \in \mathcal{P}$, the restriction of KZ cocycle to $H^1(\varepsilon^j)$ has Lyapunov exponents $\pm \lambda(j)$ (because it acts via matrices in $U(1, 1) \simeq SL(2, \mathbb{R})$). Moreover, the restriction of KZ cocycle to $H^1(\varepsilon^j)$ is conjugated to the restriction of KZ cocycle to $H^1(\varepsilon^{N-j})$, so that $\lambda(j) = \lambda(N-j)$.

\[^{41}\text{Here, we used the automorphism } T \text{ to change variables in the integral defining } B_\omega. \text{ Cf. Theorem 63.}\]
Therefore, by the discussion of the previous paragraph, we can deduce that \( \lambda_{(j)} = \lambda_{(N-j)} \) is non-zero by showing that 
the restriction of \( B_\omega \) to \( H^{1,0}(\varepsilon) \oplus H^{1,0}(\varepsilon^{N-j}) \) is not degenerate. Here, this last fact is true because \( H^{1,0}(\varepsilon) := \mathbb{C} \cdot \alpha_j := \mathbb{C} \cdot \alpha_j(b_1(j), \ldots, b_4(j)), b_k(j) := [a_k j/N], \) for \( j \in \mathcal{P} \) (cf. Lemma \ref{lem:explicit}), so that

\[
B_\omega(\alpha_j, \alpha_{N-j}) = \int_M \prod_{k=1}^4 |x - x_k|^{a_k-1}|dx|^2/|y|^N \neq 0
\]

Here, we used that \([a_k j/N] + [a_k (N-j)/N] = a_k - 1 \) for \( j \in \mathcal{P} \).

In other words, we just showed that

\textbf{Corollary 90.} If \( j \in \mathcal{P} \), the Lyapunov exponents \( \pm \lambda_{(j)} \) of the restriction of the complex KZ cocycle to \( H^1(\varepsilon) \) are non-zero.

At this point, we can say (in view of Corollaries \ref{cor:explicit} and \ref{cor:lyapunov}) that the Lyapunov spectrum of KZ cocycle over square-tiled cyclic covers is \textit{qualitatively} well-known: it can be diagonalized by blocks by restriction to \( H^1(\varepsilon) \) and zero Lyapunov exponents come precisely from blocks \( H^1(\varepsilon) \) with \( j \in \mathcal{N} \). However, in some applications\cite{Zorich}, it is important to determine \textit{quantitatively} individual exponents of KZ cocycle. In the case of cyclic covers, A. Eskin, M. Kontsevich and A. Zorich\cite{Zorich} determined the value of \( \lambda_{(j)} \) for \( j \in \mathcal{P} \). Roughly speaking, they start the computation \( \lambda_{(j)} = \lambda_{(N-j)} \) from the fact (already mentioned) that \( 2\lambda_{(j)} = \lambda_{(j)} + \lambda_{(N-j)} \) coincides with the average of the eigenvalues of \( H_\omega = B_\omega \cdot B_\omega^* \) restricted to \( H^{1,0}(\varepsilon) \oplus H^{1,0}(\varepsilon^{N-j}) \). Then, they use the fact that \( H^{1,0}(\varepsilon) \) has complex dimension 1 for \( j \in \mathcal{P} \) to reduce the calculation of the aforementioned average to the computation of the \textit{orbifold degree} of the \textit{line bundle} \( H^{1,0}(\varepsilon) \). After this, the calculation of the orbifold degree of \( H^{1,0}(\varepsilon) \) can be performed \textit{explicitly} by noticing that \( H^{1,0}(\varepsilon) \) has a \textit{global section} \( H^{1,0}(\varepsilon) = \mathbb{C} \cdot \alpha_j(b_1(j), \ldots, b_4(j)) := \mathbb{C} \cdot \alpha_j \) over the \( SL(2, \mathbb{R}) \)-invariant

\[
\{(y^N = (x - x_1)^{a_1} \ldots (x - x_4)^{a_4}), \omega \}.
\]

Since the orbifold degree is expressed as a certain integral depending on \( \alpha_j \), a sort of integration by parts argument can be used to rewrite the orbifold degree in terms of the “behavior at infinity” of \( \alpha_j \), that is, the behavior of \( \alpha_j \) when \( x_i \) approaches \( x_j \) for some \( i \neq j \). This led them to the following result:

\textbf{Theorem 91} (A. Eskin, M. Kontsevich and A. Zorich). Let \( b_k(j) = [a_k j/N] \). Then,

\[
\lambda_{(j)} = 2 \cdot \min\{b_1(j), 1 - b_1(j), \ldots, b_4(j), 1 - b_4(j)\}
\]

for any \( j \in \mathcal{P} \).

\textsuperscript{42}For instance, the precise knowledge of Lyapunov exponents of KZ cocycle for a certain \( SL(2, \mathbb{R}) \)-invariant \( \mu \)-ergodic probability supported in \( \mathcal{H}_5 \) recently permitted V. Delecroix, P. Hubert and S. Lelièvre\cite{Delecroix} to confirm a conjecture of the physicists J. Hardy and J. Weber that the so-called \textit{Ehrenfest wind-tree model} of Lorenz gases has \textit{abnormal} diffusion for typical choices of parameters.
Coming back to the qualitative analysis of KZ cocycle (and its Lyapunov spectrum) over square-tiled cyclic covers, we observe that our discussion so far shows that the neutral Oseledets bundle $E^c$ (associated to zero Lyapunov exponents) of KZ cocycle coincides with the annihilator $\text{Ann}(B^R)$ in the case of square-tiled cyclic covers. In other words, the zero Lyapunov exponents of KZ cocycle have a nice geometrical explanation in the case of square-tiled cyclic covers: they come precisely from the annihilator of the second fundamental form $B^R$! In particular, since $B^\omega$ depends continuously (real-analytically) on $\omega$, we conclude that the neutral Oseledets bundle $E^c$ of KZ cocycle is continuous in the case of square-tiled cyclic covers! Of course, this should be compared with the general statement of Oseledets theorem ensuring only measurability of Oseledets subspaces. A nice consequence of the equality $E^c = \text{Ann}(B^R)$ for square-tiled cyclic covers is the fact that, in this setting, $E^c = \text{Ann}(B^R)$ is $SL(2, \mathbb{R})$-invariant: indeed, this is a mere corollary of $g_t$-invariance of $E^c$ (coming from Oseledets theorem), $SO(2, \mathbb{R})$-invariance of $\text{Ann}(B^R)$ (coming from the definition of $B^\omega$) and the fact that $SL(2, \mathbb{R})$ is generated by the diagonal subgroup $g_t$ and the rotation subgroup $SO(2, \mathbb{R})$. Furthermore, by combining the $SL(2, \mathbb{R})$-invariance of $E^c = \text{Ann}(B^R)$ for square-tiled cyclic covers with Theorem 29, we deduce that KZ cocycle acts isometrically on $E^c = \text{Ann}(B^R)$ in this setting. In resume, this discussion of this paragraph proves the following result (from \cite{33}):

**Theorem 92.** The neutral Oseledets bundle $E^c$ of KZ cocycle over square-tiled cyclic covers is a continuous $SL(2, \mathbb{R})$-invariant subbundle of the Hodge bundle. Moreover, the KZ cocycle acts isometrically on $E^c$ because $E^c$ coincides with the annihilator of the second fundamental form $B^\omega$.

It is tempting to ask whether this scenario holds for other $SL(2, \mathbb{R})$-invariant probabilities, that is, whether the annihilator of the second fundamental form is always the explanation of zero Lyapunov exponents of KZ cocycle with respect to $SL(2, \mathbb{R})$-invariant probabilities. As it turns out, we’ll see below an example where the mechanism responsible for the neutral Oseledets bundle is not related to the second fundamental form $B^\omega$.

### 8.2. Other cyclic covers.
Consider the following family of Riemann surfaces

$$M = M_{10}(x_1, \ldots, x_6) = \{y^6 = (x - x_1) \ldots (x - x_6)\},$$

where $x_1, \ldots, x_6 \in \overline{\mathbb{C}}$ are mutually distinct, equipped with the meromorphic differential

$$\eta = (x - x_1)dx/y^3$$

By Riemann-Hurwitz formula applied to $p : M_{10}(x_1, \ldots, x_6) \to \overline{\mathbb{C}}$, $p(x, y) = x$, one can check that $M_{10}(x_1, \ldots, x_6)$ has genus 10. Furthermore, by studying $\eta$ near $x_1, \ldots, x_6$ (and $\infty$), one can see

---

43At the complex level, this can be also seen from the fact that, by Corollaries 49 and 90, $\bigoplus_{j \in \mathbb{Z}} H^1(\mathcal{E}^j)$ is the neutral Oseledets bundle of the complex KZ cocycle, and the subspaces $H^1(\mathcal{E}^j)$ vary continuously with $\omega$ because they are eigenspaces of the cohomological action of an automorphism.

44We saw this fact in the particular cases of Eierlegende Wollmilchsau (Remark 64) and Ornithorynque (Theorem 67).
that $\eta$ has a zero of order 8 over $x_1$ and 5 double zeroes over $x_2, \ldots, x_6$, i.e., $\eta \in \mathcal{H}(8, 2^5) := \mathcal{H}(8, 2, \ldots, 2)$.

We denote by $Z$ the locus of $\mathcal{H}(8, 2^5)$ determined by the family $(M_{10}(x_1, \ldots, x_6), \omega)$ where $\omega$ is the positive multiple of $\eta$ with unit area.

**Lemma 93.** $Z$ is a closed $SL(2, \mathbb{R})$-invariant locus of $\mathcal{H}(8, 2^5)$ naturally isomorphic to the stratum $\mathcal{H}^{(1)}(2)$ of unit area translation surfaces in $\mathcal{H}(2)$.

**Proof.** Let’s consider the covering $h : M_{10}(x_1, \ldots, x_6) \to M_2(x_1, \ldots, x_6)$, $h(x, y) = y^3$, where

$$M_2(x_1, \ldots, x_6) := \{ z^2 = (x - x_1) \cdots (x - x_6) \}$$

Note that $M_2(x_1, \ldots, x_6)$ is a genus 2 Riemann surface (by Riemann-Hurwitz formula). Moreover,

$$\omega = h^* (\theta)$$

where $\theta$ is the positive multiple of $(x - x_1)dx/z \in \mathcal{H}(2)$ with unit area.

Because any Riemann surface of genus 2 is hyperelliptic, it is not hard to see that the family

$$(M_2(x_1, \ldots, x_6), \theta)$$

parametrizes the entire stratum $\mathcal{H}^{(1)}(2)$ of unit area translation surfaces in $\mathcal{H}(2)$: indeed, any Riemann surface genus 2 has six Weierstrass points (in this case they are the fixed points of the hyperelliptic involution), so that it can be represented an algebraic equation of the form

$$z^2 = (x - x_1) \cdots (x - x_6)$$

Here, the six Weierstrass points are located over $x_1, \ldots, x_6$. In other words, any genus 2 Riemann surface is biholomorphic to some $M_2(x_1, \ldots, x_6)$. Also, the zero of an Abelian differential $\theta$ in $M_2(x_1, \ldots, x_6)$ must be located at one of the Weierstrass points. Thus, by renumbering the points $x_1, \ldots, x_6$ (in order to place the zero over $x_1$), we can write a $\theta \in \mathcal{H}(2)$ in $M_2(x_1, \ldots, x_6)$ as a multiple of $(x - x_1)dx/z$. Alternatively, one can show that

$$\{(M_2(x_1, \ldots, x_6), \theta) : x_1, \ldots, x_6 \in \mathbb{C} \text{ distincts} \}$$

is the entire stratum $\mathcal{H}^{(1)}(2)$ by counting dimensions. More precisely, by using Möbius transformations, one can normalize $x_1 = 0, x_2 = 1, x_3 = \infty$ and we see that the Riemann surface structure of $M_2(x_1, \ldots, x_6)$ depends on 3 complex parameters (namely, $x_4, x_5, x_6$ after normalization). Furthermore, the choice of an Abelian differential $\theta$ in $\mathcal{H}(2)$ to attach $M_2(x_1, \ldots, x_6)$ depends on 1 complex parameter in general. However, $\theta$ is normalized to have unit area, the actual choice of $\theta$ depends on 1 real parameter. Hence, the locus

$$\{(M_2(x_1, \ldots, x_6), \theta) : x_1, \ldots, x_6 \in \mathbb{C} \text{ distincts} \} \subset \mathcal{H}^{(1)}(2)$$
has real dimension 7. Now, we observe that the stratum $\mathcal{H}(2)$ has complex dimension 4, i.e., real dimension 8 (cf. Section 2), so that the locus $\mathcal{H}^{(1)}(2)$ of unit area translation surfaces in $\mathcal{H}(2)$ has real dimension 7, and a connectedness argument can be employed to show

$$\{(M_2(x_1, \ldots, x_6), \theta) : x_1, \ldots, x_6 \in \mathbb{C} \text{ distincts}\} = \mathcal{H}^{(1)}(2)$$

Therefore, we have that the locus $Z$ can be recovered from $\mathcal{H}^{(1)}(2) = \{(M_2(x_1, \ldots, x_6), \theta)\}$ by taking triple covers $p : (M_{10}(x_1, \ldots, x_6), \omega) \rightarrow (M_2(x_1, \ldots, x_6), \theta)$, $p(x, y) = (x, y^3)$. Because $SL(2, \mathbb{R})$ acts on translation surfaces by post-composition with translation charts while the covering $p$ is obtained by pre-composition with translation charts, we deduce that $Z$ is a $SL(2, \mathbb{R})$-invariant closed locus from the fact that $\mathcal{H}^{(1)}(2)$ has the same properties. $\square$

Just to get a “feeling” on how the flat structure of translation surfaces in $\mathcal{H}(2)$ look like, we notice the following facts. It is not hard to check that the flat structure associated to $(M_2(x_1, \ldots, x_6), \theta)$ is described by the following octagon (whose opposite parallel sides are identified):

Here, the vertices of this octagon are all identified to a single point corresponding to $x_1$. Moreover, since $x_2, \ldots, x_6$ are Weierstrass points of $M_2(x_1, \ldots, x_6)$, one can organize the picture in such a way that the four points $x_2, \ldots, x_5$ are located exactly at the middle points of the sides, and $x_6$ is located at the “symmetry center” of the octagon. See the picture above for an indication of the relative positions of $x_1$ (marked by a black dot) and $x_2, \ldots, x_6$ (marked by crosses). In this way, we obtain a concrete description of $\mathcal{H}(2)$.

Now, since $Z$ is defined by Abelian differentials $(M_{10}(x_1, \ldots, x_6), \omega)$ given by certain triple (ramified) covers of the Abelian differentials $(M_2(x_1, \ldots, x_6), \theta) \in \mathcal{H}(2)$, one can check that the flat structure associated to $(M_{10}(x_1, \ldots, x_6), \omega)$ is described by the following picture:
Here, we glue the half-sides determined by the vertices (black dots) and the crosses of these five pentagons in a cyclic way, so that every time we positively cross the side of a pentagon indexed by \( j \), we move to the corresponding side on the pentagon indexed \( j + 1 \) (mod 5). For instance, in the figure above we illustrated the effect of going around the singularity point over \( x_6 \).

The natural isomorphism between the locus \( \mathcal{Z} \) and \( \mathcal{H}(1)(2) \) enables us to put \( SL(2, \mathbb{R}) \)-invariant measures on \( \mathcal{Z} \): for example, by pulling back to \( \mathcal{Z} \) the Masur-Veech probability of \( \mathcal{H}(1)(2) \) we obtain a fully supported \( SL(2, \mathbb{R}) \)-invariant \( g_\mu \)-ergodic probability on \( \mu_\mathcal{Z} \).

In what follows, we will study the Lyapunov spectrum of KZ cocycle with respect to \( \mu_\mathcal{Z} \). By the reasons explained in the previous subsection, we consider the complex KZ cocycle over \( \mathcal{Z} \).

Denoting by \( T(x, y) = (x, \varepsilon y), \varepsilon = \exp(2\pi i/6) \), the automorphism of order 6 of \( M_{10}(x_1, \ldots, x_6) \) generating the Galois group of the covering \( p : M_{10}(x_1, \ldots, x_6) \to \overline{\mathbb{C}}, p(x, y) = x \), we can write

\[
H^1(M, \mathbb{C}) = \bigoplus_{j=1}^{5} H^1(\varepsilon^j)
\]

where \( H^1(\varepsilon^j) \) is the eigenspace of the eigenvalue \( \varepsilon^j \) of the cohomological action \( T^* \) of \( T \). Again, the fact that \( T \) is an automorphism implies that the complex KZ cocycle \( G^{KZ,\mathbb{C}}_t \) preserves each \( H^1(\varepsilon^j) \), that is, we can use these eigenspaces to diagonalize by blocks the complex KZ cocycle.

A direct computation reveals that

\[
\{(x - x_1)^{k-1}dx/y^j : 0 < k < j < 6\}
\]

is a basis of holomorphic differentials of \( M_{10}(x_1, \ldots, x_6) \). In particular, \( \dim_{\mathbb{C}} H^1(\varepsilon^j) = j - 1 \).

Since \( H^1(\varepsilon^j) = H^{1,0}(\varepsilon^j) \oplus H^{0,1}(\varepsilon^j) \) and \( H^{0,1}(\varepsilon^{6-j}) = H^{1,0}(\varepsilon^j) \), we deduce that the restriction of the complex KZ cocycle to \( H^1(\varepsilon^j) \) acts via matrices in the group

\[
U(j-1, 5-j)
\]

Therefore, we obtain that \( G^{KZ,\mathbb{C}}_t|_{H^1(\varepsilon^j)} \) has only zero Lyapunov exponents for \( j = 1 \) and 5 because the compactness of the groups \( U(0, 4) \) and \( U(4, 0) \).

Next, we observe that one can “reduce” the study of \( G^{KZ,\mathbb{C}}_t|_{H^1(\varepsilon^j)} \) to the complex KZ cocycle over \( \mathcal{H}(1)(2) \): more precisely, the fact that \( \varepsilon^3 = -1 \) shows that \( H^1(\varepsilon^3) = h^*(H^1(M_2(x_1, \ldots, x_6), \mathbb{C})) \).
where \( h : M_10(x_1, \ldots, x_6) \to M_2(x_1, \ldots, x_6), h(x, y) = (x, y^3) \), that is, \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \) is a copy of KZ cocycle over the stratum \( H(1)(2) \). By Remark 77, this means that we know the Lyapunov exponents of \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \), namely, they are \( \pm 1 \) and \( \pm 1/3 \).

Thus, it remains “only” to investigate \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \) and \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \). In fact, since \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \) is the complex conjugate of \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \), it suffices to study the latter cocycle.

Here, we will take advantage of the fact that \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \) acts via \( U(3, 1) \) to get \( 2 = 3 - 1 \) “automatic” zero Lyapunov exponents coming from general linear algebra:

**Proposition 94.** \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \) has 2 zero Lyapunov exponents (at least).

**Proof.** Since the cocycle \( C_t := G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \) preserves the Hodge intersection form \((.,.)\), we have

\[
(v, w) = (C_t(v), C_t(w))
\]

for all \( t \in \mathbb{R} \).

Let \( v, w \) be two vectors in some Oseledets subspaces associated to Lyapunov exponents \( \lambda, \mu \) with \( \lambda + \mu \neq 0 \). By definition of Lyapunov exponents, one has that

\[
|(C_t(v), C_t(w))| \leq C e^{(\lambda+\mu)t}\|v\|\|w\| \to 0
\]
as \( t \to +\infty \) or \( -\infty \) (depending on whether \( \lambda + \mu < 0 \) or \( \lambda + \mu > 0 \)).

Therefore, we conclude that \( (v, w) = 0 \) whenever \( v, w \) belong to Oseledets subspaces associated to Lyapunov exponents \( \lambda, \mu \) with \( \lambda + \mu \neq 0 \). In particular, denoting by \( E^u \), resp. \( E^s \), the *unstable*, resp. *stable*, Oseledets subspace associated to the *positive*, resp. *negative*, Lyapunov exponents of \( G_{t_1}^{KZ, \mathbb{C}}|_{H^1(\varepsilon^4)} \), we obtain that \( E^u \) and \( E^s \) are *isotropic* vector subspaces with respect to the Hodge form \((.,.)\), i.e.,

\[
E^u, E^s \subset \mathcal{C}
\]
where \( \mathcal{C} = \{v \in H^1(\varepsilon^4) : (v, v) = 0\} \) is the *light-cone* of the pseudo-Hermitian form \((.,.)|_{H^1(\varepsilon^4)}\) of signature \((3, 1)\).

At this point, the following general linear algebra fact is useful.

**Lemma 95.** A vector space \( V \) inside the light-cone of a pseudo-Hermitian form of signature \((p, q)\) has dimension \( \min\{p, q\} \) at most.

**Proof.** By taking adequate coordinates, we may assume that \((.,.)\) is the standard Hermitian form of signature \((p, q)\) in \( \mathbb{C}^{p+q} \):

\[
(z, w) = z_1\overline{w_1} + \cdots + z_p\overline{w_p} - z_{p+1}\overline{w_{p+1}} - z_{p+q}\overline{w_{p+q}}
\]

By symmetry, we can assume that \( p \leq q \), i.e., \( \min\{p, q\} = p \).

Suppose that \( V \) is a vector space of dimension \( \geq p + 1 \) inside the light-cone \( \mathcal{C} \). Let’s select \( v^{(1)}, \ldots, v^{(p+1)} \in V \) a collection of \( p + 1 \) linearly independent vectors and let’s define \( w^{(1)} := (v^{(1)}_1, \ldots, v^{(1)}_p), \ldots, w^{(p+1)} := (v^{(p+1)}_1, \ldots, v^{(p+1)}_p) \in \mathbb{C}^p \), where \( v^{(i)}_j \in \mathbb{C} \) is the \( j \)th coordinate of the vector \( v^{(i)} \).
Because \( w^{(1)}, \ldots, w^{(p+1)} \) is a collection of \( p+1 \) vectors in \( \mathbb{C}^p \), we can find a non-trivial collection of coefficients \( (a_1, \ldots, a_{p+1}) \in \mathbb{C}^{p+1} - \{0\} \) with

\[
\sum_{j=1}^{p+1} a_j w^{(j)} = 0
\]

Since \( w^{(j)} \) were built from the \( p \) first coordinates of \( v^{(j)} \), we deduce that

\[
V \ni v := \sum_{j=1}^{p+1} a_j v^{(j)} = (0, \ldots, 0, v_{p+1}, \ldots, v_{p+q})
\]

However, since \( v \in V \subset \mathbb{C} \), one would have

\[
0 = (v, v) = -|v_{p+1}|^2 - \cdots - |v_{p+q}|^2
\]

that is, \( v = 0 \), a contradiction with the linear independence of \( v^{(1)}, \ldots, v^{(p+1)} \in V \) because \( (a_1, \ldots, a_{p+1}) \in \mathbb{C}^{p+1} - \{0\} \). This shows that \( \dim V \leq p \) whenever \( V \subset \mathbb{C} \), as desired. \( \square \)

By applying this lemma in the context of the cocycle \( G_{t|H^1(\varepsilon^4)}^{KZ, \mathcal{C}} \), we get that \( E^u \) and \( E^s \) have dimension 1 at most because they are in the light-cone of a pseudo-Hermitian form of signature \((3, 1)\). Since \( H^1(\varepsilon^4) = E^u \oplus E^c \oplus E^s \) and \( H^1(\varepsilon^4) \) has dimension 4, we deduce that

\[
\dim(E^c) = 4 - \dim(E^u) - \dim(E^s) \geq 4 - 1 - 1 = 2,
\]

i.e., \( G_{t|H^1(\varepsilon^4)}^{KZ, \mathcal{C}} \) has 2 zero Lyapunov exponents at least. \( \square \)

In fact, by computing the restriction of form \( B_\omega \) to \( H^{1,0}(\varepsilon^2) \oplus H^{1,0}(\varepsilon^4) \) in the basis of holomorphic differentials

\[
\{dx/y^2, dx/y^4, (x-x_1)dx/y^4, (x-x_1)^2dx/y^4\}
\]

one can show that it has rank 2. By combining this with an analog of Theorem 44 to \( G_{t|H^1(\varepsilon^4)}^{KZ, \mathcal{C}} \), one can show that \( G_{t|H^1(\varepsilon^4)}^{KZ, \mathcal{C}} \) has exactly 2 zero Lyapunov exponents, a positive Lyapunov exponent \( \lambda \) and a negative Lyapunov exponent \( -\lambda \). Furthermore, the natural isomorphism between \( \mathcal{Z} \) and \( H^{(1)}(2) \) permits to compute the Siegel-Veech constant (cf. Subsection 6.4) of \( \mu_\mathcal{Z} \) and this knowledge together with Theorem 75 can be put forward to provide the explicit value \( \lambda = 4/9 \).

We refer the reader to [33] and [34] where this is discussed in details.

After this discussion, we understand completely the Lyapunov spectrum of KZ cocycle with respect to the “Masur-Veech” measure \( \mu_\mathcal{Z} \) of the locus \( \mathcal{Z} \):

**Proposition 96.** The non-negative part of the Lyapunov spectrum of KZ cocycle with respect to \( \mu_\mathcal{Z} \) is

\[
\{1 > 4/9 = 4/9 > 1/3 > 0 = 0 = 0 = 0 = 0 = 0\}
\]
Now, we pass to the study of the neutral Oseledets subspace of KZ cocycle over \( \mathcal{Z} \), or, more precisely \(^{45}\) \( G_t^{KZ,C}_{t_2} \mid_{H^1(\varepsilon^t)} \). Here, it is worth to notice that the neutral Oseledets subspace of \( G_t^{KZ,C}_{t_2} \mid_{H^1(\varepsilon^t)} \) and the intersection of the annihilator of \( B_\omega \) with \( H^{1,0}(\varepsilon^2) \oplus H^{1,0}(\varepsilon^4) \) have the same rank (namely, 2). In particular, it is natural to ask whether these subspaces coincides, or equivalently, the neutral Oseledets subspace of \( G_t^{KZ,C}_{t_2} \mid_{H^1(\varepsilon^t)} \) has a nice geometrical explanation.

This was shown not to be true in \(^{34}\) along the following lines. Since the neutral Oseledets subspace is \( g_t \)-invariant and the annihilator of \( B_\omega \) is continuous and \( SO(2,\mathbb{R}) \)-invariant, the coincidence of these subspaces would imply that the neutral Oseledets subspace of \( G_t^{KZ,C}_{t_2} \mid_{H^1(\varepsilon^t)} \) is a (rank 2) continuous \( SL(2,\mathbb{R}) \)-invariant subbundle of \( H^1(\varepsilon^4) \). This property imposes severe restrictions on the behavior \( G_t^{KZ,C}_{t_2} \mid_{H^1(\varepsilon^t)} \): for instance, by considering two periodic (i.e., pseudo-Anosov) orbits of the Teichmüller flow in the same \( SL(2,\mathbb{R}) \) associated to two Abelian differentials on the same Riemann surface, we get that the matrices \( A \) and \( B \) representing \( G_t^{KZ,C}_{t_2} \mid_{H^1(\varepsilon^t)} \) along these periodic orbits must share a common subspace of dimension 2, and this last property can be contradicted by explicitly computing with some periodic orbits. Unfortunately, while this idea is very simple, the calculations needed to implement it are somewhat long and we will not try to reproduce them here. Instead, we refer to Appendix A of \(^{34}\) where the calculation is largely detailed (and illustrated with several pictures).

Remark 97. During an exposition of this topic by the second author, Y. Guivarch asked whether \( G_t^{KZ,C}_{t_2} \mid_{H^1(\varepsilon^t)} \) still acts isometrically on its neutral subspace. This question is very natural and interesting because now that one can’t use variational formulas involving \( B_\omega \) to deduce this property (as we did in the case of square-tiled cyclic covers). As it turns out, the answer to Y. Guivarch’s question is positive by the following argument: one has that the neutral Oseledets subspace \( E^c \) is outside the light-cone \( C \) because the stable Oseledets subspace \( E^s \) has dimension 1, and so, if \( E^c \cap C \) were non-trivial, we would get a subspace \( (E^c \cap C) \oplus E^s \subset C \) of dimension at least 2 inside the light-cone \( C \) of an Hermitian form of signature (3,1), a contradiction with Lemma 95 above. In other words, the light-cone is a geometric mechanism of production of neutral Oseledets subbundles with isometric behavior genuinely different from the (also geometric) method of using the annihilator of the second fundamental form of Gauss-Manin connection of the Hodge bundle.

Of course, the fact that the neutral Oseledets subspace doesn’t coincide with the annihilator of the second fundamental form \( B_\omega \) is not the “end of the road”: indeed, by carefully inspecting the arguments of the previous paragraph one notices that it leaves open the possibility that the neutral Oseledets subspace maybe continuous despite the fact that it is not the annihilator of \( B_\omega \).

Heuristically, one strategy to “prove” that the neutral Oseledets subspace is not very smooth goes as follows: as we kneo, the Lyapunov exponents of the Teichmüller flow can be deduced from the ones of the KZ cocycle by shifting them by \( \pm 1 \); in this way, the smallest non-negative Lyapunov

\(^{45}\) Here we “excluded” the part of the neutral Oseledets bundle coming from the blocks \( H^1(\varepsilon) \) and \( H^1(\varepsilon^5) \) because the complex KZ cocycle acts via \( U(0,4) \) and \( U(4,0) \), and it is not hard to show from this that the corresponding part of the neutral Oseledets bundle is “geometrically explained” by the annihilation of the form \( B_\omega \) on these blocks.
The exponent of the Teichmüller flow is $5/9 = 1 - 4/9$; therefore, the generic points tend to be separated by Teichmüller flow by $\geq e^{5t/9}$ after time $t \in \mathbb{R}$; on the other hand, the largest Lyapunov exponent on the fiber $H^1(\varepsilon^4)$ is $4/9$, so that the angle between the neutral Oseledets bundle over two generic points grows by $\leq e^{4t/9}$ after time $t \in \mathbb{R}$; hence, in general, one can’t expect the neutral Oseledets bundle to be better than $\alpha = (4/9)/(5/9) = 4/5$ Hölder continuous.

Of course, there are several details missing in this heuristic, and currently we don’t know how to render it into a formal argument. However, in a recent work still in progress [6], A. Avila, J.-C. Yoccoz and the second author proved (among other things) that the neutral Oseledets subspace $E^c$ of $G_{KZ}^H|_{H^1(\varepsilon^4)}$ is not continuous at all (and hence only measurable by Oseledets theorem). In the sequel, we provide a brief sketch of this proof of the non-continuity of $E^c$.

As we mentioned a few times in this text, the Teichmüller flow and the Kontsevich-Zorich cocycle over (connected components of) strata can be efficiently coded by means of the so-called Rauzy-Veech induction. Roughly speaking, given a (connected component of a) stratum $C$ of Abelian differentials of genus $g \geq 1$, the Rauzy-Veech induction associates the following objects: a finite oriented graph $G(C)$, a finite collection of simplices (“Rauzy-Veech boxes”) and a finite number of copies of a Euclidean space $\mathbb{C}^{2g}$ over each vertex of $G(C)$, a (expanding) projective map between (parts of) the simplices over the vertices connected by this arrow, and a matrix between the copies of $\mathbb{C}^{2g}$ over the vertices connected by this arrow. We strongly recommend J.-C. Yoccoz’s survey [72] for more details on the Rauzy-Veech induction.

In this language, the simplices (Rauzy-Veech boxes) over the vertices of this graph represent admissible parameters determining translations surfaces (Abelian differentials on Riemann surfaces $M$) in $C$, the (expanding) projective map between (parts of) the simplices (associated to vertices connected by a given arrow) correspond to the action of the Teichmüller flow on the parameter space (after running this flow for an adequate amount of time), and the matrices (attached to the arrows) on $\mathbb{C}^{2g}$ are the action of the Kontsevich-Zorich cocycle on the first cohomology group $H^1(M, \mathbb{C}) \simeq \mathbb{C}^{2g}$.

Among the main properties of the Rauzy-Veech induction, we can highlight the fact that it permits to “simulate” almost every (with respect to Masur-Veech measure) orbit of Teichmüller flow on on $C$ in the sense that these trajectories correspond to (certain) infinite paths on the graph $G(C)$. In order words, the Rauzy-Veech induction allows to code the Teichmüller flow as a subshift of a Markov shift on countably many symbols (as one can use loops on $G(C)$ based on an arbitrarily fixed vertex as basic symbols / letters of the alphabet of our Markov subshift). Moreover, the KZ cocycle over these trajectories of Teichmüller flow can be computed by simply multiplying the matrices attached to the arrows one sees while following the corresponding infinite path on $G(C)$. Equivalently, we can think the KZ cocycle as a monoid of (countably many) matrices (as we can only multiply the matrices precisely when our oriented arrows can be concatenated, but in principle we don’t dispose of the inverses of our matrices because we don’t have the right to “revert” the orientation of the arrows).
In the particular case of $\mathcal{H}(2)$, the associated graph $\mathcal{G}(\mathcal{H}(2))$ is depicted below:

![Rauzy Diagram](image)

**Figure 18.** Schematic representation of the Rauzy diagram associated to $\mathcal{H}(2)$.

The letters near the arrows are not important here, only the 7 vertices (black dots) and the arrows between them.

Now, we observe that $Z$ was defined by taking certain triple covers of Abelian differentials of $\mathcal{H}(2)$, so that it is also possible to code the Teichmüller flow and KZ cocycle on $Z$ by the same graph and the same simplices over its vertices, but by changing the matrices attached to the arrows: in the case of $\mathcal{H}(2)$, these matrices acted on $\mathbb{C}^4$, but in the case of $Z$ they act on $\mathbb{C}^{20}$ and they contain the matrices of the case of $\mathcal{H}(2)$ as a block.

At this stage, one can prove non-continuity of the neutral Oseledets subspace $E^c$ of $G^KZ, C_{|H^1(\varepsilon^i)}$ as follows.

Firstly, one computes the restriction of KZ cocycle (or rather the matrices of the monoid) to $E^c$ on certain “elementary” loops and one checks that they have finite order. In particular, every time we can get the inverses of the matrices associated to these elementary loops by simply repeating these loops an appropriate number of times (namely, the order of the matrix minus 1). On the other hand, since these elementary loops are set up so that any infinite path (coding a Teichmüller flow orbit) is a concatenation of elementary loops, one conclude that the action (on $E^c \subset \mathbb{C}^{20}$) of our monoid of matrices is through a group! In particular, given any loop $\gamma$ (not necessarily an elementary one), we can find another loop $\delta$ such that the matrix attached to $\delta$ (i.e., the matrix obtained by multiplying the matrices attached to the arrows forming $\delta$ “in the order they show up” with respect to their natural orientation of $\delta$) is exactly the inverse of the matrix attached to $\gamma$.

Secondly, by computing with a pair of “sufficiently random” loops $\gamma_A$ and $\gamma_B$, it is not hard to see that we can choose such that their attached matrices $A$ and $B$ have distinct and/or transverse central eigenspaces $E^c_A$ and $E^c_B$ (associated to eigenvalues of modulus 1).

In this way, the periodic orbits (pseudo-Anosov orbits) of the Teichmüller flow coded by the infinite paths $...\gamma_A\gamma_A\gamma_A...$ and $...\gamma_B\gamma_B\gamma_B...$ obtained by infinite concatenation of the loops $\gamma_A$ and $\gamma_B$ have distinct and/or transverse neutral Oseldets bundle, but this is no contradiction to
continuity since the base points of these periodic orbits are not very close. However, we can use \( \gamma_A \) and \( \gamma_B \) to produce a contradiction as follows. Let \( k \gg 1 \) a large integer. Since our monoid acts by a group, we can find a loop \( \gamma_{C,k} \) such that the matrix attached to it is \( A^{-k} \). It follows that the matrix attached to the loop \( \gamma_{A,B,k} := \gamma_A \cdots \gamma_A \gamma_{C,k} \gamma_B \) is \( A^k \cdot A^{-k} \cdot B \), i.e., \( B \). Therefore, the infinite paths \( \ldots \gamma_A \gamma_A \gamma_A \ldots \) and \( \ldots \gamma_{A,B,k} \gamma_{A,B,k} \gamma_{A,B,k} \ldots \) produce periodic orbits whose neutral Oseledets bundle still are \( E_A^0 \) and \( E_B^0 \) (and hence, distinct and/or transverse), but this time their basepoints are arbitrarily close (as \( k \to \infty \)) because the first \( k \) “symbols” (loops) of the paths coding them are equal (to \( \gamma_A \)).

Remark 98. Actually, this argument is part of more general considerations in [6] on certain cyclic covers obtained by taking \( 2^n \) copies of a regular polygon with \( m \) sides, and cyclically gluing the sides of these polygons in such a way that their middle points become ramification points: indeed, \( Z \) corresponds to the case \( n = 3 \) and \( m = 5 \) of this construction.

Remark 99. It is interesting to notice that the real version of Kontsevich-Zorich cocycle over \( Z \) on \((H^1(\varepsilon^2) \oplus H^1(\varepsilon^4)) \cap H^1(M_{10}, \mathbb{R})\) is an irreducible symplectic cocycle with non-continuous neutral Oseledets bundle. In principle, this irreducibility at the real level makes it difficult to see the presence of zero exponents, so that the passage to its complex version (where we can decompose it as a sum of two complex conjugated monodromy representations by matrices in \( U(1, 3) \) and \( U(3, 1) \)) reveals a “hidden truth” not immediately detectable from the real point of view (thus confirming the famous quotation of J. Hadamard: “the shortest route between two truths in the real domain passes through the complex domain”). We believe this example has some independent interest because, to the best of our knowledge, most examples of symplectic cocycles and/or diffeomorphisms exhibiting some zero Lyapunov exponents usually have smooth neutral Oseledets bundle due to some sort of “invariance principle” (see this article of A. Avila and M. Viana [8] for some illustrations of this).

We state below two “optimistic guesses” on the features of the KZ cocycle over the support of general \( SL(2, \mathbb{R}) \)-invariant probabilities mostly based on our experience so far with cyclic covers. Notice that we call these “optimistic guesses” instead of “conjectures” because we think they’re shared (to some extent) by others working with Lyapunov exponents of KZ cocycle (and so it would be unfair to state them as “our” conjectures).

**Optimistic Guess 1.** Let \( \mu \) be a \( SL(2, \mathbb{R}) \)-invariant probability in some connected component of a stratum of Abelian differentials and denote by \( \mathcal{L} \) its support. Then, there exists a finite (ramified) cover \( \tilde{\mathcal{L}} \) such that (the lift of) the Hodge bundle \( H^1_\mathcal{L} \) over \( \tilde{\mathcal{L}} \) can be decomposed into a direct sum of continuous

\[
H^1_\mathcal{L} = L \oplus (A_1 \otimes W_1 \oplus \cdots \oplus A_m \otimes W_m) \oplus (B_1 \otimes (U_1 \oplus \overline{U}_1) \oplus \cdots \oplus B_1 \otimes (U_n \oplus \overline{U}_n))
\]

where \( W_1, \ldots, W_m, U_1, \ldots, U_n \) are distinct \( SL(2, \mathbb{R}) \)-irreducible representations admitting Hodge filtrations \( W_i = W_i^{1,0} \oplus W_i^{0,1}, U_j = U_j^{1,0} \oplus U_j^{0,1} \) such that \( W_i = \overline{W}_i, U_j \cap \overline{U}_j = \{0\}, A_i, B_j \)
are complex vector spaces (taking into account the multiplicities of the irreducible factors \( W_i, U_j \)), and \( L \) is the tautological bundle \( L = L_{1,0} \oplus L^{0,1} \), \( L_{1,0} = \mathbb{C}\omega \), \( L^{0,1} = \mathbb{C}\overline{\omega} \), \( \omega \in \mathcal{L} \). Moreover, this decomposition is unique and it can’t be further refined after passing to any further finite cover.

**Remark 100.** When \( \mu \) is the (unique) \( SL(2, \mathbb{R}) \)-invariant probability supported on a Teichmüller cover \( \mathcal{L} \) (i.e., a closed \( SL(2, \mathbb{R}) \)-orbit), the Optimistic Guess 1 is a consequence of Deligne’s semisimplicity theorem \[16\].

**Optimistic Guess 2.** In the setting of Optimistic Guess 1, denote by

\[
p_i = q_i = r_i = \dim \mathbb{C}W_{i,1}^{1,0} = \dim \mathbb{C}W_{i,1}^{0,1}
\]

and

\[
p_j = \dim \mathbb{C}U_{j,1}^{1,0}, q_j = \dim \mathbb{C}W_{j,1}^{0,1}, r_j = \min\{p_j, q_j\}
\]

Then, the Lyapunov spectrum of the KZ cocycle on \( W_i \) is simple, i.e.,

\[
\lambda_{i,1} \gg \cdots \gg \lambda_{i,r_i} \gg -\lambda_{i,r_i} \gg \cdots \gg -\lambda_{i,1}
\]

and the Lyapunov spectrum of the KZ cocycle on \( U_j \) is “as simple as possible”, i.e.,

\[
\lambda_{j,1} \gg \cdots \gg \lambda_{j,r_j} \gg 0 \gg \cdots \gg 0 \gg -\lambda_{j,r_j} \gg \cdots \gg -\lambda_{j,1}
\]

with

\[
|q_j - p_j|
\]

**Remark 101.** This “guess” is based on the general philosophy (supported by works as the ones of A. Raugi and Y. Guivarch \[37\], and I. Goldscheid and G. Margulis \[36\]) that, after reducing our cocycle to irreducible pieces, if the cocycle restricted to such a piece is “sufficiently generic” inside a certain Lie group of matrices \( G \), then the Lyapunov spectrum on this piece should look like the “Lyapunov spectrum” (i.e., collection of the logarithms of the norms of eigenvalues) of the “generic” matrix of \( G \). For instance, since a generic matrix inside the group \( U(p, q) \) has spectrum

\[
\lambda_1 \gg \cdots \gg \lambda_r \gg 0 \gg \cdots \gg 0 \gg -\lambda_r \gg \cdots \gg -\lambda_1
\]

where \( r = \min\{p, q\} \), the above guess essentially claims that, once one reduces the KZ cocycle to irreducible pieces, its Lyapunov spectrum on each piece must be as generic as possible.

**Remark 102.** Notice that the previous guess doesn’t make any attempt to compare Lyapunov exponents within distinct irreducible factors: indeed, in general non-isomorphic representations may lead to the same exponent by “pure chance” (as it happens in the case of certain genus 5 Abelian differentials associated to the “wind-tree model”, cf. \[17\]).

We close this section by mentioning that in Appendix \[D\] below it is presented some recent results on both non-simplicity and simplicity of Lyapunov spectrum of KZ cocycle in the context of *square-tiled surfaces*. 
9. Arithmetic Teichmüller curves with complementary series

During some conversations of the second author with Artur Avila and Jean-Christophe Yoccoz about the $SL(2, \mathbb{R})$ action on the moduli space of Abelian differentials and spectral gap of the corresponding $SL(2, \mathbb{R})$ unitary representations, we showed the following result:

**Theorem 103.** There are Teichmüller curves (actually $SL(2, \mathbb{R})$ orbits of square-tiled surfaces) with complementary series.

The proof of this result is not very long assuming previous knowledge of the theory of unitary representations of $SL(2, \mathbb{R})$. In particular, we’ll borrow the notations from Appendix A where the reader is quickly reminded of main results in this theory (e.g., Bargmann’s classification) and its connection to Ratner’s work \[63\] on the rates of mixing of geodesic flows. Then, we’ll combine this knowledge with a recent theorem of J. Ellenberg and D. B. McReynolds \[20\] to construct the desired square-tiled surfaces by a certain cyclic cover construction and a reverse Ratner estimate.

**Remark 104.** The algebraic part of the proof of Theorem 103 was already known by A. Selberg: in fact, as pointed out to us by N. Bergeron and P. Hubert, the same cyclic covering construction giving arbitrarily small first eigenvalue of $S = \Gamma \setminus \mathbb{H}$ (i.e., arbitrarily small spectral gap) was found by Selberg and the reader can find an exposition of this argument in the subsection 3.10.1 of Bergeron’s book \[10\]. In particular, although the “difference” between Selberg argument and the previous one is the fact that the former uses the first eigenvalue of the Laplacian $\Delta_S$ while we use the dynamical properties of the geodesic flow (more precisely the rates of mixing) and a reverse Ratner estimate, it is clear that both arguments are essentially the same.

**9.1. J. Ellenberg and D. McReynolds theorem.** Recall that a square-tiled surface is a Riemann surface $M$ obtained by gluing the sides of a finite collection of unit squares of the plane so that a left side (resp., bottom side) of one square is always glued with a right side (resp., top) of another square together with the Abelian differential $\omega$ induced by the quotient of $dz$ under these identifications. As we know, square-tiled surfaces are dense in the moduli space of Abelian differentials (because $(M, \omega)$ is square-tiled iff the periods of $\omega$ are rational) and the $SL(2, \mathbb{R})$-orbit of any square-tiled surface is a nice closed submanifold of $\mathcal{H}_g$ which can be identified with $\Gamma \setminus SL(2, \mathbb{R})$, where $\Gamma$ is the Veech group of our square-tiled surface (i.e., the finite-index subgroup $\Gamma$ of $SL(2, \mathbb{Z})$ stabilizing the $SL(2, \mathbb{R})$ of our square-tiled surface in the moduli space). Furthermore, the Teichmüller geodesic flow is naturally identified with the geodesic flow on $\Gamma \setminus \mathbb{H}$. In other words, the Teichmüller flow of the $SL(2, \mathbb{R})$-orbit of a square-tiled surface corresponds to the geodesic flow of $\Gamma \setminus \mathbb{H}$ where $\Gamma$ is a finite-index subgroup of $SL(2, \mathbb{Z})$ (hence $\Gamma$ is a lattice of $SL(2, \mathbb{R})$). In the converse direction, J. Ellenberg and D. McReynolds \[20\] recently proved that:

---

\[46\]Recall that Teichmüller curves are a shorthand for closed $SL(2, \mathbb{R})$-orbits in $\mathcal{H}_g$. See Subsection 6.3 of Section 6 for more comments on Teichmüller curves.
**Theorem 105** (Ellenberg and McReynolds). *Any finite-index subgroup \( \{\pm 1\} \subset \Gamma \) of the congruence subgroup \( \Gamma(2) \subset SL(2, \mathbb{Z}) \) is the Veech group of some square-tiled surface.*

### 9.2. Teichmüller curves with complementary series.

We are ready to prove Theorem 103 claiming that there are square-tiled surfaces such that the representation \( \rho_S \) associated to its \( SL(2, \mathbb{R}) \)-orbit \( S \) has irreducible factors in the complementary series.

Observe that the natural identification between \( SL(2, \mathbb{R}) \)-orbits \( S \) of square-tiled surfaces and the unit cotangent bundle of \( \Gamma \setminus \mathbb{H} \) where \( \Gamma \) is the corresponding Veech group permit to think of \( \rho_S \) as the regular unitary \( SL(2, \mathbb{R}) \)-representation \( \rho_\Gamma \) on the space \( L^2_0(\Gamma \setminus \mathbb{H}, \nu_\Gamma) \) of zero-mean \( L^2 \)-functions with respect to the natural measure \( \nu_\Gamma \) on \( \Gamma \setminus \mathbb{H} \).

In view of Ellenberg and McReynolds theorem, it suffices to find a finite-index subgroup \( \Gamma \subset \Gamma(2) \) such that \( \rho_\Gamma \) has complementary series. As we promised, this will be achieved by a cyclic covering procedure. Firstly, we fix a congruence subgroup \( \Gamma(m) \) such that the corresponding modular curve \( \Gamma(m) \setminus \mathbb{H} \) has genus \( g \geq 1 \), e.g., \( \Gamma(6) \). Next, we fix a homotopically non-trivial closed geodesic \( \beta \) of \( \Gamma(m) \setminus \mathbb{H} \) after the compactification of its cusps and we perform a cyclic covering of \( \Gamma(m) \setminus \mathbb{H} \) (i.e., we choose a subgroup \( \Gamma \subset \Gamma(m) \)) of high degree \( N \) such that a lift \( \beta_N \) of \( \beta \) satisfies \( \ell(\beta_N) = N \cdot \ell(\beta) \).

![Diagram](image-url)

**Figure 19.** Cyclic cover construction of a subgroup \( \Gamma(6)(2k) \) of \( \Gamma(6) \) of index \( N = 2k \).

Now, we select two small open balls \( U \) and \( V \) of area \( 1/N \) whose respective centers are located at two points of \( \beta_N \) belonging to very far apart fundamental domains of the cyclic covering \( \Gamma \setminus \mathbb{H} \), so that the distance between the centers of \( U \) and \( V \) is \( \sim N/2 \).

---

47 Recall that the (principal) congruence subgroup \( \Gamma(m) \) of \( SL(2, \mathbb{Z}) \) is
\[
\Gamma(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : a \equiv d \equiv 1 \pmod{m}, b \equiv c \equiv 1 \pmod{m} \right\}.
\]
Define $u = \sqrt{N} \cdot \chi_U$, $v = \sqrt{N} \cdot \chi_V$. Take $f = u - \int u$ and $g = v - \int v$ the zero mean parts of $u$ and $v$.

We claim that $\rho_\Gamma$ has complementary series, i.e., $\sigma(\Gamma) > -1$ (i.e., $\beta(\Gamma) > -1/4$) for a sufficiently large $N$. Actually, we will show a little bit more: $\sigma(\Gamma)$ is arbitrarily close to 0 for large $N$ (i.e., the spectral gap of $\Gamma$ can be made arbitrarily small).

Indeed, suppose that there exists some $\varepsilon_0 > 0$ such that $\sigma(\Gamma) < -\varepsilon_0$ for every $N$. By Ratner’s theorem \[109\], it follows that

$$|\langle f, \rho_\Gamma(a_t)g \rangle| \leq C(\varepsilon_0) \cdot e^{\sigma(\Gamma) \cdot t} \|f\|_{L^2} \|g\|_{L^2}$$

for any $|t| \geq 1$. On the other hand, since the distance between the centers of $U$ and $V$ is $\sim N/2$, the support of $u$ is disjoint from the image of the support of $v$ under the geodesic flow $a(t_N)$ for a time $t_N \sim N/2$. Thus, $\langle u, \rho_\Gamma(a(t_N))v \rangle = \int u \cdot v \circ a(t_N) = 0$, and, a fortiori,

$$|\langle f, \rho_\Gamma(a_t)g \rangle| = \left| \int u \cdot v \circ a(t_N) - \int u \cdot \int v \right| = \int u \cdot \int v \sim 1/N.$$

Putting these two estimates together and using the facts that $\|f\|_{L^2} \leq \|u\|_{L^2} \leq 1$ and $\|g\|_{L^2} \leq \|v\|_{L^2} \leq 1$, we derive the inequality

$$1/N \leq C(\varepsilon_0) e^{-\varepsilon_0 \cdot N/2}.$$

In particular, $\varepsilon_0 \leq C(\varepsilon_0) \cdot \frac{\ln N}{N}$, a contradiction for a sufficiently large $N$.

9.3. **Explicit square-tiled surfaces with complementary series.** The curious reader may ask whether Theorem \[103\] admits an explicit (or effective) version, that is, whether it is possible to actually exhibit square-tiled surfaces with complementary series.
Of course, the naive strategy is to make the arguments of the previous two subsections as explicit as possible. By trying to do so, we notice that there are essentially two places where one needs to pay attention:

- firstly, in Ellenberg-McReynolds theorem, given a group $\Gamma$, we need to know explicitly a square-tiled surface with Veech group $\Gamma$;
- secondly, for the explicit construction of a group $\Gamma$ with complementary series, we need explicit constants in Ratner’s theorem\(^{109}\) indeed, in terms of the notation of the previous subsection, by taking $\varepsilon_0 = 1$, we need to know the constant $C(\varepsilon_0) = C(1)$ in order to determine a value of $N$ (and hence $\Gamma$) violating the inequality $1 \leq C(1) \ln \frac{N}{N}$ (imposed by an eventual absence of complementary series).

This indicates that a straightforward implementation of the naive strategy might be tricky:

- a closer inspection of the methods of J. Ellenberg and D. McReynolds reveals that the construction of a square-tiled surface $M$ with a prescribed Veech group $\Gamma \subset \Gamma(2)$ passes by several covering processes, and, in particular, the total number of squares of $M$ tend to grow very fast as the index $[\Gamma(2) : \Gamma]$ increases;
- even though the constant $C(1)$ in Ratner’s theorem is rather explicit (after a tedious bookkeeping of constants in Ratner’s original argument one can check\(^{48}\) that $C(1) < 25$), since the function $\ln \frac{N}{N}$ decays “slowly”, the first values of $N$ violating the inequality $1 \leq C(1) \ln \frac{N}{N}$ are likely to be large: because $N$ is directly related to the index of $\Gamma$ in $\Gamma(2)$, this indicates that $\Gamma$ has large index in $\Gamma(2)$.

However, one can slightly improve the implementation of this strategy by recalling that the presence of complementary series is detected by the first eigenvalue $\lambda_1(\Gamma)$ of the Laplacian on $\Gamma \setminus \mathbb{H}$ and the so-called Cheeger-Buser inequality provide fairly good bounds on $\lambda_1(\Gamma)$ in terms of the geometry of $\Gamma \setminus \mathbb{H}$. More precisely, by denoting by $\Gamma = \Gamma_{2k}(6) \subset \Gamma(6)$ the subgroup constructed by the method indicated in Figure\(^{19}\) the Cheeger-Buser inequality says that

$$\sqrt{10} \lambda_1(\Gamma_{6}(2k)) + 1 \leq 10 h(\Gamma_{6}(2k)) + 1 \quad (9.1)$$

where $h(\Gamma_{6}(2k))$ is the Cheeger constant of the hyperbolic surface $\Gamma_{6}(2k) \setminus \mathbb{H}$:

$$h(\Gamma_{6}(2k)) := \inf_{\gamma \text{ multicurve separating } \Gamma_{6}(2k) \setminus \mathbb{H} \text{ into two connected open regions } A,B} \frac{\text{length}(\gamma)}{\min\{\text{area}(A), \text{area}(B)\}}$$

By numbering in Figure\(^{19}\) the preimages $\alpha_0, \ldots, \alpha_{2k-1} \subset \Gamma_{6}(2k) \setminus \mathbb{H}$ of the curve $\alpha \subset \Gamma(6) \setminus \mathbb{H}$ by the order we “see” them along $\beta$ and by taking $\gamma = \alpha_0 \cup \alpha_k$, we obtain a multicurve such that:

- $\text{length}(\gamma) = 2 \cdot \text{length}(\alpha) = 4 \cdot \arccosh(17)$,
- $\Gamma_{6}(2k) \setminus \mathbb{H} - \gamma = A \cup B$ and $\text{area}(A) = \text{area}(B) = k \cdot \text{area}(\Gamma(6) \setminus \mathbb{H}) = k \cdot 24\pi$.

\(^{48}\)See the blog post “Explicit constants in Ratners estimates on rates of mixing of geodesic flows of hyperbolic surfaces” at the second author’s mathematical blog\(^{19}\).
Therefore, $h(\Gamma_6(2k)) \leq \arccosh(17)/(k \cdot 6\pi)$ and one can use Cheeger-Buser inequality \[9.1\] to conclude that

$$\lambda_1(\Gamma_6(2k)) \leq 1/(2k) < 1/4$$

for $k \geq 3$. By Appendix \[A\] this means that, e.g., $\Gamma_6(6) \backslash \mathbb{H}$ has complementary series.

This reduces the problem of construction of explicit square-tiled surfaces with complementary series to find some square-tiled surface with Veech group $\Gamma_6(6)$. At this point, one can improve again over the naive strategy above by following J. Ellenberg and D. McReynolds methods only partly. More precisely, since $\lambda_1(\tilde{\Gamma} \backslash \mathbb{H}) \leq \lambda_1(\Gamma \backslash \mathbb{H})$ when $\tilde{\Gamma} \subset \Gamma$ (i.e., $\tilde{\Gamma} \backslash \mathbb{H}$ covers $\Gamma \backslash \mathbb{H}$), it suffices to construct a square-tiled surface with Veech group $\Gamma \subset \Gamma_6(6)$ and, for this purpose, we don’t have to follow \[20\] until the end: by doing so, we “save” a few “covering steps” needed when one wants the Veech group to be exactly $\Gamma_6(6)$. In other words, by “stopping” the arguments in \[20\] earlier, we get “only” a square-tiled surface $M$ with Veech group $\Gamma \subset \Gamma_6(6)$ but we “reduce” the total number of squares of $M$ because we don’t insist into taking further “coverings steps” to get Veech group equal to $\Gamma_6(6)$.

In fact, this “mildly improved” strategy was pursued on the article \[55\] by Gabriela Schmithüsen and the second author were it is shown that:

**Theorem 106.** There exists an explicit pair of permutations $h, v \in S_{576}$ in 576 elements determining a square-tiled surface $(M, \omega) \in \mathcal{H}(1, 5, 5, 5, 2, \ldots, 2)$ with 576 squares, genus 147, and Veech group $\Gamma \subset \Gamma_6(6)$. In particular, the Teichmüller curve $SL(2, \mathbb{R}) \cdot (M, \omega)$ has complementary series.

We close this section by referring the reader to \[55\] for the explicit pair of permutations $h, v$ quoted above and a complete proof of this result.

**Appendix A. Representation theory of $SL(2, \mathbb{R})$ and Ratner’s work**

Let $\rho : SL(2, \mathbb{R}) \to U(\mathcal{H})$ be an unitary representation of $SL(2, \mathbb{R})$, i.e., $\rho$ is a homomorphism from $SL(2, \mathbb{R})$ into the group $U(\mathcal{H})$ of unitary transformations of the complex separable Hilbert space $\mathcal{H}$. We say that a vector $v \in \mathcal{H}$ is a $C^k$-vector of $\rho$ if $g \mapsto \rho(g)v$ is $C^k$. Recall that the subset of $C^\infty$-vectors is dense in $\mathcal{H}$.

The Lie algebra $sl(2, \mathbb{R})$ of $SL(2, \mathbb{R})$ (i.e., the tangent space of $SL(2, \mathbb{R})$ at the identity) is the set of all $2 \times 2$ matrices with zero trace. Given a $C^1$-vector $v$ of $\rho$ and $X \in sl(2, \mathbb{R})$, the Lie derivative $L_Xv$ is

$$L_Xv := \lim_{t \to 0} \frac{\rho(\exp(tX)) \cdot v - v}{t}$$

where $\exp(X)$ is the exponential map (of matrices).

**Exercise 107.** Show that $\langle L_Xv, w \rangle = -\langle v, L_Xw \rangle$ for any $C^1$-vectors $v, w \in \mathcal{H}$ of $\rho$ and $X \in sl(2, \mathbb{R})$.

\[49\] Cf. Appendix \[C\] for more comments on this construction.
An important basis of $\mathfrak{sl}(2, \mathbb{R})$ is

$$W := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

**Exercise 108.** Show that $\exp(tW) = \begin{pmatrix} \cosh t & \sinh t \\ -\sinh t & \cosh t \end{pmatrix}$, $\exp(tQ) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and $\exp(tV) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Furthermore, $[Q, W] = 2V$, $[Q, V] = 2W$ and $[W, V] = 2Q$ where $[,]$ is the Lie bracket of $\mathfrak{sl}(2, \mathbb{R})$ (i.e., $[A, B] := AB - BA$ is the commutator).

The Casimir operator $\Omega_\rho$ is $\Omega_\rho := (L_V^2 + L_Q^2 - L_W^2)/4$ on the dense subspace of $C^2$-vectors of $\rho$. It is known that $\langle \Omega_\rho v, w \rangle = \langle v, \Omega_\rho w \rangle$ for any $C^2$-vectors $v, w \in \mathcal{H}$, the closure of $\Omega_\rho$ is self-adjoint, $\Omega_\rho$ commutes with $L_X$ on $C^3$-vectors for any $X \in \mathfrak{sl}(2, \mathbb{R})$ and $\Omega_\rho$ commutes with $\rho(g)$ for any $g \in SL(2, \mathbb{R})$.

Furthermore, when the representation $\rho$ is irreducible, $\Omega_\rho$ is a scalar multiple of the identity operator, i.e., $\Omega_\rho v = \lambda(\rho)v$ for some $\lambda(\rho) \in \mathbb{R}$ and for any $C^2$-vector $v \in \mathcal{H}$ of $\rho$.

In general, as we’re going to see below, the spectrum $\sigma(\Omega_\rho)$ of the Casimir operator $\Omega_\rho$ is a fundamental object.

### A.1. Bargmann’s classification.

We introduce the following notation:

$$r(\lambda) := \begin{cases} -1 & \text{if } \lambda \leq -1/4, \\ -1 + \sqrt{1 + 4\lambda} & \text{if } -1/4 < \lambda < 0, \\ -2 & \text{if } \lambda \geq 0. \end{cases}$$

Note that $r(\lambda)$ satisfies the quadratic equation $x^2 + 2x - 4\lambda = 0$ when $-1/4 < \lambda < 0$.

Bargmann’s classification of irreducible unitary $SL(2, \mathbb{R})$ says that the eigenvalue $\lambda(\rho)$ of the Casimir operator $\Omega_\rho$ has the form

$$\lambda(\rho) = (s^2 - 1)/4$$

where $s \in \mathbb{C}$ falls into one of the following three categories:

- **Principal series**: $s$ is purely imaginary, i.e., $s \in \mathbb{R}i$;
- **Complementary series**: $s \in (0, 1)$ and $\rho$ is isomorphic to the representation

$$\rho_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) := (cx + d)^{-1-s} f \left( \frac{ax + b}{cx + d} \right),$$

where $f$ belongs to the Hilbert space $\mathcal{H}_s := \{ f : \mathbb{R} \to \mathbb{C} : \int \int \frac{f(x)f(y)}{|x-y|^2} dx dy < \infty \}$;
- **Discrete series**: $s \in \mathbb{N} - \{0\}$.

In other words, $\rho$ belongs to the principal series when $\lambda(\rho) \in (-\infty, -1/4]$, $\rho$ belongs to the complementary series when $\lambda(\rho) \in (-1/4, 0)$ and $\rho$ belongs to the discrete series when $\lambda(\rho) = (n^2 - 1)/4$ for some natural number $n \geq 1$. 
A.2. Some examples of \( SL(2, \mathbb{R}) \) unitary representations. Given a dynamical system consisting of a \( SL(2, \mathbb{R}) \) action (on a certain space \( X \)) preserving some probability measure \( (\mu) \), we have a naturally associated unitary \( SL(2, \mathbb{R}) \) representation on the Hilbert space \( L^2(X, \mu) \) of \( L^2 \) functions of the probability space \( (X, \mu) \). More concretely, we'll be interested in the following two examples.

Hyperbolic surfaces of finite volume. It is well-known that \( SL(2, \mathbb{R}) \) is naturally identified with the unit cotangent bundle of the upper half-plane \( \mathbb{H} \). Indeed, the quotient \( SL(2, \mathbb{R})/SO(2, \mathbb{R}) \) is diffeomorphic to \( \mathbb{H} \) via

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot SO(2, \mathbb{R}) \mapsto \frac{a i + b}{c i + d}
\]

Let \( \Gamma \) be a lattice of \( SL(2, \mathbb{R}) \), i.e., a discrete subgroup such that \( M := \Gamma \backslash SL(2, \mathbb{R}) \) has finite volume with respect to the natural measure \( \mu \) induced from the Haar measure of \( SL(2, \mathbb{R}) \). In this situation, our previous identification shows that \( M := \Gamma \backslash SL(2, \mathbb{R}) \) is naturally identified with the unit cotangent bundle \( T_1 S \) of the hyperbolic surface \( S := \Gamma \backslash SL(2, \mathbb{R})/SO(2, \mathbb{R}) = \Gamma \backslash \mathbb{H} \) of finite volume with respect to the natural measure \( \nu \).

Since the action of \( SL(2, \mathbb{R}) \) on \( M := \Gamma \backslash SL(2, \mathbb{R}) \) and \( S := \Gamma \backslash \mathbb{H} \) preserves the respective probability measures \( \mu \) and \( \nu \) (induced from the Haar measure of \( SL(2, \mathbb{R}) \)), we obtain the following (regular) unitary \( SL(2, \mathbb{R}) \) representations:

\[
\rho_M(g) f(\Gamma z) = f(\Gamma z \cdot g) \quad \forall f \in L^2(M, \mu)
\]

and

\[
\rho_S(g) f(\Gamma z SO(2, \mathbb{R})) = f(\Gamma z \cdot g SO(2, \mathbb{R})) \quad \forall f \in L^2(S, \nu).
\]

Observe that \( \rho_S \) is a subrepresentation of \( \rho_M \) because the space \( L^2(S, \nu) \) can be identified with the subspace \( \mathcal{H}_\Gamma := \{ f \in L^2(M, \mu) : f \text{ is constant along } SO(2, \mathbb{R}) \text{- orbits} \} \). Nevertheless, it is possible to show that the Casimir operator \( \Omega_{\rho_M} \) restricted to \( C^2 \)-vectors of \( \mathcal{H}_\Gamma \) coincides with the Laplacian \( \Delta = \Delta_S \) on \( L^2(S, \nu) \). Also, we have that a number \(-1/4 < \lambda < 0\) belongs to the spectrum of \( \Omega_{\rho_M} \) (on \( L^2(M, \mu) \)) if and only if \(-1/4 < \lambda < 0\) belongs to the spectrum of \( \Delta = \Delta_S \) on \( L^2(S, \nu) \).

Moduli spaces of Abelian differentials. Of course, an interesting space philosophically related to the hyperbolic surfaces of finite volumes are the moduli spaces \( \mathcal{H}_g \) of Abelian differentials on Riemann surfaces of genus \( g \geq 1 \).

As we saw in the last Section 1.4 of Section 1, the case of \( Q_1 \) is particularly clear: it is well-known that \( Q_1 \) is isomorphic to the unit cotangent bundle \( SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R}) \) of the modular curve. In this nice situation, the \( SL(2, \mathbb{R}) \) action has a natural absolutely continuous (w.r.t. Haar measure) invariant probability \( \mu_{(1)} \), so that we have a natural unitary \( SL(2, \mathbb{R}) \) representation on \( L^2(Q_1, \mu_{(1)}) \).
After the works of H. Masur and W. Veech, we know that the general case has some similarities with the genus 1 situation, in the sense that connected components $C$ of strata $\mathcal{H}_g$ of $\mathcal{H}_g$ come equipped with a natural Masur-Veech measure invariant probability $\mu_C$. In particular, we get also an unitary $SL(2,\mathbb{R})$ representation on $L^2(C,\mu_C)$. More generally, there are plenty of $SL(2,\mathbb{R})$-invariant probabilities $\mu$ on $C$ (e.g., coming from square-tiled surfaces) and evidently all of them lead can be used to produce unitary $SL(2,\mathbb{R})$ representations (on $L^2(C,\mu)$).

A.3. Rates of mixing and size of the spectral gap. Once we have introduced two examples (coming from Dynamical Systems) of unitary $SL(2,\mathbb{R})$ representations, what are the possible series (in the sense of Bargmann classification) appearing in the decomposition of our representation into its irreducible factors.

In the case of hyperbolic surfaces of finite volume, we understand precisely the global picture: the possible irreducible factors are described by the rates of mixing of the geodesic flow on our hyperbolic surface.

More precisely, let $A := \{a(t) := \text{diag}(e^t, e^{-t}) \in SL(2,\mathbb{R})\}$ be the 1-parameter subgroup of diagonal matrices of $SL(2,\mathbb{R})$. It is not hard to check that the geodesic flow on a hyperbolic surface of finite volume $\Gamma \backslash \mathbb{H}$ is identified with the action of the diagonal subgroup $A$ on $\Gamma \backslash SL(2,\mathbb{R})$.

Ratner showed that the Bargmann’s series of the irreducible factors of the regular representation $\rho_\Gamma$ of $SL(2,\mathbb{R})$ on $L^2(\Gamma \backslash SL(2,\mathbb{R}))$ can be deduced from the rates of mixing of the geodesic flow $a(t)$ along a certain class of observables. In order to keep the exposition as elementary as possible, we will state a very particular case of Ratner’s results (referring the reader to [63] for more general statements). We define $\mathcal{C}(\Gamma) := \{f \in L^2(\Gamma \backslash SL(2,\mathbb{R})) : f$ is constant along $SO(2,\mathbb{R})$–orbits and $\int f = 0\}$ equipped with the usual $L^2$ inner product $\langle , \rangle$. In the sequel, we denote by

$$\mathcal{C}(\Gamma) = \sigma(\Delta_S) \cap (-1/4, 0)$$

the intersection of the spectrum of the Laplacian $\Delta_S$ with the open interval $(-1/4, 0)$,

$$\beta(\Gamma) = \sup \mathcal{C}(\Gamma)$$

with the convention $\beta(\mathcal{C}(\Gamma)) = -1/4$ when $\mathcal{C}(\Gamma) = \emptyset$ and

$$\sigma(\Gamma) = \sigma(\beta(\Gamma)) := -1 + \sqrt{1 + 4\beta(\Gamma)}.$$

We remember the reader that the subset $\mathcal{C}(\Gamma)$ detects the presence of complementary series in the decomposition of $\rho_\Gamma$ into irreducible representations. Also, since $\Gamma$ is a lattice, it is possible to show that $\mathcal{C}(\Gamma)$ is finite and, a fortiori, $\beta(\Gamma) < 0$. Because $\beta(\Gamma)$ essentially measures the distance between zero and the first eigenvalue of $\Delta_S$ on $\mathcal{H}_\Gamma$, it is natural to call $\beta(\Gamma)$ the spectral gap.

**Theorem 109.** For any $f, g \in \mathcal{H}_\Gamma$ and $|t| \geq 1$, we have

- $|\langle f, \rho_\Gamma(a(t))g \rangle| \leq C_{\beta(\Gamma)} \cdot e^{\sigma(\Gamma)t} \cdot \|f\|_{L^2} \cdot \|g\|_{L^2}$ when $\mathcal{C}(\Gamma) \neq \emptyset$;
\[ |(f, \rho_t(a(t))g)\| \leq C_{\beta(\Gamma)} \cdot e^{\sigma(\Gamma)t} \cdot \|f\|_{L^2} \cdot \|g\|_{L^2} = C_{\beta(\Gamma)} \cdot e^{-t} \cdot \|f\|_{L^2} \cdot \|g\|_{L^2} \text{ when } C(\Gamma) = 0, \]

\[ \sup(\sigma(\Delta_S) \cap (-\infty, -1/4)) < -1/4 \text{ and } -1/4 \text{ is not an eigenvalue of the Casimir operator } \Omega_{\rho_t}; \]

\[ |(f, \rho_t(a(t))g)\| \leq C_{\beta(\Gamma)} \cdot t \cdot e^{\sigma(\Gamma)t} \cdot \|f\|_{L^2} \cdot \|g\|_{L^2} = C_{\beta(\Gamma)} \cdot t \cdot e^{-t} \cdot \|f\|_{L^2} \cdot \|g\|_{L^2} \text{ otherwise, i.e., when } C(\Gamma) = 0 \text{ and either } \sup(\sigma(\Delta_S) \cap (-\infty, -1/4)) = -1/4 \text{ or } -1/4 \text{ is an eigenvalue of the Casimir operator } \Omega_{\rho_t}. \]

Here \( C_{\beta(\Gamma)} > 0 \) is a constant such that \( C_{\mu} \) is uniformly bounded when \( \mu \) varies on compact subsets of \( (-\infty, 0) \).

In other words, Ratner’s theorem relates the (exponential) rate of mixing of the geodesic flow \( a(t) \) with the spectral gap: indeed, the quantity \( |(f, \rho_t(a(t))g)| \) roughly measures how fast the geodesic flow \( a(t) \) mixes different places of phase space (actually, this is more clearly seen when \( f \) and \( g \) are characteristic functions of Borelian sets), so that Ratner’s result says that the exponential rate \( \sigma(\Gamma) \) of mixing of \( a(t) \) is an explicit function of the spectral gap \( \beta(\Gamma) \) of \( \Delta_S \).

In the case of moduli spaces of Abelian differentials, our knowledge is less complete than the previous situation: as far as we know, the best results about the “spectral gap” of the \( SL(2, \mathbb{R}) \) representation \( \rho_s \) on the space \( L^2_0(\mathcal{C}, \mu) \) of zero-mean \( L^2 \)-functions with respect to a given \( SL(2, \mathbb{R}) \)-invariant probabilities \( \mu \) on a connected component \( \mathcal{C} \) of the moduli space \( \mathcal{H}_g \) are the two results discussed in Subsection 4.3 of Section 4 namely:

**Theorem 110** (A. Ávila, S. Gouëzel, J.-C. Yoccoz). In the case of the Masur-Veech measure \( \mu = \mu_\mathcal{C} \), the unitary \( SL(2, \mathbb{R}) \) representation \( \rho_\mathcal{C} = \rho_{\mu_\mathcal{C}} \) has spectral gap in the sense that it is isolated from the trivial representation, i.e., there exists some \( \varepsilon > 0 \) such that all irreducible factors \( \rho_\mathcal{C}^{(s)} \) of \( \rho_\mathcal{C} \) in the complementary series are isomorphic to the representation \( \rho_s \) with \( s < 1 - \varepsilon \).

**Theorem 111** (A. Ávila and S. Gouëzel). Let \( \mu \) be an algebraic \( SL(2, \mathbb{R}) \)-invariant probability, and consider the integral decomposition \( L^2_0(\mathcal{C}, \mu) = \int \mathcal{H}_\xi d\lambda(\xi) \) of the unitary \( SL(2, \mathbb{R}) \) representation \( L^2_0(\mathcal{C}, \mu) \) into irreducible factors \( \mathcal{H}_\xi \). Then, for any \( \delta > 0 \), the representations \( \mathcal{H}_\xi \) of the complementary series with parameter \( s(\mathcal{H}_\xi) \in [\delta, 1] \) appear only discretely (i.e., \( \{s \in [\delta, 1] : s = s(\mathcal{H}_\xi) \text{ for some } \xi \} \) is finite) and with finite multiplicity (i.e., for each \( s \in [\delta, 1] \), \( \{\xi : s(\mathcal{H}_\xi) = s\} \) is finite). In particular, the Teichmüller geodesic flow \( g_t \) is exponentially mixing with respect to \( \mu \).

Observe that, generally speaking, the results of Avila, Gouezel and Yoccoz say that \( \rho_\mu \) doesn’t contain all possible irreducible representations of the complementary series, but it doesn’t give any hint about quantitative estimates of the “spectral gap”, i.e., how small \( \varepsilon > 0 \) can be in general. In fact, at the present moment, it seems that the only situation where one can say something more precise is the case of the moduli space \( \mathcal{H}_1 \):

\[ \text{Recall that, roughly speaking, a } SL(2, \mathbb{R}) \text{-invariant probability is algebraic whenever its support is an affine suborbifold (i.e., a suborbifold locally described, in periodic coordinates, by affine subspaces) such that, in period coordinates, } \mu \text{ is absolutely continuous with respect to Lebesgue measure and its density is locally constant.} \]
Theorem 112 (Selberg/Ratner). The representation $\rho_{H_1}$ has no irreducible factor in the complementary series and it holds $|\langle v, \rho_{H_1} w \rangle| \leq C \cdot t \cdot e^{-t}$.

In fact, using the notation of Ratner’s theorem, Selberg proved that $C(SL(2, \mathbb{Z})) = \emptyset$. Since we already saw that $H_1 = SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R})$, the first part of the theorem is a direct consequence of Selberg’s result, while the second part is a direct consequence of Ratner’s result.

In view of the previous theorem, it is natural to make the following conjecture:

Conjecture (J.-C. Yoccoz) The representations $\rho_{\mu_C}$ don’t have complementary series (where $\mu_C$ are the Masur-Veech measures).

This conjecture is currently open (to the best of the authors’ knowledge). In any case, it is worth to recall that in Section 9 we saw that this conjecture becomes false if the invariant natural measure $\mu_C$ is replaced by other invariant measures supported on smaller loci.

Appendix B. A pseudo-Anosov in genus 2 with vanishing second Lyapunov exponent

In this (short) appendix, we will indicate the construction of a periodic orbit $\gamma$ of the Teichmüller flow on $H(2)$ such that the second Lyapunov exponent of KZ cocycle over $\gamma$ vanishes. For this purpose, recall that typical orbits of the Teichmüller flow on $H(2)$ are coded by $\infty$-complete paths in the Rauzy diagram schematically depicted below:

Furthermore, it is possible to attach matrices to the arrows of Rauzy diagrams such that the KZ cocycle over a Teichmüller flow orbit represented by a certain concatenation of arrows of a $\infty$-complete path is simply given by the product of the matrices associated to the arrows in the order they are concatenated.

The reader can find detailed explanation on this in J.-C. Yoccoz’s survey [72].

For our current task, starting from the vertex at the center of the Rauzy we take the following concatenation $\gamma$ of arrows $D \to B \to B \to D \to C \to D \to A \to A \to A$, i.e., $\gamma = DB^2DCA^3$. Here, using the language of [72], we’re coding arrows by the associated winning letter. The fact that the four letters $A, B, C, D$ appear in the construction of $\gamma$ means that it is $\infty$-complete, so that $\gamma$ represents a periodic orbit of Teichmüller flow.

A direct calculation (with the formulas presented in [72]) shows that the KZ cocycle over $\gamma$ is represented by a matrix $B_\gamma$ with characteristic polynomial

\[
x^4 - 7x^3 + 11x^2 - 7x + 1
\]
By performing the substitution $y = x + 1/x$, we obtain
$$y^2 - 7y + 9$$
whose discriminant is
$$\Delta = \sqrt{13}$$
Since
$$\frac{7 - \sqrt{13}}{2} < 2$$
we see that $B_\gamma$ has a pair of complex conjugated eigenvalues of modulus 1, i.e., $\gamma$ gives a pseudo-Anosov in $\mathcal{H}(2)$ such that KZ cocycle over $\gamma$ has vanishing second Lyapunov exponent.

**Appendix C. Volumes of strata (after A. Eskin & A. Okounkov)**

This appendix contains some comments on the work [25] of A. Eskin and A. Okounkov about the computation of the volumes $\lambda_\kappa^{(1)}(\mathcal{H}(1)(\kappa))$ of strata of Abelian differentials (cf. Section 3).

In 0th order of approximation, the idea of A. Eskin and A. Okounkov: by analogy with the case of $\mathbb{R}^n$, one can hope to compute the volume of a stratum $\mathcal{H}(1)(\kappa)$ by counting *integral/rational* points. More precisely, recall that the volume of the unit sphere $S^{n-1}$ of $\mathbb{R}^n$ can be calculated by the following method. Denoting by $B(0, R)$ the ball of radius $R$ in $\mathbb{R}^n$, let
$$N(R) := \#(B(0, R) \cap \mathbb{Z}^n)$$
That is, $N(R)$ is the number of integral points of $\mathbb{R}^n$ in the ball $B(0, R)$ of radius $R$:

Because $N(R)$ is a good approximation of the volume of $B(0, R)$ for large $R$, we see that the knowledge of the asymptotic behavior of $N(R)$, i.e.,
$$N(R) \sim c(n)R^n$$
for a constant $c(n) > 0$ allows to deduce the volume of the unit sphere $S^{n-1}$ by homogeneity, i.e.,
$$\text{vol}(S^{n-1}) = \frac{d\text{vol}(B(0, R))}{dR}|_{R=1} = n \cdot c(n)$$
In the case of volume of strata of moduli spaces of Abelian differentials, the strategy is “similar”:...
firstly, one realizes that the role of integral points is played by square-tiled surfaces, so that the volume of the “ball” $\mathcal{H}^{(R)}(\kappa)$ of translation surfaces in the stratum $\mathcal{H}(\kappa)$ with total area at most $R$ is reasonably approximated by the number $N_\kappa(R)$ of square-tiled surfaces in the stratum $\mathcal{H}(\kappa)$ composed of $R$ unit squares at most;

secondly, one computes the asymptotics $N_\kappa(R) \sim c(\kappa) \cdot R^{2g+s-1}$ (recall that the stratum $\mathcal{H}(\kappa)$ has complex dimension $2g + s - 1$ when $\kappa = (k_1, \ldots, k_s)$, $2g - 2 = \sum_{j=1}^{s} k_j$);

finally, by homogeneity, one deduces that $\lambda_\kappa^{(1)}(\mathcal{H}^{(1)}(\kappa)) = (4g + 2s - 2) \cdot c(\kappa)$.

Evidently, the most difficult step here is the calculation of $c(\kappa)$. In rough terms, the main point is that one can reduce the computation of $c(\kappa)$ to a combinatorial problem about permutations and the representation theory of the symmetric group helps us in this task. However, the implementation of this idea is a hard task and it is out of the scope of these notes to present the arguments of A. Eskin and A. Okounkov. In particular, we will content ourselves to reduce the calculation of $c(\kappa)$ to a combinatorial problem and then we will simply state some of the main results of [25]. Finally, we will conclude this appendix by showing how the action of $SL(2, \mathbb{Z})$ on square-tiled surfaces translates in terms of combinatorics of permutations, so that it will be “clear” that the $SL(2, \mathbb{Z})$-action on square-tiled surfaces with a “low” number of squares can be calculated with the aid of computer programs.

The computation of $c(\kappa)$ essentially amounts to count the number of square-tiled surfaces with $N$ squares inside a given stratum $\mathcal{H}(\kappa)$. Combinatorially speaking, a square-tiled surface with $N$ squares can be coded numbering its squares from 1 to $N$ and then considering a pair of permutations $h, v \in S_N$ such that

- $h(i)$ is the number of the square to the right of the square $i$;
- $v(i)$ is the number of the square on the top of the square $i$.

For example, the $L$-shaped square-tiled surface below is coded by the pair of permutations\(^{[5]}\) $h = (1, 2)(3)$ and $v = (1, 3)(2)$.

![Diagram of an L-shaped square-tiled surface]

Logically, the codification by a pair of permutations is not unique because we can always renumber the squares without changing the square-tiled surface.

In combinatorial terms, the operation of renumbering corresponds to perform a simultaneous conjugation of $h$ and $v$, i.e., we replace the pair of permutations $(h, v)$ by $(\phi h \phi^{-1}, \phi v \phi^{-1})$. Because

\[^{[5]}\text{In what follows, we will represent permutations by their cycles.}\]
we’re interested in the square-tiled surface itself (but not on the particular codification), we will declare that

\[(h, v) \sim (h', v') \iff h' = \phi h \phi^{-1} \text{ and } \phi v \phi^{-1} \text{ for some } \phi \in S_N\]

Moreover, we see that a (connected) square-tiled surface with \(N\) squares is coded by a pair of permutations \(h, v \in S_N\) acting transitively on the set \(\{1, \ldots, N\}\) of squares.

In this language, we just saw that a connected square-tiled surface with \(N\) squares is the same as the equivalence classes of a pair of permutations acting transitively on \(\{1, \ldots, N\}\) modulo simultaneous conjugation.

Next, we observe that the stratum \(H(\kappa)\) of a square-tiled surface can be read off from a pair of permutations \((h, v)\) coding it. Indeed, we note that the commutator \([h, v] := vhv^{-1}h^{-1}\) geometrically corresponds to turn around (in the counterclockwise sense) the leftmost bottom corners of squares:

Therefore, a square-tiled surface coded by \((h, v)\) belong to the stratum \(H(d_1 - 1, \ldots, d_s - 1)\) where \(d_1, \ldots, d_s\) are the lengths of the (non-trivial) cycles of the commutator \([h, v]\) of \(h\) and \(v\).

In other words, we see that counting (connected) square-tiled surfaces with \(N\) squares in \(H(k_1, \ldots, k_s)\) is the same as the combinatorial problem of counting equivalence classes of pairs of permutations \((h, v) \in S_N \times S_N\) such that

- \((h, v)\) act transitively on \(\{1, \ldots, N\}\)
- the commutator \([h, v]\) has \(s\) (non-trivial) cycles of lengths \((k_1 + 1), \ldots, (k_s + 1)\).

modulo simultaneous conjugations.

By solving this combinatorial problem, A. Eskin and A. Okounkov \[25\] proved that
Theorem 113. The number $c(\kappa)$ and, a fortiori, the volume $\lambda_{\kappa}^{(1)}(\mathcal{H}^{(1)}(\kappa))$ is a rational multiple of $\pi^{2g}$. Moreover, the generating function

$$
\sum_{N=1}^{\infty} q^N \cdot \sum_{S \in \mathcal{H}(\kappa) \atop \text{square-tiled surface with } N \text{ squares}} \frac{1}{\# \text{Aut}(S)}
$$

is quasi-modular: indeed, it is a polynomial in the Eisenstein series $G_2(q), G_4(q)$ and $G_6(q)$.

Remark 114. The (very) attentive reader may recall that strata are not connected in general and they may have 3 connected components at most distinguished by hyperellipticity and parity of spin structure. As it turns out, the volume of individual connected components can be translated into a combinatorial problem of counting certain equivalence classes of permutations, but the new counting problem becomes slightly harder because parity of spin structure is not completely easy to read off from pairs of permutations: they have to do with the so-called theta characteristics. Nevertheless, this computation was successfully performed by A. Eskin, A. Okounkov and R. Pandharipande [26] to determine explicit formulas for volumes of connected components of strata.

Closing this appendix, let’s translate the action of $SL(2, \mathbb{Z})$ in terms of pairs of permutations. For this sake, recall that $SL(2, \mathbb{Z})$ is generated by $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Therefore, it suffices to translate the action of $S$ and $T$ in terms of pairs of permutations $(h, v)$, and this is not hard: for instance, note that $T$ acts as so that, in the language of permutations, $T(h, v) = (h, vh^{-1})$.

Similarly, one can convince himself/herself that $S(h, v) = (hv^{-1}, v)$.

In order words, by definition $T$ and $S$ act on pair of permutations $(h, v)$ via the so-called Nielsen transformations. Therefore, this combinatorial description is particularly effective to compute (by hands or with a computer program) $SL(2, \mathbb{Z})$-orbits of square-tiled surfaces: we consider a pair of permutation $(h, v)$ and we successively apply $T$ and $S$ by paying attention to the fact that we’re interested in pairs of permutations modulo simultaneous conjugations.

For instance, we invite the reader to use this approach to solve the following exercise:

Exercise 115. Show that the $SL(2, \mathbb{Z})$-orbit of the “Swiss cross” given by the pair of permutations $h = (1, 2, 3)(4)(5), v = (1)(2, 4, 5)(3)$: has cardinality 9.
This appendix is intended to briefly survey on some recent results from [58] and [57] on non-simplicity and simplicity of Lyapunov exponents of KZ cocycle over $SL(2, \mathbb{R})$-orbits of square-tiled surfaces.

D.1. Square-tiled surfaces with symmetries and multiplicity of Lyapunov exponents.

In this subsection we’ll follow closely [58]. Consider a square-tiled surface $M$ represented as a pair of permutations $(h, v) \in S_N \times S_N$ (see the previous appendix). The subgroup $G$ of $S_N$ generated by $h$ and $v$ is called monodromy group. Note that the stabilizers of the squares of $M$ form a conjugacy class of subgroups of $G$ whose intersection is trivial. Conversely, given a finite group $G$ generated by two elements $h$ and $v$, and a subgroup $H$ of $G$ whose intersection with its conjugated is trivial (i.e., $H$ doesn’t contain non-trivial normal subgroups of $G$), we recover an origami whose squares are labelled by the elements of $H \setminus G$ such that $Hgh$ is the neighbor to the right of $Hg \in H \setminus G$ and $Hgv$ is the neighbor on the top of $Hg \in H \setminus G$. For the sake of this subsection, we’ll think of a square-tiled surface $M$ as the data of $G, H, h, v$ as above.

As we explained in Section 7, the study of the non-tautological Lyapunov exponents of KZ cocycle over the $SL(2, \mathbb{R})$-orbits of a square-tiled surface $M$ amounts to understand the action of the affine group $\text{Aff}(M)$ on $H^1_0(M, \mathbb{R})$. Actually, by technical (linear algebra) reasons, we start with the action of $\text{Aff}(M)$ on $H^1_0(M, \mathbb{C})$.

In this direction, we’ll warm up with the action of the group of automorphisms $\text{Aut}(M)$ on $H^1_0(M, \mathbb{C})$. By thinking of $M$ as the data $(G, H, h, v)$, it is possible to check that $\text{Aut}(M)$ is naturally isomorphic to $N/H$ where $N$ is the normalizer of $H$ in $G$. By taking this point of view, we have that $H^1_0(M, \mathbb{C})$ is a $N/H$-module and we can ask ourselves what’s the multiplicity $\ell_\alpha$ of a given irreducible representation $\chi_\alpha$ of the finite group $N/H$ inside $H^1_0(M, \mathbb{C})$.

In [58], J.-C. Yoccoz, D. Zmiaikou and the second author show the following formula:

**Theorem 116.** One has

$$\ell_\alpha = \frac{\#G}{\#N} \dim(\chi_\alpha) - \sum_{g \in G} \frac{1}{n(g)} \dim(\text{Fix}_\alpha(g \epsilon^n(g) g^{-1}))$$
where \( c = [h, v] \) is the commutator of \( h \) and \( v \), \( n(g) > 0 \) is the smallest integer such that \( ge^{n(g)}g^{-1} \in N \), and \( \text{Fix}_a(n) \) is the subspace fixed by \( \chi_a(n) \).

**Remark 117.** An interesting consequence of this formula is the fact that the multiplicity \( \ell_a \) depend on \( h \) and \( v \) only by means of its commutator \( c = [h, v] \).

This formula is one of the ingredients towards the following result:

**Corollary 118.** The multiplicity \( \ell_a \) is never equal to 1, i.e., either \( \ell_a = 0 \) or \( \ell_a > 1 \).

Once we understand the decomposition of the \( \text{Aut}(M) \simeq N/H \)-module \( H_1^{(0)}(M, \mathbb{C}) \) into irreducible pieces, we can pass to the analysis of the \( N/H \)-module \( H_1^{(0)}(M, \mathbb{R}) \).

By general representation theory, the Galois group \( \text{Gal}(\mathbb{C}|\mathbb{R}) \) acts naturally on the set \( \text{Irr}_\mathbb{C}(N/H) \) of \( \mathbb{C} \)-irreducible representations of \( N/H \), and the \( \mathbb{R} \)-irreducible representations \( a \in \text{Irr}_\mathbb{R}(N/H) \) are precisely the \( \text{Gal}(\mathbb{C}|\mathbb{R}) \)-orbits of \( \alpha \in \text{Irr}_\mathbb{C}(N/H) \). Furthermore, given such an orbit \( a \in \text{Irr}_\mathbb{R}(N/H) \), the character \( \chi_a \) has the form

\[
\chi_a = m_a \sum_{\alpha \in a} \chi_{\alpha}
\]

where \( m_a \) is the so-called Schur index.

Concerning the multiplicities inside the \( N/H \)-modules \( H_1^{(0)}(M, \mathbb{R}) \) and \( H_1^{(0)}(M, \mathbb{C}) = \mathbb{C} \otimes H_1^{(0)}(M, \mathbb{R}) \), this means that \( \ell_a = m_a \ell_a \) for any \( \alpha \in a \). In particular, one can use Theorem 116 to determine the multiplicities \( \ell_a \) of \( \mathbb{R} \)-irreducible \( N/H \)-representations in \( H_1^{(0)}(M, \mathbb{R}) \).

Moreover, given \( V_a \) an irreducible \( \mathbb{R}(N/H) \)-module, the commuting algebra \( D_a \) of \( \mathbb{R}(N/H) \) in \( \text{End}_\mathbb{R}(V_a) \) is a skew-field of degree \( m_a^2 \) over its center \( K_a \).

This means that by denoting \( W_a \simeq V_a^{\ell_a} \) the isotypical component of \( V_a \) in \( H_1^{(0)}(M, \mathbb{R}) \), the commuting algebra of \( \mathbb{R}(N/H) \) in \( \text{End}_\mathbb{R}(W_a) \) is isomorphic to the matrix algebra \( M(\ell_a, D_a) \).

Actually, it is possible to show that there are only three types of \( a \in \text{Irr}_\mathbb{R}(N/H) \):

- \( a \) is real, i.e., \( a = \{\alpha\} \), \( m_a = 1 \) and \( D_a \simeq \mathbb{R} \);
- \( a \) is complex, i.e., \( a = \{\alpha, \overline{\alpha}\} \), \( m_a = 1 \) and \( D_a \simeq \mathbb{C} \);
- \( a \) is quaternionic, i.e., \( a = \{\alpha\} \), \( m_a = 2 \) and \( D_a \simeq \mathbb{H} \) (Hamilton’s quaternions).

In resume, the action of \( \text{Aut}(M) \simeq N/H \) on \( H_1^{(0)}(M, \mathbb{R}) \) is completely determined: \( H_1^{(0)}(M, \mathbb{R}) \) is decomposed into isotypical components \( W_a \simeq V_a^{\ell_a} \) (where the multiplicity \( \ell_a \) is explicitly computable) of real, complex or quaternionic type.

After this warm up, let’s consider the action of the affine group \( \text{Aff}(M) \) on \( H_1^{(0)}(M, \mathbb{R}) \). Note that \( \text{Aff}(M) \) acts by conjugation on \( \text{Aut}(M) \), i.e., \( AgA^{-1} \in \text{Aut}(M) \) whenever \( A \in \text{Aff}(M) \) and \( g \in \text{Aut}(M) \). In particular, \( \text{Aff}(M) \) permutes the isotypical components \( W_a \) of the \( \text{Aut}(M) \)-module \( H_1^{(0)}(M, \mathbb{R}) \), so that one can pass to an adequate finite index subgroup \( \text{Aff}_{*s}(M) \) of \( \text{Aff}(M) \) such that the elements of \( \text{Aff}_{*s}(M) \) preserves each isotypical component \( W_a \) and they act on the \( \mathbb{R}(\text{Aut}(M)) \)-module \( W_a \) via automorphisms.

In fact, we can say a little bit more about the action of \( \text{Aff}_{*s}(M) \) on isotypical components \( W_a \). Recall that \( H_1^{(0)}(M, \mathbb{R}) \) carries a symplectic intersection form \( \{\ldots\} \) preserved by \( \text{Aff}(M) \).
Moreover, it is not hard to see that the restriction \( \{ \ldots \} |_{W_a} \) to any isotypical component \( W_a \) of \( H_1^{(0)}(M, \mathbb{R}) \) is non-degenerate. In other words, \( \text{Aff}_+(M) \) acts on \( W_a \) via the group

\[
\text{Sp}(W_a) := \{ \text{automorphisms of the } \mathbb{R}(N/H) \text{- module } W_a \text{ preserving } \{ \ldots \} |_{W_a} := \{ \ldots \} |_{W_a} \}
\]

By studying each possibility for \( a \in \text{Irr}_\mathbb{R}(N/H) \), one can show that:

- if \( a \) is real, \( \ell_a \) is even and \( \text{Sp}(W_a) \) is isomorphic to the symplectic group \( \text{Symp}(\ell_a, \mathbb{R}) \);
- if \( a \) is complex, there are integers \( p_a, q_a \) with \( \ell_a = p_a + q_a \) such that \( \text{Sp}(W_a) \) is isomorphic to the group \( U_C(p_a, q_a) \) of matrices with complex coefficients preserving a pseudo-Hermitian form of signature \( (p_a, q_a) \);
- if \( a \) is quaternionic, there are integers \( p_a, q_a \) with \( \ell_a = p_a + q_a \) such that \( \text{Sp}(W_a) \) is isomorphic to the group \( U_H(p_a, q_a) \) of matrices with quaternionic coefficients preserving a pseudo-Hermitian form of signature \( (p_a, q_a) \).

From this discussion, we already can get derive some consequences for the Lyapunov exponents of KZ cocycle: indeed, the fact that \( \text{Aff}_+(M) \) acts on complex and quaternionic isotypical components \( W_a \) via the groups \( U_C(p_a, q_a) \) and \( U_H(p_a, q_a) \) can be used to ensure the presence of \( |p_a - q_a| \) zero Lyapunov exponents (at least). See e.g. [58] and/or Appendix A of [84] for more details. Moreover, by looking at the definitions it is not hard to show that Oseledets subspaces \( W_a(\theta, x) \) associated to a Lyapunov exponent \( \theta \) of the restriction of KZ cocycle (or, equivalently \( \text{Aff}_+(M) \)) to a isotypical component \( W_a \cong V_a^{\ell_a} \) at a point \( x \) in the \( SL(2, \mathbb{R}) \)-orbit of \( M \) are \( \text{Aut}(M) \)-invariant. Therefore, these Oseledets subspaces \( W_a(\theta, x) \) is a \( \mathbb{R}(\text{Aut}(M)) \)-module obtained as a finite sum of copies of \( V_a \), and, \textit{a fortiori}, the multiplicity of the Lyapunov exponent \( \theta \) (i.e., the dimension of \( W_a(\theta, x) \)) is a \textit{multiple} of \( \dim_\mathbb{R}(V_a) \).

In a nutshell, we can resume our discussion so far as follows: starting with a square-tiled surface \( M \) with a \textit{non-trivial} group of automorphism \( \text{Aut}(M) \) (in the sense that \( \text{Aut}(M) \) has a rich representation theory), \textit{usually} one finds:

- several vanishing Lyapunov exponents, mostly coming from complex and/or quaternionic isotypical components, and
- \textit{high multiplicity}, i.e., non-simplicity, of general Lyapunov exponents.

Closing this subsection, let’s point out that our discussion so far depend only on the knowledge of \( G, H \) and the \textit{commutator} \( c = [h, v] \), cf. Remark [117]. In particular, one may ask whether the Lyapunov exponents \( \theta \) depend only on \( c \). As it turns out, the answer to this question is \textit{negative}: for instance, for \( G = A_6 \) and \( H = \{ \text{id} \} \), it is possible to construct two pairs of permutations \( (h_0, v_0) \) and \( (h_1, v_1) \) generating \( G = A_6 \) with the \textit{same} commutator \( c = [h_0, v_0] = [h_1, v_1] \) such that the sum of the non-negative Lyapunov exponents of KZ cocycle over the \( SL(2, \mathbb{R}) \)-orbits of the corresponding square-tiled surfaces is \( 278/5 \) and \( 54 \) and hence \textit{distinct} Lyapunov spectra.

\[52\text{In fact, we already met this phenomenon during the proof of Proposition [94]}.\]
D.2. A criterion for the simplicity of Lyapunov exponents of square-tiled surfaces. In this subsection we will follow [57] to give a criterion for the simplicity of the Lyapunov exponents of square-tiled surfaces. Then, we will see some applications of this result in the cases of square-tiled surfaces in $\mathcal{H}(2)$ (genus 2) and $\mathcal{H}(4)$ (genus 3).

D.2.1. Avila-Viana simplicity criterion for cocycles over countable shifts. Let’s start by studying simplicity of Lyapunov exponents in the abstract setting of cocycles over countable shifts.

Let $A$ be a finite or countable alphabet. Define $\Sigma = \Lambda^\mathbb{N}$ and denote by $f : \Sigma \to \Sigma$ the natural (left) shift map on $\Sigma$. Let $\Omega = \bigcup_{n \geq 0} \Lambda^n$ the set of words of the alphabet $\Lambda$. Given $\ell \in \Omega$, let

$$\Sigma(\ell) := \{x \in \Sigma : x \text{ starts by } \ell\}$$

**Definition 119.** We say that a probability measure $\mu$ on $\Sigma$ has bounded distortion whenever there exists a constant $C(\mu) > 0$ such that

$$\frac{1}{C(\mu)} \mu(\Sigma(\ell_1)) \mu(\Sigma(\ell_2)) \leq \mu(\Sigma(\ell_1 \ell_2)) \leq C(\mu) \mu(\Sigma(\ell_1)) \mu(\Sigma(\ell_2))$$

for any $\ell_1, \ell_2 \in \Omega$.

The bounded distortion assumption says that, in some sense, $\mu$ is “not very far” from a Bernoulli measure. As an exercise, the reader can check that bounded distortion implies that $\mu$ is $f$-ergodic.

From now on, we will assume that $\mu$ has bounded distortion and we think of $(f, \mu)$ as our base dynamical system. Next, we discuss some assumptions concerning the class of cocycles we want to investigate over this base dynamics.

**Definition 120.** We say that a cocycle $A : \Sigma \to Sp(2d, \mathbb{R})$ is

- **locally constant** if $A(x) = A_{x_0}$, where $A_\ell \in Sp(2d, \mathbb{R})$ for $\ell \in \Lambda$, and $x = (x_0, \ldots) \in \Sigma$;

- **(log-)integrable** if $\int_{\Sigma} \log \|A^{\pm 1}(x)\| \, d\mu(x) = \sum_{\ell} \mu(\Sigma(\ell)) \log \|A^{\pm 1}_\ell\| < \infty$.

**Remark 121.** Following the work [7] of Avila and Viana, we’ll focus here in the case $A_\ell \in Symp(d, \mathbb{R})$, $d$ even, because we want to apply their criterion to a symplectic cocycle closely related to the Kontsevich-Zorich cocycle. However, it is not hard to see that Avila-Viana simplicity criterion below can extended to the groups $U_C(p,q)$ and $U_\mathbb{H}(p,q)$, and this particularly useful because KZ cocycle may act via these groups in some examples (as we already saw above). For more details on this, see [57].

Given $\ell = (\ell_0, \ldots, \ell_{n-1}) \in \Omega$, we write

$$A^{\ell} := A_{\ell_{n-1}} \cdots A_{\ell_0}$$

By definition, it satisfies $(f, A)^n(x) = (f^n(x), A^{\ell})$ for any $x \in \Sigma(\ell)$.

The ergodicity of $\mu$ (coming from the bounded distortion property) and the integrability of the cocycle $A$ allow us to apply the Oseledeets theorem to deduce the existence of Lyapunov exponents

$$\theta_1 \geq \cdots \geq \theta_d \geq -\theta_d \geq \cdots \geq -\theta_1$$
We denote by $G(k)$ the Grassmanian of

- isotropic $k$-planes if $1 \leq k \leq d$, and
- coisotropic $k$-planes if $d \leq k < 2d$.

At this point, we are ready to introduced the main assumptions on our cocycle $A$:

**Definition 122.** We say that the cocycle $A$ is

- pinching if there exists $\ell^* \in \Omega$ such that the spectrum of the matrix $A^{\ell^*}$ is simple.
- twisting if for each $k$ there exists $\ell(k) \in \Omega$ such that
  \[ A^{\ell(k)}(F) \cap F' = \{0\} \]
  for any $A^{\ell^*}$-invariant subspaces $F \in G(k)$, $F' \in G(2d - k)$.

**Remark 123.** Later on, we will refer to the matrices $B$ with the same property as $A^{\ell(k)}$ above as $(k)$-twisting with respect to (the pinching matrix) $A := A^{\ell^*}$.

In this language, one has the following version of the simplicity criterion of A. Avila and M. Viana [7]:

**Theorem 124.** Let $A$ be a locally constant log-integrable cocycle over a base dynamics $(f, \mu)$ consisting of a countable shift $f$ and a $f$-invariant probability measure $\mu$ with bounded distortion. Suppose that the cocycle $A$ is pinching and twisting. Then, the Lyapunov spectrum of $A$ is simple.

In the sequel, we wish to apply this result to produce a simplicity criterion for the KZ cocycle over $SL(2, \mathbb{R})$-orbits of square-tiled surfaces. For this sake, we will need to briefly discuss how to code the Teichmüller geodesic flow on $SL(2, \mathbb{R})$-orbits of square-tiled surfaces via a countable shift (closely related to the continued fraction algorithm).

The $SL(2, \mathbb{R})$-orbit of a square-tiled surface $(M, \omega)$ is a finite cover of the modular surface $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$: indeed, one has $SL(2, \mathbb{R}) \cdot (M, \omega) \simeq SL(2, \mathbb{R})/SL(M, \omega)$ in moduli space, where $SL(M, \omega)$ is the Veech group, and $SL(M, \omega)$ is a finite-index subgroup of $SL(2, \mathbb{Z})$ when $(M, \omega)$ is a square-tiled surface.

This hints that it is a nice idea to start our discussion by reviewing how the geodesic flow on the modular surface is coded by continued fraction algorithm.

**D.2.2. Coding the geodesic flow on the modular surface.** We will think of $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ as the space of normalized (i.e., unit covolume) lattices of $\mathbb{R}^2$, and we will select an appropriate fundamental domain. Here, it is worth to point out that we’re not going to consider the lift to $SL(2, \mathbb{R})$ of the “classical” fundamental domain $F = \{ z \in \mathbb{H} : |z| \geq 1, |\text{Re}z| \leq 1/2\}$ of the action of $SL(2, \mathbb{Z})$ on the hyperbolic plane $\mathbb{H}$. Indeed, as we will see below, our choice of fundamental domain is not $SO(2, \mathbb{R})$-invariant, while any fundamental domain obtained by lifting to $SL(2, \mathbb{R})$ a fundamental domain of $\mathbb{H}/SL(2, \mathbb{Z})$ must be $SO(2, \mathbb{R})$-invariant (as $\mathbb{H}/SL(2, \mathbb{Z}) = SO(2, \mathbb{R}) \backslash SL(2, \mathbb{R})/SL(2, \mathbb{Z})$).
Definition 125. A lattice \( L \subset \mathbb{R}^2 \) is irrational if \( L \) intersect the coordinate axis \( x \) and \( y \) precisely at the origin \( 0 \in \mathbb{R}^2 \). Equivalently, \( L \) is irrational if and only if the orbit \( g_t(L) \) doesn’t diverge (neither in the past nor in the future) to the cusp of \( SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \).

Our choice of fundamental domain will be guided by the following fact:

Proposition 126. Let \( L \) be a normalized irrational lattice. Then, there exists an unique basis \( \{v_1 = (\lambda_1, \tau_1), v_2 = (\lambda_2, \tau_2)\} \) of \( L \) such that exactly one of the two possibilities below occur:

- top case \(-\lambda_2 \geq 1 > \lambda_1 > 0 \) and \( 0 < \tau_2 < -\tau_1 \);
- bottom case \(-\lambda_1 \geq 1 > \lambda_2 > 0 \) and \( 0 < -\tau_1 < \tau_2 \).

The proof of this result is based on Minkowski theorem (ensuring the existence of vectors of the irrational lattice \( L \) in \( Q^+ = (0,1) \times (0,1) \) or \( Q^- = (0,1) \times (-1,0) \)) and some elementary computations. See, e.g., [57] for more details.

Below, we illustrate irrational lattices of top and bottom types:

Using this proposition, we can describe the Teichmüller geodesic flow \( g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \) on the space \( SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \) of normalized lattices as follows. Let \( L_0 \) be a normalized irrational lattice, and let \((v_1, v_2)\) be the basis of \( L_0 \) given by the proposition above, i.e., the top, resp. bottom, condition. Then, we see that the basis \( (g_t v_1, g_t v_2) \) of \( L_t := g_t L_0 \) satisfies the top, resp. bottom condition for all \( t < t^* \), where \( \lambda_1 e^{t^*} = 1 \) in the top case, resp. \( \lambda_2 e^{t^*} = 1 \) in the bottom case.

However, at time \( t^* \), the basis \( \{v_1^* = g_{t^*} v_1, v_2^* = g_{t^*} v_2\} \) of \( L_0 \) ceases to fit the requirements of the proposition above, but we can remedy this problem by changing the basis: for instance, if the basis \( \{v_1, v_2\} \) of the initial lattice \( L_0 \) has top type, then it is not hard to check that

\[
\begin{align*}
v_1' &= v_1^* \\
v_2' &= v_2^* - av_1^*
\end{align*}
\]

where \( a = \lfloor \lambda_2/\lambda_1 \rfloor \) is a basis of \( L_{t^*} \) of bottom type. This is illustrated in the picture below:

Here, we observe that the quantity \( \alpha := \lambda_1/\lambda_2 \in (0,1) \) giving the ratios of the first coordinates of the vectors \( g_t v_1, g_t v_2 \) forming a top type basis of \( L_t \) for any \( 0 \leq t < t^* \) is related to the integer
a by the formula

\[ a = \lfloor 1/\alpha \rfloor \]

Also, the new quantity \( \alpha' \) giving the ratio of the first coordinates of the vectors \( v'_1, v'_2 \) forming a bottom type basis of \( L_t \) is related to \( \alpha \) by the formula

\[ \alpha' = \lambda_2/\lambda'_1 = \{1/\alpha\} := G(\alpha) \]

where \( G \) is the so-called Gauss map. In this way, we find the classical relationship between the geodesic flow on the modular surface \( SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \) and the continued fraction algorithm.

At this stage, we’re ready to code the Teichmüller flow over the unit tangent bundle of the Teichmüller surface \( SL(2, \mathbb{R})/SL(M) \) associated to a square-tiled surface.

**D.2.3. Coding the Teichmüller flow on \( SL(2, \mathbb{R}) \)-orbits of square-tiled surfaces.** Let \( \Gamma(M) \) be the following graph: the set of its vertices is

\[ \text{Vert}(\Gamma(M)) = \{SL(2, \mathbb{Z})\text{-orbit of } M\} \times \{t, b\} = \{M = M_1, \ldots, M_r\} \times \{t, b\} \]

and its arrows are

\[ (M_i, c) \xrightarrow{\gamma_{a, t, c}} (M_j, \bar{c}) \]

where \( a \in \mathbb{N}, a \geq 1, c \in \{t, b\}, \bar{c} = b \) (resp. \( t \)) if \( c = t \) (resp. \( b \)), and

\[ M_j = \begin{cases} 
(1\ a) M_i, & \text{if } c = t \\
(0\ 1) & \\
(1\ 0) M_i, & \text{if } c = b \\
(a\ 1) \end{cases} \]

Notice that this graph has finitely many vertices but countably many arrows. Using this graph, we can code irrational orbits of the flow \( g_t \) on \( SL(2, \mathbb{R})/SL(M) \) as follows. Given \( m_0 \in SL(2, \mathbb{R}) \), let \( L_{st} = \mathbb{Z}^2 \) be the standard lattice and put \( m_0 L_{st} = L_0 \). Also, let us denote \( m_t = g_t m_0 \).
By Proposition 126, there exists a unique $h_0 \in SL(2, \mathbb{Z})$ such that $v_1 = m_0 h_0^{-1}(e_1)$, $v_2 = m_0 h_0^{-1}(e_2)$ satisfying the conditions of the proposition (here, $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$). Denote by $c$ the type (top or bottom) of the basis $\{v_1, v_2\}$ of $L_0$. We assign to $m_0$ the vertex $(M_i := h_0 M, c) \in Vert(\Gamma(M))$.

For sake of concreteness, let’s assume that $c = t$ (top case). Recalling the notations introduced after the proof of Proposition 126, we notice that the lattice $L_t^*$ associated to $m_t^*$ has a basis of bottom type $v_1' = g_t^* m_0 h_0^{-1}(e_1) = g_t^* m_0 h_1^{-1}(e_1)$ and $v_2' = g_t^* m_0 h_0^{-1}(e_2 - a e_1) = g_t^* m_0 h_1^{-1}(e_2)$ where $h_1 = h^* h_0$ and

$$h^* = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

In other words, starting from the vertex $(M_i, t)$ associated to the initial point $m_0$, after running the geodesic flow for a time $t^*$, we end up with the vertex $(M_j, b)$ where $M_j = h_* M_i$. Equivalently, the piece of trajectory from $m_0$ to $g_t^* m_0$ is coded by the arrow

$$(M_i, t) \xrightarrow{\gamma_{t, t^*}} (M_j, b)$$

Evidently, we can iterate this procedure (by replacing $L_0$ by $L_t^*$) in order to code the entire orbit $g_t m_0$ by a succession of arrows. However, this coding has the “inconvenient” (with respect to the setting of Avila-Viana simplicity criterion) that it is not associated to a complete shift but only a subshift (as we do not have the right to concatenate two arrows $\gamma$ and $\gamma'$ unless the endpoint of $\gamma$ coincides with the start of $\gamma'$).

Fortunately, this little difficulty is easy to overcome: in order to get a coding by a complete shift, it suffices to fix a vertex $p^* \in Vert(\Gamma(M))$ and consider exclusively concatenations of loops based at $p^*$. Of course, we pay a price here: since there may be some orbits of $g_t$ whose coding is not a concatenation of loops based on $p^*$, we’re throwing away some orbits in this new way of coding. But, it is not hard to see that the (unique, Haar) $SL(2, \mathbb{R})$-invariant probability $\mu$ on $SL(2, \mathbb{R})/SL(M)$ gives zero weight to the orbits that we’re throwing away, so that this new coding still captures most orbits of $g_t$ (from the point of view of $\mu$). In any case, this allows to code $g_t$ by a complete shift whose (countable) alphabet is constituted of (minimal) loops based at $p^*$.

Once we know how to code our flow $g_t$ by a complete shift, the next natural step (in view of Avila-Viana criterion) is the verification of the bounded distortion condition of the invariant measure induced by $\mu$ on the complete shift.

As we saw above, the coding of the geodesic flow (and modulo the stable manifolds, that is, the “$t$-coordinates” [vertical coordinates]) is the dynamical system

$$Vert(\Gamma(M)) \times ((0, 1) \cap (\mathbb{R} - \mathbb{Q})) \to Vert(\Gamma(M)) \times ((0, 1) \cap (\mathbb{R} - \mathbb{Q}))$$
given by \((p, \alpha) \mapsto (p', G(\alpha))\) where \(G(\alpha) = \{1/\alpha\} = \alpha'\) is the Gauss map and \(p \xrightarrow{\gamma_{0, p}} p'\) with 
\(a = \lceil 1/\alpha \rceil\).
In this language, \(\mu\) becomes (up to normalization) the Gauss measure \(dt/(1 + t)\) on each copy \(\{p\} \times (0, 1), p \in \text{Vert}(\Gamma(M))\), of the unit interval \((0, 1)\).

Now, for sake of concreteness, let us fix \(p^*\) a vertex of top type. Given \(\gamma\) a loop based on \(p^*\), i.e., a word on the letters of the alphabet of the coding leading to a complete shift, we denote by 
\(I(\gamma) \subset (0, 1)\) the interval corresponding to \(\gamma\), that is, the interval \(I(\gamma)\) consisting of \(\alpha \in (0, 1)\) such that the concatenation of loops (based at \(p^*\)) coding the orbit of \((p^*, \alpha)\) starts by the word \(\gamma\).

In this setting, the measure induced by \(\mu\) on the complete shift is easy to express: by definition, the measure of the cylinder \(\Sigma(\gamma)\) corresponding to concatenations of loops (based at \(p^*\)) starting by \(\gamma\) is the Gauss measure of the interval \(I(\gamma)\) up to normalization. Because the Gauss measure is equivalent to the Lebesgue measure (as its density \(1/(1 + t)\) satisfies \(1/2 \leq 1/(1 + t) \leq 1\) in \((0, 1)\)), we conclude that the measure of \(\Sigma(\gamma)\) is equal to \(|I(\gamma)| = \text{Lebesgue measure of } I(\gamma)\) up to a multiplicative constant.

In particular, it follows that the bounded distortion condition for the measure induced by \(\mu\) on the complete shift is equivalent to the existence of a constant \(C > 0\) such that 
\[
C^{-1}|I(\gamma_0)| \cdot |I(\gamma_1)| \leq |I(\gamma)| \leq C|I(\gamma_0)| \cdot |I(\gamma_1)|
\] (D.1)

for every \(\gamma = \gamma_0 \gamma_1\).

In resume, this reduces the bounded distortion condition to the problem of understanding the interval \(I(\gamma)\). Here, by the usual properties of the continued fraction algorithm, it is not hard to show that \(I(\gamma)\) is a Farey interval 
\[
I(\gamma) = \left(\frac{p}{q}, \frac{p + p'}{q + q'}\right)
\]
with 
\[
\left(\begin{array}{cc}
p' & p \\
q' & q
\end{array}\right) \in SL(2, \mathbb{Z})
\]
being \(t\)-reduced, i.e., \(0 < p' \leq p, q' < q\).

Consequently, from this description, we recover the classical fact that 
\[
\frac{1}{2q^2} \leq |I(\gamma)| = \frac{1}{q(q + q')} \leq \frac{1}{q^2}
\] (D.2)

Given \(\gamma = \gamma_0 \gamma_1\), and denoting by \(\left(\begin{array}{cc}
p_0' & p_0 \\
q_0' & q_0
\end{array}\right)\), resp. \(\left(\begin{array}{cc}
p_1' & p_1 \\
q_1' & q_1
\end{array}\right)\), resp. \(\left(\begin{array}{cc}
p' & p \\
q' & q
\end{array}\right)\) the matrices associated to \(\gamma_0\), resp. \(\gamma_1\), resp. \(\gamma\), it is not hard to check that 
\[
\left(\begin{array}{cc}
p' & p \\
q' & q
\end{array}\right) = \left(\begin{array}{cc}
p_0' & p_0 \\
q_0' & q_0
\end{array}\right) \left(\begin{array}{cc}
p_1' & p_1 \\
q_1' & q_1
\end{array}\right)
\]
so that \(q = q_0 q_1 + q_0 q_1\). Because these matrices are \(t\)-reduced, we have that 
\[
q_0 q_1 \leq q \leq 2q_0 q_1
\]
Therefore, in view of (D.1) and (D.2), the bounded distortion condition follows.

Once we know that the basis dynamics (Teichmüller geodesic flow on $SL(2, \mathbb{R})/SL(M)$) is coded by a complete shift equipped with a probability measure with bounded distortion, we can pass to the study of the Kontsevich-Zorich cocycle in terms of the coding.

D.2.4. Coding KZ cocycle over $SL(2, \mathbb{R})$-orbits of square-tiled surfaces. Let $(M_i, [t, \text{resp. } b]) \xrightarrow{\gamma_{i-1}} (M_j, [b, \text{resp. } t])$ be an arrow of $\Gamma(M)$ and denote by $A : M_i \to M_j$ an affine map of derivative \[
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}, \text{ resp. } \begin{pmatrix}
1 & 0 \\
a & 1
\end{pmatrix}.
\]
Of course, $A$ is only well-defined up to automorphisms of $M_i$ and/or $M_j$. In terms of translation structures, given $g \in SL(2, \mathbb{R})$ and a translation structure $\zeta$ on $M$, the identity map $\text{id} : (M, \zeta) \to (M, g\zeta)$ is an affine map of derivative $g$.

Given $\gamma$ a path in $\Gamma(M)$ obtained by concatenation $\gamma = \gamma_1 \ldots \gamma_\ell$, and starting at $(M_i, c)$ and ending at $(M_j, c')$, one has, by functoriality, $A_\gamma : M_i \to M_j$ an affine map given by $A_\gamma = A_{\gamma_\ell} \ldots A_{\gamma_1}$.

Suppose now that $\gamma$ is a loop based at $(M, c)$. Then, by definition, the derivative $A_\gamma \in SL(M)$. For our subsequent discussions, an important question is: what matrices of the Veech group $SL(M)$ can be obtained in this way?

In this direction, we recall the following definition (already encountered in the previous section):

**Definition 127.** We say that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ is

- t-reduced if $0 < a \leq b, c < d$;
- b-reduced if $0 < d \leq b, c < a$.

Observe that the product of two t-reduced (resp. b-reduced) matrices is also t-reduced (resp. b-reduced), i.e., these conditions are stable by products.

The following statement is the answer to the question above:

**Proposition 128.** The matrices associated to the loops $\gamma$ based at the vertex $(M, c)$ are precisely the c-reduced matrices of $SL(M)$.

D.2.5. Simplicity criterion for KZ cocycle over $SL(2, \mathbb{R})$-orbits of square-tiled surfaces. At this point, our discussion so far says that it suffices to check pinching and twisting conditions to obtain simplicity of the Lyapunov spectrum of square-tiled surfaces. In this direction, M. Möller, J.-C. Yoccoz and the second author [57] showed that pinching and twisting conditions (and, a fortiori, simplicity of Lyapunov spectrum) can be obtained from certain Galois theory conditions in the context of square-tiled surfaces:

**Theorem 129.** Let $M$ be a square-tiled surface. Suppose that there are two affine diffeomorphisms $\varphi_A$ and $\varphi_B$ whose linear parts $D\varphi_A$ and $D\varphi_B$ are either both t-reduced or both b-reduced, and assume that the action of $\varphi_A$ and $\varphi_B$ on the complementary subspace $H_1^{(0)}(M, \mathbb{Z})$ is given by two matrices $A, B \in Sp(2g - 2, \mathbb{Z})$ with the following properties:

i) The eigenvalues of $A$ are real.
ii) The splitting field of the characteristic polynomial $P$ of $A$ has degree $2^{g-1}(g-1)!$, i.e., the Galois group is the biggest possible.

iii) $A$ and $B^2$ don’t share a common proper invariant subspace.

Then the Lyapunov spectrum of $M$ is simple.

Finally, the condition iii) above can be verified by checking that i) and ii) hold, and the disjointness of the splitting fields of $A$ and $B$ (see Remark 130 below).

In what follows, we’ll give a sketch of proof of this theorem. We begin by noticing that the matrix $A$ verifies the pinching condition (cf. Theorem 124 and Definition 122); indeed, since the Galois group $G$ of $P$ is the largest possible, we have that $P$ is irreducible, and thus its roots are simple. By the assumption (i), all roots $\lambda_i, \lambda_i^{-1}, 1 \leq i \leq d,$ of $P$ are real, so that the pinching condition is violated by $A$ precisely when there are $i \neq j$ such that $\lambda_i = -\lambda_j^\pm$. However, this is impossible because $G$ is the largest possible: for instance, since $i \neq j$, we have an element of $G$ fixing $\lambda_i$ and exchanging $\lambda_j$ and $\lambda_j^{-1}$; applying this element to the relation $\lambda_i = -\lambda_j^\pm$, we would get that $\lambda_i = -\lambda_j$ and $\lambda_i = -\lambda_j^{-1}$, so that $\lambda_j = \pm 1$ a contradiction with the fact that $P$ is irreducible.

Remark 130. Concerning the applications of this theorem to the case of origamis, we observe that item (iii) is satisfied whenever the splitting fields $\mathbb{Q}(P_B)$ and $\mathbb{Q}(P)$ of the characteristic polynomials of $B$ and $A$ are disjoint as extensions of $\mathbb{Q}$, i.e., $\mathbb{Q}(P_B) \cap \mathbb{Q}(P) = \mathbb{Q}$. Indeed, if $E \subset \mathbb{R}^{2d}$ is invariant by $A$ and $B^2$, one has that

- $E$ is generated by eigenvectors of $A$ (as $A$ is pinching, i.e., $A$ has simple spectrum), so that $E$ is defined over $\mathbb{Q}(P)$, and
- $E$ is invariant by $B^2$, so that $E$ is also defined over $\mathbb{Q}(P_B)$.

Since $\mathbb{Q}(P)$ and $\mathbb{Q}(P_B)$ are disjoint, it follows that $E$ is defined over $\mathbb{Q}$. But this is impossible as $A$ doesn’t have rational invariant subspaces (by (i) and (ii)).

Once we know that the matrix $A$ satisfies the pinching condition, the proof of Theorem 129 is reduced to checking the twisting condition with respect to $A$ (Remark 123). Keeping this goal in mind, we introduced the following notations.

We denote by $\tilde{R}$ the set of roots of the polynomial $P$ (so that $\# \tilde{R} = 2d$), for each $\lambda \in \tilde{R}$, we put $p(\lambda) = \lambda + \lambda^{-1}$, and we define $R = p(\tilde{R})$ (so that $\# R = d$).

Given $1 \leq k \leq d$, let $\tilde{R}_k$, resp. $R_k$ be the set whose elements are subsets $\Lambda$ of $\tilde{R}$, resp. $R$ with $k$ elements, and let $\tilde{R}_k$ be the set whose elements are subsets $\Lambda$ of $\tilde{R}$ with $k$ elements such that $p|\Lambda$ is injective. In other words, $\tilde{R}_k$ consist of those $\Lambda \in \tilde{R}_k$ such that if $\lambda \in \Lambda$, then $\lambda^{-1} \notin \Lambda$.

Next, we make a choice of basis of $\mathbb{R}^{2d}$ as follows. For each $\lambda \in \tilde{R}$, we select an eigenvector $v_\lambda$ of $A$ associated to $\lambda$, i.e., $Av_\lambda = \lambda v_\lambda$. In particular, $v_\lambda$ is defined over $\mathbb{Q}(\lambda) \subset \mathbb{Q}(P)$. Then, we assume that the choices of $v_\lambda$’s are coherent with the action of the Galois group $G$, i.e., $v_{g\lambda} = g v_\lambda$ (and thus $A(v_{g\lambda}) = (g\lambda)v_{g\lambda}$) for each $g \in G$. In this way, for each $\Lambda \in \tilde{R}_k$, we can associated a multivector $v_\Lambda = v_{\lambda_1} \wedge \cdots \wedge v_{\lambda_k} \in \Lambda^k \mathbb{R}^{2d}$ (using the natural order of the elements of $\Lambda = \{\lambda_1 < \cdots < \lambda_k\}$).

By definition, $(\Lambda^k A)(v_\Lambda) = N(\Lambda) v_\Lambda$ where $N(\Lambda) := \prod \lambda_i$. 


From our assumptions (i) and (ii) on $A$, we have that:

- $v_{\lambda}$, $\lambda \in \tilde{R}_k$, is a basis of $\bigwedge^k \mathbb{R}^{2d}$
- the subspace generated by $v_{\lambda_1}, \ldots, v_{\lambda_k}$ is isotropic if and only if $\lambda = \{\lambda_1, \ldots, \lambda_k\} \in \tilde{R}_k$

Also, by an elementary (linear algebra) computation, it is not hard to check that a matrix $C$ is twisting with respect to $A$ if and only if

$C^{(k)}_{\Delta' \Delta} \neq 0$ \hspace{1cm} (D.3)

for all $\lambda, \lambda' \in \tilde{R}_k$, where $C^{(k)}_{\Delta' \Delta}$ are the coefficients of the matrix $\bigwedge^k C$ in the basis $v_{\lambda}$.

In order to organize our discussions, we observe that the condition (D.3) can be used to define an oriented graph $\Gamma_k(C)$ as follows. Its vertices $\text{Vert}(\Gamma_k(C)) = \hat{R}_k$, and we have an arrow from $\lambda_0$ to $\lambda_1$ if and only if $C^{(k)}_{\Delta_0 \Delta_1} \neq 0$. In this language, (D.3) corresponds to the fact that $\Gamma_k(C)$ is a complete graph.

Of course, the verification of the completeness of $\Gamma_k(C)$ is not simple in general, and hence it could be interesting to look for more soft properties of $\Gamma_k(C)$ ensuring completeness of $\Gamma_k(D)$ for some matrix $D$ constructed as a product of powers of $C$ and $A$. Here, we take our inspiration from Dynamical Systems and we introduce the following classical notion:

**Definition 131.** $\Gamma_k(C)$ is mixing if there exists $m \geq 1$ such that for all $\lambda_0, \lambda_1 \in \tilde{R}_k$ we can find an oriented path in $\Gamma_k(C)$ of length $m$ going from $\lambda_0$ to $\lambda_1$.

Here, we note that it is important in this definition that we can connect two arbitrary vertices by a path of length *exactly* $m$ (and not of length $\leq m$). For instance, the figure below shows a connected graph that is not mixing because all paths connecting $A$ to $B$ have odd length while all paths connecting $A$ to $C$ have even length.

As the reader can guess by now, mixing is a soft property ensuring completeness of a “related” graph. This is the content of the following proposition:

**Proposition 132.** Suppose that $\Gamma_k(C)$ is mixing with respect to an integer $m \geq 1$. Then, there is a finite family of hyperplanes $V_1, \ldots, V_t$ of $\mathbb{R}^{m-1}$ such that, for any $\ell = (\ell_1, \ldots, \ell_{m-1}) \in \mathbb{Z}^{m-1} - (V_1 \cup \cdots \cup V_{m-1})$, the matrix

$D(n) := C A^{n\ell_1} \cdots C A^{n\ell_{m-1}} C$

satisfies (D.3) for all sufficiently large $n$. 
Proof. To alleviate the notation, let’s put $D = D(n)$. By definition,

$$D_{\Delta_0 ; \Delta_m}^{(k)} = \sum_{\gamma \text{ path of length } m \text{ in } \Gamma_k(C) \text{ from } \Delta_0 \text{ to } \Delta_m} C^{(k)}_{\Delta_0 ; \Delta_1} N(\Delta_1)^{nt_1} C^{(k)}_{\Delta_1 ; \Delta_2} \cdots N(\Delta_{m-1})^{nt_{m-1}} C^{(k)}_{\Delta_{m-1} ; \Delta_m} =: \sum_{\gamma} c_\gamma \exp(nL_\gamma(\ell))$$

where $c_\gamma \neq 0$ and $L_\gamma(\ell) = \sum_{i=1}^{n-1} \ell_i \left( \sum_{\lambda \in \Delta} \log |\lambda| \right)$.

Our goal is to prove that $D_{\Delta_0 ; \Delta_m}^{(k)} \neq 0$. Of course, even though $D_{\Delta_0 ; \Delta_m}^{(k)}$ was expressed as a sum of exponentials with non-vanishing coefficients, there is a risk of getting non-trivial cancelations so that the resulting expression vanishes. The idea is to show that $\ell$ can be chosen suitably to avoid such cancelations, and the heart of this argument is the observation that, for $\gamma \neq \gamma'$, the linear forms $L_\gamma$ and $L_{\gamma'}$ are distinct. Indeed, given $\Delta \in \hat{R}_k$ and $\Delta' \in \hat{R}_k$, $\Delta' \neq \Delta$, we claim that the following coefficients of $L_\gamma$ and $L_{\gamma'}$ differ:

$$\sum_{\lambda \in \Delta} \log |\lambda| \neq \sum_{\lambda' \in \Delta'} \log |\lambda'|$$

Otherwise, we would have a relation

$$\prod_{\lambda \in \Delta} \lambda = \pm \prod_{\lambda' \in \Delta'} \lambda' := \phi$$

But, since $\Delta \in \hat{R}_k$, we have that if $\lambda \in \Delta$ then $\lambda^{-1} \notin \Delta$. In particular, by taking an element $\lambda(0) \in \Delta - \Delta'$, and by considering an element $g$ of the Galois group $G$ with $g(\lambda(0)) = \lambda(0)^{-1}$ and $g(\lambda) = \lambda$ otherwise, one would get on one hand that

$$g\phi = \prod_{\lambda \in \Delta} g\lambda = \lambda(0)^{-2}\phi$$

but, on the other hand,

$$g\phi = \pm \prod_{\lambda' \in \Delta'} g\lambda' = \pm \begin{cases} \lambda(0)^2\phi & \text{if } \lambda(0)^{-1} \in \Delta' \\ \phi & \text{otherwise} \end{cases}$$

so that $\lambda(0)^{-2}\phi = \pm \lambda(0)^2\phi$ or $\pm \phi$, a contradiction in any event (as $\lambda(0)$ is real and $\lambda(0) \neq \pm 1$ by the pinching conditions on $A$).

Now, we define $V(\gamma, \gamma') = \{ \ell : L_\gamma(\ell) = L_{\gamma'}(\ell) \}$. Since $L_\gamma$ and $L_{\gamma'}$ are distinct linear forms for $\gamma \neq \gamma'$, it follows that $V(\gamma, \gamma')$ is a hyperplane. Because there are only finitely many paths $\gamma, \gamma'$ of length $m$ on $\Gamma_k(C)$, the collection of $V(\gamma, \gamma')$ corresponds to a finite family of hyperplanes $V_1, \ldots, V_t$.

Finally, we complete the proof by noticing that if $\ell \notin V_1 \cup \cdots \cup V_t$, then

$$D_{\Delta_0 ; \Delta_m}^{(k)} = \sum_{\gamma} c_\gamma \exp(nL_\gamma(\ell)) \neq 0$$

for $n \to \infty$ sufficiently large because the coefficients $L_\gamma(\ell)$ are mutually distinct. \qed
At this point, the proof of Theorem 129 goes along the following lines:

- **Step 0**: We will show that the graphs $\Gamma_k(C)$ are always non-trivial, i.e., there is at least one arrow starting at each of its vertices.

- **Step 1**: Starting from $A$ and $B$ as above, we will show that $\Gamma_1(B)$ is mixing and hence, by Proposition 132, there exists $C$ twisting 1-dimensional (isotropic) $A$-invariant subspaces.

- By Step 1, the treatment of the case $d = 1$ is complete, so that we have to consider $d \geq 2$. Unfortunately, there is no “unified” argument to deal with all cases and we are obliged to separate the case $d = 2$ from $d \geq 3$.

- **Step 2**: In the case $d \geq 3$, we will show that $\Gamma_k(C)$ (with $C$ as in Step 1) is mixing for all $1 \leq k < d$. Hence, by Proposition 132, we can find $D$ twisting $k$-dimensional isotropic $A$-invariant subspaces for all $1 \leq k < d$. Then, we will prove that $\Gamma_d(D)$ is mixing and, by Proposition 132, we have $E$ twisting with respect to $A$, so that this completes the argument in this case.

- **Step 3**: In the special case $d = 2$, we will show that either $\Gamma_2(C)$ or a closely related graph $\Gamma_2^*(C)$ are mixing and we will see that this is sufficient to construct $D$ twisting 2-dimensional isotropic $A$-invariant subspaces.

In the sequel, the following easy remarks will be repeatedly used:

**Remark 133.** If $C \in Sp(2d, \mathbb{Z})$, then the graph $\Gamma_k(C)$ is invariant under the action of Galois group $G$ on the set $\hat{R}_k \times \tilde{R}_k$ (parametrizing all possible arrows of $\Gamma_k(C)$). In particular, since the Galois group $G$ is the largest possible, whenever an arrow $\lambda \rightarrow \lambda'$ belongs to $\Gamma_k(C)$, the inverse arrow $\lambda' \rightarrow \lambda$ also belongs to $\Gamma_k(C)$. Consequently, $\Gamma_k(C)$ always contains loop of even length.

**Remark 134.** A connected graph $\Gamma$ is not mixing if and only if there exists an integer $m \geq 2$ such that the lengths of all of its loops are multiples of $m$.

**Step 0: $\Gamma_k(C)$ is nontrivial.**

**Lemma 135.** Let $C \in Sp(2d, \mathbb{R})$. Then, each $\lambda \in \hat{R}_k$ is the start of at least one arrow of $\Gamma_k(C)$.

**Remark 136.** Notice that we allow symplectic matrices with real (not necessarily integer) coefficients in this lemma. However, the fact that $C$ is symplectic is important here and the analogous lemma for general invertible (i.e., $GL$) matrices is false.

**Proof.** For $k = 1$, since every 1-dimensional subspace is isotropic, $\hat{R}_1 = \tilde{R}$ and the lemma follows in this case from the fact that $C$ is invertible. So, let’s assume that $k \geq 2$ (and, in particular, $\hat{R}_k$ is a proper subset of $\tilde{R}_k$). Since $C$ is invertible, for each $\lambda \in \hat{R}_k$, there exists $\lambda' \in \tilde{R}_k$ with $C^{(k)}_{\lambda \lambda'} \neq 0$

Of course, one may have a priori that $\lambda' \in \tilde{R}_k - \hat{R}_k$, i.e., $\#p(\lambda) < k$, and, in this case, our task is to “convert” $\lambda'$ into some $\lambda'' \in \tilde{R}_k$ with $C^{(k)}_{\lambda \lambda''} \neq 0$. 

Evidently, in order to accomplish this task it suffices to show that if \( \# p(\Lambda') < k \) and \( C^{(k)}_{\Lambda, \Lambda'} \neq 0 \), then there exists \( \Lambda'' \) with \( C^{(k)}_{\Lambda, \Lambda''} \neq 0 \) and \( \# p(\Lambda'') = \# p(\Lambda') + 1 \). Keeping this goal in mind, we observe that \( \Lambda' \notin \hat{R}_k \) implies that we can write \( \Lambda = \{ \lambda_1', \lambda_2', \ldots, \lambda_k' \} \) with \( \lambda_1' \cdot \lambda_2' = 1 \). Also, the fact that \( C^{(k)}_{\Lambda, \Lambda'} \neq 0 \) is equivalent to say that the \( k \times k \) minor of \( C \) associated to \( \Lambda \) and \( \Lambda' \) is invertible, and hence, by writing \( \Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_k \} \), we can find \( w_1, \ldots, w_k \in \mathbb{R}^{2d} \) such that

\[
\text{span}\{w_1, \ldots, w_k\} = \text{span}\{v_{\lambda_1}, v_{\lambda_2}, \ldots, v_{\lambda_k}\}
\]

\[
C(w_i) = v_{\lambda_i'} + \sum_{\lambda \notin \Lambda'} C^*_i{\lambda} v_{\lambda}.
\]

In other words, we can make a change of basis to convert the invertible minor of \( C \) into the \( k \times k \) identity matrix.

Now, denoting by \( \{ \ldots \} \) the symplectic form, we observe that \( \{w_1, w_2\} = 0 \) because \( \Lambda \in \hat{R}_k \), i.e., the span of \( v_{\lambda_1} \) is an isotropic subspace, and \( w_1, w_2 \in \text{span}\{v_{\lambda_1}, v_{\lambda_2}, \ldots, v_{\lambda_k}\} \). On the other hand, since \( C \) is symplectic, we get that

\[
0 = \{w_1, w_2\} = \{C(w_1), C(w_2)\} = \{v_{\lambda_1'}, v_{\lambda_2'}\} + \sum_{\lambda', \lambda'' \notin \Lambda'} C^*_i{\lambda'} C^*_j{\lambda''} \{v_{\lambda'}, v_{\lambda''}\}
\]

Since \( \{v_{\lambda_1'}, v_{\lambda_2'}\} \neq 0 \) (as \( \lambda_1' \cdot \lambda_2' = 1 \)), it follows that there exists \( \lambda', \lambda'' \notin \Lambda' \) with \( C^*_i{\lambda} \neq 0 \) and \( C^*_i{\lambda''} \neq 0 \).

Then we define \( \Lambda' := (\Lambda' - \{\lambda_1'\}) \cup \{\lambda'\} \). We have that \( \# p(\Lambda'') = \# p(\Lambda') + 1 \). Furthermore, the minor \( C_{[\Lambda, \Lambda'']} \) of \( C \) associated to \( \Lambda \) and \( \Lambda'' \) is obtained from the minor \( C_{[\Lambda, \Lambda']} \) of \( C \) associated to \( \Lambda \) and \( \Lambda' \) by removing the line associated to \( v_{\lambda'i} \) and replacing it by the line associated to \( v_{\lambda''} \). By looking in the basis \( w_1, \ldots, w_k \), this means that the minor \( C_{[\hat{\Lambda}, \Lambda'']} \) differs from the identity minor \( C_{[\hat{\Lambda}, \Lambda']} \) by the fact that the line associated to \( v_{\lambda'} \) was replaced by the line associated to \( v_{\lambda''} \). In other words, in the basis \( w_1, \ldots, w_k \), one of the entries 1 of \( C_{[\hat{\Lambda}, \Lambda']} \) was replaced by the coefficient \( C^*_i{\lambda'} \neq 0 \). Thus, we conclude that the determinant \( C_{[\hat{\Lambda}, \Lambda'']}^{(k)} \) of the the minor \( C_{[\hat{\Lambda}, \Lambda'']} \) is

\[
C_{[\hat{\Lambda}, \Lambda'']}^{(k)} = C^*_i{\lambda'} \neq 0
\]

Therefore, \( \Lambda'' \) satisfies the desired properties and the argument is complete. \( \square \)

**Step 1: \( \Gamma_1(B) \) is mixing.** For \( d = 1 \), the set \( \hat{R}_1 \) consists of exactly one pair = \( \{\lambda, \lambda^{-1}\} \), so that the possible Galois invariant graphs are:

![Graphs](image_url)

(1) (2) (3)
In the first case, by definition, we have that $B(Rv_\lambda) = Rv_\lambda$ (and $B(Rv_{\lambda-1}) = Rv_{\lambda-1}$), so that $B$ and $A$ share a common subspace, a contradiction with our hypothesis in Theorem 129.

In the second case, by definition, we have that $B(Rv_\lambda) = Rv_\lambda$ and $B(Rv_{\lambda-1}) = Rv_\lambda$, so that $B^2(Rv_\lambda) = Rv_\lambda$ and thus $B^2$ and $A$ share a common subspace, a contradiction with our assumptions in Theorem 129.

Finally, in the third case, we have that the graph $\Gamma_1(B)$ is complete, and hence $B$ is 1-twisting with respect to $A$.

Now, after this “warm up”, we pass to the general case $d \geq 2$. Firstly, suppose that the sole arrows in $\Gamma_1(B)$ are of the form $\lambda \to \lambda \pm 1$. Then, $B(Rv_\lambda \oplus Rv_{\lambda-1}) = Rv_\lambda \oplus Rv_{\lambda-1}$, and, since $d \geq 2$, the subspace $Rv_\lambda \oplus Rv_{\lambda-1}$ is non-trivial. In particular, in this case, $B$ and $A$ share a common non-trivial subspace, a contradiction. Of course, this arguments breaks up for $d = 1$ (and this is why we had a separate argument for this case).

Therefore, we may assume that $\Gamma_1(B)$ has some arrow $\lambda \to \lambda'$ with $\lambda' \neq \lambda \pm 1$. Because the Galois group $G$ is the largest possible and $\Gamma_1(B)$ is invariant under the action of $G$ (see Remark 133), we have that all arrows of this type belong to $\Gamma_1(B)$. In view of Remarks 133 and 134, it suffices to construct a loop of odd length in $\Gamma_1(B)$.

Since we dispose of all arrows $\lambda \to \lambda'$ with $\lambda' \neq \lambda \pm 1$, if $d \geq 3$, we can easily construct a loop of length 3:

On the other hand, for $d = 2$, we have two possibilities. If $\Gamma_1(B)$ is the non-mixing graph invariant under the Galois group:

we get that $B(Rv_{\lambda_1} \oplus Rv_{\lambda_1}^{-1}) = Rv_{\lambda_2} \oplus Rv_{\lambda_2}^{-1}$ and $B(Rv_{\lambda_2} \oplus Rv_{\lambda_2}^{-1}) = Rv_{\lambda_1} \oplus Rv_{\lambda_1}^{-1}$, so that $B^2$ and $A$ share a common invariant subspace, a contradiction.
So, we have some extra arrow in the previous picture, say:

[Diagram]

In this case, it is not hard to see that the addition of any extra arrow allows to build up loops of length 3, so that, by Remarks 133 and 134, the argument is complete.

Therefore, in any event, we proved that $\Gamma_1(B)$ is mixing.

Step 2: For $d \geq 3$, $\Gamma_k(C)$ is mixing for $2 \leq k < d$, and $\Gamma_d(D)$ is mixing. Given $C \in Sp(2d, \mathbb{Z})$ twisting 1-dimensional $A$-invariant subspaces, we wish to prove that $\Gamma_k(C)$ is mixing for all $2 \leq k < d$ whenever $d \geq 3$. Since $\Gamma_k(C)$ is invariant under the Galois group $G$ (see Remark 133), we start by considering the orbits of the action of $G$ on $\hat{R}_k \times \hat{R}_k$.

**Proposition 137.** The orbits of the action of $G$ on $\hat{R}_k \times \hat{R}_k$ are

$$O_{\ell,\ell} = \{(\lambda, \lambda') \in \hat{R}_k \times \hat{R}_k : \#(\lambda \cap \lambda') = \ell, \#(p(\lambda) \cap p(\lambda')) = \ell\}$$

where

$$0 \leq \ell \leq k, \ell \geq 2k - d \quad (D.4)$$

We leave the proof of this proposition as an exercise to the reader. This proposition says that the orbits of the action $G$ on $\hat{R}_k \times \hat{R}_k$ are naturalized parametrized by

$$\bar{I} = \{(\ell, \ell) \text{ satisfying } (D.4)\}$$

In particular, since $\Gamma_k(C)$ is $G$-invariant, we can write $\Gamma_k(C) = \Gamma_k(\bar{J})$ for some $\bar{J} := \bar{J}(C) \subset \bar{I}$, where $\Gamma_k(J)$ is the graph whose vertices are $\hat{R}_k$ and whose arrows are

$$\bigcup_{(\ell,\ell) \in \bar{J}} O_{\ell,\ell}$$

**Proposition 138.** The graph $\Gamma_k(\bar{J})$ is not mixing if and only if

- either $k \neq d/2$ and $\bar{J} \subset \{(\ell, k) : 0 \leq \ell \leq k\}$
- or $k = d/2$ and $\bar{J} \subset \{(\ell, k) : 0 \leq \ell \leq k\} \cup \{(0, 0)\}$

Proof. Let $\bar{J} \subset \{(\ell, k) : 0 \leq \ell \leq k\}$ for $k \neq d/2$ or $\bar{J} \subset \{(\ell, k) : 0 \leq \ell \leq k\} \cup \{(0, 0)\}$ for $k = d/2$. Then, one can see that, since $k < d$, $\Gamma_k(\bar{J})$ is not mixing simply because it is not connected! For the proof of the converse statement, due to the usual space-time limitations, we’re going only to say a few words on (referring to the forthcoming article by M. Möller, J.-C. Yoccoz and C. M. for formal arguments). Essentially, one starts by converting pairs $\{(\lambda, \lambda^{-1})\}$ into a single point $p(\lambda) = p(\lambda^{-1})$, so that $\Gamma_k(\bar{J})$ becomes a new graph $\bar{\Gamma}_k(\bar{J})$. Then, one proves that, if
Coming back to the study of $\Gamma_k(C)$, $1 \leq k < d$, $d \geq 3$, we set $\tilde{J} := \tilde{J}(C)$. By the previous proposition, if $\Gamma_k(C)$ is not mixing, then $\tilde{J}(C) \subset \{(\tilde{\ell},k) : 0 \leq \tilde{\ell} \leq k\}$ for $k \neq d/2$ or $\tilde{J} \subset \{(\tilde{\ell},k) : 0 \leq \tilde{\ell} \leq k\} \cup \{(0,0)\}$ for $k = d/2$. For sake of concreteness, we will deal “only” with the case $\tilde{J} \subset \{(\tilde{\ell},k) : 0 \leq \tilde{\ell} \leq k\}$ (leaving the particular case $\tilde{J} = \{(0,0)\}$ when $k = d/2$ as an exercise to the reader). In this situation, we have an arrow $\{\lambda_1, \ldots, \lambda_k\} = \Delta \rightarrow \Delta' = \{\lambda_1', \ldots, \lambda_k'\}$ of $\Gamma_k(C)$ with $p(\Delta) = p(\Delta')$. This means that $C^{(k)}_{\Delta\Delta'} \neq 0$, and hence we can find $w_1, \ldots, w_k$ such that $\text{span}\{w_1, \ldots, w_k\} \cap \{0\} = \emptyset$ and

$$C(w_i) = v_{\lambda_i'} + \sum_{\lambda \in \Delta'} C^*_\lambda v_{\lambda}$$

In other words, as we also did in Step 0, we can use $w_1, \ldots, w_k$ to “convert” the minor of $C$ associated to $\Delta\Delta'$ into the identity.

We claim that if $\lambda, \lambda^{-1} \notin \Delta'$, then $C^*_\lambda = 0$ for all $i = 1, \ldots, k$. Indeed, by the same discussion around minors and replacement of lines, if this were not true, say $C^*_\lambda \neq 0$, we could find an arrow from $\Delta$ to $\Delta'' = (\Delta' - \{\lambda_i'\}) \cup \{\lambda\}$. Since $p(\Delta) = p(\Delta')$, we have that $(p(\Delta) \cap p(\Delta'')) = k - 1$, so that, for some $\tilde{\ell}_0$, one has $(\tilde{\ell}_0, k - 1) \in \tilde{J} \subset \{(\tilde{\ell},k) : 0 \leq \tilde{\ell} \leq k\}$, a contradiction showing that the claim is true.

From the claim above we deduce that e.g. $C(v_{\lambda_i'})$ is a linear combination of $v_{\lambda_i'}, i = 1, \ldots, k$, a contradiction with the fact that $C$ twists 1-dimensional $A$-invariant subspaces. In other words, we proved that $\Gamma_k(C)$ is mixing for each $2 \leq k < d$ whenever $C$ twists 1-dimensional $A$-invariant subspaces.

By Proposition 132 it follows that we can construct a matrix $D$ twisting $k$-dimensional isotropic $A$-invariant subspaces for $1 \leq k < d$, and we wish to show that $\Gamma_d(D)$ is mixing. In this direction, we consider the orbits of the action of the Galois group $G$ on $\hat{R}_d \times \hat{R}_d$. By Proposition 137 the orbits are

$$O_{\ell,k} = \{(\Delta, \Delta') \in \hat{R}_k \times \hat{R}_k : \#(\Delta \cap \Delta') = \ell, \#(p(\Delta) \cap p(\Delta')) = \ell\}$$

with $\ell \leq k$, $\ell \geq 2k - d$ and $k = d$. In particular, $\ell = d$ in this case, and the orbits are parametrized by the set

$$I = \{0 \leq \ell \leq d\}$$

For sake of simplicity, we will denote the orbits of $G$ on $\hat{R}_d \times \hat{R}_d$ by

$$O(\ell) = \{(\Delta, \Delta') \in \hat{R}_d \times \hat{R}_d : \#(\Delta \cap \Delta') = \ell\}$$

and we write

$$\Gamma_d(D) = \Gamma_d(J) = \bigcup_{\ell \in J} O(\ell)$$

where $J = J(D) \subset I = \{0 \leq \ell \leq d\}$. 
It is possible to show (again by the arguments with “minors” we saw above) that if $D$ is $k$-twisting with respect to $A$, then $J$ contains two consecutive integers say $\ell, \ell + 1$.

We claim that $\Gamma_d(D)$ is mixing whenever $J$ contains two consecutive integers.

Indeed, we start by showing that $\Gamma_d(J)$ is connected. Notice that it suffices to connect two vertices $\Delta_0$ and $\Delta_1$ with $\#(\Delta_0 \cap \Delta_1) = d - 1$ (as the general case of two general vertices $\Delta$ and $\Delta'$ follows by producing a series of vertices $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ with $\#(\Delta_i \cap \Delta_{i+1}) = d - 1$, $i = 0, \ldots, a - 1$). Given $\Delta_0$ and $\Delta_1$ with $\#(\Delta_0 \cap \Delta_1) = d - 1$, we select $\Delta' \subset \Delta_0 \cap \Delta_1$ with $\#\Delta' = d - \ell - 1$. Then, we consider $\Delta''$ obtained from $\Delta_0$ by replacing the elements of $\Delta'$ by their inverses. By definition, $\#(\Delta'' \cap \Delta_0) = \ell + 1$ and $\#(\Delta'' \cap \Delta_1) = \ell$ (because $\#(\Delta_0 \cap \Delta_1) = d - 1$). By assumption, $J$ contains $\ell + 1$ and $\ell$, so that we have the arrows $\Delta_0 \to \Delta''$ and $\Delta'' \to \Delta_1$ in $\Gamma_d(J)$. Thus, the connectedness of $\Gamma_d(J)$ follows.

Next, we show that $\Gamma_d(J)$ is mixing. Since $\Gamma_d(J)$ is invariant under the Galois group, it contains loops of length 2 (see Remark 133). By Remark 134, it suffices to construct some loop of odd length in $\Gamma_d(J)$. We fix some arrow $\Delta \to \Delta' \in \mathcal{O}(\tilde{\ell})$ of $\Gamma_d(J)$. By the construction “$\Delta_0 \to \Delta'' \to \Delta_1$” when $\#(\Delta_0 \cap \Delta_1) = d - 1$ performed in the proof of the connectedness of $\Gamma_d(J)$, we can connect $\Delta'$ to $\Delta$ by a path of length $2\ell$ in $\Gamma_d(J)$. In this way, we have a loop (based on $\Delta_0$) in $\Gamma_d(J)$ of length $2\ell + 1$.

**Step 3: Special case $d = 2$.** We consider the symplectic form $\{\ldots\} : \wedge^2 \mathbb{R}^4 \to \mathbb{R}$. Since $\wedge^2 \mathbb{R}^4$ has dimension 6 and $\{\ldots\}$ is non-degenerate, $K := \text{Ker}\{\ldots\}$ has dimension 5.

By denoting by $\lambda_1 > \lambda_2 > \lambda_2^{-1} > \lambda_1^{-1}$ the eigenvalues of $A$, we have the following basis of $K$:

- $v_{\lambda_1} \wedge v_{\lambda_2}, v_{\lambda_1} \wedge v_{\lambda_2}^{-1}, v_{\lambda_1}^{-1} \wedge v_{\lambda_2}, v_{\lambda_1} v_{\lambda_2}^{-1}$;
- $v_* = \frac{v_{\lambda_1} \wedge v_{\lambda_1}^{-1}}{\omega_1} - \frac{v_{\lambda_2} \wedge v_{\lambda_2}^{-1}}{\omega_2}$, where $\omega_i = \{v_{\lambda_i}, v_{\lambda_i}^{-1}\} \neq 0$.

In general, given $C \in \text{Sp}(4, \mathbb{Z})$, we can use $\wedge^2 C|_K$ to construct a graph $\Gamma_2^2(C)$ whose vertices are $\tilde{R}_2 \simeq \{v_{\lambda_1} \wedge v_{\lambda_2}, \ldots, v_{\lambda_1}^{-1} \wedge v_{\lambda_2}^{-1}\}$ and $v_*$, and whose arrows connect vertices associated to non-zero entries of $\wedge^2 C|_K$.

By definition, $\wedge^2 A|_K = v_*$, so that 1 is an eigenvalue of $\wedge^2 A|_K$. In principle, this poses a problem to apply Proposition 132 (to deduce 2-twisting properties of $C$ from $\Gamma_2^2(C)$ is mixing), but, as it turns out, the fact that the eigenvalue 1 of $\wedge^2 A|_K$ is simple can be exploited to rework the proof of Proposition 132 to check that $\Gamma_2^2(C)$ is mixing implies the existence of adequate products $D$ of powers of $C$ and $A$ satisfying the 2-twisting condition (i.e., $D$ twists 2-dimensional $A$-invariant isotropic subspaces).

Therefore, it “remains” to show that either $\Gamma_2^2(C)$ or $\Gamma_2^2(C)$ is mixing to complete this step.

We write $\Gamma_2^2(C) = \Gamma_2(J)$ with $J \subset \{0, 1, 2\}$.

- If $J$ contains 2 two consecutive integers, then one can check that the arguments of the end of the previous sections work and $\Gamma_2(J)$ is mixing.
- Otherwise, since $J \neq \emptyset$ (see Step 0), we have $J = \{0\}, \{2\}, \{1\}$ or $\{0, 2\}$. As it turns out, the cases $J = \{0\}, \{2\}$ are “symmetric”, as well as the cases $J = \{1\}, \{0, 2\}$.
For sake of concreteness, we will consider the cases $J = \{2\}$ and $J = \{1\}$ (leaving the treatment of their “symmetric” as an exercise). We will show that the case $J = \{2\}$ is impossible while the case $J = \{1\}$ implies that $\Gamma_2^*(C)$ is mixing.

We begin by $J = \{2\}$. This implies that we have an arrow $\lambda \to \lambda$ with $\lambda = \{\lambda_1, \lambda_2\}$. Hence, we can find $w_1, w_2$ with $\text{span}\{w_1, w_2\} = \text{span}\{v_{\lambda_1}, v_{\lambda_2}\}$ and
\[
C(w_1) = v_{\lambda_1} + C_{11}^* v_{\lambda_1^{-1}} + C_{12}^* v_{\lambda_2^{-1}},
\]
\[
C(w_2) = v_{\lambda_2} + C_{21}^* v_{\lambda_1^{-1}} + C_{22}^* v_{\lambda_2^{-1}}.
\]
Since $J = \{2\}$, the arrows $\lambda \to \{\lambda_1^{-1}, \lambda_2\}, \lambda \to \{\lambda_1, \lambda_2^{-1}\}$, and $\lambda \to \{\lambda_1^{-1}, \lambda_2^{-1}\}$ do not belong $\Gamma_2(J)$.

Thus, $C_{11}^* = C_{22}^* = 0 = C_{12}^* C_{21}^*$, On the other hand, because $C$ is symplectic, $\omega_1 C_{21}^* - \omega_2 C_{12}^* = 0$ (with $\omega_1, \omega_2 \neq 0$). It follows that $C_{ij}^* = 0$ for all $1 \leq i, j \leq 2$, that is, $C$ preserves the $A$-subspace spanned by $v_{\lambda_1}$ and $v_{\lambda_2}$, a contradiction with the fact that $C$ is 1-twisting with respect to $A$.

Now we consider the case $J = \{1\}$ and we wish to show that $\Gamma_2^*(C)$. We claim that, in this situation, it suffices to construct arrows from the vertex $v_*$ to $\hat{R}_2$ and vice-versa. Notice that the action of the Galois group can’t be used to revert arrows of $\Gamma_2^*(C)$ involving the vertex $v_*$, so that the two previous statements are “independent”. Assuming the claim holds, we can use the Galois action to see that once $\Gamma_2^*(C)$ contains some arrows from $v_*$ and some arrows to $v_*$, it contains all such arrows. In other words, if the claim is true, we have the following situation:

Thus, we have loops of length 2 (in $\hat{R}_2$), and also loops of length 3 (based on $v_*$), so that $\Gamma_2^*(C)$ is mixing. In particular, our task is reduced to show the claim above.

The fact that there are arrows from $\hat{R}_2$ to $v_*$ follows from the same kind of arguments involving “minors” (i.e., selecting $w_1, w_2$ as above, etc.) and we will not repeat it here.

Instead, we will focus on showing that there are arrows from $v_*$ to $\hat{R}_2$. The proof is by contradiction: otherwise, one has $\wedge^2 C(v_*) \in \mathbb{R} v_*$. Then, we invoke the following elementary lemma (whose proof is a straightforward computation):

**Lemma 139.** Let $H \subset \mathbb{R}^4$ be a symplectic 2-plane. Given $e, f$ a basis of $H$ with $\{e, f\} = 1$, we define
\[
i(H) := e \wedge f
\]
The bi-vector $i(H)$ is independent of the choice of $e, f$ as above. Denote by $H^\perp$ the symplectic orthogonal of $H$ and put

$$v(H) := i(H) - i(H^\perp) \in K$$

Then, $v(H)$ is collinear to $v(H')$ if and only if $H' = H$ or $H' = H^\perp$.

Since $v_* := i(\text{span}(v_{\lambda_1}, v_{\lambda_1}^{-1}))$, from this lemma we obtain that $\wedge^2 C(v_*) \in \mathbb{R} v_*$ implies

$$C(\text{span}(v_{\lambda_1}, v_{\lambda_1}^{-1})) = \text{span}(v_{\lambda_1}, v_{\lambda_1}^{-1})$$

or

$$\text{span}(v_{\lambda_2}, v_{\lambda_2}^{-1}),$$

a contradiction with the fact that $C$ is 1-twisting with respect to $A$.

This completes the sketch of proof of Theorem 129. Now, we will conclude this appendix with some applications of this theorem to the first two “minimal” strata $\mathcal{H}(2)$ and $\mathcal{H}(4)$.

D.3. Some applications of Theorem 129. In the sequel we will need the following lemma about the “minimal” strata:

**Lemma 140.** A translation surface in the stratum $\mathcal{H}(2g - 2)$ has no non-trivial automorphisms.

**Proof.** Let $(M, \omega) \in \mathcal{H}(2g - 2)$ and denote by $p_M \in M$ the unique zero of the Abelian differential $\omega$. Any automorphism $\phi$ of the translation structure $(M, \omega)$ satisfies $\phi(p_M) = p_M$. Suppose that there exists $\phi$ a non-trivial automorphism of $(M, \omega)$. We have that $\phi$ has finite order, say $\phi^\kappa = \text{id}$ where $\kappa \geq 2$ is the order of $\phi$. Since $\phi$ fixes $p_M$ and $\phi$ is non-trivial, $p_M$ is the sole fixed point of $\phi$. Hence, the quotient $N$ of $M$ by the cyclic group $\langle \phi \rangle \simeq \mathbb{Z}/\kappa \mathbb{Z}$ can be viewed as a normal covering $\pi : (M, p_M) \to (N, p_N)$ of degree $\kappa$ such that $\pi$ is not ramified outside $p_N := \pi(p_M)$, $\pi^{-1}(p_N) = \{p_M\}$ and $\pi$ is ramified of index $\kappa$ at $p_N$.

Now let us fix $* \in N - \{p_N\}$ a base point, so that the covering $\pi$ is given by a homomorphism $h : \pi_1(N - \{p_N\}, *) \to \mathbb{Z}/\kappa \mathbb{Z}$. In this language, we see that a loop $\gamma \in \pi_1(N - \{p_N\}, *)$ around $p_N$ based at $*$ is a product of commutators in $\pi_1(N - \{p_N\}, *)$, so that $h(\gamma) = 1$ (as $\mathbb{Z}/\kappa \mathbb{Z}$), a contradiction with the fact that $\pi$ is ramified of order $\kappa \geq 2$ at $p_N$. □

**Remark 141.** In a certain sense this lemma implies that origamis of minimal strata $\mathcal{H}(2g - 2)$ are not concerned by the discussions of Subsection D.1: indeed, there the idea was to use representation theory of the automorphism group of the origami to detect multiple (i.e., non-simple) and/or zero exponents; in particular, we needed rich groups of automorphisms and hence, from this point of view, origamis without non-trivial automorphisms are “uninteresting”.

Next, we recall that M. Kontsevich and A. Zorich classified all connected components of strata $\mathcal{H}(k_1 - 1, \ldots, k_s - 1)$, and, as an outcome of this classification, any stratum has at most 3 connected components. We will come back to this latter when discussing the stratum $H(4)$. For now, let us mention that the two strata $H(2)$ and $H(1, 1)$ of genus 2 translation structures are connected.
Once we fix a connected component \( C \) of a stratum, we can ask about the classification of \( SL(2, \mathbb{Z}) \)-orbits of primitive\(^5\) square-tiled surfaces in \( C \). In this direction, it is important to dispose of invariants to distinguish \( SL(2, \mathbb{Z}) \)-orbits. As it turns out, the monodromy group of a square-tiled surface (cf. Subsection D.1) is such an invariant, and the following result (from the PhD thesis) of D. Zmiaikou [73] shows that this invariant can take only two values when the number of squares of the origami is sufficiently large:

**Theorem 142** (D. Zmiaikou). Given an stratum \( \mathcal{H}(k_1, \ldots, k_s) \), there exists an integer \( N_0 = N_0(k_1, \ldots, k_s) \) such that any primitive origami of \( \mathcal{H}(k_1, \ldots, k_s) \) with \( N \geq N_0 \) squares has monodromy group isomorphic to \( A_N \) or \( S_N \).

**Remark 143.** The integer \( N_0 = N_0(k_1, \ldots, k_s) \) has explicit upper bounds (as it was shown by D. Zmiaikou), but we did not include it in the previous statement because it is believed that the current upper bounds are not sharp.

**Remark 144.** In order to alleviate the notation, we will refer to square-tiled surfaces as origamis in what follows.

After this (brief) general discussion, let’s specialize to the case of genus 2 origamis.

D.3.1. **Classification of \( SL(2, \mathbb{Z}) \)-orbits of square-tiled surfaces in \( H(2) \).** Denote by \( N \geq 3 \) the number of squares of a origami in \( \mathcal{H}(2) \). By the results of P. Hubert and S. Lelièvre [42], and C. McMullen [62], it is possible to show that the \( SL(2, \mathbb{Z}) \)-orbits of origamis are organized as follows

- if \( N \geq 4 \) is even, then there is exactly 1 \( SL(2, \mathbb{Z}) \)-orbit and the monodromy group is \( S_N \).
- if \( N \geq 5 \) is odd, then there are exactly 2 \( SL(2, \mathbb{Z}) \)-orbits distinguished by the monodromy group being \( A_N \) or \( S_N \).
- if \( N = 3 \), there is exactly 1 \( SL(2, \mathbb{Z}) \)-orbit and the monodromy group is \( S_3 \).

Concerning the Lyapunov exponents of the KZ cocycle, recall that M. Bainbridge [9], and A. Eskin, M. Kontsevich and A. Zorich [21] that the second (non-negative) exponent \( \lambda_2 \) is always \( \lambda_2 = 1/3 \) in \( \mathcal{H}(2) \) (independently of the \( SL(2, \mathbb{R}) \)-invariant probability considered), so that the Lyapunov spectrum of KZ cocycle is always simple in \( \mathcal{H}(2) \).

Of course, they say much more as the explicit value of \( \lambda_2 \) is given, while the simplicity in \( H(2) \) amounts only to say that \( \lambda_2 > 0 (\geq -\lambda_2) \). But, as it turns out, the arguments employed by Bainbridge and EKZ are sophisticated (involving the Deligne-Mumford compactification of the moduli space of curves, etc.) and long (both of them has \( \geq 100 \) pages). So, partly motivated by this, we will discuss in this section the application of the simplicity criterion of Theorem 129 to the case of square-tiled surfaces in \( H(2) \): evidently, this gives only the fact that \( \lambda_2 > 0 \) but it has the advantage of relying on the elementary methods developed in previous posts.

We begin by selecting the following \( L \)-shaped origami \( L(m, n) \):
In terms of the parameters $m, n \geq 2$, the total number of squares of $L(m, n)$ is $N = n + m - 1 \geq 3$. Notice that the horizontal permutation $r$ associated $L(m, n)$ is an $m$-cycle while the vertical permutation $u$ is an $n$-cycle. Thus, by our discussion so far above, one has that:

- if $m + n$ is odd, the monodromy group is $S_N$,
- if $m$ and $n$ are odd, the monodromy group is $A_N$, and
- if $m$ and $n$ are even, the monodromy group is $S_N$.

As we already know, given an origami $p : M \to \mathbb{T}^2$, we have a decomposition

$$H_1(M, \mathbb{Q}) = H_1^{st}(M, \mathbb{Q}) \oplus H_1^{(0)}(M, \mathbb{Q})$$

such that the standard part $H_1^{st}(M, \mathbb{Q})$ has dimension 2 and associated Lyapunov exponents $\lambda_1 = 1$ and $-\lambda_1 = -1$, while the part $H_1^{(0)}(M, \mathbb{Q})$ consisting of homology classes projecting to 0 under $p$ has dimension $2g - 2$ and associated Lyapunov exponents $\lambda_2, \ldots, \lambda_g$ and their opposites.

In the particular case of $L(m, n)$, in terms of the homology cycles $\sigma_1, \sigma_2, \zeta_1, \zeta_2$ showed in the previous picture, we can construct the following basis of $H_1^{(0)}(M, \mathbb{Q})$:

$$\sigma := \sigma_1 - m\sigma_2, \quad \zeta := \zeta_1 - n\zeta_2.$$ 

Now, we choose the elements $S = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ of the Veech group of $L(m, n)$.

Since $L(m, n) \in \mathcal{H}(2)$ has no automorphisms (see Lemma 140), we can also think of $S$ and $T$ as elements of the affine group of $L(m, n)$.

As the reader can check in the figures, the action of $S$ on the homology cycles $\sigma_1, \sigma_2, \zeta_1, \zeta_2$ is

- $S(\sigma_i) = \sigma_i$ for $i = 1, 2$;
- $S(\zeta_1) = \zeta_1 + \sigma_1 + (n - 1)m\sigma_2$
- $S(\zeta_2) = \zeta_2 + \sigma_1$
Therefore, $S(\sigma) = \sigma$ and $S(\zeta) = \zeta + (n-1)\sigma$. Actually, the same computation with $S$ replaced by any power $S^a$, $a \in \mathbb{N}$, gives the “same” result

$$S^a(\sigma) = \sigma, \quad S^a(\zeta) = \zeta + a(1-n)\sigma$$

By the natural symmetry between the horizontal and vertical directions in $L(m,n)$, there is no need to compute twice to get the corresponding formulas for $T$ and/or $T^b$:

$$T^b(\sigma) = \sigma + b(1-m)\zeta, \quad T^b(\zeta) = \zeta$$

By combining $S$ and $T$, we can find pinching elements acting on $H_1^{(0)}(M, \mathbb{Q})$: for instance,

$$A := ST = \begin{pmatrix} 1 + (m-1)(n-1) & (1-n) \\ (1-m) & 1 \end{pmatrix}$$

has eigenvalues given by the roots of $\lambda^2 - t\lambda + 1 = 0$ where $t = t_A = 2 + (m-1)(n-1)$ is the trace of $A$; since $|t_A| > 2$ (as $m,n \geq 2$) and $t_A^2 - 4 = (m-1)(n-1)((m-1)(n-1) + 4$ is not a square, $A$ has two real irrational eigenvalues, and hence $A$ is a pinching matrix whose eigenspaces are defined over $\mathbb{Q}(\sqrt{t_A})$.

Thus, we can apply the simplicity criterion as soon as one has a twisting matrix $B$ with respect to $A$. We take

$$B := S^2T^2$$

By applying the previous formulas for the powers $S^a$ and $T^b$ of $S$ and $T$ (with $a = b = 2$), we find that the trace of $B$ is

$$t_B = 2 + 4(m-1)(n-1)$$

so that $t_B^2 - 4 = 16(m-1)(n-1)((m-1)(n-1) + 1$ is also not a square. Hence, $B$ is also a pinching matrix whose eigenspaces are defined over $\mathbb{Q}(\sqrt{t_B})$. Furthermore, these formulas for $t_A$ and $t_B$ above show that the quadratic fields $\mathbb{Q}(\sqrt{t_A})$ and $\mathbb{Q}(\sqrt{t_B})$ are disjoint in the sense that $\mathbb{Q}(\sqrt{t_A}) \cap \mathbb{Q}(\sqrt{t_B}) = \mathbb{Q}$. So, $B$ is twisting with respect to $A$, and therefore $1 = \lambda_1 > \lambda_2 > 0$.

Remark 145. For the “other” stratum of genus 2, namely $H(1,1)$, the results of Bainbridge and EKZ show that $\lambda_2 = 1/2$. In principle, the (weaker) fact that $\lambda_2 > 0$ in this situation can be derived for particular origamis in $H(1,1)$ along the lines sketched above for $H(2)$, but it is hard to treat such origamis in a systematic way because currently there is only a conjectural classification of $SL(2,\mathbb{Z})$-orbits.

This closes our (preliminary) discussions of the application of the simplicity criterion in the (well-established) case of $H(2)$. Now, we pass to the case of the stratum $H(4)$. 

D.3.2. SL(2, ℤ)-orbits of origamis in \( \mathcal{H}(4) \). Let’s start with some general comments about the stratum \( \mathcal{H}(4) \).

**Connected components of the stratum \( \mathcal{H}(4) \).** By the work of M. Kontsevich and A. Zorich \[48\], we know that \( \mathcal{H}(4) \) has two connected components, a hyperelliptic connected component \( \mathcal{H}(4)_{\text{hyp}} \), and a odd spin connected component \( \mathcal{H}(4)_{\text{odd}} \). In this particular case, they can be distinguished as follows:

- \(-\text{id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\) belongs to the Veech group of all translation surfaces in the connected component \( \mathcal{H}(4)_{\text{hyp}} \).
- \(-\text{id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\) doesn’t belong to the Veech group of the generic translation surface of the connected component \( \mathcal{H}(4)_{\text{odd}} \).

**Weierstrass points and SL(2, ℤ)-orbits of “symmetric” origamis: an invariant of E. Kani, and P. Hubert and S. Lelièvre.** In general, we have the exact sequence

\[ 1 \to \text{Aut}(M) \to \text{Aff}(M) \to \text{SL}(M) \to 1 \]

If the origami \( M \) belongs to a minimal stratum \( \mathcal{H}(2g - 2) \) (e.g., \( \mathcal{H}(4) \)), by Lemma 140 one has \( \text{Aut}(M) = 1 \) and therefore \( \text{Aff}(M) \cong \text{SL}(M) \). In other words, given \( g \in \text{SL}(M) \), there exists an unique affine diffeomorphism with derivative \( g \).

Suppose now that \( p : M \to \mathbb{T}^2 \) is a (reduced) origami such that \(-\text{id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\) belongs to the Veech group \( \text{SL}(M) \) of \( M \). Thus, if \( M \in \mathcal{H}(2g - 2) \), there exists an unique affine diffeomorphism \( \phi \) of \( M \) with derivative \(-\text{id}\). Of course, \( \phi \) is an involution and it corresponds to a lift under \( p \) of the involution \( x \mapsto -x \) of \( \mathbb{T}^2 \). It follows that the fixed points of \( \phi \) project to the points of \( \mathbb{T}^2 \) of order 1 and 2. (Actually, it is also possible to show that the fixed points of \( \phi \) are Weierstrass points of \( M \), but we will not use this here). In the figure below we indicated the 4 points of \( \mathbb{T}^2 \) of orders 1 and 2, and we denoted by

- \( \ell_0 \) the number of fixed points of \( \phi \) over the integer point \( 0 = (0, 0) \in \mathbb{T}^2 \),
- \( \ell_1 \) the number of fixed points of \( \phi \) over the half-integer point \( (0, 1/2) \in \mathbb{T}^2 \),
- \( \ell_2 \) the number of fixed points of \( \phi \) over the half-integer point \( (1/2, 1/2) \in \mathbb{T}^2 \),
- \( \ell_3 \) the number of fixed points of \( \phi \) over the half-integer point \( (1/2, 0) \in \mathbb{T}^2 \)

Furthermore, the action of \( \text{SL}(2, ℤ) \) conjugates the involutions of the origamis in the \( \text{SL}(2, ℤ) \)-orbit of \( M \) (by an element of \( \text{SL}(2, ℤ) \)). Since \( \text{SL}(2, ℤ) \) fixes the integer point 0 while it permutes
(arbitrarily) the 3 points of order 2, we have that the list
\[ \ell_0, \{\ell_1, \ell_2, \ell_3\} \]
is an invariant of the \( SL(2, \Z) \)-orbit of \( M \) if one considers the sublist \( \{\ell_1, \ell_2, \ell_3\} \) up to permutations.

**Remark 146.** The sole zero of \( M \in H(2g-2) \) is fixed under the involution \( \phi \), so that \( \ell_0 \geq 1 \).

**Remark 147.** For \( i > 0, \ell_i \equiv N(\mod 2) \): for instance, the involution permute the squares forming \( M \) and, e.g. \( \ell_2 \) is the number of squares fixed by \( \phi \); since \( \phi \) is an involution, the \( \phi \)-orbits of squares have size 1 or 2, so that \( \ell_2 \equiv N(\mod 2) \); finally, since \( SL(2, \Z) \) acts transitively on the set of half-integer points of \( \mathbb{T}^2 \), we can “replace” \( \ell_2 \) by \( \ell_i, i > 0 \), to get the same conclusion for all \( i > 0 \).

The invariant \( [\ell_0, \{\ell_1, \ell_2, \ell_3\}] \) was introduced by E. Kani \[43\], and P. Hubert and S. Lelièvre \[42\] to study \( SL(2, \Z) \)-orbits of origamis of \( \mathcal{H}(2) \) (historically speaking this invariant came before the monodromy invariant). Since any genus 2 Riemann surface is hyperelliptic, \(-id\) belongs to the Veech group of any translation surface \( M \in \mathcal{H}(2) \), and the quotient of \( M \) by involution \( \phi \) has genus 0. In particular, by Riemann-Hurwitz formula, the involution \( \phi \) has 6 fixed points (the 6 Weierstrass points of \( M \)). For the origami \( L(m,n) \) with \( N = n + m - 1 \) squares shown during the discussion in \( \mathcal{H}(2) \), one can compute the invariant \( [\ell_0, \{\ell_1, \ell_2, \ell_3\}] \):

\[
\ell_2 = \begin{cases} 
1, & \text{if } n, m \text{ are odd} \\
2, & \text{if } n + m \text{ is odd} \\
3, & \text{if } n, m \text{ are even}
\end{cases}
\]

\[
\ell_3 \text{(resp. } \ell_1) = \begin{cases} 
0 \text{(resp. } 2), & \text{if } m \text{ is odd and } n \text{ is even} \\
1, & \text{if } n + m \text{ is even} \\
2 \text{(resp. } 0), & \text{if } m \text{ is even and } n \text{ is odd}
\end{cases}
\]

In resume, we get that

- if \( N = n + m - 1 \) is even, there is exactly one \( SL(2, \Z) \)-orbit of origamis of \( \mathcal{H}(2) \) with invariant \( [2, \{2, 2, 0\}] \).
- if \( N = n + m - 1 \) is odd, there are exactly two \( SL(2, \Z) \)-orbit of origamis of \( \mathcal{H}(2) \):
  - if \( m, n \) are odd, the monodromy group is \( A_N \) and the invariant is \( [3, \{1, 1, 1\}] \);
  - if \( m, n \) are even, the monodromy group is \( S_N \) and the invariant is \( [1, \{3, 1, 1\}] \).

Coming back to \( \mathcal{H}(4) \), after some numerical experiments of V. Delecroix and S. Lelièvre (using SAGE), currently we dispose of a conjectural classification of \( SL(2, \Z) \) orbits of origamis in \( \mathcal{H}(4) \).

**Conjecture of Delecroix and Lelièvre in \( \mathcal{H}(4) \).**

- For the hyperelliptic component \( \mathcal{H}(4)^{hyp} \), the quotient of the involution \( \phi \) of any translation surface \( M \in \mathcal{H}(4)^{hyp} \) has genus 0 and (by Riemann-Hurwitz formula) 8 fixed points. The (experimentally observed) Kani-Hubert-Lelièvre invariants for the \( SL(2, \Z) \)-orbits of reduced origamis in \( \mathcal{H}(4) \) with \( N \) squares are:
  - For \( N \geq 9 \) odd, \( [5, \{1, 1, 1\}] \), \( [3, \{3, 1, 1\}] \), \( [1, \{5, 1, 1\}] \) and \( [1, \{3, 3, 1\}] \);
For $N \geq 10$ even, $\{4, \{2, 2, 0\}\}, \{2, \{4, 2, 0\}\}, \{2, \{2, 2\}\}$

- For the “odd spin” connected component $\mathcal{H}(4)^{\text{odd}}$ and $N \geq 9$:
  - For the generic origami $M \in \mathcal{H}(4)^{\text{odd}}$ (i.e., $-\text{id} \notin SL(M)$), there are exactly two $SL(2, \mathbb{Z})$-orbits distinguished by the monodromy group $A_N$ or $S_N$;
  - For the so-called “Prym” case, i.e., $M \in \mathcal{H}(4)^{\text{odd}}$ and $-\text{id} \notin SL(M)$, the quotient of the involution $\phi$ has genus 1, and hence, by Riemann-Hurwitz formula, $\phi$ has 4 fixed points. In this situation, the $SL(2, \mathbb{Z})$-orbits were classified by a recent theorem of E. Lanneau and D.-Manh Nguyen [50] whose statement we recall below.

**Theorem 148** (E. Lanneau and D.-Manh Nguyen). In the Prym case in $\mathcal{H}(4)^{\text{odd}}$:

- if $N$ is odd, there exists precisely one $SL(2, \mathbb{Z})$-orbit whose associated Kani-Hubert-Lelievre invariant is $[1, \{1, 1, 1\}]$;
- if $N \equiv 0(\text{mod } 4)$, there exists precisely one $SL(2, \mathbb{Z})$-orbit whose associated Kani-Hubert-Lelievre invariant is $[2, \{2, 0, 0\}]$;
- if $N \equiv 2(\text{mod } 4)$, there are precisely two $SL(2, \mathbb{Z})$-orbits whose associated Kani-Hubert-Lelievre invariant are $[2, \{2, 0, 0\}]$ and $[4, \{0, 0, 0\}]$.

### Simplicity of Lyapunov spectrum of origamis in $\mathcal{H}(4)$.

Concerning Lyapunov exponents of KZ cocycle, it is possible to prove (see, e.g., [60]) that the Prym case of $\mathcal{H}(4)^{\text{odd}}$ always lead to Lyapunov exponents $\lambda_2 = 2/5$ and $\lambda_3 = 1/5$. In particular, the Lyapunov spectrum of KZ cocycle in the Prym case is simple. As a side remark, we would like to observe that, interestingly enough, the simplicity criterion developed here can’t be used in the Prym case: indeed, the presence of the involution permits to decompose or, more precisely, diagonalize by blocks, the KZ cocycle so that the twisting property is never satisfied!

In any event, in [57] the authors apply the simplicity criterion of Theorem 129 to all (non-Prym) $SL(2, \mathbb{Z})$-orbits in $\mathcal{H}(4)$ described in Delecroix-Lelièvre conjecture. The outcome is that for such $SL(2, \mathbb{Z})$-orbits, the Lyapunov spectrum of the KZ cocycle is simple except (maybe) for finitely many $SL(2, \mathbb{Z})$-orbits (corresponding to “small” values of $N$, the number of squares).

In other words, in [57] it is shown that

**Theorem 149.** If the conjecture of Delecroix-Lelièvre is true, then all but (maybe) finitely many $SL(2, \mathbb{Z})$-orbits of origamis of $\mathcal{H}(4)$ have simple Lyapunov spectrum.

Unfortunately, since there are several $SL(2, \mathbb{R})$-orbits in Delecroix-Lelièvre conjecture above, we will not be able to discuss all of them here. Moreover, as we’re going to see in a moment, the proof of the simplicity statement above involves long computer-assisted calculations, so that our plan for the rest of this post will be the following: we will select a family of $SL(2, \mathbb{Z})$-orbits and we will start some hand-made computations towards the simplicity statement; then, we will see why we run in “trouble” with the hand-made calculation (essentially it is “too naive”) and why we need some computer assistance to complete the argument; however, we will not present the computer
assisted computations as we prefer to postpone them to the forthcoming article, and we will end
this series with this “incomplete” argument as it already contains the main ideas and it is not as
technical as the complete argument.

In the case of an origami \( M \in \mathcal{H}(4) \), we have that \( \dim(H^{(0)}_1(M, \mathbb{R})) = 4 \), so that the KZ cocycle
takes its values in \( Sp(4, \mathbb{Z}) \). Given \( A, B \in Sp(4, \mathbb{Z}) \), denote by
\[
P_A(x) = x^4 + ax^3 + bx^2 + ax + 1
\]
and
\[
P_B(x) = x^4 + a'x^3 + b'x^2 + a'x + 1
\]
their characteristic polynomials.

Recall that if the roots of \( P_A \) are all real, and the Galois group is \( G \) is the largest possible, i.e.,
\( G \cong S_2 \times (\mathbb{Z}/2\mathbb{Z})^2 \), then the matrix \( A \) is pinching.

Following the usual trick to determine the roots of \( P_A \), we pose \( y = x + 1/x \), so that we get an
equation
\[
Q_A(y) = y^2 + ay + (b - 2)
\]
with \( a, b \in \mathbb{Z} \). The roots of \( Q_A \) are
\[
y_{\pm} = \frac{-a \pm \sqrt{\Delta_1}}{2}
\]
where \( \Delta_1 := a^2 - 4(b - 2) \). Then, we recover the roots of \( P_A \) by solving the equation
\[
x^2 - y_{\pm}x + 1 = 0
\]
of discriminant \( \Delta_{\pm} := y_{\pm}^2 - 4 \).

A simple calculation shows that
\[
\Delta_2 := \Delta_+ \Delta_- = (b + 2)^2 - 4a^2 = (b + 2 + 2a)(b + 2 - 2a)
\]
For later use, we denote by \( K_A \), resp. \( K_B \), the splitting field of \( P_A \), resp. \( P_B \).

We leave the following proposition as an exercise in Galois theory for the reader:

**Proposition 150.** It holds:

(a) If \( \Delta_1 \) is a square (of an integer number), then \( Q_A \) and \( P_A \) are not irreducible.

In the sequel, we will always assume that \( \Delta_1 \) is not a square.

(b) If \( \Delta_2 \) is a square, then \( [K_A : \mathbb{Q}] = 4 \) and the Galois group \( G \) is \((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})\).

(c) If \( \Delta_1 \Delta_2 \) is a square, then \( [K_A : \mathbb{Q}] = 4 \) and the Galois group is \( \mathbb{Z}/4\mathbb{Z} \).

(d) If \( (\Delta_1, \Delta_2) \) and \( \Delta_1 \Delta_2 \) are not squares, then \( [K_A : \mathbb{Q}] = 8 \), the Galois group is \( G \cong D_4 \cong S_2 \times (\mathbb{Z}/2\mathbb{Z})^2 \), \( K_A \) contains exactly 3 quadratic fields \( \mathbb{Q}(\sqrt{\Delta_1}), \mathbb{Q}(\sqrt{\Delta_2}) \) and \( \mathbb{Q}(\sqrt{\Delta_1 \Delta_2}) \),
and each intermediate field \( \mathbb{Q} \subset L \subset K_A \) contains one of these quadratic fields.

(e) If \( (\Delta_1, \Delta_2) \) and \( \Delta_1 \Delta_2 \) are not squares, and \( \Delta_1 \Delta_2 > 0 \), then the roots of \( P_A \) are real and
the Galois group of \( P_A \) is the largest possible.
This proposition establishes a sufficient criterion for $A, B \in Sp(4, \mathbb{Z})$ to fit the pinching and twisting conditions. Indeed, we can use item (e) to get pinching matrices $A$, and we can apply item (d) to produce twisting matrices $B$ with respect to $A$ as follows: recall that (cf. Remark 130) the twisting condition is true if the splitting fields $K_A$ and $K_B$ are “disjoint” (i.e., $K_A \cap K_B = \mathbb{Q}$), and, by item (d), this disjointness can be checked by computing the quantities $\Delta_1, \Delta_2, \Delta_1 \Delta_2$ associated to $A$, and $\Delta'_1, \Delta'_2, \Delta'_1 \Delta'_2$ associated to $B$, and verifying that they generated distinct quadratic fields.

Now, we consider the following specific family $M = M_n, n \geq 5$, of origamis in $\mathcal{H}(4)^{odd}$:

Here, the total number of squares is $N = n + 4$ and $-\text{id}$ doesn’t belong to the Veech group (i.e., this origami is “generic” in $\mathcal{H}(4)^{odd}$). The horizontal permutation is the product of a $n$-cycle with a 2-cycle (so its parity equals the parity of $n$), and the vertical permutation is the product of a 4-cycle with a 2-cycle (so that its parity is even). In particular, here the monodromy group is $S_N$ if $N$ is odd, and $A_N$ if $N$ is even.

In terms of the homology cycles $\sigma_1, \sigma_2, \sigma_3, \zeta_1, \zeta_2, \zeta_3$ showed in the figure above, we can select the following basis of $H^1_0(M, \mathbb{Q})$:

- $\Sigma_1 = \sigma_1 - n\sigma_3, \Sigma_2 = \sigma_2 - 2\sigma_3$,
- $Z_1 = \zeta_1 - 4\zeta_3, Z_2 = \zeta_2 - 2\zeta_3$.

Then, we take the following elements of the Veech group of $M$

$$S = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

Notice that when $n$ is even we can replace $2n$ by $n$ in the definition of $S$, but we prefer to consider $S$ directly (to avoid performing to similar discussions depending on the parity of $n$).
By direct inspection of the figure above we deduce that:

- $S(\sigma_i) = \sigma_i$, for $i = 1, 2, 3$,
- $S(\zeta_1) = \zeta_1 + 2\sigma_1 + n\sigma_2 + 4n\sigma_3$,
- $S(\zeta_2) = \zeta_1 + 2\sigma_1 + n\sigma_2$,
- $S(\zeta_3) = \zeta_1 + 2\sigma_1$.

Hence,

- $S(\Sigma_i) = \Sigma_i$, for $i = 1, 2$,
- $S(Z_1) = Z_1 - 6\Sigma_1 + n\Sigma_2$,
- $S(Z_2) = \zeta_1 - 2\Sigma_1 + n\sigma_2$,

Thus, in terms of the basis $\{\Sigma_1, \Sigma_2, Z_1, Z_2\}$ of $H^1_{1}(M, \mathbb{Q})$, the matrix of $S|_{H^1_{1}}$ has the following decomposition into $2 \times 2$ blocks:

$$S|_{H^1_{1}} = \begin{pmatrix} 1 & \mathcal{S} \\ 0 & 1 \end{pmatrix}, \text{ where } \mathcal{S} = \begin{pmatrix} -6 & n \\ -2 & n \end{pmatrix}$$

By symmetry, the matrix of $T|_{H^1_{1}}$ is:

$$T|_{H^1_{1}} = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}, \text{ where } T = \begin{pmatrix} 1 - n & -1 \\ 2 & 2 \end{pmatrix}$$

In particular,

$$(ST)|_{H^1_{1}} = \begin{pmatrix} 1 + \mathcal{ST} & \mathcal{S} \\ T & 1 \end{pmatrix}$$

Of course one can extract the characteristic polynomial $x^4 + ax^3 + bx^2 + ax + 1$ of $(ST)|_{H^1_{1}}$ directly from these formulas, but in our particular we can take the following “shortcut”: observe that $ST(v_1, v_2) = \lambda(v_1, v_2)$ (where $v_1, v_2 \in \mathbb{R}^2$) if and only if $\mathcal{ST}(v_1) = (\lambda + \lambda^{-1} - 2)v_1 = (y_+ - 2)v_1$. By computing with $\mathcal{S}$ and $T$, and using that $y_\pm$ are the solutions of $y^2 - ay + (b - 2) = 0$, one finds $a = -7n + 6$, $b = 8n^2 - 2n - 14$ and

$$\Delta_1 = 17n^2 - 76n + 100$$

By Proposition 150, we wish to know e.g. how frequently $\Delta_1$ is a square. Evidently, one can maybe produce some ad hoc method here, but since we want to verify simplicity conditions in a systematic way, it would be nice to this type of question in “general”.

Here, the idea is very simple: the fact that $\Delta_1 = 17n^2 - 76n + 100$ is a square is equivalent to get integer and/or rational solutions to

$$z^2 = 17t^2 - 76t + 100,$$

that is, we need to understand integer/rational points in this curve.

This hints the following solution to our problem: if we can replace $A = ST$ by more complicated products of (powers of) $S$ and $T$ chosen more or less by random, it happens that $\Delta_1$ becomes a polynomial $P(n)$ of degree $\geq 5$ without square factors. In this situation, the problem
of knowing whether $\Delta_1 = P(n)$ is a square of an integer number becomes the problem of finding integer/rational points in the curve
$$z^2 = P(n).$$

Since this a non-singular curve (as $P(n)$ has no square factors) of genus $g \geq 2$ (as $\deg(P) \geq 5$), we know by Faltings’ theorem (previously known as Mordell’s conjecture) that the number of rational solutions is finite.

Remark 151. In the case we get polynomials $\tilde{P}(n)$ of degree 3 or 4 after removing square factors of $P(n)$, we still can apply (and do [apply in [57]]) Siegel’s theorem saying that the genus 1 curve $z^2 = \tilde{P}(n)$ has finite many integer points, but we preferred to mention the case of higher degree polynomials because it is the “generic situation” of the argument (in some sense).

In other words, by the end of the day, we have that $\Delta_1 = P(n)$ is not the square of an integer number for all but finitely many values of $n$.

Of course, at this point, the general idea to get the pinching and twisting conditions (and hence simplicity) for the origamis $M = M_n$ for all but finitely many $n \in \mathbb{N}$ is clear: one produces “complicated” products of $S, T$ (and also a third auxiliary parabolic matrix $U$) leading to elements $A, B \in Sp(4, \mathbb{Z})$ such that the quantities $\Delta_i = \Delta_i(n), i = 1, 2, \Delta_3 = \Delta_1 \Delta_2(n)$ associated to $A$, $\Delta'_i = \Delta'_i(n), \Delta'_3 = \Delta'_1 \Delta'_2(n)$ associated to $B$ (and also the “mixed products” $\Delta_i \Delta'_j$ for $i, j = 1, 2, 3$) are polynomials of the variable $n$ of high degree (and without square factors if possible), so that these quantities don’t take square of integers as their values for all but finitely many $n \in \mathbb{N}$.

Unfortunately, it seems that there is no “systematic way” of choosing these “complicated” products of $S, T$, etc. and, as a matter of fact, the calculations are too long and that’s why we use computer-assisted calculations at this stage. Again, as mentioned by the beginning of this section, these computer-assisted computations are not particularly inspiring (i.e., there are no mathematical ideas behind them), so we will close this post by giving an idea of what kind of complicated coefficient $b = b(n)$ of the characteristic polynomial of $A$ we get in our treatment of the family $M = M_n$:

$$b(n) = 3840n^5 - 17376n^4 + 14736n^3 + 25384n^2 - 28512n - 7066$$

References


44. A. Katok, Interval exchange transformations and some special flows are not mixing, Israel J. Math. 35 (1980), 301–310.

E-mail address: gforni@math.umd.edu

CNRS, LAGA, INSTITUT GALILÉE, UNIVERSITÉ PARIS 13, 99, AV. JEAN-BAPTISTE CLÉMENT 93430, VILLETANEUSE, FRANCE.
E-mail address: matheus@impa.br