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Decidability of Chaos for Some Families of Dynamical Systems

Alexander Arbieto[1] and Carlos Matheus[2]

[1,2]IMPA
Est. D. Castorina, 110
Jardim Botânico, 22460-320
Rio de Janeiro, RJ, Brazil
alexande@impa.br
matheus@impa.br

Abstract. We show that the existence of positive Lyapounov exponents and/or SRB measures are undecidable (in the algorithmic sense) properties within some parametrized families of interesting dynamical systems: the quadratic family and Hénon maps. Because the existence of positive exponents (or SRB measures) is, in a natural way, a manifestation of "chaos," these results may be understood as saying that the chaotic character of a dynamical system is undecidable. Our investigation is directly motivated by questions asked by Carleson and Smale in this direction.

1. Introduction

The notion of decidability goes as far back as the third decade of the previous century. At the time, even before computers were actually built, an important issue was which functions can be computed by an automatic machine. Notions such as that of recursive functions and Turing machines were proposed to answer such a question. It turned out that all purposed definitions of computable functions were equivalent (Church's thesis). Then a set is called decidable if its characteristic function is computable. Problems of decidability have played an important role

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in several branches of mathematics since then (e.g., the word problem, solved by Novikov [**13**] or the classification problem for 4-manifolds [**11**]).

Recently, it has been asked by Carleson in his talk on the ICM-1990 (see p. 1246 of [**6**]) and by Smale [**15**] (included in the fourteenth problem on Smale's paper) whether the set of chaotic systems is decidable, either in a general context or within special relevant families of systems. Our main result, Theorem A, is an answer to this question in the setting of certain families of low-dimensional maps: the quadratic family and the Hénon family, where "chaos" in this setting will be positive Lyapounov exponents, Sinai–Ruelle–Bowen (SRB) measures, or positive entropy.

Definition 1.1. *We denote by* $Q_a(x) = 1 - ax^2, a, x \in \mathbb{R}$, *the quadratic family and by* $H_{a,b}(x, y) = (1 - ax^2 + y, bx), (x, y) \in \mathbb{R}^2, a, b \in \mathbb{R}$, *the Hénon maps*.

We deal with some sets associated to the maps defined above (for the precise definitions, see Section 2):

- $\exp^+(Q) := \{a \in (0, 2] : Q_a$ has positive Lyapounov exponent$\}$
- $\exp^+(H) := \{(a, b) \in (0, 2] \times (-b_0, b_0) : H_{a,b}$ has positive Lyapounov exponents$\}$ (with $b_0 > 0$ small).
- $\mathrm{srb}(Q) := \{a \in (0, 2] : Q_a$ has an SRB measure$\}$
- $\mathrm{srb}(H) := \{(a, b) \in (0, 2] \times (-b_0, b_0) : H_{a,b}$ has an SRB measure$\}$ (with $b_0 > 0$ small).
- $\mathrm{ent}^+(Q) := \{a \in (0, 2] : Q_a$ has positive topological entropy$\}$.

Then our main result is:

Theorem A. *The sets* $\exp^+(Q)$, $\exp^+(H)$, $\mathrm{srb}(Q)$, *and* $\mathrm{srb}(H)$ *are undecidable. The set* $\mathrm{ent}^+(Q)$ *is decidable*.

We define the technical notions of positive Lyapounov exponents, SRB measures, topological entropy, and decidable sets in Section 2. For instance, we make some comments on the problem of the decidability of sets associated to dynamical systems. In [**4**], Blum, Shub, and Smale present a model for computation over arbitrary ordered rings R (e.g., $R = \mathbb{R}$). For this general setting, they obtain a theory which reflects the classical theory over $\mathbb{Z}$. However, one virtue of this theory is that it forces the use of more algebraic methods, i.e., classical mathematics, than the approach from logic. In particular, [**4**] shows that most Julia sets (a set associated to the rational maps $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere $\overline{\mathbb{C}}$, just like the Mandelbrot set) are undecidable. Next, there is an important problem of the decidability of sets associated to dynamical systems, namely, Is the Mandelbrot set decidable? This problem was first proposed by Penrose [**14**], but since the classical computational theory deals with Turing machines (integers inputs), the question makes no sense. Now, with the modern theory cited above, Blum and Smale formalized and solved the question. For references and comments on this problem, see [**5**]. We observe that a result by Shishikura [**16**] says that the boundary of the Mandelbrot set has Hausdorff dimension equal to 2. In particular, the argument given in [**5**] implies that the Mandelbrot is not decidable. For the sake of completeness, we briefly recall some steps of the Blum–Smale argument. If the Mandelbrot set M is the halting set of some machine, the boundary of M is the union of sets with Hausdorff dimension at most 1. In particular, ∂M has Hausdorff dimension at most 1, a contradiction with Shishikura's theorem. These comments show that the problem of the decidability of sets associated to dynamical systems is relevant.

The paper is organized as follows. In Section 2 we define the fundamental concepts and state the results used in Section 3. After that, we give the proof of Theorem A. Finally, in the last section, we finish this paper with some final remarks on some open questions related to the subject treated here.

2. Basic Results

This section is devoted to introducing some basic concepts used in the proof of our theorem.

2.1. *Dynamical Results*

All of the concepts presented here have general statements which hold for any dynamical system. Here we will give adapted definitions for the systems considered in this paper, which are more simple.

(a) *Lyapounov Exponents*. The Lyapounov exponents for $Q_a(x)$ and $H_{a,b}(x)$ are defined by

$$\lambda(Q_a) = \limsup_{n\to\infty} \frac{1}{n} \log |DQ_a^n(1)|,$$
$$\lambda(H_{a,b}) = \limsup_{n\to\infty} \frac{1}{n} \log \|DH_{a,b}^n(z_0)(0,1)\|.$$

Here z_0 is a critical point in the sense of [**2**].

We also remark that if we take $b = 0$ in the Hénon map, then the positivity of the Lyapounov exponent of $H_{a,0}$ implies the positivity of the Lyapounov exponent of Q_a.

In general we define the Lyapounov exponents for an invariant measure μ of a differentiable transformation $T : M \to M$ like the possible values (x μ-a.e.) of

$$\limsup_{n\to\infty} \frac{1}{n} \log \|DT^n(x)v\|, \qquad v \in T_xM - \{0\}.$$

Here we have the remarkable result from Benedicks and Carleson [**1**] (see also [**7**] for another proof and some other references).

Theorem 2.1 (Benedicks and Carleson). *For a subset of positive measure $E \subset (0, 2]$, if $a \in E$, then Q_a has positive Lyapounov exponent.*

(b) *SRB Measures*. An SRB measure for $Q_a(x)$ is an absolutely continuous (with respect to Lebesgue measure) invariant measure for $Q_a(x)$. A measure μ is an SRB measure for $H_{a,b}(x)$ if there is a positive Lyapounov exponent μ-a.e and the conditional measures of μ on unstable manifolds are absolutely continuous with respect to the Lebesgue measure induced on these manifolds (for details, see [**3**] or [**2**]).

Now we have the following result by Jakobson [**9**] (see [**7**] for an alternative proof):

Theorem 2.2 (Jakobson). *For a subset of positive Lebesgue measure $\mathcal{C} \subset (0, 2]$, if $a \in \mathcal{C}$, then Q_a has an SRB measure with positive entropy.*

(c) *Topological Entropy.* Let X be a compact metric space, $f\colon X \to X$ a continuous, $n \in \mathbb{N}$ and $\varepsilon > 0$, a subset $E \subset X$ is (n, ε)-separated if, for any $x \neq y$ inside E, there is $0 \leq j \leq n$ such that $d(f^j(x), f^j(y)) > \varepsilon$. We denote $s_n(\varepsilon)$ the largest cardinality of any set $E \subset X$ which is (n, ε)-separated. Then we define

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon).$$

We refer the reader to [**7**] for the proof and references of the next result.

Theorem 2.3. *For the quadratic family Q_a, the map $a \to h_{\text{top}}(Q_a)$ is monotone.*

(d) *Attracting cycles*. An attracting cycle is a periodic point p for f (of period n) such that $|(f^n)'(p)| < 1$. Related to this notion we have the following theorem due to Graczyk and Światek [**8**] and Lyubich [**10**]:

Theorem 2.4 (Graczyk, Światek, and Lyubich). *For an open and dense set of parameters $a \in (0, 4]$, the mapping Q_a has an attracting cycle.*

2.2. *Computational Results*

Here we give some notions from computation and complexity theory. We invite the reader to see the work of Blum, Shub, and Smale [**4**] for detailed definitions.

We deal with machines over an ordered ring R (e.g., $R = \mathbb{R}$) introduced by [**4**]. Following their terminology we will call $S \subset R^n$ a *halting set* if there exists a machine M such that for any input $a \in S$ the machine halts in finite time. We also remark that the halting sets are the R.E. sets of their recursive function theory.

We say that $\Omega \subset R^n$ is a *decidable set* if Ω and its complement are halting sets. In this case there is a machine which decides for each $z \in R^n$: Is $z \in \Omega$?

A *basic semialgebraic set* is a subset of R^n defined by a set of polynomial inequalities of the form $h_i(x) < 0, i = 1, \ldots, l, h_j(x) \leq 0, j = l+1, \ldots, m$.

The following key proposition relates halting sets and basic semialgebraic sets:

Proposition 2.5. *Any R.E. set over $\mathbb{R}$ is the countable union of basic semialgebraic sets. In particular, any R.E. set over $\mathbb{R}$ has a countable number of connected components.*

The proof of this proposition is contained in [**4**]. In fact, they show that any R.E. set is the countable union of basic semialgebraic sets. Since a basic semi-algebraic set has a finite number of connected components, any R.E. set has a countable number of connected components. In the same paper, they use this proposition to prove that most Julia sets are undecidable. Here we have used similar arguments, in particular this proposition.

3. Proof of the Theorem

Proof of Theorem A. First, we consider the set $\exp^+(Q)$ of parameters of the quadratic family with positive Lyapounov exponents. By a remarkable result of Benedicks and Carleson (see Theorem 2.1 above), $\exp^+(Q)$ has positive Lebesgue measure. However, if $a \in \exp^+(Q)$, then Q_a does not have attracting periodic orbits (see [**7**]) and, by Theorem 2.4 above, the set of the parameters a such that Q_a admits an attracting periodic orbit (i.e., Q_a is hyperbolic) is an open dense set. In particular, $\text{int}(\exp^+(Q)) = \emptyset$. Thus, $\exp^+(Q)$ has an uncountable number of connected components. But, Proposition 2.5 says that any decidable set has a countable number of connected components. So, $\exp^+(Q)$ is not a decidable set.

Second, we consider the set $\exp^+(H)$. Suppose that $\exp^+(H)$ is decidable and we denote by M' the machine whose "halting" set (see Section 2) is $\exp^+(H)$. We can define a new machine M as follows. The inputs are parameters $a \in (0, 2]$. Given such an a, we define the input $(a, 0)$ for the machine M' and, by hypothesis, M' outputs either 1 (yes) if $(a, 0) \in \exp^+(H)$ or 0 (no) if $(a, 0) \notin \exp^+(H)$. Since, for $b = 0$, $H_{a,b}(x, 0) = H_{a,0}(x, 0) = (1 - ax^2, 0)$ the machine M, in fact, "decides" $\exp^+(Q)$, a contradiction with the previous paragraph. This concludes the proof for these sets.

By Jakobson's theorem (see Theorem 2.2), $\text{srb}(Q)$ has positive Lebesgue measure. Since the existence of absolutely continuous invariant measures is an obstruction to the hyperbolicity (i.e., attracting periodic orbits), $\text{srb}(Q)$ has an empty interior. In particular, it is sufficient to make a *mutatis mutandis* argument to conclude the proof for the sets $\text{srb}(Q)$ and $\text{srb}(H)$.

From the definition of decidable sets, it is easy to see that intervals are decidable. Now, Theorem 2.3 says that the topological entropy of the quadratic family is monotone. In particular, $\text{ent}^+(Q)$ is an interval. This concludes the proof of the theorem. □

Remark 3.1. In fact, if there exists a periodic orbit for $Q_a(x)$ with period s, and s is not of the form $s = 2^n$, then the topological entropy is positive. Therefore, following the period doubling cascade we can find the number $c = \sup\{a : Q_a \text{has zero topological entropy}\}$.

4. Some Questions

The fact that positive topological entropy is a decidable property in one dimension, does not permit us to directly reduce the two-dimensional case for the previous case. So, the following very interesting question remains open:

Question 1. Is the set $\{h_{\text{top}}(H_{a,b}) > 0\}$ undecidable?

In the same line we have the following question due to Milnor [**12**]:

Question 2. Given an explicit dynamical system (e.g., quadratic family or Hénon map) and $\varepsilon > 0$ is it possible to compute the topological entropy with a maximum error of ε?

An instructive model to explore the complexity of the Hénon map can be a piecewise linear (possible discontinuous) map on the interval with two parameters. In this case the topological entropy may be calculated using the approach by Widodo [**17**].

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